INFORMATION AND CONTROL 8, 143–158 (1965)

Band-Limited Functions and the Sampling Theorem

MOSHE ZAKAI*

Applied Research Laboratory, Sylvania Electronic Systems, Waltham, Massachusetts

The definition of band-limited functions (and random processes) is extended to include functions and processes which do not possess a Fourier integral representation. This definition allows a unified approach to band-limited functions and band-limited (but not necessarily stationary) processes. The sampling theorem for functions and processes which are band-limited under the extended definition is derived.

I. INTRODUCTION

The well known sampling theorem states that if $f(t)$ can be represented as

$$f(t) = \int_{-W_0}^{W_0} \psi(\omega) e^{i\omega t} \, d\omega \tag{1}$$

where $\int_{-W_0}^{W_0} | \psi(\omega) | \, d\omega < \infty$; or, more generally, if

$$f(t) = \int_{-W_0}^{W_0} e^{i\omega t} \, d\mu(\omega) \tag{1a}$$

where $\mu(\omega)$ is of bounded variation and continuous at the end points $\pm W_0$, then

$$f(t) = \sum_{n=-\infty}^{\infty} f(n\tau) \frac{\sin[(\pi/\tau)(t - n\tau)]}{(\pi/\tau)(t - n\tau)}$$

where $\tau \leqq \tau_0 = \pi/W_0$ and the sampling series converges uniformly on any bounded interval (if the real variable t is replaced by a complex variable then the sampling series converges uniformly on any bounded region in the complex plane).

Consider, for example, the function $Si(t)$ where

$$Si(t) = \int_0^t \frac{\sin x}{x} \, dx = \frac{1}{i2} \int_{-1}^{1} \frac{e^{i\omega t}}{\omega} \, d\omega \tag{2}$$

* On leave of absence from the Scientific Department, Israel Ministry of Defence, at which the major part of this paper was written.

the second integral is not absolutely convergent (this integral has to be interpreted as a principal value integral) so that the sampling theorem does not apply to $Si(t)$. Since, however, $Si(t)$ is the response of an ideal low pass filter to a step function, it seems reasonable to consider $Si(t)$ as a band-limited function and to expect that the sampling theorem is still valid in some sense for this function.

Stochastic versions of the sampling theorem have been derived for stationary processes (Balakrishnan, 1957; Beutler, 1961; Blanc-Lapierre and Frotet, 1953; Lloyd, 1959). In both cases, that of (1a) and the stochastic version, the starting point is that the function or process has a suitably restricted frequency spectrum. It should, however, be noted that the sample functions of a stationary band-limited process do not have, in general, a representation of the form (1a) since $\mu(\omega)$ may be of unbounded variation. It seems, therefore, desirable to have a unified approach to band-limited functions and to the sampling theorem which will include, as special cases, functions of the form (1a) and almost all sample functions of band-limited stationary processes as well as the function $Si(t)$ and similar functions and also nonstationary band-limited processes.

The purpose of this paper is to extend the definition of band-limited functions and processes and to derive the sampling theorem for functions and processes which are band-limited under the extended definition.

Instead of the spectral characterization of band-limited functions, it is also possible to characterize such functions in the following way. Let $\eta(t; W)$ be the inverse Fourier transform of $\gamma(\omega)$ where $\gamma(\omega) = 1$ for $|\omega| \leqq W$ and $\gamma(\omega) = 0$ for $|\omega| > W$; then, if $f(t)$ is of the form of Eq. (1) or (1a) and if $W \geqq W_0$ then $f(t)$ is, indeed, reproduced without distortion when passed through the filter $\gamma(\omega)$, namely:

$$f(t)*\eta(t; W) = \left[\int_{-\infty}^{\infty} f(\theta)\eta(t-\theta; W)\, d\theta\right] = f(t). \tag{3}$$

It is possible to use other functions, instead of $\eta(t; W)$, to characterize band-limited functions; in particular, let $H(\omega; W, \delta)$ be as described in Fig. 1. If (3) is satisfied and if $h(t; W, \delta)$ is the inverse Fourier transform of $H(\omega; W, \delta)$, then $f(t) = f(t)*h(t; W, \delta)$.

Only functions satisfying

$$\int_{-\infty}^{\infty} \frac{|f(t)|^2}{1+t^2}\, dt < \infty \tag{4}$$

will be considered in this paper. A function $f(t)$ satisfying (4) will be defined in Section III to be band-limited if it is reproduced by some

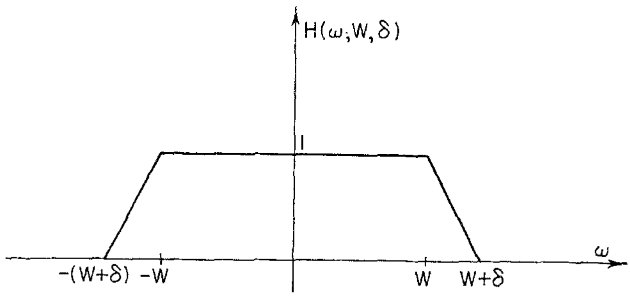

FIG. 1

$h(t; W, \delta); f(t) = f(t)*h(t; W, \delta)$, and $h(t; W, \delta)$ is the inverse Fourier transform of $H(\omega; W, \delta)$ of Fig. 1. The reason for preferring $h(t; W, \delta)$ over $\eta(t; W)$ is that $\eta(t; W)$ is $O(|t|^{-1})$ as $|t| \to \infty$ (since $\gamma(\omega)$ is discontinuous at $\omega = \pm W$) while $h(t; W, \delta)$ is $O(|t|^{-2})$ as $|t| \to \infty$, thus considerably simplifying convergence problems. Conditions under which a function $f(t)$ is band-limited will be derived in Section III (Theorems 1 and 2). The sampling theorem for functions which are band-limited under the definition of this paper is derived in Section IV.

A random process (stationary or not) is defined in Section V to be band-limited if the expectation of (4) is finite and if almost all sample functions of the process are band-limited (and by the same parameters W and δ). By the result of Section IV, the sampling theorem is, obviously, valid individually for almost all sample functions of the process. In the stochastic versions of the sampling theorem for stationary processes (Balakrishnan, 1957; Beutler, 1961; Lloyd, 1959), the convergence of the sampling series is mean convergence (with the exception of Theorem 4 of Lloyd (1959).[1] The result of this paper deals with almost sure convergence for all t: for almost all sample functions the sampling series converges to the sample function for all t. It is shown that for stationary processes the ordinary definition of a band-limited process and that of this paper are equivalent. Conditions on the autocorrelation function $R(t, t')$ under which the process is band-limited are derived.

II. PRELIMINARIES

The class of functions $f(t)$, $(-\infty < t < \infty)$, satisfying

$$\|f\|^2 = \int_{-\infty}^{\infty} \frac{|f(t)|^2}{1 + t^2} dt < \infty \tag{4}$$

[1] See also note added in proof at the end of this paper.

will be denoted as H_0. H_0 with the scalar product

$$(f, g) = \int_{-\infty}^{\infty} \frac{f(t) \cdot \bar{g}(t)}{1 + t^2} dt$$

forms a Hilbert space. (H_0 includes L_2 functions, bounded functions, and functions $f(t)$ for which $(2T)^{-1}\int_{-T}^{T} | f(t)|^2 dt$ is bounded in T (Wiener, 1933). If the upper half-plane is transformed into the unit circle by means of the bilinear transformation then H_0 corresponds to the class of functions belonging to L_2 on the circumference of the unit circle (Hoffman, 1962).) Throughout this paper, t (or x) will always denote a real variable and z will denote a complex variable.

LEMMA 1. *If $f(t)$ belongs to H_0 then*

$$\left\| f(t) * \frac{1}{1 + s^2} \right\| \leqq \pi \| f(t) \|,$$

$f(t)*1/(1 + t^2)$ is finite for finite t, and at most $O(|t|)$ as $|t| \to \infty$.

Proof:

$$\left\| f(t) * \frac{1}{1 + t^2} \right\|^2 \leqq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{| f(\alpha) | \cdot | f(\beta) |}{(1 + t^2)(1 + (t - \alpha)^2)(1 + (t - \beta)^2)} \cdot d\alpha\, d\beta\, dt,$$

since $2 | a \cdot b | \leqq | a |^2 + | b |^2$, we have

$$\left\| f(t) * \frac{1}{1 + t^2} \right\|^2 \leqq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \pi \frac{| f(\alpha) |^2}{(1 + t^2)(1 + (t - \alpha)^2)} d\alpha\, dt.$$

From the inequality

$$\frac{1}{1 + t^2} * \frac{1}{1 + t^2} = \frac{\pi}{t^2 + 4} \leqq \pi \frac{1}{t^2 + 1} \tag{5}$$

it follows that $\| f(t)*1/(1 + t^2) \| \leqq \pi \| f(t) \|$. By the Schwarz inequality:

$$\begin{aligned} \left| \int_{-\infty}^{\infty} \frac{f(\theta)}{\sqrt{1 + (t - \theta)^2}} \cdot \frac{1}{\sqrt{1 + (t - \theta)^2}} d\theta \right| & \\ \leqq \pi^{1/2} \left[\int_{-\infty}^{\infty} \frac{| f(\theta) |^2}{1 + (t - \theta)^2} d\theta \right]^{1/2} & \qquad (6) \\ \leqq \pi^{1/2} \left[\max_{-\infty<\theta<\infty} \left(\frac{1 + \theta^2}{1 + (t - \theta)^2} \right) \right]^{1/2} \cdot \| f(t) \|. & \end{aligned}$$

But, since

$$1 + \theta^2 = 1 + (t + \theta - t)^2 \leqq 1 + 2t^2 + 2(t - \theta)^2 < 2(1 + t^2)(1 + (t - \theta)^2), \tag{7}$$

it follows that

$$\left| f(t) * \frac{1}{1 + t^2} \right| < \sqrt{2\pi} \cdot \| f(t) \| \cdot \sqrt{1 + t^2}. \tag{8}$$

LEMMA 2. *Let*

$$u(z) = \int_{-\infty}^{\infty} f(\theta) \frac{\sin \alpha(z - \theta)}{z - \theta} \cdot \frac{\sin \beta(z - \theta)}{z - \theta} \, d\theta \tag{9}$$

where $z = x + iy$, α *and* β *are real and positive and* $f(t)$ *belongs to* H_0; *then:*

(a) *The integral above converges uniformly in any bounded region of the* z *plane.*

(b) $u(x)$ *is a continuous function of* x.

(c)

$$| u(z) | \leqq K \cdot \| f(t) \| \cdot (1 + x^2)^{1/2} \cdot e^{(\alpha+\beta)|y|}. \tag{10}$$

(d) *if* $f(t)$ *is continuous then* $u(z)$ *is an analytic function of* z *in the entire* z *plane; namely, an entire function.*

Proof: The proof follows from the inequality

$$\left| \frac{\sin \alpha z}{z} \right| < k_1 \frac{1}{\sqrt{1 + x^2}} e^{\alpha|y|}; \; k_1 > \max(\alpha, 1). \tag{11}$$

Let $a(z, \theta)$ be the integrand in (9); then

$$| a(z, \theta) | \leqq k_2 \frac{| f(\theta) |}{1 + \theta^2} \frac{1 + \theta^2}{1 + (x - \theta)^2} e^{(\alpha+\beta)|y|}. \tag{12}$$

If we consider now a bounded region in the z plane so that $| y | \leqq y_0$ and $| x | \leqq x_0$, then by (7)

$$| a(\theta, z) | \leqq 2k(1 + x_0^2)e^{(\alpha+\beta)y_0} | f(\theta) |/(1 + \theta^2). \tag{13}$$

The integrand of (9) is, therefore, dominated in the region $| x | \leqq x_0$ and $| y | \leqq y_0$ by the right hand side of (13) which is integrable $(-\infty, \infty)$ since

$$\left(\int_{-\infty}^{\infty} \frac{| f(\theta) |}{1 + \theta^2} \, d\theta \right)^2 \leqq \int_{-\infty}^{\infty} \frac{| f(\theta) |^2}{1 + \theta^2} \, d\theta \cdot \int_{-\infty}^{\infty} \frac{1}{1 + \theta^2} \, d\theta. \tag{14}$$

Therefore (see Titchmarsh, 1939, sec 1.51) (9) converges uniformly in any bounded region. From (13) and (14) it follows, by dominated convergence, that if $x_n \to x$ then $u(x_n) \to u(x)$; therefore $u(x)$ is a continuous function of x. Part (c) of the lemma follows from (11) and lemma 1. Part (d) follows from part (a) and theorem 2.84 of Titchmarsh (1939).

LEMMA 3. *If $\psi(\omega)$ belongs to $L_2(-W_0, W_0)$ and $0 < \tau < \pi/W_0$ then*

$$\sum_{-\infty}^{\infty} (-1)^n g(n\tau) = 0 \tag{15}$$

where

$$g(t) = \int_{-W_0}^{W_0} \psi(\omega) e^{i\omega t} \, d\omega.$$

Proof: Since $\psi(\omega)$ is $L_2(-W_0, W_0)$ it is also $L_1(-W_0, W_0)$. Since

$$\sum_{-N_1}^{N_2} (-1)^n e^{i\omega n\tau} = \frac{(-1)^{N_1} e^{-i\omega N_1 \tau} + (-1)^{N_2} e^{i\omega(N_2+1)\tau}}{1 + e^{i\omega\tau}}$$

and since $\tau < \pi/W_0$, it follows that $| 1 + e^{i\omega\tau} |^{-1} = [2(1 + \cos \omega\tau)]^{-1/2}$ is bounded in the interval $[-W_0, W_0]$. Let $\psi(\omega)(1 + e^{i\omega\tau})^{-1} = \psi_1(\omega)$; then

$$\sum_{-N_1}^{N_2} (-1)^n g(n\tau) = \int_{-W_0}^{W_0} \psi_1(\omega)[(-1)^{N_1} e^{-i\omega N_1 \tau} + (-1)^{N_2} e^{i\omega(N_2+1)\tau}] \, d\omega$$

and it follows by the Riemann-Lebesgue theorem that

$$\sum_{-\infty}^{\infty} (-1)^n g(n\tau) = \lim_{\substack{N_1 \to \infty \\ N_2 \to \infty}} \sum_{-N_1}^{N_2} (-1)^n g(n\tau) = 0. \tag{16}$$

The condition $\tau < \pi/W_0$ was necessary to make sure that $\psi_1(\omega) = \psi(\omega)(1 + e^{i\omega\tau})^{-1}$ belongs to L_1 in $[-W_0, W_0]$. Therefore, if we add the requirement that $\psi(\omega) \cdot (1 + e^{i\pi\omega/W_0})^{-1}$ [or $\psi(\omega) \cdot (\omega^2 - W_0^2)^{-1}$] belongs to L_1 in $[-W_0, W]$ then the result of Lemma 3 is also valid for $\tau = \pi/W_0$.

III. BAND-LIMITED FUNCTIONS

DEFINITION: *A function $f(t)$ satisfying* (4) *is now defined to be "band-limited (W, δ)" or "to belong to $H_1(W, \delta)$" if for almost all $t(-\infty < t < \infty)$*

$$f(t)*h(t; W, \delta) = \int_{-\infty}^{\infty} f(\theta) h(t - \theta; W, \delta) \, d\theta = f(t) \tag{17}$$

where $h(t; W, \delta)$, $(W > 0, \delta > 0)$, *is the inverse Fourier transform of* $H(\omega)$ *of Fig. 1 and is given by*

$$\begin{aligned} h(t; W, \delta) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} H(\omega; W, \delta)\, d\omega \\ &= \frac{2}{\pi\delta t^2} \sin\left(W + \frac{\delta}{2}\right) t \cdot \sin\frac{\delta}{2}\, t. \end{aligned} \tag{18}$$

By Fubini's theorem we have that if $f(t)$ belongs to H_0 then $(f(t)*h(t; W, \delta)*(h(t; W'\delta') = f(t)*(h(t; W, \delta)*h(t; W', \delta'))$. Since $h(t; W, \delta)*h(t; W', \delta') = h(t; W, \delta)$ for any $W' \geqq W + \delta$ and any $\delta > 0$, it follows that if $g(t)$ is band-limited (W, δ) then it is also band-limited (W', δ') if $W' \geqq W + \delta$ and $\delta' > 0$. A linear combination of functions satisfying (17) also satisfies (17). Moreover, let $f_n(t)$ be a Cauchy sequence of functions in H_0 converging in the H_0 norm to $f(t)$ and let all the $f_n(t)$ be band-limited (W, δ). By Lemma 1 and inequality (11)

$$\begin{aligned} \| f - f*h(\cdot; W, \delta) \| &= \| f - f_n + f_n - f*h(\cdot; W, \delta) \| \\ &\leqq \| f - f_n \| + \| h(t; W, \delta)*(f(t) - f_n(t)) \| \\ &\leqq (1 + K)\cdot \| f - f_n \| \to 0, \end{aligned}$$

therefore $f(t)$ is also band-limited (W, δ). $H_1(W, \delta)$ *as defined above is, therefore, a closed linear subspace of* H_0. Since $f(t)*h(t; W, \delta)$ is a continuous function of t (Lemma 2), it follows that any function belonging to $H_1(W, \delta)$ can be modified by changing its values on a set of measure zero so that it becomes continuous. Only this modified version of $f(t)$ will be considered in this paper. It follows from (17) and Lemma 2 that *if* $f(t)$ *is band-limited* (W, δ) *then* $f(t)$ *can be extended to the complex plane* $z = x + iy$ *by*

$$f(z) = \int_{-\infty}^{\infty} f(t) h(z - t; W, \delta)\, dt$$

where $f(z)$ *is an entire function and*

$$| f(x + iy) | \leqq k \, \| f \| \cdot \frac{2}{\pi\delta} (1 + x^2)^{1/2} e^{(W+\delta)|y|}.$$

Consider, now, the function $g(z) = [f(z) - f(0)]\cdot z^{-1}$; then $g(z)$ is also an entire function and $| g(z) | \leqq k_g e^{(W+\delta)|y|}$; moreover, $g(z)$ belongs to $L_2(-\infty, \infty)$ on the real axis. Therefore $g(z)$ satisfies the requirements of the theorem of Paley and Wiener (1934), namely; $\psi(\omega)$, the Fourier transform of $g(t)$, vanishes outside $[-(W + \delta), W + \delta]$. Since $g(t)$ is $L_2(-\infty, \infty)$, $\psi(\omega)$ is L_2 in $[-(W + \delta), W + \delta]$ and therefore also L_1 in the same interval. Conversely, if $\psi(\omega)$ is L_2 in the interval $(-A, A)$ and $g(t) = (2\pi)^{-1}\int_{-A}^{A} \psi(\omega)e^{i\omega t}\, d\omega$, then $g(t)$ belongs to $L_2(-\infty, \infty)$; therefore $c + tg(t)$ (where c is constant) belongs to H_0. Moreover, $[\sin \epsilon t/\epsilon t]\cdot t\cdot g(t)$ belongs to $L_2(-\infty, \infty)$ (and since the Fourier transform of this function is $\epsilon^{-1}[\psi(\omega - \epsilon) + \psi(\omega + \epsilon)]$), it also belongs to $H_1(W, \delta)$ for $W \geqq A + \epsilon$ and $\delta > 0$. Since $[\sin \epsilon t/\epsilon t]\cdot t\cdot g(t)$ converges in H_0 to $t\cdot g(t)$ and $H_1(W, \delta)$ is a closed subspace of H_0, it follows that $c + t\cdot g(t)$ belongs to $H_1(W, \delta)$ for any $W > A$ and $\delta > 0$. Now, let

$$g_\epsilon(t) = (2\pi)^{-1} \int_{-(A+\epsilon)}^{A+\epsilon} \psi(\omega)e^{i\omega t}\, d\omega;$$

then $t\cdot g_\epsilon(t)$ belongs to $H_1(W, \delta)$ for $W \geqq A$ and $\delta > 0$. Since $g_\epsilon(t)$ converges to $g(t)$ in the L_2 norm, it follows that $c + tg_\epsilon(t)$ converges to $c + t\cdot g(t)$ in the H_0 norm. Therefore $c + tg(t)$ belongs to $H_1(W, \delta)$ for any $W \geqq A$ and $\delta > 0$. The following theorem has, therefore, been proved.

THEOREM 1. (a) *If $f(t)$ belongs to $H_1(W, \delta)$, then $f(t) = f(0) + t\cdot g(t)$ where $g(t)$ belongs to $L_2(-\infty, \infty)$ and is band-limited in the conventional sense, namely,*

$$g(t) = \frac{1}{2\pi} \int_{-(W+\delta)}^{(W+\delta)} \psi(\omega)e^{i\omega t}\, d\omega. \tag{19}$$

(b) *If $g(t)$ belongs to $L_2(-\infty, \infty)$ and is band-limited in the conventional sense with bandwidth $A[g(t) = (2\pi)^{-1}\int_{-A}^{A} \psi(\omega)e^{i\omega t}\, d\omega]$ then $f(t) = c + tg(t)$ belongs to $H_1(W, \delta)$ for all $W \geqq A$ and $\delta > 0$.*

Theorem 1 suggests the following *definition*: *The bandwidth W_0 of a function $f(t)$ bandlimited (W, δ) is the smallest A so that the Fourier transform of $[f(t) - f(0)]t^{-1}$ vanishes for almost all ω outside $(-A, A)$.* Then, by (19), $W_0 \leqq W + \delta$ (it seems very reasonable to expect that $W_0 \leqq W$ but we have not yet found the proof) and $f(t)$ is band-limited (W, δ) for all $W \geqq W_0$ and all $\delta > 0$.

THEOREM 2. *If $f(z)$ is an entire function and $|f(z)| \leqq Be^{A|z|}$ and if $f(z)$ belongs to H_0 on the real axis then $f(t)$ is band-limited and the bandwidth W_0 of $f(t)$ satisfies $W_0 \leqq A$.*

Proof: $g(z) = [f(z) - f(0)]\cdot z^{-1}$ satisfies the conditions of the Paley-

Wiener (1934) theorem. Therefore $g(x)$ satisfies the requirements of part (b) of Theorem 1. As an application of Theorem 2, it will now be shown that $Si(t)$ which was defined by Eq. (2) is band-limited. $(\sin z)/z$ is an entire function and $| z^{-1} \sin z | \leqq e^{|z|}$, therefore

$$\left| \int_0^{z_0} \frac{\sin z}{z} \, dz \right| \leqq Be^{|z_0|} \quad \text{and} \quad \int_0^z \frac{\sin z'}{z'} \, dz'$$

is an entire function; since $Si(t)$ is bounded (for real t) it belongs to H_0 and by Theorem 2, $Si(t)$ is band-limited $(1, \delta)$ for all $\delta > 0$. It also follows from Theorem 2 that if $f(t)$ is band-limited with bandwidth W_0 and if $(f(t))^n$ (where n is a positive integer) belongs to H_0, then $(f(t))^n$ is also band-limited and with bandwidth nW_0.

It follows from Theorem 1 and the sampling theorem for functions of the form (1) that $[f(t) - f(0)]t^{-1}$ is uniquely determined by its values at the sampling points $t = n\tau$; $(n = 0, \pm 1, \pm 2, \cdots)$, where $\tau \leqq \pi W_0^{-1}$.

Therefore $f(t)$ *is uniquely determined by the samples* $f(n\tau)$, $(n = 0, \pm 1, \pm 2, \cdots)$; $\tau \leqq \pi W_0^{-1}$ *and* $f'(0)$, *the value of the derivative of* $f(t)$ *at* $t = 0$. If, instead of $\tau \leqq \pi W_0^{-1}$, we allow only $\tau < \pi W_0^{-1}$ then it is sufficient to know only $f(n\tau)$ since, by Lemma 3, $f'(0)$ is determined by $f(n\tau)(n = \pm 1, \pm 2, \cdots)$.

IV. THE SAMPLING THEOREM FOR BAND-LIMITED FUNCTIONS

THEOREM 3. *If* $f(t)$ *is band-limited with* (W, δ) *bandwidth* W_0 *and if* $\tau < \pi/W_0$ *then*

$$f(z) = \sum_{n=-\infty}^{\infty} f(n\tau) \frac{\sin[(\pi/\tau)(z - n\tau)]}{(\pi/\tau)(z - n\tau)}$$

and the convergence is uniform in any bounded region of the z *plane.*

Proof: By Theorem 1 and the sampling theorem for functions of the form (1)

$$g(z) = \frac{f(z) - f(0)}{z} = \sum_{n=-\infty}^{\infty} g(n\tau) \frac{\sin[(\pi/\tau)(z - n\tau)]}{(\pi/\tau)(z - n\tau)}$$

or

$$\begin{aligned} f(z) &= f(0) + \frac{\tau}{\pi} f'(0) \sin \frac{\pi z}{\tau} \\ &\quad + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} z \cdot [f(n\tau) - f(0)] \frac{\sin[(\pi/\tau)(z - n\tau)]}{n\pi(z - n\tau)} \end{aligned} \tag{20}$$

and the convergence is uniform in any bounded region of the z plane. Since

$$\frac{z}{n\pi(z-n\tau)} = \frac{1}{(\pi/\tau)(z-n\tau)} + \frac{1}{n\pi},$$

it follows that

$$f(z) = f(0) + \frac{\tau}{\pi} f'(0) \sin \frac{\pi z}{\tau} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left[(f(n\tau) - f(0)) \cdot \frac{\sin[(\pi/\tau)(z-n\tau)]}{(\pi/\tau)(z-n\tau)} + \frac{(-1)^n}{\pi} \frac{f(n\tau) - f(0)}{n} \sin \frac{\pi z}{\tau} \right]. \tag{21}$$

By Lemma 3

$$f'(0) + \sum_{n\neq 0} \frac{f(n\tau) - f(0)}{n\tau} (-1)^n = 0,$$

hence

$$f(z) = f(0) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (f(n\tau) - f(0)) \frac{\sin[(\pi/\tau)(z-n\tau)]}{(\pi/\tau)(z-n\tau)} \tag{22}$$

and the convergence is uniform in any bounded region of the z plane. Set in (1a)

$$\mu(\omega) = \begin{cases} \tfrac{1}{2}f(0), & \omega > 0 \\ -\tfrac{1}{2}f(0), & \omega < 0. \end{cases}$$

Then $f(z) = f(0)$ and by the sampling theorem for functions of the form (1a):

$$f(0) = \sum_{n=-\infty}^{\infty} f(0) \frac{\sin[(\pi/\tau)(z-n\tau)]}{(\pi/\tau)(z-n\tau)} \tag{23}$$

and the required result follows by substituting (23) in (22). Equations (21) and (23) are still true for $\tau \leqq \pi W_0^{-1}$ and the requirement $\tau < \pi W_0^{-1}$ comes only through Lemma 3. Therefore, if the Fourier transform of $[f(t) - f(0)] \cdot t^{-1}$ satisfies the condition given at the end of Section II. Theorem 3 remains true for $\tau = \pi W_0^{-1}$.

THEOREM 4: *Given the sequence* a_n, $n = 0, \pm 1, \pm 2, \cdots$, *satisfying* $|a_0| < \infty$ *and such that the two series*

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left| \frac{a_n}{n} \right|^2$$

and

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n \frac{a_n}{n}$$

converge, then the series

$$\sum_{n=-\infty}^{\infty} a_n \frac{\sin[(\pi/\tau)(z - n\tau)]}{(\pi/\tau)(z - n\tau)} \tag{24}$$

converges uniformly in any bounded region of the z plane to a function $f(z)$, and $f(t)$ is band-limited (W, δ) for $W = \pi/\tau$ and any $\delta > 0$.

Proof: Let

$$g_n = \frac{a_n - a_0}{n\tau} \quad \text{for} \quad n \neq 0$$

and (25)

$$g_0 = -\frac{1}{\tau} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n g_n .$$

Then $\sum_{-\infty}^{\infty} | g_n |^2 < \infty$, and since $\sin [(\pi/\tau)(t - n\tau)]/(\pi/\tau)(t - n\tau)$ is an orthonormal sequence, it follows by the Riesz-Fischer theorem that

$$g(t) = \sum_{-\infty}^{\infty} g_n \frac{\sin[(\pi/\tau)(t - n\tau)]}{(\pi/\tau)(t - n\tau)}$$

converges in the L_2 mean to an L_2 function $g(t)$. Moreover, the Fourier transform of $g(t)$ belongs to L_2 (therefore also to L_1) in $[-\pi/\tau, \pi/\tau]$ and zero a.e. outside this interval; therefore $g(n\tau) = g_n$. Therefore $f(t) = a_0 + t\cdot g(t)$ is band-limited $(\pi/\tau, \delta)$ for any $\delta > 0$ and the sampling theorem holds for any sampling interval smaller then τ. In order to show that the sampling interval may equal τ, we note that $f(t)$ has a representation of the form (21). Substituting (24) into (21) we obtain (22), and (25) is obtained by substituting (23) into (22).

V. BAND-LIMITED RANDOM PROCESSES

Let $\{f(t)\}$ be a random process and let $R(t, t')$ be the autocorrelation function of the process: $R(t, t') = E\{f(t)\cdot\bar{f}(t')\}$. We assume, from now on, that $R(t, t)$ is a continuous function of $t(-\infty < t < \infty)$, therefore the process is continuous in the mean and in probability and a measurable version of this process exists (Loève (1955), Sec. 35 E). All the processes considered in this paper are assumed to be measurable.

Since

$$E\{\| f \|^2\} = \int_{-\infty}^{\infty} (1 + t^2)^{-1} E\{| f(t) |^2\} \, dt,$$

it follows that if

$$\int_{-\infty}^{\infty} \frac{R(t, t)}{1 + t^2} \, dt < \infty \tag{26}$$

then almost all sample functions of the process belong to H_0.

DEFINITION: *A random process* $\{f(t)\}$, $(-\infty < t < \infty)$, *is defined to be band-limited* (W, δ) *if the autocorrelation function of the process,* $R(t, t')$, *satisfies* (26), *if* $R(t, t)$ *is a continuous function of* $t(-\infty < t < \infty)$, *and if almost all sample functions of the process are band-limited* (W, δ). It follows from this definition that if $\{f(t)\}$ is a random process which is band-limited (W, δ), and $f(t)$ is a sample function of the process, and if $\tau < \pi(W + \delta)^{-1}$ then, with probability 1,

$$f(t) = \sum_{n=-\infty}^{\infty} f(n\tau) \frac{\sin [(\pi/\tau)(t - n\tau)]}{(\pi/\tau)(t - n\tau)}$$

for *all* $-\infty < t < \infty$.

THEOREM 5. *Let* $R(t, t')$ *be the autocorrelation function of* $\{f(t)\}$ *and let* $R(t, t)$ *satisfy* (26). *Then a necessary and sufficient condition that* $\{f(t)\}$ *be band-limited* (W, δ) *is that* $R(t, t')$ *satisfy the condition*

$$\int_{-\infty}^{\infty} R(t, \theta) h(t' - \theta; W, \delta) \, d\theta = R(t, t') \tag{27}$$

for all t *and* t' $(-\infty < t, t' < \infty)$.

Proof: If $\{f(t)\}$ is band-limited (W, δ) then $f(t) \cdot (\bar{f}(t') * h(t'; W, \delta)) = f(t) \cdot \bar{f}(t')$ and (27) follows by taking the expectation of both sides. Conversely, since $R(t, t') = \bar{R}(t', t)$, (27) implies that $R(t, t') * h(t; W, \delta) = R(t, t') * h(t; W, \delta) * h(t'; W, \delta) = R(t, t')$. Therefore

$$E\left\{\int_{-\infty}^{\infty} \frac{| f(t) - f(t) * h(t; W, \delta) |^2}{1 + t^2} \, dt\right\} = 0.$$

In the proof of Theorem 6 it will be shown that $R(t, t')$ is analytic in the t, t' plane, it follows that $R(t, t)$ is continuous and almost all the sample functions of $\{f(t)\}$ are continuous (Loève (1955), 35.3C). Therefore for almost all the sample functions $f(t) = f(t) * h(t; W, \delta)$ for *all* t and $\{f(t)\}$ is band-limited (W, δ).

If the process is stationary, then $R(t, t)$ is a constant and (26) is indeed satisfied. For stationary processes

$$R(t, t') = \int_{-\infty}^{\infty} e^{iw(t-t')} \, dF(w)$$

where $F(\omega)$ is of bounded variation and

$$E\{[f(t)*h(t; W, \delta)] \cdot \bar{f}(t')\} = \int_{-\infty}^{\infty} H(\omega; W, \delta) e^{iw(t-t')} \, dF(\omega)$$

where $H(\omega; W, \delta)$ is as defined by Fig. 1. It follows immediately that for stationary processes the conventional definition of band-limited processes and that of this paper are equivalent: if $\{f(t)\}$ is band-limited W_0 in the sense of the conventional definition, then it is band-limited (W, δ) for all $W \geqq W_0$ and $\delta > 0$, and if $\{f(t)\}$ is band-limited (W, δ) then it is also band-limited W_0 (where $W_0 \geqq W$) in the conventional sense.

THEOREM 6. (a) *Let $\{f(t)\}$ be a band-limited process (W, δ) and let $R_g(t, t')$ be the autocorrelation function of the $\{g(t)\}$ process where* $g(t) = [f(t) - f(0)] \cdot t^{-1}$ (and $R_g(t, t') = [R(t, t') - R(0, t') - R(t, 0) + R(0, 0)] \cdot (t, t')^{-1}$). Then

$$R_g(t, t') = \int_{-(W+\delta)}^{(W+\delta)} \int_{-(W+\delta)}^{(W+\delta)} \psi(\omega, \omega') e^{i(\omega t - \omega' t')} \, d\omega d\omega'$$

where $\psi(\omega, \omega')$ is L_2 in the square $|\omega|, |\omega'| \leqq W + \delta$.

(b) *If $\{g(t)\}$ is a random process and $R_g(t, t')$, the autocorrelation function of $\{g(t)\}$, satisfies*

$$\int_{-\infty}^{\infty} R_g(t, t) \, dt < \infty \tag{28}$$

and

$$R_g(t, t') = \int_{-W_0}^{W_0} \int_{-W_0}^{W_0} \psi(\omega, \omega') e^{i(\omega t - \omega' t')} \, d\omega \, d\omega' \tag{29}$$

where $\psi(\omega, \omega')$ belongs to L_2 in $(-W_0 \leqq \omega, \omega' \leqq W_0)$, then the process $\{c + t \cdot g(t)\}$ with $\overline{c^2} < \infty$ is band-limited (W, δ) for all $W \geqq W_0$ and all $\delta > 0$.

Proof: Applying (13) to the r.h.s. of the equation $R(t, t') = R(t, t')*h(t; W, \delta)*h(t'; W, \delta)$ it follows that $R(t, t')$ is uniformly bounded in any bounded region of the t, t' plane. From Theorem 5 and

Lemma 2 it follows that $R(t, t')$ is analytic in each of the variables t and t'. Now consider $t = t'$:

$$R(t, t) = \int_{-\infty}^{\infty} R(\theta, t)h(t - \theta; W, \delta)\, d\theta \tag{30}$$

and for each t both $R(\theta, t)$ and $h(t - \theta; W, \delta)$ are analytic in θ. Since $R(t, t')$ is an autocorrelation function $| R(t, t')|^2 \leqq R(t, t) \cdot R(t', t')$, therefore

$$R(t, t) \leqq R^{1/2}(t, t) \int_{-\infty}^{\infty} R^{1/2}(\theta, \theta) \cdot | h(t - \theta; W, \delta) |^{1/2} \cdot | h(t - \theta; W, \delta)|^{1/2}\, d\theta. \tag{31}$$

Applying (13) and the Schwartz inequality to the right hand side of (31), it follows that the r.h.s. of (30) is uniformly integrable in any finite t interval and therefore $R(t, t)$ is an analytic function of t. Since $R(t, t')$ is an autocorrelation function, the analiticity of $R(t, t)$ implies that of $R(t, t')$ is the t, t' plane (section 34.2, corollary 3 of Loève (1955)). Consequently $[R(t, t') - R(0, t') - R(t, 0) + R(0, 0)]. (tt')^{-1}$ is bounded for $| t |, | t' | < 1$ and

$$\int_{-\infty}^{\infty} R_g(t, t)\, dt < \infty.$$

Since $| R_g(t, t')|^2 \leqq R_g(t, t) \cdot R_g(t', t')$, it follows that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} | R_g(t, t')|^2\, dt\, dt' < \infty.$$

Therefore $\psi(\omega, \omega')$, the Fourier transform of $R_g(t, t')$, exists in the mean, $\psi(\omega, \omega')$ is L_2 in $-\infty < \omega, \omega' < \infty$ and

$$R_g(t, t') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(\omega, \omega')e^{i(\omega t - \omega' t')}\, d\omega\, d\omega'.$$

By Theorem 1 almost all sample functions of $[g(t)]$ are band-limited $(W + \delta, \delta')$ for all $\delta' > 0$, and therefore the $[g(t)]$ process is band-limited $(W + \delta, \delta')$ for all $\delta' > 0$. By (27) and since $R_g(t, t') = \bar{R}_g(t', t)$,

$$R_g(t, t') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_g(\theta, \theta') \cdot h(t - \theta; W + \delta, \delta') \cdot h(t' - \theta'; W + \delta, \delta')\, d\theta\, d\theta'. \tag{32}$$

For each pair t, t' the function $[h(t - \theta; W + \delta, \delta') \cdot h(t' - \theta'; W + \delta, \delta')]$

belongs to L_2 in the θ, θ' plane and so does $R(\theta, \theta')$. By the Parceval theorem for functions of two variables it follows that the right hand side of (32) is equal to

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(\omega, \omega')\bar{F}(\omega, \omega')\,d\omega\,d\omega'$$

where

$$F(\omega, \omega') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t-\theta; W+\delta, \delta')h(t'-\theta; W+\delta, \delta')$$
$$\cdot e^{-i(\omega\theta-\omega'\theta')}\,d\theta\,d\theta' = H(\omega; W+\delta, \delta')\cdot H(\omega'; W+\delta, \delta')e^{i(\omega t-\omega t')}.$$

We have thus shown that for all $\delta' > 0$

$$\psi(\omega, \omega') = \psi(\omega, \omega')\cdot H(\omega; W+\delta, \delta')\cdot H(\omega'; W+\delta, \delta'),$$

which means that $\psi(\omega, \omega')$ vanishes outside $|\omega|, |\omega'| \leqq (W+\delta)$; therefore $\psi(\omega, \omega')$ is also L_1 and

$$R_g(t, t') = \int_{-(W+\delta)}^{(W+\delta)}\int_{-(W+\delta)}^{(W+\delta)}\psi(\omega, \omega')e^{i(\omega t-\omega' t')}\,d\omega\,d\omega'.$$

The $\{g(t)\}$ process is, therefore, a harmonizable process (Loève, 1955) (the $\{f(t)\}$ process may in general not be harmonizable as the function $Si(t)$ of Eq. (2), which may be considered as a degenerate nonstationary process, shows). Turning now to the proof of part (b); it follows from (28) that almost all sample functions of $\{g(t)\}$ are $L_2(-\infty, \infty)$. From (29) it follows that $R_g(t, t)$ is continuous and that almost all sample functions of $\{g(t)\}$ are band-limited (W, δ) (where $W \geqq W_0$ and $\delta > 0$). The rest follows from Theorem 1.

Acknowledgment

This work was carried under the auspices of the Scientific Department. Israel Ministry of Defence.

Note added in proof: The convergence a.s. of the sampling expansion for stationary processes was proved by Y. K. Belyaev (*Theory Probability Appl.*, **4**, 402–408 (1959)).

Received: February 4, 1964

References

Balakrishnan, A. (1957), A note on the sampling principle for continuous signals. *IRE Trans. Inform. Theory* **IT-3**, 143–146.

Beutler, F. J. (1961), Sampling theorems and bases in a Hilbert space. *Inform. Control*, **4**, 97–117.

Blanc-Lapierre, A., and Frotet, R. (1953), "Theorie de Fonctions Aleatoires," pp. 352, 454, 513. Masson, Paris.

Hoffman, K. "Banach Spaces of Analytic Functions." Prentice Hall, Englewood Cliffs, New Jersey.

Lloyd, S. P. (1959), A sampling theorem for stationary (wide sense) stochastic processes. *Trans. Am. Math. Soc.* **92,** 1–12.

Loève, M. (1955), "Probability Theory." Van Nostrand, New York.

Paley, R. E. A. C., and Wiener, N. (1934), "Fourier Transforms in the Complex Domain," Theorem X. Am. Math. Soc., New York.

Titchmarsh, E. (1939), "Theory of Functions," 2nd. ed. Oxford Univ. Press, London.

Wiener, N. (1933), "The Fourier Integral and Certain of Its Applications," Theorem 20. Cambridge Univ. Press. Reprinted by Dover, New York.