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Weak axioms of determinacy and subsystems of analysis II (Σ_2^0 games)

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Abstract

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In [10], we have shown that the statement that all Σ_1^1 partitions are Ramsey is deducible over $\mathbf{ATR}_0$ from the axiom of Σ_1^1 monotone inductive definition, but the reversal needs $\boldsymbol{\Pi}_1^1$-$\mathbf{CA}_0$ rather than $\mathbf{ATR}_0$. By contrast, we show in this paper that the statement that all Σ_2^0 games are determinate is also deducible over $\mathbf{ATR}_0$ from the axiom of Σ_1^1 monotone inductive definition, but the reversal is provable even in $\mathbf{ACA}_0$. These results illuminate the substantial differences among lightface theorems which can not be observed in boldface.

1. Introduction

In this paper, we investigate the proof-theoretic strength of Σ_2^0 determinacy from the standpoint congenial to the program of Reverse Mathematics, whose goal is to answer the following question: *What set existence axioms are needed to prove the theorems of ordinary mathematics*? For information on the program, see [1, 2, 6, 7, 8]. Although this paper has many ideas coming from its predecessor [11], in which the determinancy of Δ_0^2 games was mainly discussed, it can be read separately.

A function $\Gamma: \mathscr{P}(\omega) \to \mathscr{P}(\omega)$ is called a *monotone operator* (over ω), if $\Gamma(X) \subset \Gamma(Y)$ whenever $X \subset Y \subset \omega$. Γ is a Σ_1^1 operator if its graph $\{(x, X): x \in \Gamma(X)\}$ is Σ_1^1 without second-order parameters. The axiom of Σ_1^1 *monotone inductive definition*, denoted as (Σ_1^1-MI), asserts that for each Σ_1^1 monotone operator Γ, there exists a transfinite sequence $\langle \Gamma_\alpha: \alpha < \sigma \rangle$, $\sigma \in \mathrm{Ord}$, such that $\Gamma_\alpha = \Gamma(\bigcup \{\Gamma_\beta: \beta < \alpha\})$ for all $\alpha < \sigma$, and such that $\Gamma_\infty = \Gamma(\Gamma_\infty)$ with $\Gamma_\infty = \bigcup \{\Gamma_\alpha: \alpha < \sigma\}$.

In the context of ordinary descriptive set theory, the importance of Σ_1^1 monotone inductive definitions has been well established. It is known that many different notions (e.g., Kolmogorov's $\mathscr{R}$-operator, weakly representability in Enderton's system $\mathscr{A}$) define the same class as Σ_1^1 monotone inductive definitions (see Hinman [4]). In addition, Solovay has shown that the Σ_1^1 monotone-inductively definable sets can be characterized in terms of Σ_2^0 games (for an account, see Moschovakis [5, pp. 414–415]). In [9, p. 24], Steel mentions that Solovay's work actually shows that (Σ_2^0-Det) and (Σ_1^1-MI) are equivalent over $\mathbf{\Delta_2^1}$-$\mathbf{CA}$ + (full induction scheme). In this paper, we obtain a lightface refinement of this result.

The weakest base theory considered in this paper is $\mathbf{ACA}_0$, a second-order arithmetic based on the Arithmetical Comprehension Axiom, and equipped with the induction axiom:

$$(0 \in X \wedge \forall n\, (n \in X \rightarrow n+1 \in X)) \rightarrow \forall n\, (n \in X),$$

instead of the usual induction scheme. The system $\mathbf{ATR}_0$ is obtained from $\mathbf{ACA}_0$ by adding the boldface axiom of *Arithmetical Transfinite Recursion*: there exists a Turing jump hierarchy starting at any set along any well-ordering.

For any formula ϕ with a variable ranging over ω^ω, we associate a two-person infinite game G_ϕ (or simply ϕ), in which players I and II alternately choose natural numbers, and in which I wins iff the resulting infinite sequence satisfies ϕ. We say that the game G_ϕ is *determinate* if either I or II has a winning strategy in G_ϕ. For a class of formulas C, the axiom (C-Det) asserts that any game (associated with a formula) in C is determinate.

In this paper, we prove

$$\mathbf{ACA}_0 \vdash (\Sigma_2^0\text{-Det}) \rightarrow (\Sigma_1^1\text{-MI}),$$

$$\mathbf{ATR}_0 \vdash (\Sigma_1^1\text{-MI}) \rightarrow (\Sigma_2^0\text{-Det}).$$

And the following is conjectured

$$\mathbf{ACA}_0 \nvdash (\Sigma_1^1\text{-MI}) \rightarrow (\Sigma_2^0\text{-Det}).$$

These relations are well contrasted with the following results in [10]:

$$\mathbf{ATR}_0 \vdash (\Sigma_1^1\text{-MI}) \rightarrow (\Sigma_1^1\text{-Ram}),$$

$$\mathbf{\Pi_1^1}\text{-}\mathbf{CA} \vdash (\Sigma_1^1\text{-Ram}) \rightarrow (\Sigma_1^1\text{-MI}),$$

$$\mathbf{ATR}_0 \nvdash (\Sigma_1^1\text{-Ram}) \rightarrow (\Sigma_1^1\text{-MI}),$$

where (Σ_1^1-Ram) means that any Σ_1^1 partition $P \subseteq \mathscr{P}(\omega)$ is Ramsey, i.e., there is an infinite set $H \subseteq \omega$ such that either all infinite subsets of H are in P or all out of P. We should also remark that the boldface versions of the three statements (Σ_2^0-Det), (Σ_1^1-Ram), and (Σ_1^1-MI) can be easily shown to be pairwise equivalent over $\mathbf{ACA}_0$, or indeed $\mathbf{RCA}_0$.

2. Preliminaries

The language of second-order arithmetic consists of the following symbols: number variables $x, y, z, \ldots$; set variables $X, Y, Z, \ldots$; constants $0, 1$; binary operations $+$ and $\cdot$; binary relations $=, <, \in$. The terms and formulas are defined in the usual way. The formulas are classified as follows.

(i) ϕ is *bounded* or Π^0_0 if it is built up from atomic formulas by propositional connectives and bounded numerical quantifiers $\forall x < t$ and $\exists x < t$;

(ii) ϕ is *arithmetical* or Π^1_0 if it contains no set quantifiers;

(iii) ϕ is Σ^i_n if $\phi = \neg\psi$ where ψ is Π^1_n ($i = 0, 1$ and $n \in \omega$);

(iv) ϕ is Π^0_{n+1} if $\phi = \forall x_1 \cdots \forall x_k\, \psi$ where ψ is Σ^0_n ($n \in \omega$);

(v) ϕ is Π^1_{n+1} if $\phi = \forall X_1 \cdots \forall X_k\, \psi$ where ψ is Σ^1_n ($n \in \omega$).

The Π^i_n *Comprehension Axiom*, denoted (Π^i_n-CA), is defined to be the scheme:

$$\exists X\, \forall x\, (x \in X \leftrightarrow \phi(x)),$$

where ϕ is Π^i_n and it has no free set variables (parameters); the boldface axiom ($\boldsymbol{\Pi}^i_n$-CA) is the same scheme but allowing ϕ to involve free set variables except X.

The system $\mathbf{ACA}_0$ consists of the ordered semiring axioms for $(\mathbb{N}, +, \cdot, 0, 1, <)$ together with ($\boldsymbol{\Pi}^1_0$-CA) and the following induction axiom;

$$(0 \in X \wedge \forall x\, (x \in X \rightarrow (x+1) \in X)) \rightarrow \forall x\, (x \in X).$$

$\mathbf{ACA}_0$ is a conservative extension of first-order Peano arithmetic. The following lemma is a useful fact in $\mathbf{ACA}_0$.

Lemma 2.1. *For any Σ^1_1 formula ϕ, there exists a Π^0_0 formula R such that*

$$\mathbf{ACA}_0 \vdash \phi \leftrightarrow \exists f\, \forall x\, R(f[x]),$$

where $f[x]$ is a code for the sequence $\langle f(0), f(1), \ldots, f(x-1)\rangle$.

Proof. The usual Tarski–Kuratowski algorithm works in $\mathbf{ACA}_0$. $\square$

We next define the *Axiom of Π^i_n Transfinite Recursion* (Π^i_n-TR) and its boldface version ($\boldsymbol{\Pi}^i_n$-TR). In particular, (Π^0_1-TR) is called the Axiom of *Arithmetical Transfinite Recursion*, and denoted by (ATR). The lightface axiom (Π^i_n-TR) is the following scheme: for any recursive well-ordering $<$, there exists a set $H \subseteq \omega$ such that

(i) if b is the $<$-least element, then $(H)_b = \emptyset$,

(ii) if b is the immediate successor of a (w.r.t. $<$), then

$$\forall n\, (n \in (H)_b \leftrightarrow \phi(n, (H)_a)),$$

(iii) if b is a limit, then

$$\forall a\, \forall n\, ((n, a) \in (H)_b \leftrightarrow a \prec b \wedge n \in (H)_a),$$

where ϕ is a Π^i_n formula without second-order parameters and

$$(H)_a = \{n: (n, a) \in H\}, \qquad (n, a) = (n+a)(n+a+1)/2 + n.$$

The boldface axiom ($\boldsymbol{\Pi}^i_n$-TR) is the following: for any $X \subseteq \omega$ and for any well-ordering $\prec$, there exists a set $H \subseteq \omega$ such that

(i) if b is the $\prec$-least element, then $(H)_b = X$,

(ii) and (iii) the same as above,

where ϕ is any Π^i_n formula (which may have second-order parameters).

The system $\mathbf{ATR}_0$, which consists of $\mathbf{ACA}_0 + (\boldsymbol{\Pi}^1_0\text{-TR})$, was first introduced by Friedman [2], and has been extensively studied in Friedman–McAloon–Simpson [3].

In [11], we have shown

$$\mathbf{ACA}_0 \vdash (\text{ATR}) \leftrightarrow (\Sigma^0_1\text{-Det}) \leftrightarrow (\Delta^0_1\text{-Det}),$$

$$\mathbf{ACA}_0 \vdash (\Pi^1_1\text{-TR}) \leftrightarrow (\Delta^0_2\text{-Det}).$$

3. (Σ^1_1-MI) implies (Σ^0_2-Det)

We might as well begin with the formalities of Σ^1_1 monotone inductive definition. First of all, a function $\Gamma: \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is identified with its graph $\{(x, X): x \in \Gamma(X)\}$. So, Γ is a Σ^1_1 operator if its graph is Σ^1_1 without second-order parameters. A function $\Gamma: \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is called a *monotone operator* (over ω), if $\Gamma(X) \subset \Gamma(Y)$ whenever $X \subset Y \subset \omega$. We here notice that $\Gamma(X) \subset \Gamma(Y)$ simply means that if $x \in \Gamma(X)$ then $x \in \Gamma(Y)$, hence it does not imply the existence of $\Gamma(X)$ or $\Gamma(Y)$.

Definition 3.1. The *Axiom of Σ^1_1 Monotone Inductive Definition*, denoted (Σ^1_1-MI), is the following scheme: for any Σ^1_1 monotone operator Γ with no second-order parameters, there is a $W \subseteq \omega \times \omega$ such that

(1) W is a pre-well-ordering on its field $F \subset \omega$,

(2) for all $x \in F$, $W_x = \Gamma(W_{<x})$, where $W_x \overset{\text{def}}{=} \{y \in F: (y, x) \in W\}$ and $W_{<x} \overset{\text{def}}{=} \{y \in F: (y, x) \in W \text{ and } (x, y) \notin W\}$,

(3) $F = \Gamma(F)$.

Theorem 3.1. $\mathbf{ATR}_0 \vdash (\Sigma^1_1\text{-MI}) \to (\Sigma^0_2\text{-Det})$.

Proof. The proof is almost straightforward from Wolfe's original proof of Σ^0_2 determinacy. So we just indicate how it can be carried out in $\mathbf{ATR}_0 + (\Sigma^1_1\text{-MI})$. For other details of Wolfe's proof, see Moschovakis [5, pp. 290–292].

Let $A \subseteq \omega^\omega$ be a Σ_2^0 predicate. So there is a recursive predicate $R \subseteq \omega$ such that for all $f \in \omega^\omega$

$$A(f) \leftrightarrow \exists x\, \forall y\, R(x, f[y]),$$

where $f[y]$ encodes the sequence $\langle f(0), f(1), \ldots, f(y-1)\rangle$. We define, by ($\Sigma_1^1$-MI), a set W of sure winning positions for player I as follows: for any ordinal α,

$$u \in W_\alpha \leftrightarrow \exists x\ [\text{I has a winning strategy in the closed game } A_{u,\alpha,x} = \{f \in \omega^\omega\colon \forall y\, (R(x, (u*f)[y]) \vee (u*f)[y] \in \bigcup\{W_\beta\colon \beta < \alpha\})\}].$$

Clearly, the right-hand side of the above formula is Σ_1^1, hence the transfinite sequence $\{W_\alpha\}$ exists by (Σ_1^1-MI), and so does its limit $W = W_\infty$.

Now what we want to show is the following: if the empty sequence $\emptyset$ is in W_∞, player I has a winning strategy in the game A, and if not, player II has a winning strategy. First suppose that $u \in W_\alpha$. By the definition of W_α, there exists x such that I has a winning strategy $\sigma_{u,\alpha,x}$ for the game $A_{u,\alpha,x}$. If I plays the original game A from the position u with strategy $\sigma_{u,\alpha,x}$, then he wins A without any change of strategy or he reaches a position $u' \in W_\beta$ for some $\beta < \alpha$. In the latter case, he can switch to a winning strategy $\sigma_{u',\beta,x'}$ for some x'. Changing strategies this way, player I can eventually win A from the position u. In particular, I wins A if $\emptyset \in W_\infty$. Here we should notice that I's winning strategy σ for A is built from infinitely many strategies $\sigma_{u,\alpha,x}$. Since "$\sigma_{u,\alpha,x}$ is a winning strategy for I in the closed game $A_{u,\alpha,x}$" is a Π_1^0 relation, the construction of σ needs ($\boldsymbol{\Pi}_1^0$-AC), which is known to be deducible in $\mathbf{ATR}_0$ (see [11]).

Next suppose $u \notin W_\infty$. Fix an $x \in \omega$ arbitrarily. By the definition of W_∞ and Σ_1^0 determinacy, II has a winning strategy $\tau_{u,x}$ in the game

$$A_{u,x} = A_{u,\infty,x} = \{f \in \omega^\omega\colon \forall y\, (R(x, f[y]) \vee f[y] \in W_\infty)\}.$$

Note that II wins $A_{u,x}$ with play f iff f is not in $A_{u,x}$. If II plays the original game A from position u with strategy $\tau_{u,x}$, he always reaches a position u' such that $\neg R(x, u')$ and $u' \notin W_\infty$. At position u', he can switch to a winning strategy $\tau_{u',x'}$ for any x'. Playing in this manner, he can construct an infinite sequence $f = u * u_0 * u_1 * \cdots$ such that for all $i \in \omega$, $\neg R(i, u * u_0 * u_1 * \cdots * u_i)$, hence $\neg A(f)$, so II wins the game A. Thus II's winning strategy τ can be built from the strategies $\tau_{u,x}$ by means of the axiom of choice. However "$\tau_{u,x}$ is a winning strategy for II in the closed game $A_{u,\alpha,x}$" is a Π_1^1 relation, and ($\boldsymbol{\Pi}_1^1$-AC) is not provable in $\mathbf{ATR}_0$. To avoid using the axiom of choice, we invent the following game. Player I first chooses any $u \notin W_\infty$ and $x \in \omega$, and since then the players play natural numbers as usual. Player II wins this game iff they produce the sequence u' such that $\neg R(x, u * u')$ and $u * u' \notin W_\infty$. It is obvious that I has no winning strategy, since for any choice of $u \notin W_\infty$ and $x \in \omega$ at I's first move, II wins the game with $\tau_{u,x}$. So by $\boldsymbol{\Sigma}_1^0$ determinacy, II has a winning strategy in this game. From such a strategy, we can easily construct a winning strategy τ for the game A. This completes the proof. □

4. (Σ_2^0-Det) implies (Σ_1^1-MI)

The purpose of this section is to prove (Σ_2^0-Det)$\rightarrow$(Σ_1^1-MI) within $\mathbf{ACA}_0$. Let Γ be a Σ_1^1 monotone operator. The axiom (Σ_1^1-MI) asserts the existence of pre-well-ordering W with field F such that $W_x = \Gamma(W_{<x})$ for all $x \in F$, and such that $F = \Gamma(F)$. We will first construct a Σ_2^0 game G_1 such that player I has no winning strategy, and such that for any winning strategy τ of player II, W is Π_1^1 in τ, which suffices to get (Σ_2^0-Det)$\rightarrow$(Σ_1^1-MI) in $\mathbf{\Pi_1^1}$-$\mathbf{CA}_0$ (Lemma 4.1). Then we will modify this game to obtain a proof of (Σ_2^0-Det)$\rightarrow$(Σ_1^1-MI) in $\mathbf{ACA}_0$ (Theorem 4.2).

The outline of our proof is as follows. In the game G_1, player I starts the game with playing a number y^*, intending to raise a question whether or not y^* is in the field of W. In reply to this question, player II may either *accept* y^* or *reject* y^*. If II accepts y^*, II is requested to list the ($\leq y^*$)-segment of W and also to give certain witnesses for his assertions. The role of I is then to watch his opponent's moves, and point out a possible error in them. If II rejects y^* at the beginning, the roles are reversed. After the initial stage, the player constructing the ($\leq y^*$)-segment of W is called Pro, and the other player Con. Roughly speaking, Pro wins the game iff Con can not prove to the last that Pro makes a false or erroneous assertion. A more precise definition of game G_1 given in the proof of Lemma 4.1 will disclose that it is indeed a Σ_2^0 game. Since I has no winning strategy in this game (Sublemma 4.1.1), II must have a winning strategy, say τ, by (Σ_2^0-Det). We let $\tilde{W}$ be the set of pairs (x, y) such that the strategy τ calls for II as Pro to put (x, y) into the list for W at every meaningful position. By the series of Sublemmas 4.1.2–7, we show that the maximal well-founded initial segment of $\tilde{W}$ is actually our desired set W. This proves (Σ_2^0-Det)$\rightarrow$(Σ_1^1-MI) in $\mathbf{\Pi_1^1}$-$\mathbf{CA}_0$. In the proof of Theorem 4.2, we define the game G to be the same as G_1 but allowing Con to win the game by finding or predicting an infinite descending sequence through the preordering Pro constructs. Then if we define $\tilde{W}$ as before, we can show that $\tilde{W}$ itself is our desired set W.

Lemma 4.1. $\mathbf{\Pi_1^1}$-$\mathbf{CA}_0 \vdash$ (Σ_2^0-Det)$\rightarrow$(Σ_1^1-MI).

Proof. Pick any Σ_1^1 monotone operator Γ, and fix it throughout the proof. By Lemma 2.1, there is a recursive relation R such that for $x \in \omega$ and $X \in \mathcal{P}(\omega)$,

$$x \in \Gamma(X) \leftrightarrow \exists f\, \forall z\, R(f[z], x, X[z]),$$

where $f[z]$ and $X[z]$ encode the finite sequence $\langle f(0), f(1), \ldots, f(z-1)\rangle$ and the finite set $X \cap \{0, 1, \ldots, z-1\}$, respectively.

Let us begin to describe our game G_1 thoroughly. Player I starts the game with playing a number y^*. Player II then replies either 1 (to *accept* y^*) or 0 (to *reject* y^*). If II chooses 1 (resp. 0), then in the rest of the game, II is called Pro (resp. Con) and I is called Con (resp. Pro). We regard Pro as the first player of the rest of the game. Then players Pro and Con alternatively choose a couple of numbers as follows:

Pro	Con
$v(0), f(0)$	
	$c(0), u(0), g(0)$
$v(1), f(1)$	
	$c(1), u(1), g(1)$
$\vdots$	
	$\vdots$

where $v, u \in \{0, 1\}^{\omega}$, $c \in \{\{-1\} \cup \omega\}^{\omega}$, and $f, g \in \omega^{\omega}$.

To explain the meaning of the game, we need the standard pairing function

$$p(m, n) = (m + n)(m + n + 1)/2 + m.$$

We often write (m, n) for $p(m, n)$ and (n_0, n_1) for $p^{-1}(n)$, unless they may cause a serious confusion. Turning back to the game, Pro builds a preordering $V = \{(n_0, n_1): v(n) = 1\}$ with y^* in its field, and for each (x, y) in V, he gives a witness for $x \in \Gamma(V_{<y})$ by means of f. At a stage n, Con may make a challenge on Pro's assertion $v(m) = 0$, $m \leqslant n$, by setting $c(n) = m$ $(\neq -1)$. If he does so, he is requested to give a witness for $x \in \Gamma(V_{<y})$, using u and g.

The winning conditions (or pay-off) of the game G_1 are described as follows.

1. First of all, we require Con to make challenges along the order already constructed by Pro in the decreasing way and below y^*. Strictly speaking, at stage n, Con may challenge Pro's assertion $m = (m_0, m_1) \notin V$, if (i) it has already been stipulated by Pro that $m_1 \leqslant_V y^*$ (i.e., $v((m_1, y^*)) = 1$ and $(m_1, y^*) < n$), and if (ii) for all previous challenges $c(n') = m' = (m'_0, m'_1)$, $n' < n$, it has been stipulated by Pro before stage n that $m_1 <_V m'_1$ (i.e., $v((m_1, m'_1)) = 1$, $(m_1, m'_1) < n$, $v((m'_1, m_1)) = 0$, and $(m'_1, m_1) < n$). If Con disobeys this rule, he loses whatever the opponent plays.

Assuming that this rule has been obeyed, we describe the further conditions.

2. The case that Con makes no challenges. Then Pro wins iff (i) V is a preordering with the field $F = \{x$: there is y such that $(x, y) \in V$ or $(y, x) \in V\}$ such that $y^* \in F$, and (ii) for all m such that $v(m) = 1$,

$$\forall z\, R(f_m[z], m_0, V_{<m_1}[z]),$$

where $f_m(n) = f((m, n))$.

3. The case that Con makes a positive but finite number of challenges. Let n be the last stage such that $c(n) \neq -1$. Suppose $c(n) = m$. Then Con wins iff (i) $V_{<m_1}$ is not a preordering with y^* in its field, or (ii)

$$\forall z\, R(g^n[z], m_0, U^n[z]) \wedge U^n \subseteq V_{<m_1},$$

where $g^n(k) = g(k + n)$ and $U^n = \{k \in \omega: u(k + n) = 1\}$.

4. The case that Con makes infinitely many challenges. Then player II wins.

This completes the definition of G_1. It is then obvious that G_1 is a Σ_2^0 game.

Sublemma 4.1.1. *Player* II *has a winning strategy.*

Proof. We will show that player I has no winning strategy in game G_1, which implies by (Σ_2^0-Det) that II has a winning strategy. By way of contradiction, assume I had a winning strategy σ. Let y^* be I's first move called by σ. Since II may either accept or reject y^*, I must defend both cases. In the case II rejects y^*, I needs construct V such that y^* is in the field of V. But if such a construction is successful, II could accept y^* and win the game by constructing the same V. To be more precise, we consider the following two plays of the game p_0, p_1, both of which are consistent with I's winning strategy σ. p_0 begins with I's proposal y^* and II's reply 0 (to *reject*). p_1 begins with I's proposal y^* and II's reply 1 (to *accept*). The rest of the two plays are the same except the roles of the players, that is, I is Pro in p_0 and II is Pro in p_1. The plays are simultaneously constructed as follows. Suppose that σ calls for I to choose $v(n)$, $f(n)$ at stage n in p_0. Then player II copies these numbers to his move at stage n in p_1. Following σ, player I replies $c(n)$, $u(n)$, $g(n)$ in p_1. Again, Player II copies these three numbers to his move at stage n in p_0. Then σ specifies I's move at stage $n+1$ in p_0, and so on. In this way, they produce exactly the same sequences, which are both consistent with σ. But, it is clear from the definition of winning conditions that player I can not win in both the plays. This is a contradiction. □

Let us fix a winning strategy τ for player II throughout the proof of Lemma 4.1. A play (or a partial play) is said to be *τ-consistent* if it can be produced while player II follows τ and Con obeys rule (1) (challenging in the decreasing order). By a partial play, we here mean a finite initial segment of a whole play of the game, which at least includes the preliminary stage (where I chooses y^* and II replies 1 or 0). Given a τ-consistent partial play p ending with Pro's move at stage n, we say that at stage n (or at p), Con *can* challenge Pro's move $v(m)=0$ in p $(m \leqslant n)$ if Con can set $c(n)=m$ without breaking rule (1).

We then say that player II *asserts* $x \not\leqslant y$ in p if II is Pro in p, and if $v((x, y))=0$ occurs in p and it can be challenged at p. Player II *never asserts* $x \not\leqslant y$ if there is no τ-consistent partial play p in which II asserts $x \not\leqslant y$, yet if τ calls for II to accept I's initial proposal y^* when $y^*=y$. We also say that player II *asserts* $x \leqslant y$ in p if II is Pro in p, and both $v((x, y))=1$ and $v((y, y^*))=1$ occur in p, and further if $v((y, z))=1$ occurs in p supposing that the last challenge in p was made to (w, z) for some w (the last condition is satisfied vacuously when no challenge occurs in p).

We now define

$$\tilde{W} = \{(x, y)\text{: (1) II never asserts } x \nleqslant y \text{ \& } \\ \text{(2) for any } n, \\ \text{if II asserts } w_1 \leqslant w_0 = y \text{ in some } p_1, \\ \text{and } w_2 \leqslant w_1 \text{ in some } p_2, \\ \vdots \\ \text{and } w_n \leqslant w_{n-1} \text{ in some } p_n, \\ \text{then II never asserts } w_n \nleqslant w_i,\ i < n\},$$

where p_i ranges over the τ-consistent partial plays. Obviously, $\tilde{W}$ is Π^0_1 in τ, and so it exists by (Π^0_1-CA). Our goal is to show that the well-founded part of $\tilde{W}$ is the desired set.

We need some other new notion. Let $(x, y) \in \tilde{W}$. If I plays $y^* = y$ at the initial stage, then τ tells II to accept y^* since Player II never asserts $x \nleqslant y$ (clause (1) in the definition of $\tilde{W}$). We now consider a τ-consistent play p in which I proposes y^*, then II accepts it, and no challenge occurs to the last. We call such a play a *simple y-play*. For any play p, let V^p be the set constructed by Pro in p, i.e.,

$$V^p = \{(x, y)\text{: } v((x, y)) = 1 \text{ occurs in } p\}.$$

We then prove

Sublemma 4.1.2. *Let $(x, y) \in \tilde{W}$. Let p be a simple y-play. Then the $(\leqslant y)$-segment of $\tilde{W}$ is exactly the same as the $(\leqslant y)$-segment of V^p.*

Proof. Assume $(x, y) \in \tilde{W}$, and let p be a simple y-play. We first show $\tilde{W}_y = V^p_y$ without regard to the orders. If $(w, y) \in \tilde{W}$, then II never asserts $w \nleqslant y$ and hence II asserts $w \leqslant y$ in p, i.e., $(w, y) \in V^p$. Conversely, assume $(w, y) \in V^p$. By clause (2) in the definition of $\tilde{W}$ applied to $(x, y) \in \tilde{W}$, II never asserts $w \nleqslant y$ since II asserts $w \leqslant y$ in p. Thus $(w, y) \in \tilde{W}$, since clause (1) is just shown and clause (2) is the same as clause (2) applied to $(x, y) \in \tilde{W}$.

Now choose any (v, w) in the $(\leqslant y)$-segment of $\tilde{W}$. Then $(v, w) \in \tilde{W}$ and $(w, y) \in \tilde{W}$. By the above paragraph, $(w, y) \in \tilde{W}$ implies that II asserts $w \leqslant y$ in p, and so (v, w) could be challenged at p if he asserts $v \nleqslant w$ in p. However, by clause (1) in the definition of $\tilde{W}$ for $(v, w) \in \tilde{W}$, II never asserts $v \nleqslant w$. So II must assert $v \leqslant w$ in p, that is, (v, w) is in the $(\leqslant y)$-segment of V^p.

Conversely, assume $(v, w) \in V^p$ and $(w, y) \in V^p$. By clause (2) in the definition of $\tilde{W}$ for $(x, y) \in \tilde{W}$, II never asserts $v \nleqslant w$, since II asserts $v \leqslant w$ and $w \leqslant y$ in p. So clause (1) for $(v, w) \in \tilde{W}$ is satisfied. For clause (2), assume that $w_0 = w$ and II asserts $w_{i+1} \leqslant w_i$ somewhere for all $i \leqslant n$. Since II asserts $w \leqslant y$ in p, by clause (2) for $(x, y) \in \tilde{W}$, II never asserts $w_{n+1} \leqslant w_i$ for all $i \leqslant n$, which satisfies the clause (2) for $(v, w) \in \tilde{W}$. Therefore, we have $(v, w) \in \tilde{W}$. □

Sublemma 4.1.3. *$\tilde{W}$ is transitive and reflexive.*

Proof. Assume $(v, w) \in \tilde{W}$ and $(w, y) \in \tilde{W}$. Let p be a simple y-play. By Sublemma 4.1.2, we have $(v, w) \in V^p$ and $(w, y) \in V^p$. Since V^p is a preordering (rule (2)), $(v, y) \in V^p$. Again by the previous sublemma, $(v, y) \in \tilde{W}$. Hence $\tilde{W}$ is transitive. The reflectivity can be shown similarly. □

Sublemma 4.1.4. *Let p be a τ-consistent play with* II *as* Pro *such that a positive but finite number of challenges are made. Let y be such that* I*'s last challenge in p is made against (x, y) for some x. Then the $(<y)$-segment of V^p is a subset of the $(<y)$-segment of $\tilde{W}$. Furthermore, if y is in the well-founded part of $\tilde{W}$, the $(<y)$-segment of V^p is the same as the $(<y)$-segment of $\tilde{W}$.*

Proof. Let p and y be as in the above statement. In the same way as the proof of Sublemma 4.1.2, we can prove that if (v, w) is in the $(\leqslant y)$-segment of V^p, then it is in $(\leqslant y)$-segment of $\tilde{W}$. Moreover, if $(w, y) \in V^p$ and $(y, w) \notin V^p$, then II asserts $y \nleqslant w$ in p, so we can not have $(y, w) \in W$, which finishes the first part of the sublemma.

By the way of contradiction, we assume the second part is false, and let y be minimal with respect to (the well-founded part of) $\tilde{W}$ such that the second conclusion fails for y. By the first part, we have $V^p_{<y} \subseteq \tilde{W}_{<y}$. We first show that $V^p_{<y}$ is unbounded in $\tilde{W}_{<y}$ with respect to $\tilde{W}$, i.e., $\forall w \in \tilde{W}_{<y}\, \exists z \in V^p_{<y}\, (w, z) \in \tilde{W}$. By Sublemma 4.1.2, the $(\leqslant y)$-segment of $\tilde{W}$ is the same as the $(\leqslant y)$-segment of V^s with a simple y-play s, so it is a preordering. Hence, if $V^p_{<y}$ is bounded in $\tilde{W}_{<y}$, there exists a $u \in \tilde{W}_{<y}$ such that $V^p_{<y} \subseteq \tilde{W}_{<u}$. Fix such a u. Since $u \in \tilde{W}_{<y}$, we have $(y, u) \notin \tilde{W}$, and so II asserts $y \nleqslant u$ in some τ-consistent partial play q. Consider the τ-consistent play p' in which I challenges (y, u) at q, and after that, he plays $U = \tilde{W}_{<u}$ and a witness for $y \in \Gamma(\tilde{W}_{<u})$. Remark that $y \in \Gamma(\tilde{W}_{<u})$ holds since $y \in \Gamma(V^p_{<y})$ and $V^p_{<y} \subseteq \tilde{W}_{<u}$. Since y is chosen as a minimal element for which the second part of the sublemma fails, the claim of the sublemma must hold for $u \in \tilde{W}_{<y}$, so we have $\tilde{W}_{<u} = V^{p'}_{<u}$. Thus I wins with p', a contradiction. Therefore, $V^p_{<y}$ is unbounded in $\tilde{W}_{<y}$.

Now, let (v, w) belong to the $(<y)$-segment of $\tilde{W}$. Since $V^p_{<y}$ is unbounded in $\tilde{W}_{<y}$, we can choose a $z \in V^p_{<y}$ such that $(w, z) \in \tilde{W}$. If $(w, z) \notin V^p$, then II asserts $w \nleqslant z$ in p, which contradicts $(w, z) \in \tilde{W}$. If $(w, z) \in V^p$ and $(v, w) \notin V^p$, then II asserts $v \nleqslant w$ in p, which contradicts $(v, w) \in \tilde{W}$. Therefore, $(v, w) \in V^p$, that is, (v, w) belongs to the $(<y)$-segment of V^p. □

We define W as the maximal well-founded initial segment of $\tilde{W}$.

Sublemma 4.1.5. *For each y in the field of W, $W_y = \Gamma(W_{<y})$.*

Proof. Choose any y in the field of W. Let p be a simple y-play. By Sublemma 4.1.2, $W_y = V^p_y \subseteq \Gamma(V^p_{<y}) = \Gamma(W_{<y})$. Next suppose $x \in \Gamma(W_{<y}) - W_y$. Then II asserts $x \nleqslant y$ in some τ-consistent partial play q. We consider the τ-consistent

play p in which I challenges (x, y) at q, and after that, plays $U = W_{<y}$ and a witness that $x \in \Gamma(W_{<y})$. By Sublemma 4.1.4, we get the contradiction that p is a win for I. $\square$

Sublemma 4.1.6. *W is a pre-well-ordering.*

Proof. We have shown $\tilde{W}$ is transitive and reflexive in Sublemma 4.1.3. Hence, W is also transitive and reflexive. The well-foundedness of W is straightforward from the definition. So we need only show that W is connected. By way of contradiction, assume there were x and y in W such that $(x, y) \notin W$ and $(y, x) \notin W$. Choose a W-minimal such x. Then we must have $W_{<x} \subset W_{<y}$ (by the transitivity). By using Sublemma 4.1.5, we have $W_x = \Gamma(W_{<x}) \subset \Gamma(W_{<y}) = W_y$. So, by the reflexivity, $(x, y) \in W$, which contradicts with the assumption. $\square$

Sublemma 4.1.7. *Let F be the field of W. Then $F = \Gamma(F)$.*

Proof. By way of contradiction, we assume $\Gamma(F) - F \neq \emptyset$, and choose a $y \in \Gamma(F) - F$. We will finally get the contradiction that the $(\leqslant y)$-segment of $\tilde{W}$ is well-founded and properly extends W. The kernel of the proof is to show that if $x \in \Gamma(F)$, then $(x, y) \in \tilde{W}$. For this sake, we first show

(i) for any $x \in \Gamma(F)$, II never asserts $x \nleqslant y$, and
(ii) for any $x \notin \Gamma(F)$, II never asserts $x \leqslant y$.

To prove (i), we choose an $x \in \Gamma(F)$. For a contradiction, we assume that II asserts $x \nleqslant y$ in a τ-consistent partial play q. Then consider the τ-consistent play p in which I makes a challenge to (x, y) at q, and plays $U = F$ and a witness for $x \in \Gamma(F)$. Since p is not a win for I, F is not a subset of $V^p_{<y}$. Then take an $x' \in F - V^p_{<y}$. Since $y \in \Gamma(V^p_{<y})$ and $y \notin \Gamma(W_{<x'})$, there must be a $y' \in V^p_{<y} - W_{<x'}$. Let I challenge (x', y') at an appropriate $q' \subset p$, and play $U = W_{<x'}$ and a witness for $x' \in \Gamma(W_{<x'})$. Call this τ-consistent play p'. Since I can not win with p', $W_{<x'}$ is not a subset of $V^p_{<y'}$. Continuing in this way, we obtain an infinite descending sequence through W, a contradiction.

To prove (ii), we choose any $x \notin \Gamma(F)$. Again by way of contradiction, assume that II asserts $x \leqslant y$ in a τ-consistent partial play q. Then consider the τ-consistent play p extending q with no more challenges. Since $x \in \Gamma(V^p_{<y})$ and $x \notin \Gamma(F)$, there must exist a $y' \in V^p_{<y} - F$. Let I challenge (y, y') at an appropriate $q' \subset p$, and play $U = F$ and a witness for $y \in \Gamma(F)$. Call this τ-consistent play p'. Since I does not win with p', there must be an $x' \in F - V^{p'}_{<y'}$. Since $y' \in \Gamma(V^{p'}_{<y'})$ and $y' \notin \Gamma(W_{<x'})$, there must be a $y'' \in V^{p'}_{<y'} - W_{<x'}$. Then let I challenge (x', y'') at an appropriate $q'' \subset p'$, and play $U = W_{<x'}$ and a witness for $x' \in \Gamma(W_{<x'})$. Continuing as in the previous case, we get a contradiction.

By using (i) and (ii), we next show that

$$(x, y) \in \tilde{W} \Leftrightarrow x \in \Gamma(F).$$

The direction $\Rightarrow$ is obvious. To show the other direction, assume that $x \in \Gamma(F)$. By (i), II never asserts $x \nleq y$, which is exactly the clause (1) in the definition of $\tilde{W}$. For clause (2), suppose that II asserts $w_1 \leq w_0 = y$ in some p_1, and $w_2 \leq w_1$ in some $p_2, \ldots$ and $w_n \leq w_{n-1}$ in some p_n. It is easy to see by (ii) that $w_i \in \Gamma(F)$ for all $i \leq n$. If $w_i \in F$, then II never asserts $w_n \nleq w_i$ by the definitions of $\tilde{W}$ and F. If $w_i \in \Gamma(F) - F$, then by (i), II never asserts $w_n \nleq w_i$. So in any case, clause (2) is satisfied by (x, y). Hence $(x, y) \in \tilde{W}$.

From the equivalence we have just shown, we can easily deduce

$$(x, y) \in \tilde{W} \ \&\ (y, x) \in \tilde{W} \Leftrightarrow x \in \Gamma(F) - F.$$

So if x is strictly below y (with respect to $\tilde{W}$), then $x \in F$. Thus the $(\leq y)$-segment of $\tilde{W}$ is well-founded and properly extends W, which contradicts with the definition of W. □

Now, the proof of Lemma 4.1 is completed. □

Theorem 4.2. $\mathbf{ACA}_0 \vdash (\Sigma^0_2\text{-Det}) \rightarrow (\Sigma^1_1\text{-MI})$.

Proof. To begin with, we recall that in the above proof of lemma, ($\boldsymbol{\Pi}^1_1$-CA) is used only to obtain the well-founded part W from $\tilde{W}$. We here add machinery to the game G_1 so that W can be obtained directly from a winning strategy without using ($\boldsymbol{\Pi}^1_1$-CA). Roughly speaking, this can be done by asking the players whether a given number y^* is in the well-founded part or not. The details will be given below.

The new game G starts like G_1. Player I proposes a number y^*, then player II replies either 1 (to *accept* y^*) or 0 (to *reject* y^*). If II chooses 1 (resp. 0), then in the rest of the game, II is called Pro (resp. Con) and I is called Con (resp. Pro). Pro and Con plays as in G_1 except that in the new game, Con is given another chance to win, that is, he can win by finding or predicting an infinite descending sequence d through the preordering Pro constructs. Strictly, they play as follows:

Pro	Con
$v(0), f(0)$	
	$c(0), u(0), g(0), d(0)$
$v(1), f(1)$	
	$c(1), u(1), g(1), d(1)$
$\vdots$	
	$\vdots$

where $v, u \in \{0, 1\}^\omega$, $c \in \{\{-1\} \cup \omega\}^\omega$, and $f, g, d \in \omega^\omega$.

The winning conditions of G are the same as those of G_1 except for (2) the case that Con makes no challenges. In this case, Pro wins G iff (i) V is a preordering with y^* in its field, (ii) for all m such that $v(m)=1$, $\forall z\, R(f_m[z], m_0, V_{<m_1}[z])$, where $f_m(n)=f((m,n))$, and additionally (iii) the sequence d is not a descending sequence through $V_{<y^*}$. Clearly, G is still a Σ_2^0 game.

All the terminology introduced in the proof of Lemma 4.1 (e.g., "player II asserts $x \nleq y$") can be used in the present context without any essential change. All the Sublemmas 4.1.1–7 can be shown for the new game in the analogous way. In addition, we can prove that $\tilde{W}$ is well-founded. Suppose that $\{x_i\}$ were an infinite descending sequence through $\tilde{W}$. If I plays $y^*=x_0$, II must accept y^*. Let p be a simple x_0-play in which I (=Con) plays $d(i)=x_{i+1}$ for each i. By the fact corresponding to Sublemma 4.1.2, we get the contradiction that this play is a win for I. Thus, $\tilde{W}$ is well-founded. This completes the proof. □

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