The Block Structure and Ext of $p$-Soluble Groups

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Communicated by Walter Feit

Received November 18, 1985

In [L] the following result was proved (where $\mathbb{N} = \{0, 1, 2, \ldots\}$):

Let $p \geq 5$ be a prime, let $G$ be a finite $p$-soluble group, and let $k$ be a field of characteristic $p$. If $V$ is a simple $kG$-module in the principal block of $kG$, then there exists $n \in \mathbb{N}$ such that $n < |G|/p$ and $H^2(G, V) \neq 0$.

In this article we shall extend this result and prove the following:

**Theorem.** Let $p \geq 5$ be a prime, let $G$ be a finite $p$-soluble group, and let $k$ be a field of characteristic $p$. If $M$ and $N$ are simple $kG$-modules in the same block of $kG$, then there exists $n \in \mathbb{N}$ such that $n < |G/O_p(G)|/p$ and $\text{Ext}^2_{kG}(M, N) \neq 0$.

We note that for non-$p$-soluble groups $G$, the conclusion of the theorem is not in general true. Suppose $G$ is the simple group $L_2(2)$, and $k$ is a field containing the field with $2^2$ elements. If $M$ and $N$ are the two $2$-dimensional simple modules of $kG$ (see Sect. 30 of [BN]), then $M^* \otimes N$ is a simple projective $kG$-module (the Steinberg module). However $M$ and $N$ are in the principal block of $kG$, yet $\text{Ext}^m_{kG}(M, N) \cong \text{Ext}^m_{kG}(k, M^* \otimes N) = 0$ for all $m \in \mathbb{N}$. We would like to thank the referee for this example, which generalizes our original example.

**Proof of the Theorem.** We shall use induction on $|G/O_p(G)|$. Set $R = O_p(G)$. It is well known (see, e.g., [St, Proposition 6.3]) that $M \downarrow _R$ and

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0021-8693/87 $3.00
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$N \downarrow_R$ have a common simple constituent, say $A$. Let $T$ be the inertia subgroup of $A$ in $G$.

First suppose $T \not\subseteq G$. Using Theorem V.2.5 of [F], there exist simple $kT$-modules $\tilde{M}$, $\tilde{N}$ in the same block with $M = \tilde{M} \uparrow^G$, $N = \tilde{N} \uparrow^G$ and $M \downarrow_R$, $N \downarrow_R$ $A$-isotypic. By an obvious extension of Lemma 7.1 of [St] we have for $m \in \mathbb{N}$,

$$\text{Ext}^m_{kG}(M, N) \cong \text{Ext}^m_{kT}(\tilde{M}, \tilde{N})$$

and the result follows by induction.

Therefore we may assume that $M \downarrow_R$ and $N \downarrow_R$ are $A$-isotypic. Since we may also assume that $k$ is algebraically closed, we can now use Fong's reduction (see, e.g., [F, p. 411]). There exists a central extension $G^*$ of $G$ by a cyclic $p'$-group $K$ such that $G^*$ contains $K \times R$ as a normal subgroup, with the following property: there exist simple $k[G^*/R]$-modules $\tilde{M}$, $\tilde{N}$ in the same block of $G^*/R$ and a $kG^*$-module $\tilde{X}$ with $\tilde{X} \downarrow_R = A$ such that $M = \tilde{X} \otimes \tilde{M}$, $N = \tilde{X} \otimes \tilde{N}$. Then

$$\text{Ext}^m_{kG}(M, N) \cong \text{Ext}^m_{k[G^*/R]}(\tilde{M}, \tilde{N})$$

by an obvious extension of Lemma 7.2 of [St],

$$\cong H^m(G^*/R, \tilde{M}^* \otimes \tilde{N})$$

$$\cong H^m(G^*/(K \times R), \tilde{M}^* \otimes_{kK} \tilde{N})$$

since $K$ is a $p'$-group.

Now $\tilde{M}^* \otimes_{kK} \tilde{N} \neq 0$ because $\tilde{M}$ and $\tilde{N}$ are in the same block of $G^*/R$ (use [St, Proposition 6.3]), $O_{p'}(G^*/(K \times R)) \cong O_{p'}(G/R) = 1$, and $O_{p'}(G^*/R)$ acts trivially on $\tilde{M}$ and $\tilde{N}$. Therefore we may apply Corollary 1.5 of [L] to deduce that there exists $n \in \mathbb{N}$ such that $n < |G^*/(K \times R)|/p = |G/R|/p$ and $H^{2n}(G^*/(K \times R), \tilde{M}^* \otimes_{kK} \tilde{N}) \neq 0$. This completes the proof of our theorem.

We conclude with the remark that the above argument yields the same result in the case $p = 2, 3$ and $G$ is of $p$-length 1 with $n < |G/R|$ (where $R = O_p(G)$). This follows by using Theorem 3 of [DS] (see also [D]) and the fact that $(G/R)/O_p(G/R)$ is a $p'$-group so that the modules over this group are semisimple.

**ACKNOWLEDGMENT**

The first author would like to thank the Alexander von Humboldt-Stiftung for their financial support.
REFERENCES


Printed in Belgium