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Free lunch and arbitrage possibilities in a financial market model with an insider

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Abstract

We consider financial market models based on Wiener space with two agents on different information levels: a regular agent whose information is contained in the natural filtration of the Wiener process W, and an insider who possesses some extra information from the beginning of the trading interval, given by a random variable L which contains information from the whole time interval. Our main concern are variables L describing the maximum of a pricing rule. Since for such L the conditional laws given by the smaller knowledge of the regular trader up to fixed times are not absolutely continuous with respect to the law of L, this class of examples cannot be treated by means of the enlargement of filtration techniques as applied so far. We therefore use elements of a Malliavin and Itô calculus for measure-valued random variables to give criteria for the preservation of the semimartingale property, the absolute continuity of the conditional laws of L with respect to its law, and the absence of arbitrage. The master example, given by $\sup_{t\in[0,1]} W_t$, preserves the semimartingale property, but allows for free lunch with vanishing risk quite generally. We deduce conditions on drift and volatility of price processes, under which we can construct explicit arbitrage strategies. © 2001 Elsevier Science B.V. All rights reserved.

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0. Introduction

Financial markets with economic agents possessing different information levels have been studied in a number of publications. There are essentially two main approaches.

Quite a number of different, mostly discrete models resulted predominantly from research in economics oriented papers (see O'Hara, 1995). Kyle (1985) investigates a

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time-discrete auction trading model with three agents: a market maker, a noise trader, and a risk neutral insider, to whom the final (Gaussian) value of one stock is known in advance. The existence of a unique equilibrium, shown by Kyle in this case, was extended by Back (1992,1993) to the time-continuous setting. Thereby the law of the insider's additional information was also considerably generalized. In these models the insider's actions may have an effect on the pricing rules. For more information on this and related classes of models and important techniques used in this area such as control theory, Malliavin's calculus and backwards stochastic differential equations, see for example, Karatzas and Ocone (1991), El Karoui and Quenez (1997), Cho and El Karoui (1998), and the recent thesis of Wu (1999).

The approach from the point of view of martingale theory, which we shall take in this paper, originated in the conceptual paper by Duffie and Huang (1986). Still closer to the setting of this paper are Karatzas and Pikovsky (1996), and Pikovsky (1999). They study a continuous-time model on a Wiener space, in which the insider possesses some extra information stored in a random variable L from the beginning of the trading interval, not available to the regular agent. They discuss questions like the additional utility of the insider with respect to particular utility functions, and martingale representation properties in the insider's filtration, thereby introducing the powerful technique of *grossissement de filtrations* to this economical context. While their L was kept within the Gaussian domain, the use of Malliavin's calculus in problems related to the enlargement of filtrations (see Imkeller, 1996,1997), eventually led to more complex additional information variables on Wiener space and beyond. The method of grossissement de filtrations was developed in a series of deep works, e.g. Yor (1985a,b,c,d), Jeulin (1980), Jacod (1985). Observations making martingale representation techniques in enlarged filtrations more easily accessible and giving a criterion for the absence of arbitrage in the insider model were made in Föllmer and Imkeller (1993). In Grorud and Pontier (1998), and Denis et al. (1998) Malliavin's calculus resp. the abstract theory of Dirichlet forms were correspondingly used to study on stochastic bases with increasing complexity – Wiener and Wiener-Poisson – admissible hedging strategies for insiders. Criteria for optimality were given. Work on the question, how an insider can be detected from his actions by statistical testing was begun in Grorud and Pontier (1998), and continued in Grorud and Pontier (1997). Techniques of enlargement of filtrations are by no means restricted to special stochastic bases. Their working area is quite general semimartingale theory. This fact was beautifully underlined in the thesis by Amendinger (1999), in which many results about martingale representations and utility optimization of an insider obtained a natural formulation in a rather general framework. In particular, in the case of logarithmic utility functions, the expected additional utility of the insider was identified in Amendinger et al. (1998) with the entropy of the law of the initially known additional information L.

Though a much more general setting had been studied in Jeulin (1980), in all the applications of the techniques of *initial* enlargement of filtrations to problems of insider trading known so far, a technical hypothesis made by Jacod (1985) plays a crucial role. Let L denote the extra information of the insider known from start. While the regular trader's evolution of knowledge is described by a filtration $\boldsymbol{F} = (\mathscr{F}_t)_{t\in[0,1]}$, L is just

$\mathscr{F}_1$-measurable, so that the evolution of information viewed by the insider, given by $\mathscr{G}_t = \bigcap_{s>t} \mathscr{F}_s \vee \sigma(L)$ augmented by the P-zero sets, $t \in [0,1]$, is essentially richer. The additional information may even destroy one of the usual requirements needed to be able to work with stochastic analysis, the *stochastic integrator property* with respect to the bigger filtration of martingales of the smaller filtration such as W or price processes S which are Ito semimartingales of W. Hence, if one wants to retain e.g. the linearity of the map $\theta \mapsto \int \theta \, \mathrm{d}S$ associating gains from trade to a trading strategy θ, one has to impose additional conditions on the insider's extra information. The crucial property to be fulfilled, Jacod's (1985) hypothesis (H′), is sufficient for the preservation of the semimartingale (integrator) property when passing to the filtration $\boldsymbol{G} = (\mathscr{G}_t)_{t \in [0,1]}$. It states that the regular conditional law of L given $\mathscr{F}_t$ is P-a.s. absolutely continuous with respect to the law of L as a common reference measure for all $t \in [0,1]$. Unfortunately, very interesting extra informations such as the maximum of a stock price over the trading interval, which could for example be modeled by $L = \sup_{t \in [0,1]} W_t$ for a Wiener process W, do not satisfy (H′).

One of the main objectives of this paper is to remove this difficulty. To generalize the hypothesis (H′), one could work well in the very general framework of Jeulin (1980). To have concise and more transparent expressions for the decompositions of semimartingales in the larger filtration for the purposes of this paper, we chose the framework of Malliavin's calculus on canonical Wiener space. In this context, our main observation generalizing Jacod's hypothesis can be paraphrased as follows. Let D denote Malliavin's gradient, and let $P_s(.,\mathrm{d}x)$ be the regular conditional law of L given $\mathscr{F}_s$, $s \in [0,1]$. The pseudodrift generated by the extra information, to be subtracted from an $\boldsymbol{F}$-Wiener process to create a $\boldsymbol{G}$-Wiener process is given by the Radon–Nikodym derivative of $D_s P_s(.,\mathrm{d}x)$ with respect to $P_s(.,\mathrm{d}x)$, evaluated at L. In case (H′) is fulfilled and $p_s(.,x)$ denotes a density of $P_s(.,\mathrm{d}x)$ with respect to the law of L, this quantity boils down to the quantity appearing in Jacod (1985), or the trace of a *logarithmic Malliavin derivative* of $p_s(.,L)$ appearing in Imkeller (1996), or Grorud and Pontier (1998).

So the technical basis we choose for this paper is a calculus within which the quantity $D_s P_s(.,\mathrm{d}x)$ becomes meaningful as a random measure. This leads us to elements of a measure-valued Malliavin calculus, developed in the appendix.

The crucial representation of the measure-valued martingale $P_s(.,\mathrm{d}x)$, $s \in [0,1]$, in terms of a generalized Clark–Ocone formula figuring a stochastic integral of the reasonably understood quantity $D_s P_s(.,\mathrm{d}x)$ is given in Section 1. In Section 2, we may then replace the criterion of Jacod (H′) by the following more general and natural one in the setting of Wiener space:

$$\text{(AC)} \quad D_s P_s(.,\mathrm{d}x) \text{ is absolutely continuous with respect to } P_s(.,\mathrm{d}x)$$
$$P\text{-a.s.} \quad \text{for } s \in [0,1].$$

This condition is in particular fulfilled for our master example, the maximum L of the Wiener process over the unit interval. Let $g_s(.,L)$ denote the Radon–Nikodym derivative under (AC), taken at L. We first show in Theorem 2.1, that the semimartingale property is preserved when passing from $\boldsymbol{F}$ to $\boldsymbol{G}$, if $g(.,L)$ is P-a.s. integrable over $[0,1]$. If

$g(.,L)$ is even square integrable P-a.s., we are able to see in the main result of the section, Theorem 2.2, that (H′) is regained, at the same time as the finiteness of the relative entropy of the regular conditional laws with respect to the law of L. An example shows that just square integrability of $g(.,L)$ is not enough to obtain equivalence of the regular conditional laws and the law of L. For this purpose, one needs stronger integrability conditions. We show in the final Theorem 2.3 of Section 2 that an exponential integrability of the Novikov type is sufficient. We finally prove that for our master example, $g(.,L)$ is not square integrable on a set of positive probability. Hereby a Bes(3) process will make its first crucial appearance. It was observed already in Delbaen and Schachermayer (1995a) to lead to arbitrage opportunities in a quite different setting (see also Karatzas and Shreve, 1998).

According to a result of Amendinger (1999) the equivalence of regular conditional laws and the law of L implies that the insider model is arbitrage free. In the light of this result, the observation made at the end of Section 2 leads us in Section 3 to look for free lunch or even arbitrage opportunities in case $L = \sup_{t\in[0,1]} W_t$ in simple one-dimensional models of security markets. In Theorem 3.1, we show along beaten paths that the lacking square integrability of $g(.,L)$ leads to *free lunches with vanishing risk* in the sense of Delbaen and Schachermayer (1994). In Theorems 3.2 and 3.3 we even single out two conditions on the drift b and volatility σ of the price process under which we are able to construct concrete arbitrage strategies. If $\sigma = 1$ and $b = \frac{1}{2}$, then knowing L in advance just means to know the maximal stock price in advance. The first strategy, which works if b/σ is bounded below, is rather obvious, and essentially tells the insider to invest as long as W is below its maximum, and to stop at the moment it is reached. The second one is less obvious. It applies if the positive part of b/σ is p-integrable for some $p > 2$. It takes advantage of a very subtle observation the insider can make due to the fact that he knows the maximum of W. After running through the maximum at time τ, W behaves locally as the negative of a Bes(3) process. Therefore, it decays essentially stronger than the drift in a small random interval after τ, the upper end of which is given by an insider stopping time μ. The insider then simply has to sell stocks at time τ and stop selling at time μ to exercise arbitrage. This strategy is in particular applicable if the drift b is continuous. It would be more realistic to suppose that the additional information of the insider consists in knowing just when the maximum of W or more generally S appears, and not how high it is, i.e. to take $L=\tau\wedge 1$. However, to deal with S instead of W adds considerably to the technical complexity of the mathematical presentation, and random times such as τ fit better into a framework where *progressive enlargements* are treated. See Jeanblanc and Rutkowski (1999) for a class of financial problems where this type of enlargements enters the scene, and Yor (1997) for a theoretical background. These interesting subjects will be dealt with in a forthcoming paper.

The extension of the results of this paper to the multi-dimensional setting should not pose essentially new problems. To keep the notational level low and transparence high, we stuck to the one-dimensional framework. It should equally be possible to dispense with the particularities of the Wiener space setting, and pass to general semimartingale theory.

1. Stochastic integral representations of conditional laws

Our basic probability space is the one-dimensional canonical Wiener space $(\Omega, \mathscr{F}, P)$, equipped with the canonical Wiener process $W = (W_t)_{t \in [0,1]}$. More precisely, $\Omega = C([0,1]; \mathbb{R})$ is the set of continuous functions on $[0,1]$ starting at 0, $\mathscr{F}$ the σ-algebra of Borel sets with respect to uniform convergence on $[0,1]$, P Wiener measure and W the coordinate process. The natural filtration $(\mathscr{F}_t)_{t \in [0,1]}$ of W is assumed to be completed by the sets of P-measure 0.

Guided by our prototypical example $L = \sup_{0 \leqslant t \leqslant 1} W_t$, in this section we will give integral representations of the conditional densities of random variables L with respect to the σ-algebras $\mathscr{F}_t$, $t \in [0,1]$, of the small filtration. For the more technical basic facts of measure-valued Malliavin calculus we refer the reader to the appendix. Let L be an $\mathscr{F}_1$-measurable random variable, and $P_t(.,\mathrm{d}x)$ a version of the regular conditional law of L given $\mathscr{F}_t$, $t \in [0,1]$. We know that the process $P_t(.,\mathrm{d}x)$, $t \in [0,1]$, is a measure valued martingale: for any $f \in C_{\mathrm{b}}(\mathbb{R})$, the process $\langle P_t(.,\mathrm{d}x), f\rangle$, $t \in [0,1]$, is a real valued continuous martingale which, provided L is smooth enough in the sense of Malliavin's calculus, can be represented by the formula

$$\langle P_t(.,\mathrm{d}x), f\rangle = \langle P_0(.,\mathrm{d}x), f\rangle + \int_0^t E(D_s\langle P_{s^+}(.,\mathrm{d}x), f\rangle | \mathscr{F}_s)\,\mathrm{d}W_s$$

(see Imkeller, 1996 for the setting, where all measures are absolutely continuous with respect to a joint reference measure). As follows from the martingale representation theorem in the Wiener filtration, in order to be able to write the stochastic integral in this formula, one of course does not need Malliavin differentiability of $P_t(.,\mathrm{d}x)$ on the whole interval $[0,t]$, but just the existence of a well-behaved *trace-type* object $D_tP_{t^+}(.,\mathrm{d}x) = \lim_{s\downarrow t} D_tP_s(.,\mathrm{d}x)$ in the sense of weak $*$ convergence in $L^2(\Omega \times [0,1]^2)$. Not to restrict generality too much from the start, we shall work with smooth approximations of $P_t(.,\mathrm{d}x)$ and take limits only for the trace-type objects.

Let $L \in \boldsymbol{D}_{1,2}$, and N an additional $\mathrm{N}(0,1)$-variable on our probability space which is independent of $\mathscr{F}_1$. For $\varepsilon > 0$, let

$$L_\varepsilon = L + \sqrt{\varepsilon}N,$$

and $P_t^\varepsilon(.,\mathrm{d}x)$ a version of the regular conditional law of L_ε given $\mathscr{F}_t$, $0 \leqslant t \leqslant 1$. In this section we shall work under the hypothesis

$$\text{(H)} \quad |D_tL| \leqslant M, \quad 0 \leqslant t \leqslant 1$$

for some random variable M the maximal function of which is p-integrable for any $p \geqslant 1$, i.e. $M^* = \sup_{0 \leqslant t \leqslant 1} |E(M|\mathscr{F}_t)| \in L_p$. Denote by p_ε the probability density of $\sqrt{\varepsilon}N$. Then for $f \in C_{\mathrm{b}}(\mathbb{R})$ we have

$$\begin{aligned}
\langle P_t^\varepsilon(.,\mathrm{d}x), f\rangle &= E(f(L_\varepsilon)|\mathscr{F}_t) \\
&= E\left(\int_{\mathbb{R}} p_\varepsilon(y - L) f(y)\,\mathrm{d}y | \mathscr{F}_t\right) \\
&= \int_{\mathbb{R}} E(p_\varepsilon(y-L)|\mathscr{F}_t) f(y)\,\mathrm{d}y \\
&= \langle E(p_\varepsilon(y-L)|\mathscr{F}_t)\,\mathrm{d}y, f\rangle.
\end{aligned}$$

Moreover, for $0 \leqslant r \leqslant t,\ y \in \mathbb{R}$,

$$D_r E(p_\varepsilon(y-L)|\mathscr{F}_t) = E(D_r\, p_\varepsilon(y-L)|\mathscr{F}_t)$$
$$= E\left(D_r L\,\frac{L-y}{\varepsilon}\, p_\varepsilon(y-L)|\mathscr{F}_t\right).$$

(H) allows to apply the Clark–Ocone formula, and we obtain for $f \in C_{\mathrm{b}}(\mathbb{R}),\ t\in[0,1]$

$$\langle P_t^\varepsilon(.,\mathrm{d}x), f\rangle = \langle P_0^\varepsilon(.,\mathrm{d}x), f\rangle + \int_0^t \left\langle E\left(D_s L\,\frac{L-y}{\varepsilon}\, p_\varepsilon(y-L)\,|\mathscr{F}_s\right)\mathrm{d}y, f\right\rangle \mathrm{d}W_s.$$

Now define

$$h_s^\varepsilon(.,x) = E\left(D_s L\,\frac{L-x}{\varepsilon}\, p_\varepsilon(x-L)\,|\mathscr{F}_s\right),$$

$$k_s^\varepsilon(.,\mathrm{d}x) = h_s^\varepsilon(.,x)\,\mathrm{d}x,$$

$\varepsilon>0,\ s\in[0,1],\ x\in\mathbb{R}$. Then, due to the boundedness of $x \mapsto x p_\varepsilon(x)$ and (H), we obtain a constant c_ε such that

$$|h_s^\varepsilon(.,x)| \leqslant c_\varepsilon\ E(|D_s L|\,|\mathscr{F}_s) \leqslant c_\varepsilon\ E(M|\mathscr{F}_s) \leqslant c_\varepsilon M^*,$$

$\varepsilon>0,\ s\in[0,1],\ x\in\mathbb{R}$. Therefore for $f\in C_{\mathrm{b}}(\mathbb{R}),\ p\geqslant 1$

$$E\left(\left[\int_0^1 \langle k_s^\varepsilon(.,\mathrm{d}x), f\rangle^2\,\mathrm{d}s\right]^{p/2}\right) \leqslant E\left(\left[\int_0^1 |k_s^\varepsilon(.,\mathrm{d}x)|^2\,\mathrm{d}s\right]^{p/2}\right)\|f\|^p$$
$$\leqslant c_\varepsilon^p E((M^*)^p)\,\|f\|^p.$$

Hence we have

$$\sup_{f\in C_{\mathrm{b}}(\mathbb{R}),\,\|f\|\leqslant 1} E\left(\left[\int_0^1 \langle k_s^\varepsilon(.,\mathrm{d}x), f\rangle^2\,\mathrm{d}s\right]^{p/2}\right) < \infty. \tag{1}$$

Since moreover for any $\varepsilon>0,\ t\in[0,1], p\geqslant 1$ by an easier argument

$$\sup_{f\in C_{\mathrm{b}}(\mathbb{R}),\,\|f\|\leqslant 1} E(\langle P_t^\varepsilon(.,\mathrm{d}x), f\rangle^p) < \infty, \tag{2}$$

Proposition A.1 allows us to write

$$P_t^\varepsilon(.,\mathrm{d}x) = P_0^\varepsilon(.,\mathrm{d}x) + \int_0^t k_s^\varepsilon(.,\mathrm{d}x)\,\mathrm{d}W_s \tag{3}$$

for $\varepsilon>0,\ t\in[0,1]$. We aim at passing to the limit $\varepsilon\to 0$ in (3), thereby keeping track of the convergence of the measure-valued processes $k_t^\varepsilon(.,\mathrm{d}x),\ t\in[0,1]$. To gain a better insight into which aspects are essential, let us first treat our prototypical example. Our treatment shares some aspects with Jeulin's (1980), but in contrast is based on Malliavin's calculus.

Example 1. Let $L=\sup_{0\leqslant t\leqslant 1} W_t$, N be a $\mathrm{N}(0,1)$-variable independent of $\mathscr{F}_1$. For $t\in[0,1],\ 0\leqslant h\leqslant 1-t$ denote by $S_t=\sup_{0\leqslant s\leqslant t} W_s$, and $\beta_h=\sup_{0\leqslant k\leqslant h}(W_{k+t}-W_t)$. Then we have

$$L = S_t \vee (W_t+\beta_{1-t}),\quad 0\leqslant t\leqslant 1,$$

with β_{1-t} independent of W_t, S_t. Denote by f_{1-t} the density of the law of β_{1-t}. We have

$$f_{1-t}(z) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{1-t}} \exp\left(-\frac{1}{2(1-t)} z^2\right) 1_{[0,\infty[}(z),$$

$z \in \mathbb{R}$. It is well known (see Nualart and Vives, 1988) that $S_t \in \boldsymbol{D}_{1,p}$ for all $0 \leqslant t \leqslant 1$, and, if τ_t denotes the (P-a.s. uniquely defined) random time at which W takes its maximum on the interval $[0,t]$, we have

$$D_s S_t = 1_{[0,\tau_t]}(s), \quad s \in [0,1]. \tag{4}$$

In particular, if we omit the subscript for $t = 1$, we have

$$D_s L = 1_{[0,\tau]}(s), \quad s \in [0,1]. \tag{5}$$

Hence for $t \in [0,1]$, $\varepsilon > 0$, $y \in \mathbb{R}$ we obtain

$$\begin{aligned} E(p_\varepsilon(y-L)|\mathscr{F}_t) &= E(p_\varepsilon(y - S_t)\, 1_{\{S_t > W_t + \beta_{1-t}\}}|\mathscr{F}_t) \\ &\quad + E(p_\varepsilon(y - (W_t + \beta_{1-t}))\, 1_{\{S_t < W_t + \beta_{1-t}\}}|\mathscr{F}_t) \\ &= p_\varepsilon(y - S_t) \int_0^{S_t - W_t} f_{1-t}(y)\, \mathrm{d}y \\ &\quad + \int_{-\infty}^{y - S_t} p_\varepsilon(v)\, f_{1-t}(y - W_t - v)\, \mathrm{d}v. \end{aligned}$$

Hence for $r \in [0,t]$

$$\begin{aligned} D_r E(p_\varepsilon(y-L)|\mathscr{F}_t) &= \frac{S_t - y}{\varepsilon} D_r S_t\, p_\varepsilon(y - S_t) \int_0^{S_t - W_t} f_{1-t}(y)\, \mathrm{d}y \\ &\quad + p_\varepsilon(y - S_t) f_{1-t}(S_t - W_t) D_r(S_t - W_t) \\ &\quad + p_\varepsilon(y - S_t) f_{1-t}(S_t - W_t)(-D_r S_t) \\ &\quad - \int_{-\infty}^{y - S_t} p_\varepsilon(v)\, \frac{y - W_t - v}{1-t} f_{1-t}(y - W_t - v)\, \mathrm{d}v. \end{aligned}$$

This in turn implies that with the above notation for $t \in [0,1]$, $x \in \mathbb{R}$

$$\begin{aligned} h_t^\varepsilon(.,x) &= D_t E(p_\varepsilon(x - L)|\mathscr{F}_{t^+}) \\ &= -p_\varepsilon(x - S_t)\, f_{1-t}(S_t - W_t) \\ &\quad - \int_{-\infty}^{x - S_t} p_\varepsilon(v)\, \frac{x - W_t - v}{1-t} f_{1-t}(x - W_t - v)\, \mathrm{d}v. \end{aligned} \tag{6}$$

Now what happens as $\varepsilon \to 0$? Let $f \in C_\mathrm{b}(\mathbb{R})$. Then for any $t \in [0,1]$ pointwise

$$\langle [p_\varepsilon(x - S_t)\, \mathrm{d}x - \delta_{S_t}(\mathrm{d}x)], f\rangle = \int_\mathbb{R} p_\varepsilon(x - S_t)\, [f(x) - f(S_t)]\, \mathrm{d}x \to 0, \tag{7}$$

as well as

$$\begin{aligned} &\int_\mathbb{R} \int_{-\infty}^{x - S_t} p_\varepsilon(v) \left[\frac{x - W_t - v}{1-t} f_{1-t}(x - W_t - v)\right. \\ &\quad \left. - 1_{[S_t,\infty[}(x)\, \frac{x - W_t}{1-t}\, f_{1-t}(x - W_t)\right] \mathrm{d}v\, f(x)\, \mathrm{d}x \to 0 \end{aligned} \tag{8}$$

as $\varepsilon \to 0$. Since $\sup_{y\in\mathbb{R}}|yf_{1-t}(y)| < \infty$, this convergence is bounded by constants depending only on t and f, and the constants are bounded on intervals $[0,t]$ for $t<1$. Hence dominated convergence shows

$$E\left(\int_0^t \langle [k_s^\varepsilon(.,\mathrm{d}x) - k_s(.,\mathrm{d}x)], f\rangle^2 \,\mathrm{d}s \to 0 \tag{9}$$

as $\varepsilon \to 0$, for $0 \leqslant t < 1$, where

$$k_s(.,\mathrm{d}x) = -\delta_{S_t}(\mathrm{d}x)\, f_{1-t}(S_t - W_t) - 1_{[S_t,\infty[}(x)\,\frac{x-W_t}{1-t}\, f_{1-t}(x-W_t)\,\mathrm{d}x. \tag{10}$$

The following convergence is obvious, due to continuity. So for all $0\leqslant t\leqslant 1$, $f\in C_{\mathrm{b}}(\mathbb{R})$, we have

$$\begin{aligned}\langle P_t^\varepsilon(.,\mathrm{d}x), f\rangle &= E(f(L_\varepsilon)|\mathscr{F}_t)\\ &\to E(f(L)|\mathscr{F}_t)\\ &= \langle P_t(.,\mathrm{d}x, f\rangle.\end{aligned}$$

Hence (10) yields the equation, valid for any $f\in C_{\mathrm{b}}(\mathbb{R})$, $t\in[0,1]$

$$\langle P_t(.,\mathrm{d}x), f\rangle - \langle P_0(.,\mathrm{d}x), f\rangle = \int_0^t \langle k_s(.,\mathrm{d}x), f\rangle\,\mathrm{d}W_s. \tag{11}$$

The $\boldsymbol{M}$-valued (for the notation see Appendix) process $k(.,\mathrm{d}x)$ even satisfies

$$E\left(\int_0^t |k_s(.,\mathrm{d}x)|^2\,\mathrm{d}s\right) < \infty, \quad 0\leqslant t<1. \tag{12}$$

Therefore Proposition A.4 immediately implies

Theorem 1.1. *Let*

$$k_t(.,\mathrm{d}x) = -\delta_{S_t}(\mathrm{d}x)\, f_{1-t}(S_t - W_t) - 1_{[S_t,\infty[}(x)\,\frac{x-W_t}{1-t}\, f_{1-t}(x-W_t)\,\mathrm{d}x,$$

$t\in[0,1]$. *Then for* $0\leqslant t\leqslant 1$ *we have*

$$P_t(.,\mathrm{d}x) = P_0(.,\mathrm{d}x) + \int_0^t k_s(.,\mathrm{d}x)\,\mathrm{d}W_s.$$

Hence for our main example there is an integral representation of the process of regular conditional densities of L.

We denote $k_t(.,\mathrm{d}x)$ also by $D_t P_{t^+}(.,\mathrm{d}x)$, $t\in[0,1]$.

We now return to the general setting. It is clear that we just have to follow the ideas needed in the treatment of Example 1 to obtain a more general result, which we also formulate in a weak version.

Theorem 1.2. *Suppose that there exists an* $\boldsymbol{M}$*-valued process* $k_t(.,\mathrm{d}x)$, $t\in[0,1]$ *such that for any* $t\in[0,1]$, $f\in C_{\mathrm{b}}(\mathbb{R})$ *we have*

$$E\left(\int_0^t \langle [k_s^\varepsilon(.,\mathrm{d}x) - k_s(.,\mathrm{d}x)], f\rangle^2 \,\mathrm{d}s \to 0 \tag{13}$$

as $\varepsilon \to 0$. *Then for any* $t \in [0,1]$, $f \in C_b(\mathbb{R})$

$$\langle P_t(.,\mathrm{d}x), f\rangle = \langle P_0(.,\mathrm{d}x), f\rangle + \int_0^t \langle k_s(.,\mathrm{d}x), f\rangle \,\mathrm{d}W_s.$$

If in addition

$$\sup_{f \in C_b(\mathbb{R}), \|f\| \leqslant 1} E\left(\int_0^t \langle k_s(.,\mathrm{d}x, f\rangle^2 \,\mathrm{d}s\right) < \infty, \tag{14}$$

then for any $t \in [0,1]$

$$P_t(.,\mathrm{d}x) = P_0(.,\mathrm{d}x) + \int_0^t k_s(.,\mathrm{d}x)\,\mathrm{d}W_s.$$

2. The semimartingale property and relative entropy of the conditional laws

Let us now ask the question under which conditions on L martingales in the small filtration remain semimartingales in the enlarged filtration. We shall formulate the question and our answer in the terminology of the preceding section. We shall show that in case $k_t(.,\mathrm{d}x) \ll P_t(.,\mathrm{d}x)$ P-a.s. for any $t \in [0,1]$, the answer can be given in terms of integrability properties of the density $g_t(.,x)$. We shall then proceed to formulate conditions on this density, under which the relative entropy of the conditional laws with respect to the law of L is finite and positive. As it turns out, this is closely related to the question whether the conditional laws are equivalent to the law of L. So, for the whole section, we shall work under the hypothesis

$$\text{(AC)} \quad k_t(.,\mathrm{d}x) \ll P_t(.,\mathrm{d}x) \quad P\text{-a.s. for } t \in [0,1].$$

Let $g_t(.,x)$, $x \in \mathbb{R}$, be a measurable density of $k_t(.,\mathrm{d}x)$ with respect to $P_t(.,\mathrm{d}x)$, $t \in [0,1]$. To fix the ideas, let us again first have a look at our main example.

Example 2. Let $L = \sup_{0 \leqslant t \leqslant 1} W_t$. Then in the notation of the preceding section for $\varepsilon > 0$, $t \in [0,1]$

$$\begin{aligned} P_t^\varepsilon(.,\mathrm{d}x) &= E(p_\varepsilon(x-L)|\mathscr{F}_t)\,\mathrm{d}x \\ &= \left[p_\varepsilon(x-S_t)\int_0^{S_t-W_t} f_{1-t}(y)\,\mathrm{d}y + \int_{-\infty}^{x-S_t} p_\varepsilon(v)\,f_{1-t}(x-W_t-v)\,\mathrm{d}v\right]\mathrm{d}x. \end{aligned}$$

Letting $\varepsilon \to 0$, we obtain

$$P_t(.,\mathrm{d}x) = \delta_{S_t}(\mathrm{d}x)\int_0^{S_t-W_t} f_{1-t}(y)\,\mathrm{d}y + 1_{[S_t,\infty[}(x)\,f_{1-t}(x-W_t)\,\mathrm{d}x.$$

Consequently, we see that (AC) is satisfied, and that the density is given by

$$g_t(.,x) = -\frac{f_{1-t}(S_t-W_t)}{\int_0^{S_t-W_t} f_{1-t}(y)\,\mathrm{d}y} 1_{\{S_t\}}(x) - 1_{[S_t,\infty[}(x)\,\frac{x-W_t}{1-t},$$

$t \in [0,t]$, $x \in \mathbb{R}$.

This formula has also been derived in Jeulin (1980), with a different approach. Jeulin (1980) also proves that $g_t(.,L)$, which is P-a.s. finite, serves as the density of the compensator of W in the enlarged filtration. Let us now show more generally that, provided $g_t(.,L)$ is well behaved, the semimartingale property is preserved when passing from the small to the enlarged filtration.

Theorem 2.1. *Suppose that* (AC) *is satisfied, and that*

$$\int_0^t |g_s(.,L)|\,\mathrm{d}s < \infty$$

P-a.s. for any $0 \leqslant t < 1$. *Then the process*

$$\tilde{W} = W - \int_0^{\cdot} g_s(.,L)\,\mathrm{d}s$$

is a $\mathscr{G}_t$*-Wiener process.*

Proof. Fix $0 \leqslant t < 1$. By localization, we may suppose that $\int_0^t |g_s(.,L)|\,\mathrm{d}s$ is bounded. Let $s_1, s_2 \in [0,t]$, $s_1 \leqslant s_2$, $F \in \mathscr{F}_{s_1}$, and $h \in C_\mathrm{b}(\mathbb{R})$. Then

$$\begin{aligned} E(1_F\, h(L)\,(W_{s_2} - W_{s_1})) &= E(1_F\, \langle P_{s_2}(.,\mathrm{d}x), h\rangle\,(W_{s_2} - W_{s_1})) \\ &= E\left(1_F \int_{s_1}^{s_2} \langle k_u(.,\mathrm{d}x), h\rangle\,\mathrm{d}u\right) \\ &= E\left(1_F \int_{s_1}^{s_2} \langle P_u(.,\mathrm{d}x), g_u(.,.)\,h\rangle\,\mathrm{d}u\right) \\ &= E\left(1_F\, h(L) \int_{s_1}^{s_2} g_u(.,L)\,\mathrm{d}u\right). \end{aligned}$$

A standard measure theoretic argument is now used to extend this equation to bounded and measurable h. Since sets of the form $1_{F\times\{L\in B\}}1_{]s_1,\,s_2]}$, $F \in \mathscr{F}_{s_1}$, $B \in \boldsymbol{B}$, generate the previsible sets for the filtration $(\mathscr{G}_t)_{t\in[0,1]}$, this implies the assertion. □

Theorem 2.1 states that the conservation of the semimartingale property is related to integrability of $g_t(.,L)$ on subintervals of $[0,1]$. We shall next investigate the square integrability of $g_t(.,L)$ on subintervals of $[0,1]$, and elaborate on the relationship with equivalence of the conditional laws of L given the small filtration and the law of L.

For this purpose fix $0 \leqslant t \leqslant 1$, and a nested 0-sequence of partitions $(B_i^n)_{i\in\mathbb{N}}$, $n \in \mathbb{N}$. More precisely, the partition is increasing with respect to inclusion, and the mesh tends to 0 as $n \to \infty$. For $n \in \mathbb{N}$, let $\mathscr{H}_n = \sigma(B_i^n\colon i \in \mathbb{N})$,

$$M^n(.,x) = \sum_{i\in\mathbb{N}} \frac{P_t(.,B_i^n)}{P_L(B_i^n)}\, 1_{B_i^n}(x),$$

$x \in \mathbb{R}$. Then M^n, $n \in \mathbb{N}$, is a martingale with respect to $(\mathscr{H}_n)_{n\in\mathbb{N}}$ and P_L on $\boldsymbol{B}$. We emphasize at this place that we are here and in the sequel referring to a spatial martingale property which is valid for any $\omega \in \Omega$ (P-a.s.). Now suppose that

$$E\left(\int_0^1 g_t(.,L)^2\,\mathrm{d}t\right) < \infty. \tag{15}$$

Note that (15) can be rephrased as

$$\int_0^1 E(\langle P_t(.,\mathrm{d}x), g_t^2(.,.)\rangle)\,\mathrm{d}t < \infty. \tag{16}$$

For $s\in[0,1]$ let next

$$S_s^n(.,x) = \sum_{i\in\mathbb{N}} \frac{D_s P_{s^+}(.,B_i^n)}{P_s(.,B_i^n)}\, 1_{B_i^n}(x),$$

$x\in\mathbb{R}$. Then, for the same reason as above, S_s^n, $n\in\mathbb{N}$, is a martingale with respect to $(\mathscr{H}_n)_{n\in\mathbb{N}}$ and $P_s(.,\mathrm{d}x)$, which is uniformly integrable and convergent to $g_s(.,.)$. We emphasize at this point that uniform integrability of the preceding and the two following martingales is due to the absolute continuity condition (AC) and (16). Moreover, $(S_s^n)^2$, $n\in\mathbb{N}$, is a submartingale, and we have

$$(S_s^n)^2 \leqslant \sum_{i\in\mathbb{N}} \frac{1}{P_s(.,B_i^n)} \int_{B_i^n} g_s^2(.,x)\,P_s(.,\mathrm{d}x)\,1_{B_i^n} = T_s^n,$$

$n\in\mathbb{N}$. But by (16), T_s^n, $n\in\mathbb{N}$, is a uniformly integrable martingale. Hence $(S_s^n)^2$, $n\in\mathbb{N}$, is a uniformly integrable non-negative submartingale which converges in $L^1(\mathbb{R},\boldsymbol{B},P_s(.,\mathrm{d}x))$ to $g_s^2(.,.)$. But also $\int_0^1 T_s^n(L)\,\mathrm{d}s$ is uniformly integrable for the measure P, and therefore

$$\int_0^1 (S_s^n(L))^2\,\mathrm{d}s \to \int_0^1 g_s^2(.,L)\,\mathrm{d}s \tag{17}$$

in $L^1(\Omega,\mathscr{F},P)$. Now note that

$$\begin{aligned}\sum_{i\in\mathbb{N}} \int_0^{\cdot} \frac{D_s P_{s^+}(.,B_i^n)}{P_s(.,B_i^n)}\,\mathrm{d}W_s\,1_{B_i^n}(L)\\ = \sum_{i\in\mathbb{N}} \int_0^{\cdot} \frac{D_s P_{s^+}(.,B_i^n)}{P_s(.,B_i^n)}\,\mathrm{d}\tilde{W}_s\,1_{B_i^n}(L) + \int_0^{\cdot} \frac{D_s P_{s^+}(.,B_i^n)}{P_s(.,B_i^n)} g_s(.,L)\,\mathrm{d}s\,1_{B_i^n}(L),\end{aligned}$$

with the Brownian motion $\tilde{W}$ for the filtration $(\mathscr{G}_t)_{t\in[0,1]}$ appearing in Theorem 2.1. Hence by standard martingale inequalities (this time in the temporal sense) and the above-stated convergence result we obtain

$$\sum_{i\in\mathbb{N}} \int_0^{\cdot} \frac{D_s P_{s^+}(.,B_i^n)}{P_s(.,B_i^n)}\,\mathrm{d}\tilde{W}_s\,1_{B_i^n}(L) \to \int_0^{\cdot} g_s(.,L)\,\mathrm{d}\tilde{W}_s$$

in $L^2(\Omega,\mathscr{F},P)$. We finally write Itô's formula for the processes $\ln P_t(.,B_i^n)$, which makes sense due to the above calculations. The result is

$$\begin{aligned}\sum_{i\in\mathbb{N}} \ln M^n(L)\,1_{B_i^n}(L) &= \sum_{i\in\mathbb{N}} \int_0^{\cdot} \frac{D_s P_{s^+}(.,B_i^n)}{P_s(.,B_i^n)}\,\mathrm{d}W_s\,1_{B_i^n}(L)\\ &\quad - \frac{1}{2}\int_0^{\cdot} \frac{D_s P_{s^+}(.,B_i^n)^2}{P_s(.,B_i^n)^2}\,\mathrm{d}s\,1_{B_i^n}(L).\end{aligned} \tag{18}$$

Hence the P-supermartingale $\ln M^n(L)$, $n\in\mathbb{N}$, converges in $L^1(\Omega,\mathscr{F},P)$. Since

$$E\left(\sum_{i\in\mathbb{N}} \ln M^n(L)\,1_{B_i^n}(L)\right) = E\left(\sum_{i\in\mathbb{N}} \ln \frac{P_t(.,B_i^n)}{P_L(B_i^n)} P_t(.,B_i^n)\right),$$

we obtain

$$\sum_{i\in\mathbb{N}} \ln M^n(L)\, 1_{B_i^n}(L) \to H(P_t(.,\mathrm{d}x)|P_L) \quad \text{in } L^1(\Omega,\mathscr{F},P),$$

where $H(P_t(.,\mathrm{d}x)|P_L)$ is the relative entropy of the conditional law $P_t(.,\mathrm{d}x)$ with respect to P_L. In particular, $H(P_t(.,\mathrm{d}x)|P_L)$ is P-a.s. finite. So we have finally proved

Theorem 2.2. *Suppose that* (AC) *is satisfied, and that*

$$E\left(\int_0^1 g_s^2(.,L)\,\mathrm{d}s\right) < \infty. \tag{19}$$

Then for $t\in[0,1]$

$$H(P_t(.,\mathrm{d}x)|P_L) < \infty \quad P\text{-a.s.},$$

and this random variable is integrable. In particular, $P_t(.,\mathrm{d}x) \ll P_L$.

The following example shows that under the hypothesis of Theorem 2.2 equivalence of $P_t(.,\mathrm{d}x)$ and P_L is too much to hope for, in general.

Example 3. Let

$$\sigma_1 = \inf\{t:\ W_t = 1\}, \quad \sigma = \sigma_1 \wedge 1, \quad A = \{\sigma = 1\}, \quad L = 1_A.$$

Then

$$\begin{aligned} P_t(.,\mathrm{d}x) &= P(L=1|\mathscr{F}_t)\,\delta_{\{1\}} + P(L=0|\mathscr{F}_t)\,\delta_{\{0\}} \\ &= P(\sigma=1|\mathscr{F}_t)\,\delta_{\{1\}} + (1-P(\sigma=1|\mathscr{F}_t))\,\delta_{\{0\}}. \end{aligned}$$

This expression can be explicitly given, by means of the following computation:

$$\begin{aligned} P(\sigma=1|\mathscr{F}_t) &= P(S_t \vee (W_t+\beta_{1-t}) < 1|\mathscr{F}_t) \\ &= 1_{\{S_t<1\}} P(\beta_{1-t} < 1-W_t|\mathscr{F}_t) \\ &= 1_{\{t<\sigma\}} \int_0^{1-W_t} f_{1-t}(y)\,\mathrm{d}y. \end{aligned} \tag{20}$$

Similarly,

$$\begin{aligned} P_L(\mathrm{d}x) &= P(L=1)\,\delta_{\{1\}} + P(L=0)\delta_{\{0\}} \\ &= \int_0^1 f_1(y)\,\mathrm{d}y\,\delta_{\{1\}} + \left(1-\int_0^1 f_1(y)\,\mathrm{d}y\right)\,\delta_{\{0\}}. \end{aligned}$$

Let

$$h_t(x) = \int_0^x f_{1-t}(y)\,\mathrm{d}y.$$

It follows from (20) that $1_{\{t<\sigma\}}h_t(1-W_t)$ is a martingale and so we have

$$1_{\{t<\sigma\}}h_t(1-W_t) = -1_{\{t<\sigma\}}\int_0^{t\wedge\sigma} h_s'(1-W_s)\,\mathrm{d}W_s,$$

where h_s' denotes the derivative of h_s with respect to x. Hence

$$\begin{aligned} P_t(.,\{1\}) &= -1_{\{t<\sigma\}}\int_0^t h_s'(1-W_s)\,\mathrm{d}W_s,\\ k_t(.,\{1\}) &= -1_{\{t<\sigma\}}h_t'(1-W_t),\\ g_t(.,1) &= -1_{\{t<\sigma\}}\frac{h_t'(1-W_t)}{h_t(1-W_t)}. \end{aligned}$$

It is easy to see that

$$g_t(.,1)^2 = 1_{\{t<\sigma\}}\frac{f_{1-t}(1-W_t)^2}{(\int_0^{1-W_t} f_{1-t}(y)\,\mathrm{d}y)^2} = \frac{1}{|1-W_t|^2}\,1_{\{t<\sigma\}} + \mathrm{O}(t).$$

Let us denote the density function of σ_1 by g. Since the Brownian motion before reaching level 1 behaves like a Bes (3) process ρ (see Revuz and Yor, 1999), we can conclude that

$$\begin{aligned} E\left(\int_0^1 g_s^2(.,L)\,1_{\{L=1\}}\,\mathrm{d}s\right) &= \int_0^1\int_0^\infty E(g_s^2(.,1)|\sigma_1 = 1+h)g(1+h)\,\mathrm{d}h\,\mathrm{d}s\\ &= \int_0^1\int_0^\infty E\left(\frac{1}{|\rho_{h+1-s}|^2}\right)g(1+h)\,\mathrm{d}h\,\mathrm{d}s + \mathrm{O}(1)\\ &= \int_0^1\int_0^\infty \frac{1}{h+1-s}\,g(1+h)\,\mathrm{d}h\,\mathrm{d}s + \mathrm{O}(1) < \infty. \end{aligned}$$

One can prove in the same way that $E(\int_0^1 g_s^2(.,L)\,1_{\{L=0\}}\,\mathrm{d}s) < \infty$. Thus (19) is satisfied, while $P_t(.,\{1\}) = 0$ for $\sigma < t$ and $P_L(\{1\}) \neq 0$.

For the equivalence of the conditional laws and the law of L exponential integrability in the form of the following Novikov-type condition is sufficient.

Theorem 2.3. *If*

$$E\left(\exp\left(\frac{1}{2}\int_0^1 g_s^2(.,L)\,\mathrm{d}s\right)\right) < \infty, \tag{21}$$

then $P_t(.,\mathrm{d}x)$ is equivalent to P_L a.s. for each $0\leqslant t\leqslant 1$.

Proof. First of all, (AC) and (21) allow us to apply Theorem 2.1 directly. Hence

$$\tilde W = W - \int_0^\cdot g_s(.,L)\,\mathrm{d}s$$

is a $\boldsymbol{G}$-Wiener process with respect to the law P. Eq. (21) in addition permits to define the Doléans exponential $\boldsymbol{G}$-martingale

$$\mathrm{d}Z_t = -Z_t g_t(.,L)\,\mathrm{d}\tilde W_t,\quad Z_0 = 1,\ t\in[0,1].$$

Therefore, we may also define the equivalent probability measure on $\mathscr{G}_t$:

$$Q|_{\mathscr{G}_t} = Z_t\cdot P,\quad t\in[0,1].$$

Now Girsanov's theorem gives that

$$W = \tilde W + \int_0^\cdot g_s(.,L)\,\mathrm{d}s$$

is a $\boldsymbol{G}$-Wiener process with respect to the law Q. Moreover, W is also $\boldsymbol{F}$-adapted with independent increments, and $W_t - W_0$, $t \in [0,1]$, is independent of $\mathscr{G}_0$ since for all $t \in [0,1]$, $\mathscr{F}_t$ and $\mathscr{G}_0$ are independent under Q (see Föllmer and Imkeller, 1993). Note that this is hypothesis (H3) of Grorud and Pontier (1998) which also characterizes the equivalence of P^L and $P_t^L(.,\mathrm{d}x)$. $\square$

Remark. The hypothesis (21) of Theorem 2.3 could be relaxed to the condition $E_P(Z_t) = 1$ for all $t \in [0,1]$, if Z is the exponential martingale appearing in the proof.

If $P_t(.,\mathrm{d}x)$ is P-a.s. equivalent with P_L, the *insider model*, i.e. the model based on the information flow $(\mathscr{G}_t)_{t \in [0,1]}$ allows for no *arbitrage*. This is shown in the thesis of Amendinger (1999). See also Amendinger et al. (1998).

We now show that the situation in our main example is quite different, and suggests that there are arbitrage opportunities.

Example 4. Let $L = \sup_{0 \leqslant t \leqslant 1} W_t$, and fix $0 < T \leqslant 1$. Recall the notation of the above treatments of this example. The random time τ at which the maximum L is taken, is known to have a absolutely continuous law on $[0,1[$, with one obvious atom on $\{1\}$. By the formula representing $g_t(.,L)$ we have

$$\int_0^T g_t^2(.,L)\,\mathrm{d}s \geqslant \int_\tau^T \left(\frac{f_{1-t}(L-W_t)}{\int_0^{L-W_t} f_{1-t}(y)\,\mathrm{d}y} \right)^2 \mathrm{d}t.$$

Now let

$$\tau_0 = \inf\{t \geqslant \tau \colon W_t = 0\}.$$

Then we may estimate further

$$\int_0^T g_t^2(.,L)\,\mathrm{d}t \geqslant \int_\tau^{\tau_0 \wedge T} \left(\frac{f_{1-t}(L-W_t)}{\int_0^{L-W_t} f_{1-t}(y)\,\mathrm{d}y} \right)^2 \mathrm{d}t.$$

Now by Revuz and Yor (1999, Proposition VI.3.13, p. 238), the Brownian motion between τ and τ_0 has the law of a Bes (3). So, conditionally on the event $T/4 < \tau < T/2$, which has positive probability, we may estimate the law of the above lower bound by the law of the random variable

$$\int_{T/4}^{\sigma} \left(\frac{f_{1-t}(\rho_t)}{\int_0^{\rho_t} f_{1-t}(y)\,\mathrm{d}y} \right)^2 \mathrm{d}t, \tag{22}$$

where ρ_t, $0 \leqslant t \leqslant 1$, is a Bes (3), and σ the first time it hits $T/4$. By the well-known path properties of the Bessel process, we obtain that (22) is infinite P-a.s., and hence that

$$\int_0^T g_s^2(.,L)\,\mathrm{d}s = \infty \tag{23}$$

on a set of positive probability.

3. Arbitrage possibilities

In this section we shall see that our prototypical example provides possibilities of arbitrage. We shall use the terminology of Karatzas and Shreve (1998) and Delbaen and Schachermayer (1994). The financial markets considered will be based on the Brownian motion in its augmented natural filtration, and will be described by simple one-dimensional models. At this place we should mention that to really create an arbitrage we tacitly assume the presence of another security (the *numeraire*) not explicitly appearing in our description, whose price is set to unity. Our financial market model (b,σ) thus consists of a progressively measurable mean rate of return process b which satisfies $\int_0^1 |b_t|\,\mathrm{d}t < \infty$ P-a.s. and of a progressively measurable volatility process σ satisfying $\int_0^1 \sigma_t^2\,\mathrm{d}t < \infty$, $\sigma^2 > 0$ P-a.s. They determine a (stock) price process given by

$$\frac{\mathrm{d}S_t}{S_t} = \int_0^t b_u\,\mathrm{d}u + \int_0^t \sigma_u\,\mathrm{d}W_u.$$

For convenience, we let $S_0 = 1$. A progressively measurable process π is called a portfolio process if

$$\int_0^1 |\pi_t\, b_t|\,\mathrm{d}t < \infty \quad P\text{-a.s.}$$

and

$$\int_0^1 |\pi_t\,\sigma_t|^2\,\mathrm{d}t < \infty \quad P\text{-a.s.}$$

The excess yield process R and gains process G are given by the formulas

$$\mathrm{d}R_t = \frac{\mathrm{d}S_t}{S_t} \tag{24}$$

and

$$G_t = \int_0^t \pi_u\,\mathrm{d}R_u, \tag{25}$$

$0 \leqslant t \leqslant 1$. The portfolio process is said to be *tame* if there is some constant $c \in \mathbb{R}$ such that $G_t \geqslant c$ for all $0 \leqslant t \leqslant 1$. Let

$$K_0 = \left\{ G_1 = \int_0^1 \pi_s\,\mathrm{d}R_s\colon \pi \text{ is tame} \right\}$$

and let C_0 denote the cone of functions dominated by elements of K_0, i.e. $C_0 = K_0 - L_+^0$. Set $C = C_0 \cap L_\infty$. The semimartingale R is said to satisfy the condition of *no arbitrage* (NA) if $C \cap L_+^\infty = \{0\}$, the condition of *no free lunch with vanishing risk* (NFLVR) if for the closure $\bar{C}$ of C in L^∞ we have $\bar{C} \cap L_+^\infty = \{0\}$.

Taking up the topics of the preceding sections we next consider $L = \sup_{0 \leqslant t \leqslant 1} W_t$, the enlarged filtration

$$\mathscr{G}_t = \bigcap_{s>t} \mathscr{F}_s \vee \sigma(L)$$

augmented by the P-zero sets, $t \in [0,1]$. This particular choice requires some comments. In the special case $\sigma = 1$, $b = \frac{1}{2}$, the price process will be given just by the exponential $S_t = \exp(W_t)$, $t \in [0,1]$. Since exp is increasing, the knowledge of $\sup_{0 \leqslant t \leqslant 1} W_t$ is just equivalent to the knowledge of $\sup_{0 \leqslant t \leqslant 1} S_t$. So in this case we are dealing with an insider who knows the maximal stock price in advance. And one can figure out that there are more cases in which our L determines the maximal stock price. Though not very realistic from the point of view of applications, this assumption is still more realistic that knowing $\sup_{0 \leqslant t \leqslant 1} W_t$. This of course raises the question why not from start we work with the additional knowledge $L = \sup_{0 \leqslant t \leqslant 1} S_t$? The answer might be disappointing from the point of view of someone having applicability of our model in mind: we make this assumption for simple technical reasons. Malliavin's calculus for the conditional laws of the maximal stock price is possible, but would add considerably to the technicality of this paper, and thus obscure the essential mathematical steps of our approach. We therefore decided to set aside technical issues like this one now, and to postpone the treatment of this interesting subject to a forthcoming paper.

Another compromise made here for purely mathematical reasons, which affects the issue of whether our additional piece of information is realistic is clearly addressed by the question: why should the insider know, in addition to the time when the maximum is achieved, how high it is? Would it not be enough for him to just know the time, in order to exercise arbitrage? This is clearly a very relevant question. And the answer for the time τ at which the maximum of W is taken, is positive. To show this, however, the mathematical framework of this paper is not quite appropriate. Here we enlarge the filtration by fixed *spatial* variables at the beginning of the action interval. The natural framework for the enlargement with random times is the well-studied *progressive enlargement* (see Yor, 1985c). Now in the terminology of time reversal of Markov processes, our random time τ is of the same type as the well-known *honest times*. In a forthcoming paper (Imkeller, 2000) we deal with progressive enlargements by times of this type in a more systematic way, and show that they allow for ample arbitrage opportunities. Now the progressive enlargement by τ is strictly smaller than the initial enlargement by this time. So Imkeller (2000) yields an even stronger statement on arbitrage possibilities than we would obtain here.

Using the decomposition of Theorem 2.1

$$W = \tilde{W} + \int_0^{\cdot} g_t(.,L)\,\mathrm{d}t,$$

we obtain a new financial market $(\tilde{b}, \tilde{\sigma})$ with $\tilde{b}_t = b_t + \sigma_t\, g_t(.,L)$, $\tilde{\sigma}_t = \sigma_t$, $t \in [0,1]$, with respect to the $\boldsymbol{G}$-Brownian motion $\tilde{W}$. In the following statements we refer to the point of view of the insider, i.e. we argue for the $\boldsymbol{G}$-Brownian motion $\tilde{W}$, and the financial market $(\tilde{b}, \tilde{\sigma})$. For the convenience of the reader, let us recall the notation of equivalent martingale measures. Given a semimartingale R_t, $t \in [0,1]$, with respect to a filtration and the measure P, a probability measure Q is called an *equivalent martingale measure* (with respect to P), if P and Q are mutually absolutely continuous and R_t, $t \in [0,1]$, is a local martingale with respect to the same filtration and Q.

Theorem 3.1. *R does not satisfy the condition* ($NFLVR$).

Proof. In a more general setting, Delbaen and Schachermayer (1996) proved that there is an equivalent martingale measure which makes R a local martingale if and only if R satisfies (NFLVR). In case the model is based on a Wiener process, hence in our case, there exists at most one equivalent martingale measure Q which, if it exists, has the form

$$\frac{dQ}{dP} = \exp\left(-\int_0^1 \alpha_t \, dM_t - \frac{1}{2}\int_0^1 \alpha_t^2 d\langle M\rangle_t \right), \tag{26}$$

if R possesses the Doob–Meyer decomposition

$$R = M + \int_0^{\cdot} \alpha_t d\langle M\rangle_t. \tag{27}$$

Now suppose that R satisfies (NFLVR). Comparing (24) and (27) we can see that

$$M_t = \int_0^t \sigma_s d\tilde{W}_s, \quad \alpha_t = \frac{\tilde{b}_t}{\sigma_t^2}.$$

Hence there exists a progressively measurable process θ such that $\tilde{b}_t = \sigma_t \, \theta_t, \ t \in [0,1]$. See for example Theorem 4.2. in Karatzas and Shreve (1998). This fact combines with (26) to give the formula

$$\frac{dQ}{dP} = \exp\left(-\int_0^1 \theta_t \, d\tilde{W}_t - \frac{1}{2}\int_0^1 \theta_t^2 \, dt \right). \tag{28}$$

We shall show in the subsequent lemma that $\int_0^1 \theta_t^2 \, dt = \infty$ on a set of positive measure. Consequently, (28) implies that $dQ/dP = 0$ on a set of positive measure (see for example Delbaen and Schachermayer, 1995b, or Revuz and Yor, 1999, p. 157). Hence P and Q cannot be equivalent. This completes the proof. □

Lemma 3.1. *On a set of positive probability we have*

$$\int_0^1 \theta_t^2 \, dt = \infty.$$

Proof. Recalling our assumption $\sigma \neq 0$, write $c_t = b_t/\sigma_t$. We have to prove that $\int_0^1 (c_t + g_t(.,L))^2 \, dt = \infty$ on a set of positive probability. Let us assume that on the contrary

$$\int_0^1 (c_t + g_t(.,L))^2 \, dt < \infty \tag{29}$$

P-a.s. Let τ be as in Section 3 the random time when the maximum of W before time 1 is reached. For $\varepsilon > 0$ let

$$\sigma(\varepsilon) = \sup\{t \geqslant \tau \colon S_t - W_t < \varepsilon\}.$$

From (23) we know that for any $\varepsilon > 0$ on the set $\{\tau + \sigma(\varepsilon) < 1\}$

$$\int_{[\tau,\tau+\sigma(\varepsilon)]} g_t(.,L)^2 \, dt = \infty, \tag{30}$$

while by definition of $\sigma(\varepsilon)$

$$\int_{[\tau,\tau+\sigma(\varepsilon)]^c} g_t(.,L)^2 \, dt < \infty. \tag{31}$$

Since this set has positive probability for ε small enough, we have (30) on a set of positive probability. Furthermore, we obtain the following estimate:

$$\begin{aligned}|c_t| &\leqslant |g_t(.,L)|\, 1_{[\tau,\tau+\sigma(\varepsilon)]}(t) + |c_t + g_t(.,L)| + |g_t(.,L)|\, 1_{[\tau,\tau+\sigma(\varepsilon)]^c}(t)\\ &= |g_t(.,L)|\, 1_{[\tau,\tau+\sigma(\varepsilon)]}(t) + v_t.\end{aligned}$$

We know, moreover, that for $t \in [\tau, \tau+\sigma(\varepsilon)]$

$$g_t(.,L) = -\frac{f_{1-t}(L-W_t)}{\int_0^{L-W_t} f_{1-t}(y)\,\mathrm{d}y} - \frac{L-W_t}{1-t}$$

so that our estimate is seen to give

$$|c_t| \leqslant E\left(1_{[\tau,\tau+\sigma(\varepsilon)]}(t)\ \frac{f_{1-t}(L-W_t)}{\int_0^{L-W_t} f_{1-t}(y)\,\mathrm{d}y}\middle|\mathscr{F}_t\right) + E(V_t|\mathscr{F}_t), \tag{32}$$

by $\boldsymbol{F}$-adaptedness of b and σ, where $V_t = v_t + (L-W)_t/(1-t)$. Let us next fix $T<1$ and $\varepsilon > 0$ such that the set $\{\tau + \sigma(\varepsilon) < T\}$ still has positive probability. Now (29) and (31) clearly imply that we have

$$\int_0^T E(V_t^2|\mathscr{F}_t)\,\mathrm{d}t < \infty. \tag{33}$$

Let us consider the first term in the estimate given by (32). We have

$$\begin{aligned}&E\left(1_{[\tau,\tau+\sigma(\varepsilon)]}(t)\ \frac{f_{1-t}(L-W_t)}{\int_0^{L-W_t} f_{1-t}(y)\,\mathrm{d}y}\middle|\mathscr{F}_t\right)\\ &\leqslant E\left(\frac{f_{1-t}(L-W_t)}{\int_0^{L-W_t} f_{1-t}(y)\,\mathrm{d}y}\middle|\mathscr{F}_t\right)\\ &= \int_0^\infty \frac{f_{1-t}(S_t \vee (W_t+z) - W_t)}{\int_0^{S_t\vee(W_t+z)-W_t} f_{1-t}(y)\,\mathrm{d}y} f_{1-t}(z)\,\mathrm{d}z\\ &= f_{1-t}(S_t - W_t) + \int_{S_t-W_t}^\infty \frac{f_{1-t}(z)^2}{\int_0^z f_{1-t}(y)\,\mathrm{d}y}\,\mathrm{d}z.\end{aligned} \tag{34}$$

Obviously, the first term in the last line of (34) is square integrable over the interval $[0,T]$ P-a.s. To estimate the second term, we write

$$\begin{aligned}&\int_0^T E\left(\left(\int_{S_t-W_t}^\infty \frac{f_{1-t}^2(z)}{\int_0^z f_{1-t}(y)\,\mathrm{d}y}\,\mathrm{d}z\right)^2\right)\mathrm{d}t\\ &= \int_0^T \int_0^\infty \left(\int_x^\infty \frac{f_{1-t}^2(z)}{\int_0^z f_{1-t}(y)\,\mathrm{d}y}\,\mathrm{d}z\right)^2 p_t(x)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{35}$$

where p_t denotes the density of $S_t - W_t$. Now we may find constants a_T, A_T such that for any $t \in [0,T], 0 \leqslant z \leqslant 1$ we have

$$a_T \leqslant f_{1-t}(z) \leqslant A_T, \quad a_T\, z \leqslant \int_0^z f_{1-t}(y)\,\mathrm{d}y \leqslant A_T\, z.$$

We therefore find constants α_T, β_T such that

$$\int_x^\infty \frac{f_{1-t}^2(z)}{\int_0^z f_{1-t}(y)\,\mathrm{d}y}\,\mathrm{d}z \leqslant \int_x^1 \frac{f_{1-t}^2(z)}{\int_0^z f_{1-t}(y)\,\mathrm{d}y}\,\mathrm{d}z + \alpha_T$$
$$\leqslant \int_x^1 \frac{\beta_T}{z}\mathrm{d}z + \alpha_T$$
$$= \beta_T|\ln x| + \alpha_T.$$

Hence the right-hand side of (35) can be estimated by a constant multiple of the obviously finite quantity

$$\int_0^T \int_0^\infty (\ln x)^2\, p_t(x)\,\mathrm{d}x\,\mathrm{d}t.$$

Hence (32), (33), and (34) imply that

$$\int_0^T |c_t|^2\,\mathrm{d}t < \infty \quad P\text{-a.s.}$$

Hence (30) has the consequence

$$\int_0^T (c_t + g_t(.,L))^2\,\mathrm{d}t = \infty \quad \text{on the set} \quad \{\tau + \sigma(\varepsilon) < T\}.$$

This is in contradiction with (29), and thus provides the desired conclusion. □

Theorem 3.1 proves that the (NFLVR) condition is violated for all possible choices of b and σ. We now construct explicit arbitrage possibilities in various cases. This will show in addition that in these cases even the (NA) condition is violated.

Theorem 3.2. *Suppose that* $\int_0^1 |b_s/\sigma_s|\,\mathrm{d}s < \infty$ *and there is* $c > 0$ *such that for any* $s \in [0,1]$ *we have* $b_s/\sigma_s \geqslant -c$ *P-a.s. Then arbitrage possibilities exist.*

Proof. Let

$$T_t = \exp\left(\int_0^t \frac{b_s}{\sigma_s}\,\mathrm{d}s + W_t - \frac{1}{2}t\right), \quad t \in [0,1].$$

We define the tame strategy

$$\pi_t = 1_{[0,\tau]}(t)\,\frac{T_t}{\sigma_t}1_{\{L>c+1/2\}}, \quad t \in [0,1].$$

Note that the set $\{L > c + \frac{1}{2}\}$ has positive probability. Hence for the gains process the following estimation is valid. We have for all $t \in [0,1]$

$$\begin{aligned} G_t &= \int_0^t \pi_s\,\mathrm{d}R_s \\ &= \int_0^{t\wedge\tau} T_s\frac{\mathrm{d}T_s}{T_s}1_{\{L>c+1/2\}} \\ &= (T_{t\wedge\tau} - T_0)1_{\{L>c+1/2\}} \\ &\geqslant -1 \end{aligned} \tag{36}$$

and

$$
\begin{aligned}
G_1 &= \left[\exp\left(L + \int_0^1 \frac{b_s}{\sigma_s}\,\mathrm{d}s - \frac{1}{2}\tau\right) - 1\right] 1_{\{L>c+1/2\}} \\
&\geqslant \left[\exp\left(L - c - \frac{1}{2}\right) - 1\right] 1_{\{L>c+1/2\}} \\
&\geqslant 0 \quad \text{and} \quad > 0
\end{aligned} \tag{37}
$$

with positive probability. □

In case the positive part of b/σ is bounded, the type of strategy of the preceding theorem curiously does not work. We have to take resort to a different strategy. As we shall see, in this case the drift may destroy any tendency in the development of W before or after the time at which it takes its maximum. But in an eventually very short interval just *after* the maximum is taken, provided the drift is sufficiently well integrable, the Wiener process will decrease at a too fast rate, so that the insider can take advantage of the corresponding decay of the price. Note that the following theorem includes the *dual* statement of the preceding theorem, i.e. that there exists a constant c such that for all $s \in [0,1]$ we have $b_s/\sigma_s \leqslant c$, but that the arbitrage strategies are quite different.

We denote by f^+ resp. f^- the positive resp. negative part of a function f.

Theorem 3.3. *Suppose that there exists* $p > 2$ *such that* $(b/\sigma)^+ \in L_p([0,1])$. *Then there is arbitrage.*

Proof. Let q be the conjugate exponent to p. Hölder's inequality gives

$$
\int_\tau^t \left(\frac{b_s}{\sigma_s}\right)^+ \mathrm{d}s \leqslant (t-\tau)^{1/q} \left(\int_\tau^t \left(\left(\frac{b_s}{\sigma_s}\right)^+\right)^p \mathrm{d}s\right)^{1/p}. \tag{38}
$$

The crucial point for our argument is the following observation. Given L, the process $(\rho_t = L - W_{\tau+t}\colon 0 \leqslant t \leqslant 1-\tau)$ is a Bes(3) process. Moreover, due to Pitman's Theorem (see Revuz and Yor, 1999, p. 253), ρ has the same law as the initial piece of the process $(2M_t - B_t\colon 0 \leqslant t \leqslant 1)$, where B is a one-dimensional Brownian motion with maximum process M. Since – as is easy to see by a standard Borel–Cantelli argument – $\lim_{t\to 0} M_t/t^{1/q} = \infty$, P-a.s., and hence a fortiori $\lim_{t\to 0} \rho_t/t^{1/q} = \infty$, we may write

$$
\lim_{t\downarrow\tau} \frac{L - W_t}{(t-\tau)^{1/q}} = \infty \quad P\text{-a.s. for } q < 2. \tag{39}
$$

Eqs. (38) and (39) together imply that on a random, small but nontrivial time interval just after τ, we have

$$
\int_\tau^t \left(\frac{b_s}{\sigma_s}\right)^+ \mathrm{d}s < L - W_t.
$$

Hence the $\boldsymbol{G}$-stopping time

$$
\mu = \inf\left\{t \geqslant \tau\colon\ \int_\tau^t \left(\frac{b_s}{\sigma_s}\right)^+ \mathrm{d}s + \frac{1}{2}(t-\tau) > \frac{L - W_t}{2}\right\} \wedge 1 \tag{40}
$$

is strictly greater than τ as long as $\tau < 1$ P-a.s. Using this stopping time, we may define our tame strategy. Let

$$\pi_t = -1_{[\tau,\mu]}(t)\frac{T'_t}{T'_\tau \sigma_t}, \quad t \in [0,1],$$

where

$$T'_t = \exp\left(-\int_0^t \frac{b_s}{\sigma_s}\,\mathrm{d}s - W_t - \frac{1}{2}t\right), \quad t \in [0,1].$$

Then we obtain for $t \in [0,1]$

$$\begin{aligned} G_t &= \int_\tau^{t\wedge\mu} \frac{T'_s}{T'_\tau}\frac{\mathrm{d}T'_s}{T'_s} \\ &= \left[\exp\left(-\int_\tau^{t\wedge\mu} \frac{b_s}{\sigma_s}\,\mathrm{d}s + (L - W_{t\wedge\mu}) - \frac{1}{2}(t\wedge\mu - \tau)\right) - 1\right] \\ &\geqslant -1 \end{aligned}$$

and

$$\begin{aligned} G_1 &\geqslant \left[\exp\left(\frac{L - W_\mu}{2}\right) - 1\right] \\ &\geqslant 0 \quad \text{and} \quad > 0 \end{aligned}$$

on the set $\{\tau < 1\}$ which has positive measure. This completes the proof. □

Corollary 3.1. *Suppose that there is a random variable ξ such that we have $\sup_{t\in[0,1]} b_t/\sigma_t \leqslant \xi$. Then there is arbitrage. In particular, there is arbitrage if b/σ is continuous.*

Proof. The condition clearly implies the p-integrability of the quotient for any $p > 2$. □

Remark. The conditions on σ and b formulated in the preceding theorems are not restrictive enough not to leave room for speculations. One could for example imagine that σ and especially the drift b could be chosen violent enough to defeat any possibility of arbitrage. So far, however, we cannot give an example.

For further reading

The following references are also of interest to the reader: Blanchet-Scaillet and Jeanblanc, 2000; Choulli, et al., 1998; Choulli, et al., 1999; Chaleyat-Maurel and Jeulin, 1985; Elliott, et al., 1999; Karatzas, et al., 1991; Meyer, 1978

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Appendix A. A calculus for measure-valued random variables

It is well known from the classical works on the enlargement of filtrations (see Jacod, 1985; Yor, 1985a; Jeulin, 1980) that for initial enlargements by random variables L properties of the conditional laws of L given the σ-algebras of the small filtration play a dominant role. From our point of view, these conditional laws will be considered as appropriate provided the conditional laws of a standard regularization given by the perturbation of L with an independent Gaussian variable are smooth in the sense of Malliavin's calculus. For this purpose, we shall consider the conditional laws as random variables with values in the space of signed measures. The regularization will allow us to work essentially with the Banach space topology on this space induced by the total variation norm $|\,.\,|$, though to define stochastic integral representations, the weak* topology will be sufficient. The reason for requiring these smoothness properties lies in the fact, that throughout this paper, we restrict to the Clark–Ocone formula for representing martingales, especially measure-valued ones, as stochastic integrals with respect to the Wiener filtration.

Let us briefly recall the basic concepts of Malliavin's calculus needed. We refer to Nualart (1995) for a more detailed treatment.

Let $\mathscr{S}$ be the set of smooth random variables on $(\Omega,\mathscr{F},P)$, i.e. of random variables of the form

$$F=f(W_{t_1},\dots,W_{t_n}),\quad f\in C_0^\infty(\mathbb{R}^n),\ t_1,\dots,t_n\in[0,1].$$

For $F\in\mathscr{S}$ we may define the Malliavin derivative

$$(DF)_s=D_sF=\sum_{i=1}^n\frac{\partial}{\partial x_i}f(W_{t_1},\dots,W_{t_n})1_{[0,t_i]}(s),\quad s\in[0,1].$$

We may regard DF as a random element with values in $L^2([0,1])$, and then define the Malliavin derivative of order k by k fold iteration of the above derivation. It will be denoted by $D^{\otimes_k}F$, and is a random element with values in $L^2([0,1]^k)$. Its value at $(s_1,\dots,s_k)\in[0,1]^k$ is written $D^{\otimes_k}_{s_1,\dots,s_k}$.

If $p\geqslant 1$ and $k\in\mathbb{N}$, we denote by $\boldsymbol{D}_{p,k}$ the Banach space given by the completion of $\mathscr{S}$ with respect to the norm

$$\|F\|_{p,k}=\|F\|_p+\sum_{1\leqslant j\leqslant k}E\left(\left[\int_0^1(D^{\otimes_j}_{s_1,\dots,s_j}F)^2\,\mathrm{d}s_1,\dots,\mathrm{d}s_j\right]^{p/2}\right)^{1/p},\quad F\in\mathscr{S}.$$

More generally, if H is a Hilbert space and $\mathscr{S}_H$ the set of linear combinations of tensor products of elements of $\mathscr{S}$ with elements of H, $\boldsymbol{D}_{p,k}(H)$ will denote the closure of $\mathscr{S}_H$ w.r. to the norm

$$\|F\|_{p,k}=\|\,|F|_H\|_p+\sum_{1\leqslant j\leqslant k}E\left(\left[\int_0^1|D^{\otimes_j}_{s_1,\dots,s_j}F|_H^2\,\mathrm{d}s_1\dots\mathrm{d}s_j\right]^{p/2}\right)^{1/p},\quad F\in\mathscr{S}_H,$$

where the Malliavin derivatives of smooth functions are given in an obvious way, and $|\,.\,|_H$ denotes the norm on H induced by the scalar product. These definitions are consistent. For example,

$$\|F\|_p + \|DF\|_{p,k-1} = \|F\|_{p,k}, \quad F \in \boldsymbol{D}_{p,k},$$

if $H = L^2([0,1])$.

Let $\boldsymbol{M}$ be the space of signed measures on $(\mathbb{R}, \boldsymbol{B})$, the real line equipped with the Borel sets. The variation norm is denoted by $|\mu|$ for $\mu \in \boldsymbol{M}$. $\boldsymbol{M}$ is endowed with the Banach space topology induced by this norm. At places it will be convenient to use a weaker topology: the weak* topology which is induced by the space $C_{\mathrm{b}}(\mathbb{R})$ of continuous bounded functions with the supremum norm $\|.\|$. Endowed with the latter topology, $\boldsymbol{M}$ is a locally convex space which, due to the separability of $C_{\mathrm{b}}(\mathbb{R})$, is separable. For $\mu \in \boldsymbol{M}, f \in C_{\mathrm{b}}(\mathbb{R})$, we denote $\langle \mu, f \rangle = \int_{\mathbb{R}} f \,\mathrm{d}\mu$. We may choose a dense sequence $(f_i)_{i \in \mathbb{N}} \subset C_{\mathrm{b}}(\mathbb{R})$, to use the standard embedding of $\boldsymbol{M}$ into an infinite dimensional metrizable space

$$\begin{aligned} \Phi : \boldsymbol{M} &\to \mathbb{R}^{\mathbb{N}}, \\ \mu &\mapsto (\langle \mu, f_i \rangle)_{i \in \mathbb{N}}. \end{aligned} \tag{41}$$

Note that μ is actually mapped into the compact cube $\prod_{i \in \mathbb{N}} [-|\mu\| \, |f_i\|, |\mu\| \, |f_i\|]$. We shall subsequently use Φ to define Malliavin derivatives, stochastic integrals, and conditional expectations of $\boldsymbol{M}$-valued objects, which finally leads us to formulate a type of measure valued martingale representation formula. For $h \in L^2([0,1])$, let $W(h) = \int_0^1 h(s) \,\mathrm{d}W_s$. We first define the smooth cylinder functions. Let

$$\begin{aligned} \mathscr{S}(\boldsymbol{M}) = \{F : F = g(W(h_1), \ldots, W(h_k), x) \,\mathrm{d}x, \ g \in C_c^{\infty}(\mathbb{R}^{k+1}), \\ h_1, \ldots, h_k \in L^2([0,1]), \ k \in \mathbb{N}\}. \end{aligned}$$

For $g \in C_c^{\infty}(\mathbb{R}^k)$ denote by $\partial_i g$ the partial derivative of g in direction i, $1 \leqslant i \leqslant k$. So we may define the Malliavin derivative for smooth cylinder functions by

$$D_s F = \sum_{i=1}^{k} \partial_i g(W(h_1), \ldots, W(h_k), x) \,\mathrm{d}x h_i(s), \quad s \in [0,1].$$

We consider DF as an element of $L^2(\Omega \times [0,1], \boldsymbol{M})$ with respect to the Banach space topology. We obviously have in terms of the Malliavin derivative of real-valued functions

$$\langle DF, f \rangle = D\langle F, f \rangle, \quad f \in C_{\mathrm{b}}(\mathbb{R}), \tag{42}$$

and hence also

$$DF = \Phi^{-1}((D\langle F, f_i \rangle)_{i \in \mathbb{N}}). \tag{43}$$

We next introduce a norm on $\mathscr{S}(\boldsymbol{M})$. For $F \in \mathscr{S}(\boldsymbol{M})$ let

$$\|F\|_{1,2} = [E(|F|^2)^{1/2} + E(\| \, |DF| \, \|_2^2)^{1/2}]. \tag{44}$$

Note that by definition, we have indeed $\|F\|_{1,2} < \infty$ for $F \in \mathscr{S}(\boldsymbol{M})$. Hence the closure $\boldsymbol{D}_{1,2}(\boldsymbol{M})$ of $\mathscr{S}(\boldsymbol{M})$ with respect to $\|.\|_{1,2}$ is well defined and nontrivial.

In a similar way, we may define $\|.\|_{1,p}$, $p \geqslant 1$, and $\boldsymbol{D}_{1,p}(\boldsymbol{M})$ by replacing the 2-norm by the p-norm, as well as for higher derivatives the norms $\|.\|_{k,p}$ and spaces $\boldsymbol{D}_{k,p}(\boldsymbol{M})$, $k \in \mathbb{N}, p \geqslant 1$. We obviously have the following property.

Proposition A.1. *For $F \in \boldsymbol{D}_{1,2}(\boldsymbol{M})$ and $f \in C_b(\mathbb{R})$ we have $\langle F, f\rangle \in \boldsymbol{D}_{1,2}$ and*

$$\langle DF, f\rangle = D\langle F, f\rangle.$$

Proof. Let $F \in \mathscr{S}(\boldsymbol{M})$, $f \in C_b(\mathbb{R})$. Then by definition with a suitable constant c

$$E(\langle F, f\rangle^2)^{1/2} + E\left(\int_0^1 D_s\langle F, f\rangle^2 \, \mathrm{d}s\right)^{1/2} \leqslant c\|F\|_{1,2}\,\|f\|.$$

The asserted property therefore follows from (42) and the closability of the operator D for measure and real-valued functions. □

The following property extends (43).

Proposition A.2. *For $F \in \boldsymbol{D}_{1,2}(\boldsymbol{M})$ we have*

$$DF = \Phi^{-1}((D\langle F, f_i\rangle)_{i \in \mathbb{N}}).$$

Proof. This is a consequence of the continuity of Φ^{-1} for the weak* topology and (43). □

We next need Itô integrals for random elements with values in $\boldsymbol{M}$ with respect to the weak* topology at least. Consider an $\boldsymbol{M}$-valued *adapted* process $(F_t)_{t \in [0,1]}$ with respect to the augmented Wiener filtration $(\mathscr{F}_t)_{t \in [0,1]}$. This amounts to say that for any $i \in \mathbb{N}$ the real-valued process $(\langle F_t, f_i\rangle)_{t \in [0,1]}$ is adapted, since due to (43) we have $F_t = \Phi^{-1}((\langle F_t, f_i\rangle)_{i \in \mathbb{N}})$. Note that in this case we also have adaptedness of $(\langle F_t, f\rangle)_{t \in [0,1]}$ for any $f \in C_b(\mathbb{R})$. Now suppose that in addition we have

$$\sup_{f \in C_b(\mathbb{R}), \|f\| \leqslant 1} E\left(\int_0^1 \langle F_t, f\rangle^2 \, \mathrm{d}t\right) < \infty. \tag{45}$$

Denote

$$\|F\|_2 = \sup_{f \in C_b(\mathbb{R}), \|f\| \leqslant 1} E\left(\int_0^1 \langle F_t, f\rangle^2 \, \mathrm{d}t\right)^{1/2}.$$

In a similar fashion, we can define $\|F\|_p$ for $p \geqslant 1$. Then for any $f \in C_b(\mathbb{R})$ the Itô integral process $\int_0^\cdot \langle F_t, f\rangle \, \mathrm{d}W_t$ is well defined and continuous on $[0,1]$. As above, we may use the embedding to define the Itô integral for the measure-valued process $(F_t)_{t \in [0,1]}$. For this purpose for any $i \in \mathbb{N}$ let $\int_0^\cdot \langle F_t, f_i\rangle \, \mathrm{d}W_t$ be a continuous version of the Itô integral of $(\langle F_t, f_i\rangle)_{t \in [0,1]}$. Let then

$$\int_0^\cdot F_t \, \mathrm{d}W_t = \Phi^{-1}\left(\left(\int_0^\cdot \langle F_t, f_i\rangle \, \mathrm{d}W_t\right)_{i \in \mathbb{N}}\right).$$

Let now $f \in C_b(\mathbb{R})$ be arbitrary. Let, moreover, $(g_i)_{i \in \mathbb{N}}$ be a subsequence of $(f_i)_{i \in \mathbb{N}}$ which converges to f. Then we have by definition

$$E\left(\int_0^1 \langle F_s, f - g_i\rangle^2 \, \mathrm{d}s\right) \leqslant \|F\|_2\,\|f - g_i\| \to 0$$

as $i \to \infty$. But we may use Doob's inequality to write

$$E\left(\sup_{t\in[0,1]}\left|\int_0^t \langle F_s, f-g_i\rangle \,\mathrm{d}W_s\right|^2\right) \leqslant 4E\left(\int_0^1 \langle F_s, f-g_i\rangle^2 \,\mathrm{d}s\right)$$

for any $i \in \mathbb{N}$. Hence for any $f \in C_\mathrm{b}(\mathbb{R})$, $\int_0^\cdot \langle F_s, f\rangle \,\mathrm{d}W_s$ is well defined as a continuous process. This way we obtain

Proposition A.3. *Let $(F_t)_{t\in[0,1]}$ be an $\boldsymbol{M}$-valued adapted process such that $\|F\|_2 < \infty$. Then there exists a uniquely (up to equivalence of processes) process $\int_0^\cdot F_t \,\mathrm{d}W_t$ on $\boldsymbol{M}$ which is continuous with respect to the weak* topology such that we have for any $f \in C_\mathrm{b}(\mathbb{R})$*

$$\left\langle \int_0^\cdot F_t \,\mathrm{d}W_t, f\right\rangle = \int_0^\cdot \langle F_t, f\rangle \,\mathrm{d}W_t.$$

Moreover,

$$\|F\|_2 = \sup_{i\in\mathbb{N}} E\left(\int_0^1 \left\langle F_t, \frac{f_i}{\|f_i\|}\right\rangle^2 \mathrm{d}t\right)^{1/2}.$$

The process $\int_0^\cdot F_t \,\mathrm{d}W_t$ is called *Itô integral process* of $F = (F_t)_{t\in[0,1]}$.

Of course, if the measure-valued process is L^2-bounded in the variation norm, the Itô integral process exists, and our statement becomes more stringent.

Proposition A.4. *Let $(F_t)_{t\in[0,1]}$ be an $\boldsymbol{M}$-valued adapted process such that*

$$E\left(\int_0^1 |F_s|^2 \,\mathrm{d}s\right) < \infty.$$

Then the Itô integral process for F exists and is continuous in the strong topology induced by the total variation norm.

Proof. Since for a measure $\mu \in \boldsymbol{M}$ we have

$$|\mu| = \sup_{f\in C_\mathrm{b}(\mathbb{R}), \|f\|\leqslant 1} \langle \mu, f\rangle,$$

we know that the process $(|F_t|)_{t\in[0,1]}$ is measurable and adapted. For any $f \in C_\mathrm{b}(\mathbb{R})$, $\|f\| \leqslant 1$, we obviously have

$$E\left(\int_0^1 \langle F_t, f\rangle^2 \,\mathrm{d}t\right) \leqslant E\left(\int_0^1 |F_t|^2 \,\mathrm{d}t\right).$$

The argument starting with (45) can now be replaced by a simpler one based upon

$$\int_0^1 \langle F_t, f-g\rangle^2 \,\mathrm{d}t \leqslant \int_0^1 |F_t|^2 \,\mathrm{d}t \;\|f-g\|^2,$$

which derives the existence of continuous integral processes $\int_0^\cdot \langle F_t, f\rangle \,\mathrm{d}W_t$ directly from the embedding. $\square$

The classical Clark–Ocone formula states that for $F \in \boldsymbol{D}_{1,2}$ one has

$$F = E(F) + \int_0^t E(D_t F|\mathscr{F}_t)\,\mathrm{d}W_t.$$

Our next aim is to derive a measure-valued version of this formula. For this purpose, we first have to extend the operation of conditional expectation to measure-valued quantities.

Let F be an $\boldsymbol{M}$-valued random variable, which is $\mathscr{F}$-measurable. This means that $\langle F, f_i\rangle$ is $\mathscr{F}$-measurable for any $i \in \mathbb{N}$. Then also $\langle F, f\rangle$ is $\mathscr{F}$-measurable for $f \in C_{\mathrm{b}}(\mathbb{R})$. Let $\mathscr{G} \subset \mathscr{F}$ be a σ-algebra. Then for any $i \in \mathbb{N}$ the conditional expectation $E(\langle F, f_i\rangle|\mathscr{G})$ is well defined, up to P-a.s. equality. Let then

$$E(F|\mathscr{G}) = \Phi^{-1}((E(\langle F, f_i\rangle)_{i \in \mathbb{N}}).$$

Then the conditional expectation is defined up to P-a.s. equality, and by definition we have

$$\langle E(F|\mathscr{G}), f_i\rangle = E(\langle F, f_i\rangle|\mathscr{G})$$

for $i \in \mathbb{N}$. Now suppose in addition

$$\|F\|_1 = \sup_{f \in C_{\mathrm{b}}(\mathbb{R}), \|f\| \leqslant 1} E(|\langle F, f\rangle|) < \infty. \tag{46}$$

Let $f \in C_{\mathrm{b}}(\mathbb{R})$ be arbitrary, and choose a subsequence $(g_i)_{i \in \mathbb{N}}$ of $(f_i)_{i \in \mathbb{N}}$ such that $g_i \to f$. Then we know that

$$|E(\langle F, f - g_i\rangle)| \leqslant \|F\|_1\, \|f - g_i\| \to 0$$

as $i \to \infty$. By using Jensen's inequality in addition, we also obtain

$$E(|E(\langle F, f - g_i\rangle|\mathscr{G})|) \leqslant \|F\|_1\, \|f - g_i\| \to 0$$

as $i \to \infty$. Therefore we get

Proposition A.5. *Let F be $\mathscr{F}$-measurable with values in $\boldsymbol{M}$, $\mathscr{G} \subset \mathscr{F}$ a σ-algebra. Assume* (46). *Then there exists a uniquely* (*up to P-a.s. equality*) *determined $\mathscr{G}$-measurable random variable $E(F|\mathscr{G})$ with values in $\boldsymbol{M}$ such that for any $f \in C_{\mathrm{b}}(\mathbb{R})$ we have*

$$\langle E(F|\mathscr{G}), f\rangle = E(\langle F, f\rangle|\mathscr{G}).$$

If we know that F is integrable in the variation norm, the statement of the preceding proposition becomes more direct.

Proposition A.6. *Let F be $\mathscr{F}$-measurable with values in $\boldsymbol{M}$, $\mathscr{G} \subset \mathscr{F}$ a σ-algebra. Suppose that $E(|F|) < \infty$. Then there exists a uniquely* (*up to P-a.s. equality*) *determined $\mathscr{G}$-measurable random variable $E(F|\mathscr{G})$ with values in $\boldsymbol{M}$ such that for any $f \in C_{\mathrm{b}}(\mathbb{R})$ we have*

$$\langle E(F|\mathscr{G}), f\rangle = E(\langle F, f\rangle|\mathscr{G}).$$

We are ready to formulate a measure-valued Clark–Ocone formula.

Theorem A.1. *Let $F \in \boldsymbol{D}_{1,2}(\boldsymbol{M})$. Then we have*

$$F = E(F) + \int_0^1 E(D_s F|\mathscr{F}_s)\,\mathrm{d}W_s.$$

Proof. For $i \in \mathbb{N}$ the classical formula yields

$$\langle F, f_i\rangle = E(\langle F, f_i\rangle) + \int_0^1 E(D_s\langle F, f_i\rangle|\mathscr{F}_s)\,\mathrm{d}W_s. \tag{47}$$

But by Propositions A.6, A.2 and definition, we have for any $t \in [0,1]$

$$E(D_t F|\mathscr{F}_t) = \Phi^{-1}(E(D_t\langle F, f_i\rangle|\mathscr{F}_t)_{i\in\mathbb{N}}).$$

Hence by Proposition A.4 and definition

$$\begin{aligned}\int_0^1 E(D_t F|\mathscr{F}_t)\,\mathrm{d}W_t &= \Phi^{-1}\left(\left(\int_0^1 \langle E(D_t F|\mathscr{F}_t), f_i\rangle\,\mathrm{d}W_t\right)_{i\in\mathbb{N}}\right)\\ &= \Phi^{-1}\left(\left(\int_0^1 E(D_t\langle F, f_i\rangle|\mathscr{F}_t)\,\mathrm{d}W_t\right)_{i\in\mathbb{N}}\right),\end{aligned}$$

where the measurability is again due to the measurability in the scalar case and the embedding. Since similar, but simpler identifications are valid for F and $E(F)$ (the conditional expectation with respect to the trivial σ-algebra), (47) may be applied to yield the desired formula. This completes the proof. □

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