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The Jacobson Radical of Graded PI-Rings and Related Classes of Rings

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1. INTRODUCTION

Let S be a semigroup. An associative ring $R = \oplus_{s \in S} R_s$ is said to be *S-graded* if $R_s R_t \subseteq R_{st}$ for all $s, t \in S$. An ideal I of R is *homogeneous* if $I = \oplus_{s \in S} I_s$, where $I_s = I \cap R_s$. The Jacobson radical of R is denoted by $\mathcal{J}(R)$. It is well known that many important properties of the radicals can be deduced with the use of homogeneity (see [4, 6]).

The problem of when the Jacobson radical is homogeneous has been investigated by several authors (see [6]). In general this question is very difficult. It is related to the famous semiprimitivity problem for group algebras (see [13]). Indeed, if the radical is homogeneous in every algebra of characteristic zero graded by a group, then all group algebras of this group over fields of characteristic zero are semiprimitive.

It would be interesting to obtain positive results under natural additional assumptions. Most of the theorems obtained so far deal with restrictions on the underlying group and show that the radical is homogeneous in rings graded by groups of various special types. It is also interesting to determine whether important ring-theoretic conditions imposed on a graded ring force the Jacobson radical of the ring to be homogeneous. Our paper is devoted to this question for a large class of

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rings, including in particular PI-rings and rings with left or right Krull dimension.

The first positive results on homogeneity of the Jacobson radical were obtained by Bergman [1] and Cohen and Montgomery [3]. In particular, [3, Theorem 4.4(3)] tells us that in characteristic zero every algebra graded by a finite group has homogeneous radical.

In this paper we shall prove that the radical is homogeneous in all PI-algebras of characteristic zero graded by cancellative semigroups. The question of whether the same is true of every right Noetherian algebra of characteristic zero seems to be quite difficult. It is not even known whether every right Noetherian group algebra over a field of characteristic zero is semiprimitive. On the other hand, for every non-cancellative semigroup S there exists an S-graded algebra whose radical is not homogeneous (see [2]).

The other main result of this paper deals with the following question, addressed in [5] and recorded in [4]: Is the Jacobson radical of every ring graded by a u.p.-semigroup homogeneous? A semigroup S is a *u.p.-semigroup* if, for any two non-empty finite subsets X, Y of S, there exists at least one element uniquely expressed in the form xy, where $x \in X$, $y \in Y$. Obviously, every u.p.-semigroup is cancellative. The investigation of rings graded by u.p.-semigroups is motivated by earlier results on group algebras of u.p.-groups (see [13]). Previous facts related to this question are discussed in the survey [4] and the monograph [6]. The answer is not known even in the case of rings graded by u.p.-groups.

We shall prove that the answer to this question is positive for a large class of graded rings. It will follow that the radical of a ring with right Krull dimension, or more generally a ring that is right Goldie modulo the prime radical, or a PI-ring or a semilocal ring graded by a u.p.-semigroup is homogeneous.

2. MAIN RESULTS

Let us begin with PI-algebras of characteristic zero.

THEOREM 2.1. *Let S be a cancellative semigroup, R an S-graded PI-algebra over a field of characteristic zero. Then the Jacobson radical of R is homogeneous.*

Easy examples of group algebras of finite groups show that the restriction on characteristic cannot be removed from Theorem 2.1.

Further, we investigate rings graded by u.p.-semigroups. Let $R = \oplus_{s \in S} R_s$ be an S-graded ring, and let $H(R) = \bigcup_{s \in S} R_s$ be the set of all *homogeneous* elements of R. The largest homogeneous ideal contained in $\mathcal{J}(R)$ will be denoted by $\mathcal{J}_{\mathrm{gr}}(R)$ and will be called the homogeneous radical of R. Our second main theorem includes three classes of rings.

THEOREM 2.2. *Let S be a u.p.-semigroup, and let R be an S-graded ring such that at least one of the following conditions is satisfied*:

(i) *all nil subsemigroups of $H(R/\mathcal{J}_{\mathrm{gr}}(R))$ are locally nilpotent*;

(ii) *every nil subsemigroup of every right primitive homomorphic image of R is locally nilpotent*;

(iii) *for every minimal prime ideal P of R, the ring R/P is a domain or embeds into a matrix ring over a skew field*.

Then the Jacobson radical of R is homogeneous.

The class of rings satisfying (i) contains all rings R such that in all homomorphic images of R all multiplicative nil subsemigroups are locally nilpotent. This applies to all PI-rings, left or right Noetherian rings, and, more generally, all rings with left or right Krull dimension (see [9, 6.3.5, 13.4.2]).

The class of rings satisfying (ii) contains, beyond the classes mentioned above, all semilocal rings.

Condition (iii) concerns all rings which are "nice" modulo the Baer radical. In particular, this covers the case where $R/\mathcal{B}(R)$ is a right Goldie ring.

3. NOTATION AND PRELIMINARIES

For the previous results on radicals of graded rings we refer to [6]. Let R be an S-graded ring. If $r \in R$ and $r = \sum_{s \in S} r_s$, where $r_s \in R_s$, then the elements r_s are called the *homogeneous components* of r. Put $\operatorname{supp}(r) = \{s \mid r_s \neq 0\}$. We assume that $\operatorname{supp}(0) = \emptyset$. By the *length* $|r|$ of r we mean $|\operatorname{supp}(r)|$. Let $H(r) = \{r_s \mid r_s \neq 0\}$ be the set of all homogeneous components of r. Put $H(0) = 0$. For $P \subseteq R$, let $H(P) = \{H(r) \mid r \in P\}$. If $T \subseteq S$, define $R_T = \oplus_{s \in T} R_s$ and $r_T = \sum_{s \in T} r_s$.

Let $R = \oplus_{s \in S} R_s$ be an S-graded ring, and let I be a homogeneous ideal of R. Then $R/I = \oplus_{s \in S} R_s/I_s$ is S-graded. If $I \subseteq \mathcal{J}(R)$, then $\mathcal{J}(R)$ is homogeneous if and only if $\mathcal{J}(R/I)$ is homogeneous. If $I \subseteq \mathcal{B}(R)$, then $\mathcal{B}(R)$ is homogeneous if and only if $\mathcal{B}(R/I)$ is homogeneous.

The following lemma was obtained by Cohen and Montgomery [3, Theorem 4.4, Corollaries 4.2, 5.4, and 6.4].

LEMMA 3.1 [3]. *Let G be a finite group of order n, and let R be a G-graded ring. Then*

$$\mathcal{J}(R_e) = R_e \cap \mathcal{J}(R), \qquad H\left(\mathcal{J}(R)^n\right) \subseteq \mathcal{J}(R),$$
$$\mathcal{B}(R_e) = R_e \cap \mathcal{B}(R), \qquad H\left(\mathcal{B}(R)^n\right) \subseteq \mathcal{B}(R).$$

Next, we include all the necessary information on the structure of the full matrix semigroups in one lemma (these facts are well known; see [10, Lemma 1.4, Theorems 1.3, 1.6] which use the concepts of completely 0-simple semigroups and Rees matrix semigroups).

LEMMA 3.2. *Let F be a skew field, F_n the set of all $n \times n$ matrices over F. For $i = 0, 1, \ldots, n$, denote by M_i the set of all matrices of rank $\leq i$. Then*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = F_n$$

are the only ideals of the multiplicative semigroup F_n. For every $i = 1, \ldots, n$, the set $M_i \setminus M_{i-1}$ is a disjoint union of subsets $G_{\alpha\beta}$, indexed by the elements α, β of a certain set Λ_i, and such that, for all $\alpha, \beta, \gamma, \delta \in \Lambda_i$,

- (i) *either $G_{\alpha\beta}$ is a group, or $G_{\alpha\beta}^2 \subseteq M_{i-1}$;*
- (ii) $G_{\alpha\beta} F_n G_{\gamma\delta} \subseteq G_{\alpha\delta} \cup M_{i-1}$;
- (iii) *$G_{\alpha *} \cup M_{i-1}$ is a right ideal of F_n, where $G_{\alpha *} = \bigcup_{\lambda \in \Lambda_i} G_{\alpha\lambda}$;*
- (iv) *$G_{*\beta} \cup M_{i-1}$ is a left ideal of F_n, where $G_{*\beta} = \bigcup_{\lambda \in \Lambda_i} G_{\lambda\beta}$;*
- (v) *$G_{\alpha\beta} \cup M_{i-1}$ is a left ideal of $G_{\alpha *} \cup M_{i-1}$;*
- (vi) *$G_{\alpha\beta} \cup M_{i-1}$ is a right ideal of $G_{*\beta} \cup M_{i-1}$.*

4. TECHNICAL PROPOSITION

This section contains our main technical proposition generalizing [5, Theorem 3.2; 4, Theorem 4.2], which played the key roles in most of the previous results on rings graded by t.u.p.-semigroups. A semigroup S is called a *t.u.p.-semigroup* if, for any two nonempty finite subsets X, Y of S such that $|X| + |Y| > 2$, there exist at least two elements $s \in S$ uniquely expressed in the form $s = xy$, where $x \in X$, $y \in Y$. Obviously, every t.u.p.-semigroup is a u.p.-semigroup. However, there are u.p.-semigroups which are not t.u.p.-semigroups; see [10, Example 10.13]. After we shall have proved the new stronger result, many of the known theorems will automatically transfer from t.u.p.- to u.p.-semigroups. Note that even in the case of rings graded by t.u.p.-semigroups our proposition tells more than earlier results mentioned above. Also, our proof uses ideas different from those used in earlier proofs.

We shall say that an element r of a graded ring R is *rigid* if $xry = 0 \Leftrightarrow xr_t y = 0$ for each $t \in \mathrm{supp}(r)$ and all $x, y \in H(R)$. It is routine to verify that if the ring R is graded by a cancellative semigroup and I is an ideal of R, then all elements of minimal positive length in I are rigid. In particular, all homogeneous elements of I are rigid.

If M is a subset of a ring R and $r \in R$, then adopt the convention that $rM^1 = rM \cup \{r\}$ and $M^1 rM^1 = MrM \cup rM \cup Mr \cup \{r\}$.

PROPOSITION 4.1. *Let S be a u.p.-semigroup with identity e, R an S-graded ring, r a rigid element of R, and let M be the multiplicative semigroup generated by $H(r)$. Then*

(i) *if $r \notin R_e$ and $M^1 rM^1$ consists of quasiregular elements, then M is nilpotent*;

(ii) *if $H(R)^1 rH(R)^1$ consists of quasiregular elements, then $r_e \in \mathcal{J}(R_e)$, and if, additionally, $r \notin R_e$, then r_e belongs to the nilradical of R_e.*

In order to prove this proposition, let us begin with the following lemma, which will allow us to consider rings and semigroups with identities.

LEMMA 4.2. (i) *Let S be a u.p.- or t.u.p.-semigroup without identity element. Then the semigroup S^e with identity e adjoined is also u.p. or t.u.p., respectively.*

(ii) *Let $R = \oplus_{s \in S} R_s$ be an S-graded ring, and let R' be the ring with identity* 1 *adjoined in the usual way. Denote by R_e the subring generated in R' by* 1. *Then $R' = \oplus_{s \in S^e} R_s$ is S^e-graded.*

Proof. The assertion (ii) is obvious. In (i) we shall only consider the case where S is a t.u.p.-semigroup, since the proof for u.p.-semigroups is similar.

Take two nonempty finite subsets $X, Y \in S^e$ with $|X| + |Y| > 2$, and any elements $a, b \in S$. The sets aX and Yb are contained in S, and $|aX| + |Yb| > 2$ because S^e is cancellative. Therefore there exist distinct elements u', v' uniquely expressed in the form $u' = axyb$, $v' = aztb$, where $ax, az \in aX$ and $yb, tb \in Yb$. Put $u = xy$, $v = zt$. By the cancellativity of S these representations of u and v as products of elements from X and Y are unique. Thus S^e is a t.u.p.-semigroup. ∎

LEMMA 4.3. *Let S be a cancellative semigroup, R an S-graded ring, r a rigid element of R, and let M be the multiplicative semigroup generated by $H(r)$ in R. If M contains* 0, *then M is nilpotent.*

Proof. Suppose that M contains 0. This means that $y_1 \cdots y_m = 0$ for some $y_1, \dots, y_m \in H(r)$. Choose m and the $y_1, \dots, y_m$ such that $q = y_1 \cdots$

$y_{m-1} \neq 0$, and consider the product $z = qr$. Since q is homogeneous, z is also a rigid element of R. Given that S is cancellative, we see that qy_m is a homogeneous component of z. Therefore $qy_m = 0$ implies $qr_s = 0$ for all $s \in S$; whence $z = 0$ and, moreover, $y_1 \cdots y_{m-1}H(r) = 0$. Similarly, considering that $y_1 \cdots y_{m-2}rH(r) = 0$, we get $y_1 \cdots y_{m-2}(H(r)^2) = 0$. Repeating this m times, we conclude that $(H(r))^m = 0$. Thus M is nilpotent. ∎

LEMMA 4.4. *Let S be a cancellative semigroup, R an S-graded ring, I the ideal of nonunits of S, and $G = S \setminus I$ (if S has no identity, then $S = I$). Assume that $x + y = yx$ for some $x \in R_I$, $y \in R$. Then $y \in K$, where K is the subring generated by $H(x)$.*

Proof. We will show that $y_g \in K$, for each $g \in I$. The case where $y_g = 0$ is trivial, and so we assume $g \in \operatorname{supp}(y)$.

First, note that $y \in R_I$ because R_I is an ideal of R. If two elements s and t of I generate the same right ideal, then $s = t$. Therefore there exists a maximum positive integer n such that $\operatorname{supp}(z_1)I^1 \supset \cdots \supset \operatorname{supp}(z_n)I^1$, for some $z_1, \dots, z_n \in H(y)$, where $z_n = y_g$. We call n the depth of y_g. We proceed by induction on the depth of y_g.

Assume first that the depth of y_g is 1. This means that gI^1 is maximal in the set of principal right ideals of I generated by the elements of $\operatorname{supp}(y)$. Hence $g = st$ implies $y_s = 0$, and so $y_g \in H(x) \subseteq K$.

Next, assume that the depth of y_g is $n > 1$. Since $x + y = yx$, we get $y_g = \sum_{st=g} y_s x_t - x_g$. If $st = g$ and $y_s \neq 0$, then the depth of y_s is less than n, and by the inductive assumption $y_g \in K$. It follows that $y_g \in K$, as claimed. ∎

Proof of Proposition 4.1. By Lemma 4.2 we may assume that R has an identity 1. Every u.p.-group is t.u.p. (see [10, Chap. 10]), and so the group of units G of S is a t.u.p.-group. It is routine to verify that $I = S \setminus G$ is an ideal of S.

Assume that $r \notin R_e$ and M^1rM^1 consists of quasiregular elements. By Lemma 4.3 in order to show that M is nilpotent it suffices to prove that $0 \in M$.

First, consider the case where $r \in R_G$. In view of [6, Corollary 22.9], all elements of M^1rM^1 are quasiregular in R_G. Since G is t.u.p., we can follow the argument used in [5]. Replacing r by $r_g r r_h$ for some g, $h \in \operatorname{supp}(r)$, without loss of generality we may assume that all elements of M^1rM^1 are quasiregular, and that $e \notin \operatorname{supp}(r)$. Then $|1 - r| > 1$ and $1 - r$ is a unit of R_G. Therefore $(1 - r)b = 1$ for some $b \in R_G$, and so $c^k(1 - r)b = c^k$ for any $c \in H(r)$ and any k. Choose $c \in H(r)$, $k > 1$, and b such that $|b|$ is minimal among the lengths of all elements b satisfying $c^k(1 - r)b = c^k$ for some c, k. If $b = 0$, then $0 = c^k \in M$ and we are done. Suppose $b \neq 0$. Since S is t.u.p. and $|1-r| > 1$, we can find an element $w \in S$, $w \neq \mathrm{supp}(c^k)$, uniquely expressed in the form $w = uv$, where $u \in \mathrm{supp}(c^k(1-r))$, $v \in \mathrm{supp}(b)$. Then $c^k(1-r)b = c^k$ implies $[c^k(1-r)]_u b_v = 0$. Since r is rigid and $c \in H(r)$, it follows easily that $c^{k+1}(1-r)b_v = 0$. Therefore $c^{k+1}(1-r)(b-b_v) = 0$, contradicting the choice of b, and proving the claim.

Second, suppose that $r \notin R_G$. Then there exists $h \in I$ such that $r_h \neq 0$. Given that I is an ideal of S, we get $b = r_h r \in R_I$. Obviously, b inherits the hypothesis imposed on r, and so we may assume that from the very beginning $r \in R_I$.

Suppose that the semigroup M does not contain 0. Denote by T the subsemigroup generated in S by $\mathrm{supp}(r)$. Then for every $t \in T$ we have $M \cap R_t \neq 0$.

Suppose that T is not a right Ore semigroup. Let ρ be the left reversive congruence on T defined in [11]. Then there exists $t \in T$ such that the set $t\,\mathrm{supp}(r)$ is ρ-separated in the sense of [12] and $(h, e) \notin \rho$ for every $h \in t\,\mathrm{supp}(r)$. Choose a nonzero element $b \in M \cap R_t$. We know that $x = br \neq 0$ and $x + y = xy = yx$ for some $y \in R_I$. By Lemma 4.4, $y \in R_T$.

The left cancellativity of ρ implies that $x' + y' = x'y' = y'x'$ for every ρ-class in T and the corresponding ρ-components x', y' of x, y. Let X be the semigroup generated by the support of x. Replacing x by any cxd, where $c, d \in M$, we see that the element $cx'd$ has a quasiinverse $u \in R_I$. Again, we know that $u \in R_X$. Therefore x' and R_X inherit the hypotheses on x and R. Proceeding in this way we eventually come to an element z whose support is in a single ρ_Z-class, where ρ_Z is the left reversive congruence on the semigroup Z generated by the support of z.

From [11, Lemma 3], we know that Z is a right Ore semigroup. Hence Z is a t.u.p.-semigroup (cf. [10, Theorem 10.6]). Since z inherits the hypotheses on r, we can apply the first paragraph of the proof to see that 0 is contained in the semigroup generated by the components of z. But $z \in M$, so this contradicts the supposition that $0 \notin M$. If T is a right Ore semigroup, then it is t.u.p., and we get a contradiction again. It follows that $0 \in M$, which completes the proof of (i).

Further, we show that (ii) easily follows from (i). Indeed, assume that $H(R)^1 r H(R)^1$ consists of quasiregular elements.

If $r \in R_e$, then $R_e r R_e \subseteq H(R)^1 r H(R)^1$ consists of quasiregular elements. By [6, Corollary 22.9], all these elements are quasiregular in R_e, as well. Therefore $r_e \in \mathcal{J}(R_e)$.

Assume that, in addition, $r \notin R_e$. Given that all elements in $H(R)^1 r H(R)^1$ are quasiregular, clearly (i) applies to every nonzero xry, where $x, y \in H(R)^1$. Therefore $L = \bigcup_{s \in S} H(R)^1 r_s H(R)^1$ is a nil ideal of $H(R)$. In particular, r_e generates a nilideal of R_e. This completes the proof. ∎

5. PROOFS OF THE MAIN THEOREMS

A semigroup S is said to be *permutational* if there exists $n > 1$ such that, for any n elements $x_1, \ldots, x_n$ of S, their product can be rearranged as $x_1 \cdots x_n = x_{\sigma 1} \cdots x_{\sigma n}$ for a nontrivial permutation σ. We shall use the following properties of permutational semigroups (see [10, Theorem 19.8, Corollary 19.13]).

LEMMA 5.1. *Every permutational group is finite-by-abelian-by-finite. A finitely generated permutational group is abelian-by-finite. Every permutational cancellative semigroup has a permutational group of fractions.*

LEMMA 5.2. *Let R be an S-graded PI-ring, and let T be a multiplicative subsemigroup of $H(R)$. If T does not contain zero, then* $\operatorname{supp}(T)$ *is a permutational subsemigroup of S.*

Proof. Let $H = \operatorname{supp}(T)$. Every PI-ring (or PI-algebra) satisfies a multilinear identity, i.e., an identity of the form

$$x_1 \cdots x_n + \sum_{1 \neq \sigma \in S_n} k_\sigma x_{\sigma 1} \cdots x_{\sigma n} = 0, \tag{1}$$

where S_n is the symmetric group, k_σ are integers (elements of the field in the case of algebras; cf. [16]). Let n be the degree of a multilinear identity (1) satisfied in R.

Take any elements $t_1, t_2, \ldots, t_n$ in T. Suppose that $t_i \in R_{h_i}$, for $i = 1, \ldots, n$. Applying (1) to the elements $t_1, \ldots, t_n$ we get

$$t_1 \cdots t_n \in R_{h_1 \cdots h_n} \cap \sum_{1 \neq \sigma \in S_n} R_{h_{\sigma 1} \cdots h_{\sigma n}}.$$

Given that T does not contain 0 it follows that $t_1 \cdots t_n \neq 0$. Therefore

$$0 \neq t_1 \cdots t_n \in R_{h_1 \cdots h_n} \cap R_{h_{\sigma 1} \cdots h_{\sigma n}}$$

for some $\sigma \neq 1$. Hence $h_1 \cdots h_n = h_{\sigma 1} \cdots h_{\sigma n}$. This means that the semigroup H is permutational, as claimed. ∎

LEMMA 5.3. *Let S be a permutational cancellative semigroup, and let R be an S-graded PI-algebra over a field of characteristic zero. Then $\mathcal{J}(R)$ is homogeneous.*

Proof. Lemma 5.1 says that S has a permutational group of fractions Q. Put $R_q = 0$ for $q \in Q \setminus S$. Then R is Q-graded and $R = R_S$. It suffices to prove that $\mathcal{J}(R_T)$ is homogeneous for every finitely generated subgroup T of Q (because then it will follow that $\mathcal{J}(R)$ is homogeneous, too; cf. [6, Lemma 30.27]).

Lemma 5.1 shows that T is abelian-by-finite. This means that T has a normal abelian subgroup A such that T/A is finite. Hence A is also finitely generated [15, 1.6.11], and so it contains a torsion-free subgroup of finite index. Therefore we may assume that A is torsion-free itself. It follows from [6, Theorem 30.28] that $\mathcal{J}(R_A)$ is homogeneous. It is easily seen that R_T is T/A-graded. Lemma 3.1 implies $\mathcal{J}(R_A) = R_A \cap \mathcal{J}(R_T)$. Therefore $\mathcal{J}(R_A)$ generates a homogeneous ideal I in R_T contained in $\mathcal{J}(\mathcal{R}_T)$. We can factor out I and assume that $\mathcal{J}(R_A) = 0$.

Take any element x in $H(\mathcal{J}(R))$. There exists $r \in \mathcal{J}(R)$ such that $x = r_g$. Given that T/A is a finite group, it follows from [3, Theorem 4.4] that $\mathcal{J}(R)$ is T/A-homogeneous. Therefore we may assume that $\operatorname{supp}(r)$ is contained in one coset gA of T/A (otherwise we could replace r by its T/A-homogeneous component involving r_g). Let n be a positive integer such that $g^n \in A$. Then $g^{n-1} \operatorname{supp}(r) \subseteq g^n A = A$, and so $x^{n-1} r \in R_A$. It follows from [6, Corollary 22.9] that $x^{n-1} r \in \mathcal{J}(R_A) = 0$; whence $x^{n-1} r = 0$. Given that G is cancellative, x^n is a homogeneous component of $x^{n-1} r$, and so $x^n = 0$.

Thus $H(\mathcal{J}(R))$ consists of nil elements. By [16, Theorem 1.6.36], $H(\mathcal{J}(R)) \subseteq \mathcal{B}(R) \subseteq \mathcal{J}(R)$. It follows that $\mathcal{J}(R_T)$ is homogeneous, as required. ∎

We are now ready for the proofs of the main theorems. First, note that the hypotheses of Theorems 2.1 and 2.2(i), (ii) are inherited by the ring $R/\mathcal{J}_{\mathrm{gr}}(R)$. Hence, for these proofs one can factor out $\mathcal{J}_{\mathrm{gr}}(R)$ and assume that $\mathcal{J}(R)$ has no nonzero homogeneous elements.

In all proofs we will suppose to the contrary that the Jacobson radical $\mathcal{J}(R)$ is not homogeneous. Then we can choose an element r of minimal length in $\mathcal{J}(R)$ with $r_s \notin \mathcal{J}(R)$ for some $s \in S$. Let $W = H(R)^1 r H(R)^1$, and let $V = H(W)$. Denote by A the additive subgroup generated in R by V. Since S is a cancellative semigroup, it is routine to verify that V is an ideal of $H(R)$, and so A is an ideal of R.

Proof of Theorem 2.1. We know that $r_s \notin P$ for some right primitive ideal P of R and some $s \in S$. Let V' be the image of V in $R' = R/P$. Since S is cancellative, every element of V is a homogeneous component of an element of $\mathcal{J}(R)$. We know that $R/P \cong F_n$ for a skew field F. Choose $a \in V$ such that its image $a' \in V'$ is of minimal positive rank as a matrix in F_n. We can choose a that is not nilpotent, since otherwise $a'H(R)'$ is a right ideal of $H(R)'$ that consists of nilpotents, so it is nilpotent, and therefore $a'R'$ is a nilpotent right ideal of R', contradicting the fact that $a' \neq 0$.

Lemma 3.2 tells us that $a'H(R)'a' \subseteq G \cup 0$ for a maximal subgroup of the multiplicative semigroup of F_n. Therefore it has no zero divisors. Replacing a by some a^N we can also assume that every projection of a

onto a right primitive homomorphic image of R lies in a maximal subgroup of this image.

It is easily verified that $\mathcal{J}(R) \cap aRa \subseteq \mathcal{J}(aRa)$. The choice of a implies that the image of aRa in every right primitive homomorphic image of R is a matrix ring over a skew field. Therefore it follows that $\mathcal{J}(R) \cap aRa = \mathcal{J}(aRa)$.

Note that aRa is an S-graded subring such that $H(aRa) \subseteq (P \cap H(R)) \cup B$ (a disjoint union) for the subsemigroup $B = \{x \in aH(R)a \mid x' \in G\}$. Let K be the additive subgroup generated by $P \cap H(R)$. Clearly, K is a homogeneous ideal of R and $a \notin K$. Moreover, R/P is a homomorphic image of R/K. Hence, to come to a contradiction, we can assume that $K = 0$. Then $H(aRa) = B \cup \{0\}$. From Lemma 5.2 we know that $\operatorname{supp}(aRa)$ is a permutational subsemigroup of S. Therefore, Lemma 5.3 implies that $\mathcal{J}(aRa)$ is homogeneous. Since $aza \in \mathcal{J}(R)$ for some $z \in R$ with a homogeneous component equal to a (because $a \in V$), it follows that $a^3 \in \mathcal{J}(aRa) \subseteq \mathcal{J}(R)$. Thus, $a^3 \in P$, which contradicts the choice of a. This completes the proof of the theorem. ∎

Remark. All the steps of the proof, with some simple modifications, apply to the Baer radical. Therefore, for any cancellative semigroup S, the Baer radical of every S-graded PI-algebra over a field of characteristic zero is homogeneous.

Proof of Theorem 2.2. Let S be a u.p.-semigroup, and let $R = \oplus_{s \in S} R_s$ be an S-graded ring.

(i) Given that S is a u.p.-semigroup, [5, Theorem 2.2] tells us that the Levitzki radical $\mathcal{L}(R)$ is homogeneous. Since $\mathcal{J}_{\mathrm{gr}}(R) = 0$, we must have $\mathcal{L}(R) = 0$.

Every nonzero w in $W = H(R)^1 rH(R)^1$ also belongs to $\mathcal{J}(R)$ and is of the same length. Let M be the multiplicative semigroup generated by $H(w)$. Obviously, $M^1 wM^1 \subseteq \mathcal{J}(R)$, and so all elements of $M^1 wM^1$ are quasiregular. Proposition 4.1 shows that M is nilpotent. In particular, all the elements of $H(w)$ are nilpotent. Therefore the set $V = H(W)$ consists of homogeneous nilpotent elements. As above, V is a multiplicative ideal of $H(R)$.

Suppose that all nil subsemigroups of $H(R/\mathcal{J}_{\mathrm{gr}}(R))$ are locally nilpotent. Then V is locally nilpotent, and so the ideal A of R generated by V is locally nilpotent. Therefore $V \subseteq \mathcal{L}(R) = 0$. This contradicts the choice of $r \neq 0$ and shows that $\mathcal{J}(R) = 0$, as desired.

(ii) Suppose that every nil subsemigroup of every right primitive homomorphic image of R is locally nilpotent.

As above we see that V is a nil semigroup. By the assumption, every image of V in a right primitive homomorphic image of R is locally nilpotent. Hence, the corresponding image of A is a locally nilpotent ideal, so it is zero. It follows that $A \subseteq \mathcal{J}(R)$ and therefore $A \subseteq \mathcal{J}_{\mathrm{gr}}(R) = 0$. This contradicts the fact that $0 \neq r \in A$.

(iii) Suppose that, for every minimal prime ideal P of R, the ring R/P is a domain or embeds into a matrix ring over a skew field. We must prove that $\mathcal{J}(R) = \mathcal{J}_{\mathrm{gr}}(R)$.

If J is a right primitive ideal of R, then J contains a minimal prime ideal P of R. Moreover, P is a homogeneous ideal by [5]. Therefore, it is enough to show that the radical of every R/P is homogeneous. So, we may assume that R is a domain or a subring of F_n for a skew field F and some $n \geq 1$.

As above, suppose that the Jacobson radical $\mathcal{J}(R)$ is not homogeneous, choose an element r of minimal length in $\mathcal{J}(R) \setminus \mathcal{J}_{\mathrm{gr}}(R)$, and introduce $W = H(R)^1 r H(R)^1$, $V = H(W)$ and the additive subgroup A generated in R by V. We shall prove that A is quasiregular. This will imply $H(r) \subseteq A \subseteq \mathcal{J}(R)$, and will give a contradiction with $r \notin \mathcal{J}_{\mathrm{gr}}(R)$.

Fix any element c in W, say $c = arb$, where $a, b \in H(R)^1$. Choose any $t \neq e$ in S and consider the homogeneous component $v = c_t$ of c. Clearly, $R/\mathcal{J}_{\mathrm{gr}}(R)$ is graded by the u.p.-semigroup S. By the choice of r, the image d of c in $R/\mathcal{J}_{\mathrm{gr}}(R)$ is a rigid element. Since $t \neq e$, Proposition 4.1 shows that $d_t^n = 0$ for some $n > 1$. Hence $v^n \in \mathcal{J}(R)$. Given that S is u.p., we get $t^n \neq e$. Therefore applying Proposition 4.1 to $v^n \in \mathcal{J}(R)$ we see that v^n is nilpotent. Hence c_t is nilpotent.

First, we consider the easier case, where R is a domain. Then we get $c_t = 0$. This implies that $c = c_e$. Hence $W = W_e$, where $W_e = \{c_e \mid c \in W\}$. Denote by A_e the additive subgroup generated in R by W_e. Then $A = A_e$. Proposition 4.1 implies that $W_e \subseteq \mathcal{J}(R_e)$, and so $A_e \subseteq \mathcal{J}(R_e)$. Clearly, A_e is an ideal of R_e. Therefore $A = A_e$ is quasiregular, as claimed.

Second, consider the case where R is embedded in a matrix ring F_n, for a skew field F and a positive integer n. Assume that $n > 1$, since otherwise $F_n = F$ is a domain again. For $i = 0, 1, \ldots, n$, denote by M_i the set of all matrices of rank $\leq i$ in F_n. Given that V is an ideal of $H(R)$, it follows from Lemma 3.2 that $V_i = V \cap M_i$ is an ideal of $H(R)$. Let A_i be the additive subgroup generated by V_i in R. Then $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = A$ is an ideal chain of the ring R.

We shall use induction on k to prove that all A_k are quasiregular. Obviously, $A_0 = 0$ is quasiregular. Assume that A_{k-1} is quasiregular for some $0 < k \leq n$. For simplicity, put $D = V_{k-1}$, $E = V_k$, $N = A_{k-1}$, and $I = A_k$. By the inductive assumption N is quasiregular. We shall prove that I/N is quasiregular.

For $\alpha, \beta \in \Lambda = \Lambda_k$, consider subsets $G_{\alpha\beta}$ and $G_{\alpha *}$ of $M_k \setminus M_{k-1}$ introduced in Lemma 3.2. Define the auxiliary sets $E_{\alpha\beta} = E \cap G_{\alpha\beta}$ and $E_{\alpha *} = E \cap G_{\alpha *}$. Let $I_{\alpha\beta}$ and $I_{\alpha *}$ be the additive subgroups generated in R by

$E_{\alpha\beta} \cup D$ and $E_{\alpha *} \cup D$, respectively. Lemma 3.2 shows that the semigroup E/D is the union of its right ideals $(E_{\alpha *} \cup D)/D$, and each $(E_{\alpha *} \cup D)/D$ is the union of its left ideals $(E_{\alpha\beta} \cup D)/D$. It follows that I/N is the sum of its right ideals $I_{\alpha *}/N$, and each $I_{\alpha *}/N$ is the sum of its left ideals $I_{\alpha\beta}/N$, where $\alpha, \beta \in \Lambda$.

Since quasiregularity is preserved by sums of one-sided ideals, it remains to check that all $I_{\alpha\beta}/N$ are quasiregular. Fix $\alpha, \beta \in \Lambda$, and put $Q = I_{\alpha\beta}/N$. We shall prove that Q is quasiregular.

First, consider the case where $G_{\alpha\beta}$ is not a group. Then $G_{\alpha\beta}^2 \subseteq M_{k-1}$ by Lemma 3.2(i). It follows that $E_{\alpha\beta}^2 \subseteq D$. Therefore $Q^2 = 0$, and so Q is quasiregular.

Second, assume that $G_{\alpha\beta}$ is a group. Let $L = E_{\alpha\beta}^3$. Pick any nonzero l in L. There exist $a, b \in E_{\alpha\beta}$, $u \in W$, and $s \in S$ such that $l = (aub)_s$. Put $c = aub$. Choose any t in $\operatorname{supp}(c) \setminus \{e\}$. We have seen above that c_t must be nilpotent. On the other hand, in view of Lemma 3.2, $H(l) \subseteq D \cup E_{\alpha\beta}$. Since $0 \notin G_{\alpha\beta} \supset E_{\alpha\beta}$, we see that $E_{\alpha\beta}$ has no nilpotent elements. Therefore $l_t \in D$ for all $t \neq e$. It follows that $s = e$ and the images of c and $l = c_e$ in Q coincide. Hence $c - l \in N = \mathscr{J}(N)$.

Given that $c \in \mathscr{J}(R)$, we get $l \in \mathscr{J}(R)$. Therefore $L \subseteq \mathscr{J}(R)$. But Q^3 is the additive group generated by L (viewed as a subset of Q). Hence $Q^3 \subseteq \mathscr{J}(I/N)$. Since Q is a left ideal of a right ideal of I/N, it follows that $Q^3 \subseteq \mathscr{J}(Q)$. So Q is quasiregular, as desired.

We have proved that all $I_{\alpha\beta}/N$ are quasiregular for all $\alpha, \beta \in \Lambda$, and hence I/N is quasiregular. As we have seen, this implies that all A_k are quasiregular. Thus $A = A_n$ is a quasiregular ideal of R, as claimed. This contradicts the choice of r and completes the proof of the theorem. ∎

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