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Spectral Decomposition and Invariant Manifolds for Some Functional Partial Differential Equations

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We study the integrodifferential convolution equation

$$\frac{d}{dt}(x+\mu * x) - Ax - \nu * x = f \qquad \text{on } [0, +\infty),$$

$$x = \phi \qquad \text{on } (-\infty, 0],$$

as well as a nonlinear perturbation of the corresponding homogeneous equation. Here A is the generator of an analytic semigroup on a Hilbert space H, and μ and ν are operator-valued dominated measures with values in $L(H)$ and $L(\mathscr{D}(A), H)$ respectively. Under the assumption that the operator given by the Laplace transform of the left-hand side of the equation is boundedly invertible on some right half-plane and on a line in the left half-plane, parallel to the imaginary axis, we decompose the solutions into components with different exponential growth rates. We construct projectors onto the stable and unstable subspaces, which are then used for the construction of stable and unstable manifolds for the nonlinear equation, which can have a fully nonlinear character. The results are applied to two equations of parabolic type. Moreover, the spectrum of the generator of the translation semigroup in various weighted spaces is determined, including the stable and unstable subspaces of our problem.

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1. INTRODUCTION

In the present paper, we study asymptotic properties of solutions to the problem

$$\begin{aligned} Dx \equiv \frac{d}{dt}(x+\mu * x) - Ax - \nu * x = f \qquad &\text{on} \quad \mathbf{R}^+ = [0, +\infty), \\ x = \phi \qquad &\text{on} \quad \mathbf{R}^- = (-\infty, 0], \end{aligned} \tag{1}$$

as well as a nonlinear perturbation of the corresponding homogeneous equation

$$\frac{d}{dt}(x+\mu * x) - Ax - \nu * x = F(x). \tag{2}$$

Here A is the generator of an analytic semigroup on a Hilbert space H, and μ and ν are operator-valued dominated measures with values in $L(H)$ and $L(\mathscr{D}(A), H)$ respectively. The nonlinear perturbation function F is supposed to vanish together with its Fréchet derivative at zero and to map a neighbourhood of zero in the space $W^{1,2}(\mathbf{R}, H) \cap L^2(\mathbf{R}, \mathscr{D}(A))$ into $L^2(\mathbf{R}^+, H)$ with appropriate weights, so the problem can have a fully nonlinear character.

The linear equation (1) was treated in Staffans [11] as an example of more general functional equations which generate semigroups. It was proved that (1) generates a strongly continuous semigroup of translation type on the weighted space $Z = W^{1,2}_\alpha(\mathbf{R}^-, H) \cap L^2_\alpha(\mathbf{R}^-, \mathscr{D}(A)) \otimes L^2_\alpha(\mathbf{R}^+, H)$, namely $T(t)(\phi, f) = (x_t, f^t)$, where x is a solution of (1) and $x_t(s) = x(t+s)$, $s \in \mathbf{R}^-$, $f^t(s) = f(t+s)$, $s \in \mathbf{R}^+$. (Here we shall see that $T(t)$ is, in fact, a group.)

Existence of stable and unstable manifolds for equations of parabolic type has been studied by many authors using methods developed in ordinary differential equations and the theory of analytic semigroups. The fully nonlinear parabolic equation was treated by Da Prato and Lunardi [2] in the space of continuous functions on $\mathbf{R}^-$ with values in the space of Hölder continuous functions, which is an interpolation space where the problem enjoys the maximal regularity property. The existence of invariant manifolds for equations of the type (2) in a finite dimensional space H was proved by Staffans [10] as an application of a general theory of convolution equations in fading memory spaces. The same problem for the semilinear equation (2) with $\mu = 0$ was treated by Milota [7]. A strongly continuous semigroup generated by (1) with $\mu = 0$ was constructed by Petzeltová [8] in spaces similar to those in [2]. The existence of stable, unstable, and center manifolds for the nonlinear perturbation (2) was proved in [8] and [9], but the assumptions, namely the requirement on the decay of the Laplace transform of ν, did not allow, e.g., the presence of point masses in the measure ν. Here we take advantage of the Hilbert space setting to enlarge the set of possible kernels, and to treat also neutral equations where $\mu \neq 0$. Two examples demonstrating this extension are given in Section 6. Our results can also be applied to models of a continuum of diffusively coupled oscillators which are described by partial neutral differential equations with finite delay, i.e. with measures supported on some finite interval. These equations have been studied by J. K. Hale in [4, 5], where the existence of the global attractor and behaviour of solutions near an equilibrium point are discussed.

In this paper, we do not use the semigroup directly in the construction of stable and unstable manifolds, but in the spirit of [11] we decompose the space Z into its stable and unstable parts: we call a pair $\binom{\phi}{f}$ consisting of the initial function and the right hand side an element of the stable subspace of the problem (1) iff the corresponding solution belongs to a solution space with a suitable decay rate at infinity. The decomposition can be performed provided that there is a line parallel to the imaginary axis which does not intersect the spectrum of the problem, and where the operators $(\lambda - A + \lambda\hat{\mu}(\lambda) - \hat{\nu}(\lambda))^{-1}: H \to \mathscr{D}(A)$ obtained from D by the Laplace transform are uniformly bounded. It is proved that the restrictions of the semigroup to the stable and unstable subspaces are groups that are similar to translation groups on $L^2_\alpha(\mathbf{R}, H)$ and $\mathscr{N}(D)$, respectively. Projectors onto the stable and unstable subspaces are constructed with the help of a right inverse of D in the space of stable solutions, obtained by the bilateral Laplace transform method and a Paley–Wiener theorem. Using the (unilateral) Laplace transform we get a right inverse also of the restriction of D to the space of functions vanishing on $\mathbf{R}^-$. These inverse operators enjoy the maximal regularity property, so the corresponding solution formulas enable us to solve even the fully nonlinear equation (2).

We prove an existence of a stable and an unstable manifold of solutions to (2). These are attractive for large negative and positive times respectively. Restricting the solutions in the stable (unstable) manifold to $\mathbf{R}^-$ we obtain the initial-valued version of the stable (unstable) manifold theorem. Our approach here is similar to that used in [10].

Finally, in the last section, we examine the spectrum of the generator of the translation semigroup in various weighted spaces, including the stable and unstable subspaces of our problem.

2. PRELIMINARIES

We consider the problem

$$Dx(t) = f(t), \qquad t \in \mathbf{R}^+, \tag{3}$$

$$x(t) = \phi(t), \qquad t \in \mathbf{R}^-, \tag{4}$$

where

$$Dx \equiv \frac{d}{dt}(x + \mu * x) - Ax - \nu * x, \tag{5}$$

$$A \text{ is the generator of an analytic semigroup } e^{At} \text{ on a Hilbert space } H, \tag{6}$$

$$\mu \in M(\mathbf{R}^+, L(H), e^{-\gamma t}), \qquad \nu \in M(\mathbf{R}^+, L(\mathscr{D}(A), H), e^{-\gamma t}), \qquad \gamma < 0. \tag{7}$$

Here $M(\mathbf{R}^+, L(X, Y), e^{-\gamma t})$ is a set of $L(X, Y)$ dominated measures on $\mathbf{R}^+$, satisfying

$$\|\mu\| = \int_{\mathbf{R}^+} e^{-\gamma t}\, d\,|\mu|(t) < \infty,$$

and the convolutions in (5) are defined by

$$(\eta * x)(t) = \int_{-\infty}^{t} d\eta(t-s)\, x(s).$$

We treat the equation in the weighted L^2 spaces

$$Y_\alpha = L^2_\alpha(\mathbf{R}, H) = L^2(\mathbf{R}, H, e^{-\alpha t}), \qquad \alpha \geqslant \gamma,$$

$$Y_\alpha = \left\{ x\colon \mathbf{R} \to H \text{ is strongly measurable, and } \|x\|_\alpha = \left[\int_{\mathbf{R}} \|x(t)\|_H^2\, e^{-2\alpha t}\, dt\right]^{1/2} < \infty \right\}.$$

Without loss of generality, we can assume that the order of the semigroup e^{At} is strictly less than $\min\{\alpha, 0\}$. (Otherwise we replace A by $A - aI$ and ν by $\nu + a\delta$, where a is a sufficiently large number and δ is the Dirac measure concentrated at zero.) Then

$$G = \mathscr{D}(A) \text{ endowed with the norm } \|x\|_G = \|Ax\|_H$$

is again a Hilbert space. We define the *solution spaces*

$$W_\alpha = W_\alpha^{1,2}(\mathbf{R}, H) \cap L_\alpha^2(\mathbf{R}, G), \qquad \|x\|_{W_\alpha} = \|x\|_\alpha + \|\dot{x}\|_\alpha + \|Ax\|_\alpha.$$

For $x \in Y_\alpha$, define $\pi^+ x$, $\pi^- x$ to be the restriction of x on $\mathbf{R}^+$, $\mathbf{R}^-$, respectively, and let $Y_\alpha^\pm$, $W_\alpha^\pm$, denote the corresponding restrictions of the spaces Y_α, W_α. We call Y_α^+ the *forcing function space* and W_α^- the *initial function space.*

We denote both the restricted functions and the functions extended by zero to $\mathbf{R}^\pm$ by $\pi^\pm f$ and we add functions belonging to Y_α^-, Y_α^+, Y_α in the obvious way. For $t \in \mathbf{R}$ and $x \in Y_\alpha$ we denote by τ_t the shift operator and by x_t and x^t the restrictions of the shifted function to $\mathbf{R}^-$ and $\mathbf{R}^+$:

$$\tau_t x(s) = x(s+t) \qquad \text{for} \quad s \in \mathbf{R}, \qquad x_t = \pi^- \tau_t x, \quad x^t = \pi^+ \tau_t x. \tag{8}$$

We shall also use spaces with different weights on $\mathbf{R}^-$, $\mathbf{R}^+$ respectively, namely

$$Y_{\alpha,\beta} = L^2(\mathbf{R}, H, \eta_{\alpha,\beta}), \qquad \text{where} \quad \eta_{\alpha,\beta} = \begin{cases} e^{-\alpha t} & \text{on } \mathbf{R}^-, \\ e^{-\beta t} & \text{on } \mathbf{R}^+, \end{cases}$$

as well as the corresponding solution spaces

$$W_{\alpha,\beta} = W^{1,2}(\mathbf{R}, H, \eta_{\alpha,\beta}) \cap L^2(\mathbf{R}, G, \eta_{\alpha,\beta}).$$

The operator E is called *causal* if $\phi(t) = 0$ for $t \in \mathbf{R}^-$ implies $E\phi(t) = 0$ for $t \in \mathbf{R}^-$.

We call E *autonomous* or *time-invariant* if $E\tau_t x = \tau_t Ex$ for every $x \in \mathscr{D}(E)$ and $t \in \mathbf{R}$.

The following lemma is an easy consequence of the foregoing definitions.

LEMMA 1. *The operator D, given by* (5), (6), and (7) *is a causal autonomous operator from $W_{\alpha,\beta}$ into $Y_{\alpha,\beta}$. The shift operator τ_t is a group on both W_α and Y_α and on both these spaces $\|\tau_t\| = e^{\alpha t}$.*

3. A SOLUTION FORMULA AND EXISTENCE THEOREMS

We will use the Laplace transform method to get a solution of a linear equation

$$\begin{aligned} Dx(t) &= f(t) \qquad t \in \mathbf{R}^+, \\ x &= 0 \qquad t \in \mathbf{R}^-. \end{aligned} \tag{9}$$

As $\widehat{Dx} = (\lambda - A + \lambda\hat{\mu}(\lambda) - \hat{v}(\lambda))\hat{x}(\lambda)$, it makes sense to define the distribution Laplace transform $\hat{D}(\lambda)$ of D to be

$$\hat{D}(\lambda) = \lambda - A + \lambda\hat{\mu}(\lambda) - \hat{v}(\lambda), \qquad \operatorname{Re}\lambda \geqslant \gamma,$$

and to examine the inverse $\hat{D}(\lambda)^{-1}$ to get a solution of (9) as the inverse Laplace transform of $\hat{D}(\lambda)^{-1}\hat{f}(\lambda)$. In the case of a nonhomogeneous initial condition we can incorporate the convolutions $(d/dt)(\mu * \phi) + v * \phi$ into the right handside.

The following lemma is based on a generalization of the Paley–Wiener theorem to the Laplace transform of Hilbert space valued L^2 functions on $\mathbf{R}^+$:

LEMMA 2. *Let $x \in L^2(\mathbf{R}^+, H, e^{-\gamma t})$. Then x is analytic for* $\operatorname{Re} z > \gamma$ *and*

$$\sup_{\sigma > \gamma} \int_{\mathbf{R}} \|\hat{x}(\sigma + i\omega)\|^2 \, d\omega = 2\pi \int_{\mathbf{R}} \|e^{-\gamma t}x(t)\|^2 \, dt < \infty.$$

Conversely, to every H-valued function ϕ, which is defined and analytic on $\operatorname{Re}\lambda > \gamma$, *and satisfies*

$$\sup_{\sigma > \gamma} \int_{\mathbf{R}} \|\phi(\sigma + i\omega)\|^2 \, d\omega < \infty,$$

there is a unique function $x \in L^2(\mathbf{R}^+, H, e^{-\gamma t})$, satisfying $\hat{x}(\lambda) = \phi(\lambda)$ for $\operatorname{Re}\lambda > \gamma$.

With the aid of this lemma, Staffans [11] proved the following:

LEMMA 3. *Suppose that there is a constant $\beta > \gamma$ such that*

$$\sup_{\operatorname{Re}\lambda > \beta} \|\hat{D}(\lambda)^{-1}\|_{L(H,G)} < \infty, \qquad \sup_{\operatorname{Re}\lambda > \beta} (1 + |\lambda|)\, \|\hat{D}(\lambda)^{-1}\|_{L(H)} < \infty. \tag{10}$$

For each $f \in Y_\beta^+ = L^2(\mathbf{R}^+, H, e^{-\beta t})$, let $R_\beta f$ be the function in $L^2(\mathbf{R}^+, G, e^{-\beta t})$ satisfying

$$(\widehat{R_\beta f})(\lambda) = \hat{D}(\lambda)^{-1}\hat{f}(\lambda) \qquad \textit{for} \quad \operatorname{Re}\lambda > \beta$$

(cf. Lemma 2). Then the operator R_β maps Y_β^+ linearly and continuously into $W_\beta^+ = L_\beta^2(\mathbf{R}^+, G) \cap W_\beta^{1,2}(\mathbf{R}^+, H)$, and it has a unique extension to a continuous, linear, causal and autonomous operator from Y_β into W_β.

See Foures and Segal [3] for a closely related result.

Remark 1. It follows from Lemma 3 that $(R_\beta f)(0)=0$ and that R_β is a right inverse operator to D:

$$DR_\beta = I \quad \text{on } Y_\beta, \quad \text{and} \quad R_\beta Df = f \quad \text{if} \quad f \in W_\beta \quad \text{and} \quad \pi^- f = 0. \tag{11}$$

To derive a solution formula for (3) and (4) with arbitrary $\phi \in W_\alpha^-$, we need the following "extension lemma".

LEMMA 4. *There is a continuous "extension mapping" E that maps*

$$W_\alpha^- = W_\alpha^{1,2}(\mathbf{R}^-, H) \cap L_\alpha^2(\mathbf{R}^-, G) \qquad \textit{into} \quad W_\alpha^{1,2}(\mathbf{R}, H) \cap L_\alpha^2(\mathbf{R}, G) = W_\alpha.$$

Proof. Define

$$E\phi(t) = \begin{cases} \phi(t), & \text{for} \quad t \leqslant 0, \\ e^{2\alpha t}\phi(-t), & \text{for} \quad t > 0. \end{cases}$$

Then $e^{-\alpha \cdot}\phi \in L^2(\mathbf{R}^-, G) \cap W^{1,2}(\mathbf{R}^-, H)$, so ψ: $\psi(t) = e^{\alpha t}\phi(-t)$ belongs to $L^2(\mathbf{R}^+, G) \cap W^{1,2}(\mathbf{R}^+, H)$, and $E\phi \in W_\alpha$. ▮

PROPOSITION 1. *Let* (10) *hold for some $\beta > \alpha$ and let x be a solution of* (3), (4). *Then*

$$x = E\phi + R_\beta \pi^+(f - DE\phi), \tag{12}$$

where E is an arbitrary extension mapping.

Proof. Suppose that (10) holds for some $\beta > \alpha$ and let E be an extension mapping. Then $\phi \in W_\alpha^-$ implies $DE\phi \in Y_\alpha$ and, if x satisfies (3) and (4), then

$$D(x - E\phi) = \begin{cases} 0 & \text{on } \mathbf{R}^-, \\ \pi^+(f - DE\phi) & \text{on } \mathbf{R}^+. \end{cases}$$

It follows from Lemma 3 that $x - E\phi = R_\beta \pi^+(f - DE\phi)$ on $\mathbf{R}^+$, $x - E\phi = 0$ on $\mathbf{R}^-$, and the solution formula follows. The solution defined by (12) automatically fulfils the initial condition (4). ▮

It should be pointed out that, due to the causality of D and R_β, x does not depend on the choice of operator E. For two extensions E_1, E_2 we have $\pi^-(E_1\phi - E_2\phi) = 0$ and, according to (11),

$$R_\beta \pi^+ D(E_1\phi - E_2\phi) = R_\beta D(E_1\phi - E_2\phi) = E_1\phi - E_2\phi,$$

so the corresponding solutions in (12) coincide.

Also $\pi^- DE\phi$ does not depend on the choice of E, so in the following we will simply write $\pi^- D\phi$ instead of $\pi^- DE\phi$.

The following existence theorem was proved in [11].

THEOREM 1. *Let D satisfy* (10) with some $\beta > \alpha$, *or, equivalently*

$$\inf_{\|x\|_H < 1,\ \mathrm{Re}\,\lambda > \beta} \|[I + \hat{\mu}(\lambda) - (\hat{\nu}(\lambda) A^{-1} - \hat{\mu}(\lambda)) A(\lambda I - A)^{-1}] x\|_H > 0. \tag{13}$$

Then for each $\phi \in W_\alpha^-$, $f \in Y_\beta^+$ the problem (3), (4) *has a unique solution $u \in W_{\alpha,\beta}$, which depends linearly and continuously on* (ϕ, f). *Moreover,*

$$T(t)(\phi, f) = (u_t, f^t) \tag{14}$$

((x_t, x^t) *defined in* (8)) *is a strongly continuous semigroup in the space $W_\alpha^- \otimes Y_\beta^+$ with*

$$\|T(t)\| \leqslant Me^{\beta t}.$$

Remark 2. As we shall see below, $T(t)$ is even a group.

Remark 3. The condition (13), which is equivalent to (10), is satisfied (with β large enough) when D is given by (5), (6), and (7) and μ, ν have no point masses at zero. Even if ν has a point mass at zero which belongs to $L(H)$, condition (13) is still fulfilled (see [11, p. 189]).

Now we can use the solution formula (12) together with the global contraction principle in the space $W_{\alpha,\beta}$ and the local or global implicit function theorem to prove two existence theorems for the nonlinear perturbed equation (2).

THEOREM 2. *Let* (13) *be fulfilled, let $\phi \in W_\alpha^-$ and let F be a Lipschitz continuous mapping from $W_{\alpha,\beta}$ into Y_β^+ with*

$$\|F(x) - F(y)\|_\beta \leqslant L_F \|x - y\|_{W_{\alpha,\beta}}, \qquad L_F \|R_\beta\|_{L(Y_\beta, W_\beta)} < 1.$$

Then there is a unique solution to the problem (2), (4) in $W_{\alpha,\beta}$.

THEOREM 3. *Let* (13) *be fulfilled, let Ω be a neighbourhood of zero in $W_{\alpha,\beta}$ and let $F \in C^1(\Omega, Y_\beta^+)$ satisfy $F(0) = 0$ and* (*the Fréchet derivative*) *$F'(0) = 0$. Then there exist neighbourhoods U and V of zero in W_α^- and $W_{\alpha,\beta}$, respectively, such that for each $\phi \in U$, Eq.* (2) *has a unique solution in V satisfying the initial condition* (4).

4. STABLE AND UNSTABLE SUBSPACES

In this section, we suppose that $\alpha \geqslant -\gamma$ and that the line $\operatorname{Re} \lambda = \alpha$ is noncritical, i.e., $\hat{D}^{-1}$ exists for $\operatorname{Re} \lambda = \alpha$ and

$$\sup_{\operatorname{Re} \lambda = \alpha} \|\hat{D}(\lambda)^{-1}\|_{L(H,G)} < \infty, \qquad \sup_{\operatorname{Re} \lambda = \alpha} (1 + |\lambda|)\, \|\hat{D}(\lambda)^{-1}\|_{L(H)} < \infty. \tag{15}$$

With the help of the bilateral Laplace transform we obtain, in the same way as in Lemma 3, the existence of another right inverse operator to D, which this time may be noncausal.

Lemma 5. *Suppose that the line* $\operatorname{Re} \lambda = \alpha$ *is noncritical, i.e., condition* (15) *holds. Then D has a* (*noncausal*) *continuous inverse operator R_α that maps* $Y_\alpha = L^2_\alpha(\mathbf{R}, H)$ *onto* $W_\alpha = L^2_\alpha(\mathbf{R}, G) \cap W^{1,2}_\alpha(\mathbf{R}, H)$. *The bilateral Laplace transform of $R_\alpha f$ exists on* $\operatorname{Re} \lambda = \alpha$ *and satisfies*

$$(\widehat{R_\alpha f})(\lambda) = \hat{D}(\lambda)^{-1} \hat{f}(\lambda) \qquad \textit{for} \quad \operatorname{Re} \lambda = \alpha.$$

Remark 4. The boundedness of $\hat{D}(\lambda)^{-1}$ on the line $\operatorname{Re} \lambda = \alpha$ given in (15) is equivalent to (13) with the infimum taken over $\operatorname{Re} \lambda = \alpha$. This together with (6) and (7) yield the existence of $\alpha_1 > \alpha$ such that

$$\inf_{\|x\|_H < 1,\ \alpha \leqslant \operatorname{Re} \lambda \leqslant \alpha_1} \|[I + \hat{\mu}(\lambda) - (\hat{v}(\lambda) A^{-1} - \hat{\mu}(\lambda)) A(\lambda I - A)^{-1}] x\|_H > 0. \tag{16}$$

Then

$$f \in Y_{\alpha_1, \alpha} \Rightarrow R_\alpha f \in W_{\alpha_1, \alpha} \qquad \text{and} \qquad f \in Y_{\alpha, \alpha_1} \Rightarrow R_\alpha f \in W_{\alpha, \alpha_1}.$$

Suppose that $\phi \in W^-_\alpha$, $f \in Y^+_\alpha$ are such that the solution x of (3), (4) belongs to W_α. Then it makes sense to call (ϕ, f) the element of the stable subspace of the problem (3), (4).

Definition 1. The *stable subspace* $\mathscr{S}$ consists of

$$\left\{ \begin{pmatrix} \pi^- x \\ \pi^+ D x \end{pmatrix}; x \in W_\alpha \right\}.$$

Lemma 6. *The stable subspace can be characterized as*

$$\left\{ \begin{pmatrix} \pi^- R_\alpha f \\ \pi^+ f \end{pmatrix}; f \in L^2_\alpha(\mathbf{R}, H) \right\}.$$

Proof. Define $x = R_\alpha f$. Then $f = Dx$ and the assertion follows. ∎

Given an arbitrary element

$$\begin{pmatrix}\phi\\ f\end{pmatrix}\in\begin{pmatrix}W_\alpha^-\\ Y_\alpha^+\end{pmatrix},$$

define

$$x_\alpha = R_\alpha(\pi^- D\phi + f). \tag{17}$$

Then

$$x_\alpha \in W_\alpha, \qquad \text{and} \qquad Dx_\alpha = \pi^- D\phi + f. \tag{18}$$

Define

$$P_{\mathscr{S}}\begin{pmatrix}\phi\\ f\end{pmatrix}=\begin{pmatrix}\pi^- x_\alpha\\ f\end{pmatrix}.$$

This is a continuous projection onto the stable subspace and, consequently, $\mathscr{S}$ is a closed subspace of $W_\alpha^- \otimes Y_\alpha^+$.

It follows from (17) that x_α is a solution of (3) with the initial condition $\pi^- x_\alpha$. Hence x_α is a solution of (3), (4) iff $\binom{\phi}{f}\in\mathscr{S}$.

The unstable subspace $\mathscr{U}$ is defined as the image of the complementary projection $P_{\mathscr{U}} = I - P_{\mathscr{S}}$:

$$P_{\mathscr{U}}\begin{pmatrix}\phi\\ f\end{pmatrix}=\begin{pmatrix}\phi-\pi^- R_\alpha(\pi^- D\phi + f)\\ 0\end{pmatrix}.$$

Definition 2. The *unstable subspace* consists of $\{\binom{\phi}{0};\ \pi^- R_\alpha \pi^- D\phi = 0\}$.

Lemma 7. *$\mathscr{U}$ is a closed subspace of $W_\alpha^- \otimes Y_\alpha^+$ consisting of those elements $\binom{\phi}{0}$ for which the corresponding solution x of* (3), (4) *satisfies $Dx = 0$ both on $\mathbf{R}^+$ and on $\mathbf{R}^-$. Thus, $x\in\mathscr{N}(D)$, where D is considered to be an operator mapping $W_{\alpha,\beta}$ into $Y_{\alpha,\beta}$ and satisfying* (13). *If $\binom{\phi}{0}\in\mathscr{U}$, then $\phi\in W_{\alpha_1}^-$ for some $\alpha_1 > \alpha$ for which* (16) *holds.*

Proof. If $Dx = 0$ on all of $\mathbf{R}$, then

$$P_{\mathscr{U}}\begin{pmatrix}\phi\\ 0\end{pmatrix}=\begin{pmatrix}\phi-\pi^- R_\alpha(0)\\ 0\end{pmatrix}=\begin{pmatrix}\phi\\ 0\end{pmatrix},$$

so $\binom{\phi}{0}\in\mathscr{U}$.

Conversely, suppose that $\binom{\phi}{0} \in \mathscr{U}$. Then $\binom{\phi}{0} = P_{\mathscr{U}}\binom{\phi}{0}$, which implies that $\pi^- R_\alpha(\pi^- D\phi) = 0$ and, since $\pi^-\phi = \pi^- x$ and D is causal, also $\pi^- R_\alpha(\pi^- Dx) = 0$. From Eq. (3) we get $\pi^+ Dx = f = 0$, so

$$\pi^- R_\alpha(Dx) = \pi^- R_\alpha(\pi^- Dx + \pi^+ Dx) = 0.$$

Hence also $DR_\alpha Dx = 0$ on $\mathbf{R}^-$ and, since D is an inverse of R_α, $Dx = 0$ on $\mathbf{R}^-$. Together with $Dx = 0$ on $\mathbf{R}^+$, this implies $Dx = 0$ on all of $\mathbf{R}$.

To prove that $\phi = \pi^- x \in W^-_{\alpha_1}$, we multiply x by a smooth cut-off function e, $e(t) = 1$ for $t \in \mathbf{R}^-$, $e(t) = 0$ for $t > 1$. Then $ex \in W_\alpha$, $\pi^- Dex = 0$, so $Dex \in W_{\alpha', \alpha}$ with $\alpha' > \alpha$. The bilateral Laplace transform of ex is given by

$$\widehat{ex}(\lambda) = D(\lambda)^{-1}\, D(\lambda)\, \widehat{ex}(\lambda) = D(\lambda)^{-1}\, \widehat{Dex}(\lambda).$$

Thus according to (16) the bilateral Laplace transform of ex is bounded in the strip $\alpha \leqslant \operatorname{Re} \lambda \leqslant \alpha_1$ and, consequently, $ex \in W_{\alpha_1, \alpha}$. Hence $\phi = \pi^- ex \in W^-_{\alpha_1}$. ∎

Now, we examine the semigroup T restricted to the stable and unstable subspaces.

THEOREM 4. *The restriction of the semigroup T to the stable subspace $\mathscr{S}$ is a group that is similar to the shift on $L^2_\alpha(\mathbf{R}, H)$.*

Proof. For

$$\begin{pmatrix} \phi \\ f \end{pmatrix} \in \begin{pmatrix} W^-_\alpha \\ Y^+_\alpha \end{pmatrix}, \qquad \begin{pmatrix} \psi \\ g \end{pmatrix} \in \begin{pmatrix} Y^-_\alpha \\ Y^+_\alpha \end{pmatrix}$$

define

$$B\begin{pmatrix} \phi \\ f \end{pmatrix} = \begin{pmatrix} \pi^- Dx_\alpha \\ f \end{pmatrix}, \qquad \mathscr{B}\begin{pmatrix} \phi \\ f \end{pmatrix} = \pi^- Dx_\alpha + f, \qquad C\begin{pmatrix} \psi \\ g \end{pmatrix} = \begin{pmatrix} \pi^- R_\alpha(\psi + g) \\ g \end{pmatrix},$$

where x_α is given by (17). Then $\mathscr{B}\binom{\phi}{f} \in L^2_\alpha(\mathbf{R}, H)$,

$$C\begin{pmatrix} \pi^- Dx_\alpha \\ f \end{pmatrix} = \begin{pmatrix} \pi^- R_\alpha(\pi^- Dx_\alpha + f) \\ f \end{pmatrix} = \begin{pmatrix} \pi^- R_\alpha(\pi^- D\phi + f) \\ f \end{pmatrix} = \begin{pmatrix} \pi^- x_\alpha \\ f \end{pmatrix}.$$

Consequently, C is the inverse to B on $\mathscr{S}$, and (17) and (18) imply that

$$T(t)\begin{pmatrix} \phi \\ f \end{pmatrix} = \mathscr{B}^{-1}\tau_t \mathscr{B}\begin{pmatrix} \phi \\ f \end{pmatrix} \qquad \text{for} \quad \begin{pmatrix} \phi \\ f \end{pmatrix} \in \mathscr{S},$$

where the shift operator is defined by (8) and the semigroup T by (14). ∎

Remark 5. The restricted semigroup is also similar to the shift on the space $L^2_\alpha(\mathbf{R}, G) \cap W^{1,2}_\alpha(\mathbf{R}, H)$.

THEOREM 5. *The restriction of the semigroup T to $\mathscr{U}$ is also a group, which is similar to the translation group on the null space of D.*

Proof. Given $\binom{\phi}{0} \in \mathscr{U}$, let x be the solution of

$$\begin{aligned} Dx(t) &= 0, && t \in \mathbf{R}^+, \\ x(t) &= \phi(t), && t \in \mathbf{R}^-, \end{aligned} \tag{19}$$

Translate x left or right, and let the new ϕ be the restriction of the translated x to $\mathbf{R}^-$:

$$T(t)\begin{pmatrix}\phi\\0\end{pmatrix} = \begin{pmatrix}x_t\\0\end{pmatrix}, \qquad t \in \mathbf{R}. \quad \blacksquare$$

COROLLARY 1. *T is a group.*

See Section 7 for a discussion of the spectrum of the generator of T.

Remark 6. Initial and forcing function semigroup versions of the group T can also be constructed. The method is the same as the one used in [12] in a finite-dimensional setting.

We also obtain a relation between the solution of (3), (4) and its stable and unstable parts as follows. Let v, v_s, v_u be solutions of (3), (4) corresponding to

$$\begin{pmatrix}\phi\\f\end{pmatrix} \in W^-_\alpha \times Y^+_\alpha, \qquad \begin{pmatrix}\phi_s\\f\end{pmatrix} = P_{\mathscr{S}}\begin{pmatrix}\phi\\f\end{pmatrix}, \qquad \begin{pmatrix}\phi_u\\0\end{pmatrix} = P_{\mathscr{U}}\begin{pmatrix}\phi\\f\end{pmatrix},$$

respectively. Then

$$v_s = R_\alpha D v, \qquad v_u = (I - R_\alpha D) v,$$

$R_\alpha D$ being a projector of the space of all solutions in $W_{\alpha,\beta}$ of (3), (4) with $\binom{\phi}{f} \in W^-_\alpha \times Y^+_\alpha$ onto the space of stable solutions, i.e., W_α. In fact, $v_s = R_\alpha(f + \pi^- D\phi_s)$ while v is given by (12). Now $Dv = f + \pi^- D\phi$ and a straightforward computation gives $R_\alpha \pi^- D\phi_s = R_\alpha \pi^- D\phi$.

In order to construct stable and unstable manifolds for a nonlinear perturbation of the homogeneous equation we will also need projections of the space $W_{\alpha,\beta}$ onto the stable and unstable subspaces of the homogeneous equation. Any solution of (3), (4) with $\binom{\phi}{f} \in W^-_\alpha \otimes Y^+_\beta$ can be decomposed into the solution of the homogeneous equation (19) and the solution of (9).

The solution of (19) can be further decomposed into its stable and unstable parts. Let

$$\mathscr{S}^I = \{x \in W_\alpha;\ \pi^+ Dx = 0\}$$

be the space of stable solutions of the homogeneous equation. (It can be identified with the stable initial functions subspace: $x \in \mathscr{S}^I$ iff $\binom{\pi^- x}{0} \in \mathscr{S}$.) Then we can decompose the space $W_{\alpha,\beta}$ and its elements as follows.

$$W_{\alpha,\beta} = \mathscr{S}^I \oplus W_\beta^+ \oplus \mathscr{N}(D),$$

$$x = P_\alpha x + P_\beta x + y,$$

where P_α, P_β are projectors of $W_{\alpha,\beta}$ onto $\mathscr{S}^I$, W_β^+ respectively:

$$\begin{aligned} P_\alpha x &= R_\alpha \pi^- Dx, \\ P_\beta x &= R_\beta \pi^+ Dx, \\ Px &= P_\alpha x + P_\beta x, \\ y &= (I - P)x. \end{aligned} \tag{20}$$

The operator $I - P$ is a projector of $W_{\alpha,\beta}$ onto $\mathscr{N}(D)$ and D is invertible on $\mathscr{R}(P)$.

5. STABLE AND UNSTABLE MANIFOLDS

Now, we turn our attention to the nonlinear equation (2). We shall treat it as a perturbation of a homogeneous equation. That is, we shall suppose F to be "small" in the neighbourhood of zero, namely its value and its Fréchet derivative at zero are supposed to be zero,

$$F(0) = 0, \qquad F'(0) = 0. \tag{21}$$

The following theorems generalize Theorems 7.2 and 7.3 in [10] to the infinite-dimensional case.

THEOREM 6. *Let Ω be a neighbourhood of zero in W_α, $F \in C^1(\Omega, Y_\alpha^+)$ and let* (15) *and* (21) *hold. Then there are neighbourhoods V_1 and V_2 of zero in $\mathscr{S}^I$ and Ω, respectively, such that for every $y \in V_1$ there is a unique solution $x(y)$ of* (2) *in V_2 with $P_\alpha x(y) = y$. This solution x is continuously differentiable in y, and $x = y$ when $F \equiv 0$.*

Proof. The idea of the proof is the following. First, we prove that there is a neighbourhood V^- of zero in W_α^- and another neighbourhood $V_2 \subset \Omega$ of zero, such that for each $f \in V^-$, the equation

$$Dx = F(x) + f \tag{22}$$

has a solution $x \in V_2$. We denote by R the operator which to every f assigns the solution of (22) and then, given y, we solve the equation

$$P_\alpha R(f) = y \tag{23}$$

to obtain $f(y)$. Finally, we define $x(y) = R(f(y))$ to get the desired solution.

We begin the proof by applying the implicit function theorem to the mapping

$$\Pi(x, f) = x - R_\alpha(F(x) + f).$$

We observe that Π maps $\Omega \otimes W_\alpha^-$ into W_α with $\Pi(0, 0) = 0$ and $\Pi'_x(0, 0) = \mathrm{Id}$. So we get open sets V^- and V_2 and an operator $R\colon V^- \to V_2 \subset W_\alpha$ such that $x = R(f)$ is a solution of (22) with $R(0) = 0$ and $R'(0) = R_\alpha$.

A solution f of the Eq. (23) can be obtained in the following way: First, we realize that, according to (20),

$$P_\alpha R_\alpha f = R_\alpha f.$$

Then, we rewrite Eq. (23) as

$$R_\alpha f + P_\alpha(R(f) - R_\alpha f) - y = 0,$$

which is equivalent to

$$f + DP_\alpha(R(f) - R_\alpha f) - Dy = 0.$$

Now, we apply the implicit function theorem once more and obtain a function $f(y)$, and, consequently, $x(y) = R(f(y))$ and the conclusion of the theorem follows. ∎

Definition 3. We call the manifold

$$\mathscr{S}_F = \{x(y),\ y \in V_1 \subset \mathscr{S}^I\}$$

the stable manifold of solutions of (2).

It follows from the construction of $\mathscr{S}_F$ and the assumption $F(0)=0$, $F'(0)=0$ that $\mathscr{S}_F$ is tangent to $\mathscr{S}^I$ at zero.

To get an existence of an unstable manifold we have to suppose that not only does F map the space W_α into Y_α, but also $W_{\alpha,\beta}$ into $Y_{\alpha,\beta}$. In many applications this assumption is not satisfied. A typical unstable manifold theorem primarily describes the behaviour of solutions for negative time, i.e., it asserts the existence of initial functions that belong to W_α^- or $W_{\alpha'}^-$, $\alpha<\alpha'<\alpha_1$ respectively, and satisfy the nonlinear equation themselfs. Fortunately, it is often possible to redefine the original operator F for functions that are "large" at infinity in such a way that our assumptions are met. This can be done with the help of some cut-off function in the same way as in standard proofs of local center manifolds for ODE's. The assertion of the theorem is then valid only for those solutions of the original equation that are "small" at infinity. (See our first example.)

At this point, because of the presence of the right hand side of (2) in the space Y_β^+, we have to use the projection operator

$$Px=R_\alpha\pi^- Dx+R_\beta\pi^+ Dx, \qquad x\in W_{\alpha,\beta}.$$

constructed at the end of Section 4. The complementary projector $I-P$ maps $W_{\alpha,\beta}$ onto $\mathscr{N}(D)\subset W_{\alpha_1,\beta}$, where α_1 is given in Lemma 7.

THEOREM 7. *Let Ω be a neighbourhood of zero in $W_{\alpha,\beta}$, let F be a C^1 mapping of Ω into $Y_{\alpha,\beta}$ and let* (13), (15), *and* (21) *hold. Then there are neighbourhoods U_1 and U_2 of zero in $\mathscr{N}(D)$ and Ω, respectively, such that for every $z\in U_1$ there is a unique solution $x(z)\in U_2$ of* (2) *with $(I-P)\,x(z)=z$. This solution $x(z)$ is continuously differentiable in z, and $x(z)=z$ when $F\equiv 0$. Moreover, $x\in W_{\alpha',\beta}$, whenever F maps $W_{\alpha',\beta}$ into $Y_{\alpha',\beta}$ and $\alpha\leqslant\alpha'\leqslant\alpha_1$.*

Proof. Let z be a solution of the equation

$$Dz=0 \qquad \text{on } \mathbf{R}.$$

Find a function $x\in W_{\alpha,\beta}$ such that

$$x=R_\alpha\pi^- F(x+z)+R_\beta\pi^+ F(x+z) \tag{24}$$

(this is just another application of the implicit function theorem). The solution of (24) satisfies

$$Dx=F(x+z).$$

Moreover, $Px = x$, as D is an inverse to both R_α and R_β. Now, $x(z) = x + z$ satisfies the equation

$$Dx(z) = F(x(z)) \qquad \text{on } \mathbf{R},$$
$$(I - P)\,x(z) = (I - P)z = z,$$

and the first assertion of the theorem follows.

If F maps $W_{\alpha', \beta}$ into $Y_{\alpha', \beta}$, then the same argument with α replaced by α' shows that the solution x belongs to the space $W_{\alpha', \beta}$. ∎

Definition 4. We call the manifold

$$\mathcal{U}_F = \{x(z),\ z \in \mathcal{N}(D)\}$$

the unstable manifold of solutions of (2).

It follows from the construction of $\mathcal{U}_F$ and the assumption $F(0) = 0$, $F'(0) = 0$ that $\mathcal{U}_F$ is tangent to $\mathcal{N}(D)$ at zero. Note that $\mathcal{N}(D)$ can be identified with $\mathcal{U}$.

Next, we shall prove the initial function versions of Theorems 6 and 7.

Definition 5. We define the local stable manifold of initial functions as

$$\mathcal{S}^I_F = \{\pi^- x,\ x \in \mathcal{S}_F\}.$$

We are going to show that there exists a neighbourhood V_1^- of zero in $\mathcal{S}^-$ and a mapping $h \in C^1(V_1^-, \mathcal{U}^-)$ such that

$$\mathcal{S}^I_F = \{\phi + h(\phi),\ \phi \in V_1^- \subset \mathcal{S}^-\}.$$

To this end, we decompose the space W_α^- with the help of the projections $P_{\mathcal{S}}$ and $P_{\mathcal{U}}$ restricted to W_α^-. We have

$$P_{\mathcal{S}}\pi^- = \pi^- P_{\mathcal{S}}\pi^- = \pi^- P_\alpha \pi^- = \pi^- P_\alpha, \qquad P_{\mathcal{U}}\pi^- = \pi^-(I - P_{\mathcal{S}})\pi^-.$$

Denote

$$P_{\mathcal{S}} W_\alpha^- = \mathcal{S}^-, \qquad P_{\mathcal{U}} W_\alpha^- = \mathcal{U}^-.$$

Now, for each $\phi \in \mathcal{S}^-$ we have $y \in \mathcal{S}^I$, where y is the solution of

$$Dy = 0 \qquad \text{on } \mathbf{R}^+, \qquad y = \phi \qquad \text{on } \mathbf{R}^-.$$

For ϕ sufficiently close to zero, Theorem 6 yields the existence of $x \in W_\alpha$, a solution of (2) such that $P_\alpha x = y$. It follows that

$$P_{\mathscr{S}} \pi^- x = \pi^- y = \phi,$$

so we can write

$$\pi^- x = \phi + h(\phi), \qquad \text{where} \quad h(\phi) = \pi^- x - P_{\mathscr{S}} \pi^- x \in \mathscr{U}^-.$$

The function h is differentiable, as it was obtained through a differentiable process.

Having an initial condition $\phi \in \mathscr{S}_F^I$, we get a global solution of (2) in W_α. The manifold $\mathscr{S}_F^I$ is invariant with respect to solutions of (2) in the sense that $u_t \in \mathscr{S}_F^I$ whenever $\phi \in \mathscr{S}_F^I$ (u_t was defined in (8)). In fact, the equation is autonomous and the operator R_α is autonomous too, as the Laplace transform commutes with the shifts.

The same reasoning leads to the definition of unstable manifold of initial functions.

Definition 6. We define the unstable manifold of initial functions as

$$\mathscr{U}_F^I = \{\pi^- x, x \in \mathscr{U}_F\}.$$

Again, we obtain a differentiable function k mapping a neighbourhood of zero in $\mathscr{U}^-$ into a neighbourhood of zero in $\mathscr{S}^-$ such that the initial function $\phi + k(\phi)$ satisfies the nonlinear Eq. (2) on $\mathbf{R}^-$.

$$\mathscr{U}_F^I = \{\phi + k(\phi), \phi \in U_1^- \subset \mathscr{U}^-\}.$$

Our intention is to show that solutions which belong neither to the stable nor to the unstable manifold are close to stable ones at minus infinity and to unstable ones at plus infinity.

Any initial condition can be expressed as a sum of elements from $\mathscr{S}_F^I$ and $\mathscr{U}$:

$$\phi = P_{\mathscr{S}} \phi + P_{\mathscr{U}} \phi = (P_{\mathscr{S}} \phi + h(P_{\mathscr{S}} \phi)) + (P_{\mathscr{U}} \phi - h(P_{\mathscr{S}} \phi)),$$

so a general solution of (2) is "close" to the stable solution at minus infinity (since elements of $\mathscr{U}^-$ belong to $W_{\alpha_1}^-$, $\alpha_1 > \alpha$).

We can prove that x is close to an unstable solution at plus infinity if we are able to solve the equation

$$D(x+y) = F(x+y) \tag{25}$$

for $y \in W_\alpha$. Then $x+y \in \mathscr{U}_F$, as $z=(I-P)(x+y) \in \mathscr{N}(D)$, and we can apply Theorem 6. Now $x=x+y-y$, $y \in W_\alpha$ and the assertion follows. To prove the existence of $y \in W_\alpha$, satisfying (25), we rewrite (25) in the form

$$Dy = F(x+y) - F(x),$$

which is equivalent to

$$y = R_\alpha(F(x+y) - F(x)),$$

provided that $F(x+y) - F(x) \in Y_\alpha$ for $y \in W_\alpha$. Moreover, we want to use the implicite function theorem again, so we need to suppose that there exists a neighbourhood V of zero in $C^1(U, Y_\alpha)$, U being a neighbourhood of zero in W_α, such that the mapping G_x: $G_x(y) = F(x+y) - F(x)$ belongs to V:

$$G_x \in V \qquad \text{for} \quad x \in \Omega, \quad x \text{ solution of (2).} \tag{26}$$

Under this assumption, there exists a neighbourhood U of zero in W_α such that each solution of (2), which belongs to U, has to be in the stable manifold $\mathscr{S}_F$. Otherwise it would be close to the unstable solution $x+y$.

We can summarize these considerations into the following theorem.

THEOREM 8. *Let D be given by* (5), (6), *and* (7). *Let F be a C^1 mapping of a neighbourhood Ω of zero in W_α (respectively $W_{\alpha,\beta}$) into Y_α (respectively $Y_{\alpha,\beta}$), and let* (10), (15), *and* (21) *hold. Then there are two differentiable functions h and k mapping neighbourhoods V_1^- (respectively U_1^-) of zero in $\mathscr{S}^-$ (respectively $\mathscr{U}^-$) into $\mathscr{U}^-$ (respectively S^-), such that setting*

$$\mathscr{S}_F^I = \{\psi + h(\psi), \psi \in V_1^- \subset \mathscr{S}^-\},$$

$$\mathscr{U}_F^I = \{\psi + k(\psi), \psi \in U_1^- \subset \mathscr{U}^-\},$$

we obtain the following conclusions:

(i) *$\mathscr{S}_F^I(\mathscr{U}_F^I)$ is tangent to $\mathscr{S}^-(\mathscr{U}^-)$ at the origin.*

(ii) *For any $\phi \in \mathscr{S}_F^I$ ($\phi \in \mathscr{U}_F^I$) there is a solution of the problem* (2), (4) *in W_α ($W_{\alpha,\beta}$).*

(iii) *If x is a solution of* (2), (4) *with $\phi \in \mathscr{S}_F^I$, then $x_t \in \mathscr{S}_F^I$ for all $t \in \mathbf{R}^+$, where x_t is defined by* (8).

(iv) *Each $\phi \in \mathscr{U}_F^I$, satisfies* (2) *with x replaced by ϕ on $\mathbf{R}^-$.*

(v) *$\phi \in \mathscr{U}_F^I$ belongs to $W_{\alpha'}$ whenever F maps $W_{\alpha',\beta}$ into $Y_{\alpha',\beta}$ and $\alpha \leqslant \alpha' \leqslant \alpha_1$.*

(vi) *There exists a neighbourhood U of zero in $W_{\alpha,\beta}$ such that if $x \in U$ is a solution of* (2) *and $x \notin \mathcal{S}_F$, $x \notin \mathcal{U}_F$, then*

(a) $\pi^- x = \phi_S + \phi_U$, *where* $\phi_S \in \mathcal{S}_F^I$ *and* $\phi_U \in \mathcal{U}^- \subset W_{\alpha_1}^-$,

(b) $x = x_1 + x_2$, *where* $x_1 \in \mathcal{U}_F$ *and* $x_2 \in W_\alpha$ *provided that* (26) *holds.*

Remark 7. (vi)(b) implies that each solution of (2) which belongs to $U \cap W_\alpha$ is in $\mathcal{S}_F$.

Remark 8. As $L^2(\mathbf{R}^+, G) \cap W^{1,2}(\mathbf{R}^+, H)$ is imbedded in the space $C(\mathbf{R}^+, [G, H]_{1/2})$, where $[G, H]_{1/2}$ denotes the half-way interpolation space between G and H, we obtained exponential decay of solutions in the stable manifold in the norm of the space $[G, H]_{1/2}$ as $t \to +\infty$ whenever $\alpha < 0$, and a similar exponential dexay of solutions in the unstable manifold as $t \to -\infty$.

6. EXAMPLES

Consider the problem

$$\begin{aligned} &u_t(t,x) - u_{xx}(t,x) + u(t,x) - u^p(t,x) \\ &\qquad - \sum_{i=1}^{\infty} a_i u_{xx}(t - t_i, x) + (l * g(u))(t,x) = 0, \qquad (27) \\ &\qquad t \in \mathbf{R}^+, \quad x \in \mathbf{R}, \quad p > 1 \\ &u(t,x) = \phi(t,x) \qquad \text{for} \quad t \in \mathbf{R}^-, \quad x \in \mathbf{R}. \end{aligned}$$

Here

$$g \in C^2(\mathbf{R}), \qquad |g(z)| \leqslant C\,|z|,$$

$$\sum_{i=1}^{\infty} a_i e^{\gamma t_i} < \infty \qquad \text{for some} \quad \gamma > 0, \qquad a_i \geqslant 0,$$

$$\int_0^\infty |l(t)|\, e^{\gamma t}\, dt < \infty, \qquad \int_0^\infty l(t)\, dt = 0.$$

It is known (see e.g. [1]) that the equation

$$-v_{xx} + c(v - v^p) = 0 \qquad (28)$$

has a unique solution in $L^2(\mathbf{R})$, which is smooth, radially symmetric and radially exponentially decreasing. This solution is called a “ground state”.

Let w be such a solution for the Eq. (28) with $c = 1/(1 + \sum a_i)$. Then, due to the assumption on l, $w(t, x) = w(x)$ satisfies

$$w_t - w_{xx} + w - w^p - k * w_{xx} + l * g(w) = 0, \tag{29}$$

where $k = \sum_{i=1}^{\infty} a_i \delta_{t_i}$; here δ_b denotes the Dirac function concentrated at b.

Denoting $z = u - w$ and subtracting (27) and (29) we get the equation for z,

$$z_t - z_{xx} + z - a(x)z - k * z_{xx} + b(x) l * z = F(z),$$
$$z(t, x) = \phi(t, x) - w(x), \qquad t \in \mathbf{R}^-,$$

with

$$a = pw^{p-1}, \qquad b = g'(w), \qquad F(z)(t) = f(z(t)) + l * q(z)(t),$$
$$f(0) = f'(0) = q(0) = q'(0) = 0.$$

This equation can be rewritten in the form (2) in the space $H = L^2(\mathbf{R})$ with

$$A : \mathscr{D}(A) = W^{2,2}(\mathbf{R}), \qquad Az = z_{xx} - z + a(x)z,$$
$$\mu = 0, \qquad \nu = k(A + 1 - a) - b(x)l.$$

It follows from the theory of the Schrödinger operators with rapidly decreasing potentials (see [6]) that the spectrum of A is of the form

$$\sigma(A) = (-\infty, -\lambda_1] \cup \{0, \lambda_2\}, \lambda_i > 0.$$

To show that (15) holds for some $\gamma < \alpha < 0$, we have to examine

$$\hat{D}(\lambda) = (\lambda - A - \hat{k}(\lambda) A + \hat{k}(\lambda)(a-1) + \hat{l}(\lambda) b)$$
$$= (1 + \hat{k}(\lambda)) \left(\frac{\lambda}{1 + \hat{k}(\lambda)} - A + \frac{\hat{k}(\lambda)(a-1) + \hat{l}(\lambda) b}{1 + \hat{k}(\lambda)} \right).$$

Now, if we take k, l small enough, we can consider $A - (\hat{k}(\lambda)(a-1) + \hat{l}(\lambda) b)/(1 + \hat{k}(\lambda))$ to be a small perturbation of A by a commuting bounded operator, so that the spectra are close (see [6]). Further, $(1 + \hat{k}(\lambda))$ then remains close to 1, so we can find a line $\{\lambda \in \mathbf{C};\ -\lambda_1 < \operatorname{Re} \lambda = \alpha < 0\}$ such that the curve $(\lambda/(1 + \hat{k}(\lambda)))$ does not intersect the spectrum $\sigma(A - (\hat{k}(\lambda)(a-1) + \hat{l}(\lambda) b)/(1 + \hat{k}(\lambda)))$ for any λ, $\operatorname{Re} \lambda = \alpha$ and (15) holds.

Now, we can apply Theorem 6 if we realize that the space $W^{1,2}(\mathbf{R}, H) \cap L^2(\mathbf{R}, W^{2,2})$ is embedded in $C(\mathbf{R}, W^{1,2})$, which is further embedded in the space of continuous functions on $\mathbf{R}^2$; this implies that F is C^1 mapping of

W_0 into $Y_0 = L^2(\mathbf{R}\times\mathbf{R})$. Because of the form of the function f and the sublinear growth of g the mapping F maps the weighted space W_α into Y_α^+. We have obtained a local stable manifold at the nontrivial eqilibrium of (27).

Theorem 7 cannot be applied directly, as the nonlinearity does not map $W_{\alpha,\beta}$ into $Y_{\alpha,\beta}$. We can solve the equation for a nonlinearity

$$F_r\colon F_r(z)(t) = f_r(e_{(\alpha/p),(\beta/p)}(t)\,z(t)),$$

where r is a suitable positive number,

$$f_r(y) = f\left(\psi\left(\frac{y}{r}\right)y\right), \qquad e_{\alpha,\beta} = \begin{cases} e^{-\alpha t} & \text{on } \mathbf{R}^-, \\ e^{-\beta t} & \text{on } \mathbf{R}^+ \end{cases},$$

and ψ is a smooth cut-off function,

$$\psi\colon G\to\mathbf{R}, \qquad |\psi(y)|<1,$$
$$\psi(y)=1 \qquad \text{iff} \quad \|y\|_G\leqslant 1,$$
$$\psi(y)=0 \qquad \text{for} \quad \|y\|_G\geqslant 2.$$

In this way obtain the assertion of Theorem 7 for those solutions of (27) that remain in an appropriate neighbourhood of zero in $W_{(\alpha/p),(\beta/p)}$.

As another example we consider the equation

$$\frac{d}{dt}\left(u(t,x)+\int_{-\infty}^{t} d\mu(t-s)\,u(s,x)\right) - \Delta u(t,x) - \int_{-\infty}^{t} d\nu(t-s)\,\Delta u(s,x)$$
$$= \int_{-\infty}^{t} d\eta(t-s)\,f(u(s,x), Du(s,x), D^2u(s,x)), \qquad t>0, \quad x\in\Omega,$$
$$\frac{\partial u(t,x)}{\partial n} = 0 \qquad \text{for} \quad t\in\mathbf{R}, \qquad x\in\partial\Omega,$$
$$u(t,x)=\phi(t,x), \qquad \text{for} \quad t\in\mathbf{R}^-, \quad x\in\Omega,$$

where $\Omega\subset\mathbf{R}^n$ is a bounded open set with a smooth boundary, $\partial/\partial n$ denotes the normal derivative, and $Du = ((\partial u/\partial x_1), ..., (\partial u/\partial x_n))$. We suppose that f is a smooth function vanishing at zero together with its first derivatives,

$$|f(p,q,r)|\leqslant C(|p|+|q|+|r|) \qquad \text{for} \quad p\in\mathbf{R}, \quad q\in\mathbf{R}^n, \quad r\in\mathbf{R}^{n^2},$$
$$\mu,\nu,\eta\in M(\mathbf{R}^+,\mathbf{R},e^{\gamma t}) \qquad \text{for some} \quad \gamma>0.$$

Now A is the Laplacian in the Hilbert space $H=L^2(\Omega)$, $G=W^{2,2}(\Omega)$. The spectrum of A consists of a sequence of nonpositive numbers $0=\mu_0>\mu_1>\mu_2>\cdots$ The operator $\hat{D}(\lambda)$ now has the form

$$\hat{D}(\lambda)=(1+\hat{\nu}(\lambda))\left(\lambda\frac{1+\hat{\mu}(\lambda)}{1+\hat{\nu}(\lambda)}-A\right).$$

It follows that $\lambda=0$ belongs to the spectrum of the problem. Let $\mu_1<\alpha<0$ and let

$$\hat{\nu}(\lambda)\neq-1 \qquad \text{for} \quad \operatorname{Re}\lambda=\alpha. \tag{30}$$

Suppose that there is $r>0$ such that

$$\frac{1+\hat{\mu}(\lambda)}{1+\hat{\nu}(\lambda)}\in\left\{z\in\mathbf{C};\ |\arg z|<\frac{\pi}{2},\ |z|>r\right\} \qquad \text{for} \quad \operatorname{Re}\lambda=\alpha.$$

Then there is a constant $\rho>0$ such that the curve

$$\lambda\frac{1+\hat{\mu}(\lambda)}{1+\hat{\nu}(\lambda)}, \qquad \operatorname{Re}\lambda=\alpha$$

belongs to the set $\{\lambda\in\mathbf{C};\ |\arg\lambda|<\pi\}\cup B_\rho(0)\backslash\{0\}$ provided that α is small enough. Hence

$$\begin{aligned}\|\hat{D}(\lambda)^{-1}\|_{L(H,G)}&\leqslant\left|\frac{1}{1+\hat{\nu}(\lambda)}\right|\left\|\left(\lambda\frac{1+\hat{\mu}(\lambda)}{1+\hat{\nu}(\lambda)}-A\right)^{-1}\right\|_{L(H,G)}\\&\leqslant C\left\|-I+\lambda\frac{1+\hat{\mu}(\lambda)}{1+\hat{\nu}(\lambda)}\left(\lambda\frac{1+\hat{\mu}(\lambda)}{1+\hat{\nu}(\lambda)}-A\right)^{-1}\right\|_{L(H,H)}\leqslant M\end{aligned}$$

for $\operatorname{Re}\lambda=\alpha$, and (15) holds.

Condition (30) is fulfilled, e.g., with $\mu=e^{-a_1t}$, $\nu=e^{-a_2t}$, $a_i>\gamma$. In this case, any λ with positive real part belongs to the resolvent set and Theorem 8 applies with any β positive and α negative, sufficiently small.

We can also take $\mu=c_1\delta_a$, $\nu=c_2\delta_b$ with small constants c_1, c_2 to get the same result. On the other hand, a simple choice $\mu=0$, $\nu=\delta_1$ gives $1+\hat{\nu}(\lambda)=1+e^{-\lambda}=0$ whenever $\lambda=i(2k+1)\pi$, a straightforward computation gives (10) for any $\beta>0$, but if $\lambda=\lambda_1+i\lambda_2$ with $\lambda_1<0$, then, choosing λ_2 such that $e^{-\lambda_1}\cos\lambda_2=0$ we get for $\lambda=\lambda_k=\lambda_1+i(\lambda_2+2k\pi)$

$$\operatorname{Re}\frac{\lambda_k}{1+e^{-\lambda_k}}\to\infty, \qquad \operatorname{Im}\frac{\lambda_k}{1+e^{-\lambda_k}}\leqslant C,$$

so that the $L(H, G)-$norm of $\hat{D}^{-1}(\lambda)$ cannot be estimated by a constant on the line $\operatorname{Re}\lambda=\lambda_1$. In this case, the operator $D: G\to H$ is not invertible and Theorem 8 cannot be applied.

7. THE SPECTRUM OF THE GENERATOR OF A TRANSLATION SEMIGROUP

According to Theorems 4 and 5, the restriction of the semigroup given by the problem (3), (4) to the stable and unstable subspaces is similar to the shift operator on $Y_\alpha=L^2_a(R, H)$ and $\mathcal{N}(D)$, respectively. In this section we examine the spectrum of the generator of the shift on Y_α, $\mathcal{N}(D)$ and, for the sake of completeness, also on the spaces $Y^\pm_\alpha$, $Y_{\alpha,\beta}$. In the following, we will use the symbol X for anyone of these spaces. The shift operator and its restrictions are defined in (8), the resolvent set of an operator A is denoted by $\rho(A)$, the spectrum and its point, continuous and residual parts are denoted by $\sigma(A)$, $\sigma_p(A)$, $\sigma_c(A)$, and $\sigma_r(A)$ respectively.

THEOREM 9. *Let τ_t be a left-shift operator on a space X and let A be the generator of the semigroup τ. Then*

$$Ax=\frac{d}{dt}x, \qquad x\in\mathcal{D}(A),$$

where the domain $\mathcal{D}(A)$ and the spectrum of A depend on the space X in the following way:

(i) $X=L^2_\alpha(\mathbf{R}, H)$:

$$\mathcal{D}(A)=W^{1,2}_\alpha(\mathbf{R}, H),$$
$$\sigma(A)=\sigma_c(A)=\{\lambda\in\mathbf{C};\ \operatorname{Re}\lambda=\alpha\}.$$

(ii) $X=L^2_\alpha(\mathbf{R}^-, H)$:

$$\mathcal{D}(A)=\{\phi\in W^{1,2}_\alpha(\mathbf{R}^-, H);\ \phi(0)=0\},$$
$$\sigma(A)=\{\lambda\in\mathbf{C};\ \operatorname{Re}\lambda\leqslant\alpha\},$$
$$\sigma_c(A)=\{\lambda\in\mathbf{C};\ \operatorname{Re}\lambda=\alpha\},$$
$$\sigma_r(A)=\{\lambda\in\mathbf{C};\ \operatorname{Re}\lambda<\alpha\}=\sigma_p(A^*).$$

(iii) $X=L^2_\alpha(\mathbf{R}^+, H)$:

$$\mathcal{D}(A)=W^{1,2}_\alpha(\mathbf{R}^+, H),$$
$$\sigma(A)=\{\lambda\in\mathbf{C};\ \operatorname{Re}\lambda\leqslant\alpha\},$$
$$\sigma_p(A)=\{\lambda\in\mathbf{C};\ \operatorname{Re}\lambda<\alpha\},$$
$$\sigma_c(A)=\{\lambda\in\mathbf{C};\ \operatorname{Re}\lambda=\alpha\}.$$

(iv) $X = L^2_{\alpha,\beta}(\mathbf{R}, H)$, $\alpha < \beta$: $\mathscr{D}(A) = W^{1,2}_{\alpha,\beta}(\mathbf{R}, H)$,

$\sigma(A) = \{\lambda \in \mathbf{C};\ \alpha \leqslant \operatorname{Re} \lambda \leqslant \beta\}$,

$\sigma_p(A) = \{\lambda \in \mathbf{C};\ \alpha < \operatorname{Re} \lambda < \beta\}$,

$\sigma_c(A) = \{\lambda \in \mathbf{C};\ \operatorname{Re} \lambda = \alpha \vee \operatorname{Re} \lambda = \beta\}$.

(v) $X = L^2_{\beta,\alpha}(\mathbf{R}, H)$, $\alpha < \beta$: $\mathscr{D}(A) = W^{1,2}_{\alpha,\beta}(\mathbf{R}, H)$,

$\sigma(A) = \{\lambda \in \mathbf{C};\ \alpha \leqslant \operatorname{Re} \lambda \leqslant \beta\}$,

$\sigma_c(A) = \{\lambda \in \mathbf{C};\ \operatorname{Re} \lambda = \alpha \vee \operatorname{Re} \lambda = \beta\}$,

$\sigma_r(A) = \{\lambda \in \mathbf{C};\ \alpha < \operatorname{Re} \lambda < \beta\}$.

(vi) $X = \{\phi \in L^2_\alpha(\mathbf{R}^-, G) \cap W^{1,2}_\alpha(\mathbf{R}^-, H);\ D\phi \equiv 0\}$: $\mathscr{D}(A) = \{\phi \in X;\ \phi' \in X\}$,

$\sigma(A) = \{\lambda \in \mathbf{C};\ \operatorname{Re} \lambda > \alpha,\ \hat{D}(\lambda)$ is not invertible$\}$,

$\sigma_{p(c,r)}(A) = \{\lambda \in \sigma(A);\ 0 \in \sigma_{p(c,r)} \hat{D}(\lambda)\}$.

Proof. It is well known that the operator B: $\mathscr{D}(B) = W^{1,2}(\mathbf{R}, H)$, $Bx = i(dx/dt)$ is selfadjoint in $L^2(\mathbf{R}, H)$ with

$$\sigma(B) = \sigma_c(B) = \mathbf{R}. \tag{31}$$

The assertion (i) now follows for $X = L^2_\alpha$ with $\alpha = 0$ and, for $\alpha \neq 0$, by multiplying functions in the equation

$$(A - \lambda I) f = g \Leftrightarrow f' - \lambda f = g \tag{32}$$

by $e^{\alpha t}$. The inverse of $(A - \lambda I)$ is given by

$$(A - \lambda I)^{-1} g = -\int_t^{+\infty} e^{\lambda(t-s)} g(s)\, ds = e_\lambda^- * g \qquad \text{for} \quad \operatorname{Re} \lambda > \alpha, \tag{33}$$

where

$$e_\lambda^-(t) = \begin{cases} -e^{\lambda t} & \text{for} \quad t < 0, \\ 0 & \text{for} \quad t > 0, \end{cases} \tag{34}$$

$e_\lambda^- \in L^1_\alpha(\mathbf{R})$ for $\operatorname{Re} \lambda > \alpha$ which implies $e_\lambda^- * g \in W^{1,2}_\alpha(\mathbf{R}, H)$ whenever $g \in L^2_\alpha(\mathbf{R}, H)$. Similarly, e_λ^+ defined as

$$e_\lambda^+(t) = \begin{cases} 0 & \text{for} \quad t < 0, \\ e^{\lambda t} & \text{for} \quad t > 0 \end{cases} \tag{35}$$

belongs to $L^1_\alpha(\mathbf{R})$ if $\operatorname{Re}\lambda < \alpha$ and

$$(A-\lambda I)^{-1} g = e^+_\lambda * g \qquad \text{for} \quad \operatorname{Re}\lambda < \alpha. \tag{36}$$

If $\operatorname{Re}\lambda = \alpha$, then $e^\pm_\lambda * g$ need not belong to $W^{1,2}_\alpha$, so $\lambda \in \sigma(A)$ and, due to (31),

$$\mathscr{R}(A-\lambda I) \qquad \text{is dense in} \quad L^2_\alpha(\mathbf{R}, A). \tag{37}$$

In the same way, using (33)–(37) we prove the assertions on the resolvent sets of A as well as the boundary lines of the spectra in (i)–(v).

As to the inner part of the spectrum, we shall treat each case separately.

(ii) If $\operatorname{Re}\lambda < \alpha$, then we have an additional condition for solution of (33), namely

$$f(0) = 0, \qquad \text{i.e.} \quad (e^+_\lambda * g)(0) = \int_{-\infty}^0 e^{-\lambda t} g(t)\, dt = 0,$$

so the range of $(A-\lambda I)$ is closed, but not the whole space. It follows that $\lambda \in \sigma_r(A)$. In this case, $e^{-\lambda t}x$, $x \in H$ is the eigenfunction of the adjoint operator $A^\star = -d/dt$, $\mathscr{D}(A^\star) = W^{1,2}_\alpha$. Thus λ is an eigenvalue of A^* with infinite multiplicity, equal to the dimension of H.

(iii) If $\operatorname{Re}\lambda < \alpha$, then $e^{\lambda t}x$ is an eigenfunction of A for each $x \in H$. It follows that λ is an eigenvalue with infinite multiplicity.

(iv) Let $\alpha < \operatorname{Re}\lambda < \beta$. Then $e^{\lambda t}x \in W^{1,2}_{\alpha,\beta}$ is an eigenfunction of A, and λ is an eigenvalue of infinite multiplicity.

(v) Let $\alpha < \operatorname{Re}\lambda < \beta$. This time $e^{\lambda t} \notin L^2_{\beta,\alpha}$. As $f \in X$ implies $f \in L^1_\eta(\mathbf{R})$ for every $\eta \in (\alpha, \beta)$ we can multiply (33) by $e^{-\lambda t}$ and integrate to get a condition on the the right hand side:

$$\int_{-\infty}^{\infty} e^{-\lambda t} g(t)\, dt = \int_{-\infty}^{\infty} \frac{d}{dt}(f(t)e^{-\lambda t})\, dt = 0.$$

Thus $A-\lambda I$ is not onto, $\mathscr{R}(A-\lambda I)$ is closed and $\lambda \in \sigma_r(A)$.

To prove (vi) we solve (33) on $\mathbf{R}^-$ with $Dg = 0$ and we require that $Df = Df' = 0$. Since D commutes with d/dt, it suffices to show that $Df = 0$.

For λ with $\operatorname{Re}\lambda < \alpha$ the solution is given by

$$\begin{aligned} f &= \pi^- e^+_\lambda * g, \\ \pi^- Df &= \pi^- D(e^+_\lambda * g) = \pi^- e^+_\lambda * Dg = 0, \end{aligned} \tag{38}$$

so this λ belongs to the resolvent set of A. The same argument applies if $\operatorname{Re}\lambda=\alpha$, because, according to Lemma 7, $g\in X\Rightarrow g\in W^-_{\alpha'}$ with some $\alpha'>\alpha$, so f is given by (38) belongs to X.

For λ with $\operatorname{Re}\lambda>\alpha$ we obtain the solution of (33) as

$$f(t)=e^{\lambda t}f(0)-\int_t^0 e^{\lambda(t-s)}g(s)\,ds=e^{\lambda t}f(0)+e_\lambda^- * g, \qquad t\leqslant 0.$$

In this case,

$$Df(t)=e^{\lambda t}\hat{D}(\lambda)\,f(0)+e^{\lambda t}h(\lambda,g)-\int_t^0 e^{\lambda(t-s)}(Dg)(s)\,ds,$$

where

$$e^{\lambda t}h(\lambda,g)=e_\lambda^- * \pi^+D\pi^-g=e^{\lambda t}(\pi^+D\pi^-g)\,\hat{}\,(\lambda), \qquad t\leqslant 0.$$

The operator $\pi^+D\pi^-$ is expressed with the help of the Dirac measure δ_0 by

$$\pi^+D\pi^-=(g(0)+\mu * g(0))\delta_0+\pi^+\mu * \pi^-g'+\pi^+\nu * \pi^-g.$$

So

$$h(\lambda,g)=g(0)+(\mu * g)(0)-\lambda(\pi^+\mu * g)\,\hat{}\,(\lambda)+(\pi^+\nu * g)\,\hat{}\,(\lambda).$$

It follows that $f\in X$ iff

$$\hat{D}(\lambda)\,f(0)=-h(\lambda,g).$$

Hence each λ such that $\hat{D}(\lambda)$ is invertible belongs to the resolvent set.

It is clear that x is in the null space of $\hat{D}(\lambda)$ iff $e^{\lambda t}x$ is the eigenvector of A.

To complete the proof we show that $\hat{D}(\lambda)$ has a full range whenever $\lambda\in\rho(A)$. To this end, take $q=e^{\gamma t}x$ with γ such that $\alpha<\operatorname{Re}\gamma<\operatorname{Re}\lambda$ and $\hat{D}(\gamma)$ is invertible. Equation (33) can be solved for $g=P_{\mathscr{U}}\binom{q}{0}$, so we have some $f\in X$ such that

$$\hat{D}(\lambda)\,f(0)-h\left(\lambda,P_{\mathscr{S}}\begin{pmatrix}q\\0\end{pmatrix}\right)=-h(\lambda,q)=\frac{1}{\gamma-\lambda}(\hat{D}(\lambda)-\hat{D}(\gamma))x.$$

Now, we realize that $h(\lambda,P_{\mathscr{S}}\binom{q}{0})\in\mathscr{R}(\hat{D}(\lambda))$. In fact, the Laplace transform of a solution of the equation $Du=0$, $\pi^-u=\phi$, with $\binom{\phi}{0}\in\mathscr{S}$ is defined for $\operatorname{Re}\lambda>\alpha$ and

$$0=\widehat{Du}(\lambda)=\hat{D}(\lambda)\,\hat{u}(\lambda)-h(\lambda,\phi).$$

Then

$$\hat{D}(\lambda)((\lambda-\gamma)(f(0)-y)-x)=\hat{D}(\gamma)x, \tag{39}$$

with y satisfying $\hat{D}(\lambda)y=h(\lambda, P_{\mathscr{S}}(\begin{smallmatrix} g \\ 0 \end{smallmatrix}))$. If $\hat{D}(\gamma)$ is invertible, then (39) gives $\mathscr{R}(\hat{D}(\lambda))=H$.

The correspondence of the particular parts of the spectrum follows from the continuity of h. ∎

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