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No fermionic wigs for BPS attractors in 5 dimensions

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ABSTRACT

We analyze the *fermionic wigging* of 1/2-BPS (electric) extremal black hole attractors in $\mathcal{N}=2$, $D=5$ ungauged Maxwell–Einstein supergravity theories, by exploiting anti-Killing spinors supersymmetry transformations. Regardless of the specific data of the real special geometry of the manifold defining the scalars of the vector multiplets, and differently from the $D=4$ case, we find that there are *no corrections* for the near-horizon attractor value of the scalar fields; an analogous result also holds for 1/2-BPS (magnetic) extremal black string. Thus, the attractor mechanism receives *no fermionic corrections* in $D=5$ (at least in the BPS sector).

1. Introduction

The question concerning the presence or absence of hairs of any kind around a black hole is very compelling and, of course, it has been studied from several points of view. Nonetheless, recently some of the authors of the present work re-posed the question by considering possible fermionic hairs (first in [1], and then in a series of papers [2]) for non-extremal, as well as BPS black holes. The first paper[1] on the subject is due to Aichelburg and Embacher [4]. They considered asymptotically flat black hole solution in $\mathcal{N}=2$, $D=4$ supergravity without vector multiplets and computed iteratively the supersymmetric variations of the background in terms of the flat-space Killing spinors. In that paper, they were able to compute some of the physical quantities such as the corrections to the angular momentum, while other interesting properties cannot be seen at that order of the expansion. Afterwards, the works [1] and [5] applied their technique to some examples of BPS black hole, up to the fourth order in the supersymmetry transformation.

In particular, for extremal black hole solutions, the *attractor mechanism* [6] is a very interesting and important physical property; essentially, it states that the solution at the horizon depends only on the conserved charges of the system, and is independent of the value of the matter fields at infinity. This is related to the *no-hair theorem*, under which, for example, a BPS black hole solution depends only upon its mass, its angular momentum and other conserved charges. As said, the authors of [5] addressed the question whether the attractor mechanism has to be modified in the presence of fermions. The conclusion was that, at the level of approximation of their computations, in the case of double-extremal BPS solutions, the mechanism is unchanged. In [1] $\mathcal{N}=2$, $D=5$ AdS black holes were investigated, and it was found that the solution, as well as its asymptotic charges, get modified at the second order due to fermionic contributions. However, in [1] the attractor mechanism and its possible modifications was not considered.

In [7], the fermionic wig for asymptotically flat BPS black holes in $\mathcal{N}=2$, $D=4$ supergravity coupled to matter was investigated. There, it has been shown that the attractor mechanism gets modified at the fourth order even in the case of double extremal solutions in the simplest example of $\mathcal{N}=2$ supergravity coupled to a single matter field (*minimally coupled* vector multiplet). The surprising result is that to the lower orders all corrections vanish for the BPS solution, while at the fourth order, despite several cancellations due to special geometry identities, some terms do survive, and thus the attractor gets modified.

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[1] For further subsequent studies on various BPS objects (black holes, M2-branes and BPS monopoles), see *e.g.* [3].

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It has also been noticed that there are situations in which some combinations of charges render the attractor modifications null; this led to the conjecture that, in those $D = 4$ models admitting an uplift to 5 dimensions, the attractor mechanism is unmodified by the fermionic wig. That motivated us to study in full generality the $D = 5$ case, by means of the same techniques; we found that *there is no modification to the attractor mechanism up to forth order* for all the ungauged $\mathcal{N} = 2$, $D = 5$ supergravity models coupled to vector multiplets. This is a rather strong result, and it has been obtained for a *generic* real special geometry of the manifold defined by the scalars of the vector multiplets. The cancellations appear to be due to identities of the special geometry, as well as to the extremal black hole solutions taken into account (*cf.* Eq. (5.1)).

We should point out that the wigging is computed by performing a perturbation of the unwigged purely bosonic BPS extremal black hole solution keeping the radius of the event horizon unchanged. The complete analysis, including the study of the fully-backreacted wigged black hole metric, will be presented elsewhere.

The plan of the paper is as follows. In Section 2 we recall some basics of $\mathcal{N} = 2$, $D = 5$ ungauged Maxwell–Einstein supergravity. The fermionic wigging is then presented in Section 3, and its evaluation on the purely bosonic background of an extremal BPS black hole is performed in Section 4. The near-horizon conditions are applied in Section 5, obtaining the *universal* result of vanishing wig corrections to the attractor value of the scalar fields of the vector multiplets in the near-horizon geometry. The *universality* of this result resides in its *independence* on the data of the real special geometry endowing the scalar manifold of the supergravity theory. Comments on this result and further remarks and future directions are given in Section 6.

Appendices A–C, specifying notations and containing technical details on the wigging procedure, are presented.

2. Ungauged $\mathcal{N} = 2$, $D = 5$ MESGT

Following [8–10], we consider $\mathcal{N} = 2$, $D = 5$ *ungauged* Maxwell–Einstein supergravity theory (MESGT), in which the $\mathcal{N} = 2$ gravity multiplet $\{e^a_\mu,\ \psi^i_\mu,\ A_\mu\}$ is coupled to n_V Abelian vector multiplets[2] $\{A_\mu, \lambda^{xi}, \phi^x\}$, with neither hyper nor tensor multiplets[3]:

$$\delta e_\mu{}^a = \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu, \tag{2.1a}$$

$$\begin{aligned}\delta\psi^i_\mu = {} & D_\mu(\hat{\omega})\epsilon^i + \frac{\mathrm{i}}{4\sqrt{6}}h_I\tilde{F}^I_{\nu\rho}\left(\gamma_\mu{}^{\nu\rho} - 4\delta^\nu_\mu\gamma^\rho\right)\epsilon^i \\ & - \frac{1}{6}\epsilon_j\bar{\lambda}^{ix}\gamma_\mu\lambda^j_x + \frac{1}{12}\gamma_{\mu\nu}\epsilon_j\bar{\lambda}^{ix}\gamma^\nu\lambda^j_x \\ & - \frac{1}{48}\gamma_{\mu\nu\rho}\epsilon_j\bar{\lambda}^{ix}\gamma^{\nu\rho}\lambda^j_x + \frac{1}{12}\gamma^\nu\epsilon_j\bar{\lambda}^{ix}\gamma_{\mu\nu}\lambda^j_x,\end{aligned} \tag{2.1b}$$

$$\delta h^I = -\frac{1}{\sqrt{6}}\mathrm{i}\bar{\epsilon}\lambda^x h^I_x, \tag{2.1c}$$

$$\delta\phi^x = \frac{1}{2}\mathrm{i}\bar{\epsilon}\lambda^x, \tag{2.1d}$$

$$\delta A^I_\mu = -\frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^x h^I_x - \frac{\sqrt{6}}{4}ih^I\bar{\epsilon}\psi_\mu, \tag{2.1e}$$

$$\begin{aligned}\delta\lambda^{xi} = {} & -\frac{\mathrm{i}}{2}\widehat{\not{\mathcal{D}}}\phi^x\epsilon^i - \delta\phi^y\Gamma^x_{yz}\lambda^{zi} + \frac{1}{4}\gamma\cdot\tilde{F}^I h^x_I\epsilon^i \\ & + \frac{1}{4\sqrt{6}}T^{xyz}\Big[3\epsilon_j\bar{\lambda}^i_y\lambda^j_z - \gamma^\mu\epsilon_j\bar{\lambda}^i_y\gamma_\mu\lambda^j_z \\ & - \frac{1}{2}\gamma^{\mu\nu}\epsilon_j\bar{\lambda}^i_y\gamma_{\mu\nu}\lambda^j_z\Big],\end{aligned} \tag{2.1f}$$

where

$$\mathcal{F}^I_{\mu\nu} = 2\partial_{[\mu}A^I_{\nu]}, \tag{2.2a}$$

$$\tilde{F}^I_{\mu\nu} = \mathcal{F}^I_{\mu\nu} + \bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^x h^I_x + \frac{i\sqrt{6}}{4}\bar{\psi}_\mu\psi_\nu h^I, \tag{2.2b}$$

$$T_{xyz} = C_{IJK}h^I_x h^J_y h^K_z, \tag{2.2c}$$

$$\Gamma^w_{xy} = h^w_I h^I_{x,y} + \sqrt{\frac{2}{3}}T_{xyz}g^{zw}. \tag{2.2d}$$

From the Vielbein postulate, the $\mathcal{N} = 2$ spin connection reads

$$\hat{\omega}^{ab}_\mu = \frac{1}{2}e_{c\mu}\left[\Omega^{abc} - \Omega^{bca} - \Omega^{cab}\right] + K^a{}_\mu{}^b, \tag{2.3}$$

where $\Omega^{abc} := e^{\mu a}e^{\nu b}(\partial_\mu e^c_\nu - \partial_\nu e^c_\mu)$ and $K^a{}_\mu{}^b := -\frac{1}{2}\bar{\psi}^{[b}\gamma^{a]}\psi_\mu - \frac{1}{4}\bar{\psi}^b\gamma_\mu\psi^a$. The covariant derivatives are defined as

$$\widehat{\mathcal{D}}_\mu\phi^x = \partial_\mu\phi^x - \frac{1}{2}\mathrm{i}\bar{\psi}_\mu\lambda^x, \tag{2.4a}$$

$$\mathcal{D}_\mu h^I = \partial_\mu h^I = -\sqrt{\frac{2}{3}}h^I_x\partial_\mu\phi^x = -\sqrt{\frac{2}{3}}h^I_x\mathcal{D}_\mu\phi^x, \tag{2.4b}$$

$$\mathcal{D}_\mu\lambda^{xi} = \partial_\mu\lambda^{xi} + \partial_\mu\phi^y\Gamma^x_{yz}\lambda^{zi} + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\lambda^{xi}, \tag{2.4c}$$

$$\mathcal{D}_\mu\psi^i_\nu = \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\right)\psi^i_\nu, \tag{2.4d}$$

and ([8]; see also *e.g.* Eq. (C.10) of [10])

$$\nabla_y h^I_x = -\sqrt{\frac{2}{3}}\left(h^I g_{xy} + T_{xyz}h^{Iz}\right), \tag{2.5a}$$

$$\nabla_y h_{Ix} = \sqrt{\frac{2}{3}}\left(h_I g_{xy} + T_{xyz}h^z_I\right). \tag{2.5b}$$

Note that only ω^{ab}_μ (and not $\hat{\omega}^{ab}_\mu$) occurs in the covariant derivative of the gravitino. Furthermore, it holds that[4] (see also *e.g.* [11–13])

$$h^I_x \equiv -\sqrt{\frac{3}{2}}\partial_x h^I, \qquad h_{Ix} \equiv a_{IJ}h^J_x, \tag{2.6a}$$

$$a_{IJ} = -2C_{IJK}h^K + 3h_I h_J, \tag{2.6b}$$

$$C_{IJK}h^I h^J h^K = 1, \qquad h_I h^I = 1. \tag{2.6c}$$

It is worth pointing out that in $D = 5$ Lorentzian signature no chirality is allowed, and the smallest spinor representation of the Lorentz group is given by symplectic Majorana spinors; for further details, see Appendix A.

3. Fermionic wigging

We now proceed to perform the *fermionic wigging*, by iterating the supersymmetry transformations of the various fields generated by the *anti-Killing spinor* ϵ (for a detailed treatment and further details, *cf. e.g.* [15,7]); schematically denoting all wigged fields as

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[2] $i = 1, 2$ of the fundamental **2** of $USp(2) \sim SU(2)$ $\mathcal{R}$-symmetry, $x = 1, \ldots, n_V$ and $I = 0, 1, \ldots, n_V$, where the 0 index pertains to the $D = 5$ graviphoton. Note that γ_μ denote the $D = 5$ gamma matrices. Moreover, we adopt the convention $\kappa = 1$ (*cf. e.g.* Appendix C of [10]).

[3] When not indicated, spinor indices are contracted using the standard $SU(2)$ metric ε^{ij} (see Appendix A).

[4] In the present treatment, C_{IJK} denotes the $\mathcal{C}_{IJK}$ of [14], their difference being just a rescaling factor.

$\widehat{\Phi}$ and the original bosonic configuration by Φ, the following expansion holds:

$$\widehat{\Phi} = e^{\delta}\Phi = \Phi + \delta\Phi + \frac{1}{2}\delta^{(2)}\Phi + \frac{1}{3!}\delta^{(3)}\Phi + \frac{1}{4!}\delta^{(4)}\Phi, \tag{3.1}$$

where, as in [4], the expansion truncates at the fourth order because of the 4-Grassmannian degrees of freedom that ϵ contains.[5]

3.1. Second order

In order to give an idea on the structure of the iterated supersymmetry transformations on the massless spectrum of the theory under consideration, we present below the second order transformation rules[6] (general results on supersymmetry iterations at the third and fourth order are given in Appendices B and C, respectively):

$$\left(\delta^{(2)}e^a_\mu\right) = \frac{1}{2}\bar{\epsilon}\gamma^a\left(\delta^{(1)}\psi_\mu\right), \tag{3.2}$$

$$\begin{aligned}\left(\delta^{(2)}\psi^i_\mu\right) &= \left(\delta^{(1)}\mathcal{D}_\mu\right)\epsilon^i - \frac{1}{6}\epsilon_j\bar{\lambda}^{ix}\gamma_\mu\left(\delta^{(1)}\lambda^j_x\right)\\
&\quad + \frac{1}{12}\gamma_{\mu\nu}\epsilon_j\bar{\lambda}^{ix}\gamma^\nu\left(\delta^{(1)}\lambda^j_x\right)\\
&\quad - \frac{1}{48}\gamma_{\mu\nu\rho}\epsilon_j\bar{\lambda}^{ix}\gamma^{\nu\rho}\left(\delta^{(1)}\lambda^j_x\right) + \frac{1}{12}\gamma^\nu\bar{\lambda}^{ix}\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^j_x\right)\\
&\quad - \frac{1}{6}\epsilon_j\bar{\lambda}^{ix}\gamma_a\lambda^j_x\left(\delta^{(1)}e^a_\mu\right) - \frac{1}{6}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^{ix}\right)\gamma_\mu\lambda^j_x\\
&\quad + \frac{1}{12}\gamma_{ab}\epsilon_j\bar{\lambda}^{ix}\gamma^b\lambda^j_x\left(\delta^{(1)}e^a_\mu\right)\\
&\quad + \frac{1}{12}\gamma_{\mu\nu}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^{ix}\right)\gamma^\nu\lambda^j_x\\
&\quad - \frac{1}{48}\gamma_{abc}\epsilon_j\bar{\lambda}^{ix}\gamma^{bc}\lambda^j_x\left(\delta^{(1)}e^a_\mu\right)\\
&\quad - \frac{1}{48}\gamma_{\mu\nu\rho}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^{ix}\right)\gamma^{\nu\rho}\lambda^j_x\\
&\quad + \frac{1}{12}\gamma^b\epsilon_j\bar{\lambda}^{ix}\gamma_{ab}\lambda^j_x\left(\delta^{(1)}e^a_\mu\right)\\
&\quad + \frac{1}{12}\gamma^\nu\epsilon_j\left(\delta^{(1)}\bar{\lambda}^{ix}\right)\gamma_{\mu\nu}\lambda^j_x\\
&\quad + \frac{i}{4\sqrt{6}}h_I\tilde{F}^I_{\nu\rho}\left[\left(\delta^{(1)}e^a_\mu\right)e^\nu_b e^\rho_c + e^a_\mu\left(\delta^{(1)}e^\nu_b\right)e^\rho_c\right.\\
&\quad \left. + e^a_\mu e^\nu_b\left(\delta^{(1)}e^\rho_c\right)\right]\left(\gamma_a{}^{bc} - 4\delta^b_a\gamma^c\right)\epsilon^i\\
&\quad + \frac{i}{12}h_{Iz}\left(\delta^{(1)}\phi^z\right)\tilde{F}^I_{\nu\rho}\left(\gamma_\mu{}^{\nu\rho} - 4\delta^\nu_\mu\gamma^\rho\right)\epsilon^i\\
&\quad + \frac{i}{4\sqrt{6}}h_I\left(\delta^{(1)}\tilde{F}^I_{\nu\rho}\right)\left(\gamma_\mu{}^{\nu\rho} - 4\delta^\nu_\mu\gamma^\rho\right)\epsilon^i,\end{aligned} \tag{3.3}$$

$$\left(\delta^{(2)}\phi^x\right) = \frac{i}{2}\bar{\epsilon}\left(\delta^{(1)}\lambda^x\right), \tag{3.4}$$

$$\begin{aligned}\left(\delta^{(2)}A^I_\mu\right) &= -\frac{1}{2}\bar{\epsilon}\gamma_\mu\left(\delta^{(1)}\lambda^x\right)h^I_x - \frac{1}{2}\left(\delta^{(1)}e^a_\mu\right)\bar{\epsilon}\gamma_a\lambda^x h^I_x\\
&\quad - \frac{i}{2}\sqrt{\frac{3}{2}}\bar{\epsilon}h^I\left(\delta^{(1)}\psi_\mu\right) + \frac{i}{2}h^I_x\left(\delta^{(1)}\phi^x\right)\bar{\epsilon}\psi_\mu\\
&\quad - \frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^x\nabla_y h^I_x\left(\delta^{(1)}\phi^y\right),\end{aligned} \tag{3.5}$$

$$\begin{aligned}\left(\delta^{(2)}\lambda^{ix}\right) &= -\frac{i}{2}\left(\delta^{(1)}e^\mu_a\right)\gamma^a\widehat{\mathcal{D}}_\mu\phi^x\epsilon^i - \frac{i}{2}\gamma^\mu\left(\delta^{(1)}\widehat{\mathcal{D}}_\mu\phi^x\right)\epsilon^i\\
&\quad - \frac{1}{4\sqrt{6}}T^{xyz}\gamma^\mu\epsilon_j\bar{\lambda}^i_y\gamma_\mu\left(\delta^{(1)}\lambda^j_z\right)\\
&\quad + \frac{1}{4}\sqrt{\frac{3}{2}}T^{xyz}\epsilon_j\bar{\lambda}^i_y\left(\delta^{(1)}\lambda^j_z\right)\\
&\quad - \frac{1}{8\sqrt{6}}T^{xyz}\gamma^{\mu\nu}\epsilon_j\bar{\lambda}^i_y\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^j_z\right)\\
&\quad - \left(\delta^{(1)}\phi^y\right)\Gamma^x_{yz}\left(\delta^{(1)}\lambda^{zi}\right) - \left(\delta^{(2)}\phi^y\right)\Gamma^x_{yz}\lambda^{zi}\\
&\quad + \frac{1}{4}\sqrt{\frac{3}{2}}T^{xyz}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^i_y\right)\lambda^j_z\\
&\quad - \frac{1}{8\sqrt{6}}T^{xyz}\gamma^{\mu\nu}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^i_y\right)\gamma_{\mu\nu}\lambda^j_z\\
&\quad - \frac{1}{4\sqrt{6}}T^{xyz}\gamma^\mu\epsilon_j\left(\delta^{(1)}\bar{\lambda}^i_y\right)\gamma_\mu\lambda^j_z\\
&\quad + \frac{1}{4\sqrt{6}}\nabla_t T^{xyz}\left(\delta^{(1)}\phi^t\right)\left[3\epsilon_j\bar{\lambda}^i_y\lambda^j_z - \gamma^\mu\epsilon_j\bar{\lambda}^i_y\gamma_\mu\lambda^j_z\right.\\
&\quad \left. - \frac{1}{2}\gamma^{\mu\nu}\epsilon_j\bar{\lambda}^i_y\gamma_{\mu\nu}\lambda^j_z\right] - \left(\delta^{(1)}\phi^y\right)\nabla_t\Gamma^x_{yz}\left(\delta^{(1)}\phi^t\right)\lambda^{zi}\\
&\quad + \frac{1}{4}\gamma\cdot\tilde{F}^I\nabla_t h^x_I\left(\delta^{(1)}\phi^t\right)\epsilon^i + \frac{1}{4}\gamma\cdot\left(\delta^{(1)}\tilde{F}^I\right)h^x_I\epsilon^i\\
&\quad + \frac{1}{2}\left(\delta^{(1)}e^\mu_a\right)e^\nu_b + e^\mu_a\left(\delta^{(1)}e^\nu_b\right)\gamma^{ab}\tilde{F}^I_{\mu\nu}h^x_I\epsilon^i,\end{aligned} \tag{3.6}$$

with

$$\begin{aligned}\left(\delta^{(1)}\tilde{F}^I_{\mu\nu}\right) &= \left(\delta^{(1)}\mathcal{F}^I_{\mu\nu}\right) + \left(\delta^{(1)}\bar{\psi}_{[\mu}\right)\gamma_{\nu]}\lambda^x h^I_x + \bar{\psi}_{[\mu}\gamma_{\nu]}\left(\delta^{(1)}\lambda^x\right)h^I_x\\
&\quad + \bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^x\nabla_y h^I_x\left(\delta^{(1)}\phi^y\right) + \frac{i\sqrt{6}}{4}\left(\delta^{(1)}\bar{\psi}_\mu\right)\psi_\nu h^I\\
&\quad + \frac{i\sqrt{6}}{4}\bar{\psi}_\mu\left(\delta^{(1)}\psi_\nu\right)h^I + \bar{\psi}_{[\mu}\left(\delta^{(1)}e^a_{\nu]}\right)\gamma_a\lambda^x h^I_x\\
&\quad - \frac{i}{2}\bar{\psi}_\mu\psi_\nu h^I_x\left(\delta^{(1)}\phi^x\right),\end{aligned} \tag{3.7}$$

$$\left(\delta^{(1)}\mathcal{D}_\mu\right) = \frac{1}{4}\left(\delta^{(1)}\omega^{ab}_\mu\right)\gamma_{ab}, \tag{3.8}$$

$$\begin{aligned}\left(\delta^{(1)}\omega^{ab}_\mu\right) &= \frac{1}{2}\left(\delta^{(1)}e_{c\mu}\right)\left[\Omega^{abc} - \Omega^{bca} - \Omega^{cab}\right]\\
&\quad + \frac{1}{2}e_{c\mu}\left[\left(\delta^{(1)}\Omega^{abc}\right) - \left(\delta^{(1)}\Omega^{bca}\right) - \left(\delta^{(1)}\Omega^{cab}\right)\right]\\
&\quad + \left(\delta^{(1)}K^a{}_\mu{}^b\right),\end{aligned} \tag{3.9}$$

$$\begin{aligned}\left(\delta^{(1)}\Omega^{abc}\right) &= \left[\left(\delta^{(1)}e^{\mu a}\right)e^{\nu b} + e^{\mu a}\left(\delta^{(1)}e^{\nu b}\right)\right]\left(\partial_\mu e^c_\nu - \partial_\nu e^c_\mu\right)\\
&\quad + e^{\mu a}e^{\nu b}\left[\partial_\mu\left(\delta^{(1)}e^c_\nu\right) - \partial_\nu\left(\delta^{(1)}e^c_\mu\right)\right],\end{aligned} \tag{3.10}$$

$$\begin{aligned}\left(\delta^{(1)}K^a{}_\mu{}^b\right) &= \frac{1}{2}\left[\left(\delta^{(1)}\bar{\psi}_\rho\right)e^{\rho[a}\gamma^{b]}\psi_\mu + \bar{\psi}_\rho\left(\delta^{(1)}e^{\rho[a}\right)\gamma^{b]}\psi_\mu\right.\\
&\quad + \bar{\psi}^{[a}\gamma^{b]}\left(\delta^{(1)}\psi_\mu\right) + \frac{1}{2}\left(\delta^{(1)}\bar{\psi}_\rho\right)e^{\rho a}\gamma_\mu\psi^b\\
&\quad + \frac{1}{2}\bar{\psi}_\rho\left(\delta^{(1)}e^{\rho a}\right)\gamma_\mu\psi^b + \frac{1}{2}\bar{\psi}^a\gamma_a\left(\delta^{(1)}e^a_\mu\right)\psi^b\\
&\quad \left. + \frac{1}{2}\bar{\psi}^a\gamma_\mu\psi_\rho\left(\delta^{(1)}e^{\rho b}\right) + \frac{1}{2}\bar{\psi}^a\gamma_\mu\left(\delta^{(1)}\psi_\rho\right)e^{\rho b}\right],\end{aligned} \tag{3.11}$$

$$\left(\delta^{(1)}\widehat{\mathcal{D}}_\mu\phi^x\right) = \partial_\mu\left(\delta^{(1)}\phi^x\right) - \frac{i}{2}\left(\delta^{(1)}\bar{\psi}_\mu\right)\lambda^x - \frac{i}{2}\bar{\psi}_\mu\left(\delta^{(1)}\lambda^x\right). \tag{3.12}$$

[5] In the present paper we will deal with a BPS background so just half of the supersymmetries are preserved.

[6] By exploiting Eq. (3.16) of [13], both $\nabla_t T^{xyz}$ and $\nabla_t \Gamma^x_{yz}$ can be related to the covariant derivative of the Riemann tensor R_{xyzt}; this latter is known to satisfy the so-called *real special geometry constraints* (see *e.g.* Eq. (2.12) of [13]).

4. Evaluation on purely bosonic background

Next, we proceed to evaluate the *fermionic wigging* on a *purely bosonic* background (characterized by setting $\psi = \lambda = 0$ identically, and denoted by $|_{\mathrm{bg}}$ throughout). This results in a dramatic simplification of previous formulæ; in particular, all covariant quantities, such as the $\widetilde{E}$-tensor [13], characterizing the real special geometry of the scalar manifold (*cf.* Appendices B and C), do not occur anymore after evaluation on such a background.

4.1. First order

At the first order, the non-zero supersymmetry variations are:

$$\left(\delta^{(1)}\psi^i_\mu\right)\big|_{\mathrm{bg}} = D_\mu(\hat\omega)\epsilon^i + \frac{i}{4\sqrt{6}}h_I\mathcal{F}^I_{\nu\rho}\left(\gamma_\mu{}^{\nu\rho} - 4\delta^\nu_\mu\gamma^\rho\right)\epsilon^i, \quad (4.1a)$$

$$\left(\delta^{(1)}\lambda^{xi}\right)\big|_{\mathrm{bg}} = -\frac{i}{2}\partial\!\!\!/\phi^x\epsilon^i + \frac{1}{4}\gamma\cdot\mathcal{F}^I h^x_I\epsilon^i. \quad (4.1b)$$

Moreover, the supercovariant field strength collapses to the ordinary field strength and the covariant derivative on ϕ^x reduces to an ordinary (flat) derivative.

4.2. Second order

$$\left(\delta^{(2)}e^a_\mu\right)\big|_{\mathrm{bg}} = \frac{1}{2}\bar\epsilon\gamma^a\left(\delta^{(1)}\psi_\mu\right)\big|_{\mathrm{bg}}, \quad (4.2a)$$

$$\left(\delta^{(2)}\phi^x\right)\big|_{\mathrm{bg}} = \frac{i}{2}\bar\epsilon\left(\delta^{(1)}\lambda^x\right)\big|_{\mathrm{bg}}, \quad (4.2b)$$

$$\left(\delta^{(2)}A^I_\mu\right)\big|_{\mathrm{bg}} = -\frac{i}{2}\sqrt{\frac{3}{2}}\bar\epsilon h^I\left(\delta^{(1)}\psi_\mu\right)\big|_{\mathrm{bg}} - \frac{1}{2}\bar\epsilon\gamma_\mu\left(\delta^{(1)}\lambda^x\right)\big|_{\mathrm{bg}}h^I_x. \quad (4.2c)$$

The supercovariant field strength, the covariant derivative on ϕ^x and the variation of the spin connection ω^{ab}_μ all collapse to zero.

4.3. Third order

At the third order, one obtains the following results:

$$\begin{aligned}\left(\delta^{(3)}\psi^i_\mu\right)\big|_{\mathrm{bg}} ={}& \left(\delta^{(2)}\mathcal{D}_\mu\right)\big|_{\mathrm{bg}}\epsilon^i - \frac{1}{3}\epsilon_j\left(\delta^{(1)}\bar\lambda^{ix}\right)\big|_{\mathrm{bg}}\gamma_\mu\left(\delta^{(1)}\lambda^j_x\right)\big|_{\mathrm{bg}}\\
&+ \frac{1}{6}\gamma_{\mu\nu}\left(\delta^{(1)}\bar\lambda^{ix}\right)\big|_{\mathrm{bg}}\gamma^\nu\left(\delta^{(1)}\lambda^j_x\right)\big|_{\mathrm{bg}}\\
&- \frac{1}{24}\gamma_{\mu\nu\rho}\epsilon_j\left(\delta^{(1)}\bar\lambda^{ix}\right)\big|_{\mathrm{bg}}\gamma^{\nu\rho}\left(\delta^{(1)}\lambda^j_x\right)\big|_{\mathrm{bg}}\\
&+ \frac{1}{6}\gamma_{\mu\nu}\epsilon_j\left(\delta^{(1)}\bar\lambda^{ix}\right)\big|_{\mathrm{bg}}\gamma^\nu\left(\delta^{(1)}\lambda^j_x\right)\big|_{\mathrm{bg}}\\
&+ \frac{i}{4\sqrt{6}}h_I\mathcal{F}^I_{\nu\rho}\left[\left(\delta^{(2)}e^a_\mu\right)\big|_{\mathrm{bg}}e^\nu_b e^\rho_c + e^a_\mu\left(\delta^{(2)}e^\nu_b\right)\big|_{\mathrm{bg}}e^\rho_c\right.\\
&\left.+ e^a_\mu e^\nu_b\left(\delta^{(2)}e^\rho_c\right)\big|_{\mathrm{bg}}\right]\left(\gamma_a{}^{bc} - 4\delta^b_a\gamma^c\right)\epsilon^i\\
&+ \frac{i}{12}h_{Iz}\left(\delta^{(2)}\phi^z\right)\big|_{\mathrm{bg}}\mathcal{F}^I_{\nu\rho}\left(\gamma_\mu{}^{\nu\rho} - 4\delta^\nu_\mu\gamma^\rho\right)\epsilon^i\\
&+ \frac{i}{4\sqrt{6}}h_I\left(\delta^{(2)}\tilde F^I_{\nu\rho}\right)\big|_{\mathrm{bg}}\left(\gamma_\mu{}^{\nu\rho} - 4\delta^\nu_\mu\gamma^\rho\right)\epsilon^i, \end{aligned} \quad (4.3a)$$

$$\begin{aligned}\left(\delta^{(3)}\lambda^{ix}\right)\big|_{\mathrm{bg}} ={}& -\frac{i}{2}\left(\delta^{(2)}e^\mu_a\right)\big|_{\mathrm{bg}}\gamma^a\partial_\mu\phi^x\epsilon^i - \frac{i}{2}\gamma^\mu\left(\delta^{(2)}\widehat{\mathcal{D}}_\mu\phi^x\right)\big|_{\mathrm{bg}}\epsilon^i\\
&- 2\left(\delta^{(2)}\phi^y\right)\big|_{\mathrm{bg}}\Gamma^x_{yz}\left(\delta^{(1)}\lambda^{zi}\right)\big|_{\mathrm{bg}}\\
&+ \frac{1}{2}\sqrt{\frac{3}{2}}T^{xyz}\epsilon_j\left(\delta^{(1)}\bar\lambda^i_y\right)\big|_{\mathrm{bg}}\left(\delta^{(1)}\lambda^j_z\right)\big|_{\mathrm{bg}}\\
&- \frac{1}{4\sqrt{6}}T^{xyz}\gamma^{\mu\nu}\epsilon_j\left(\delta^{(1)}\bar\lambda^i_y\right)\big|_{\mathrm{bg}}\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^j_z\right)\big|_{\mathrm{bg}}\\
&- \frac{1}{2\sqrt{6}}T^{xyz}\gamma^\mu\epsilon_j\left(\delta^{(1)}\bar\lambda^i_y\right)\big|_{\mathrm{bg}}\gamma_\mu\left(\delta^{(1)}\lambda^j_z\right)\big|_{\mathrm{bg}}\\
&+ \frac{1}{4}\gamma\cdot\mathcal{F}^I\nabla_t h^x_I\left(\delta^{(2)}\phi^t\right)\big|_{\mathrm{bg}}\epsilon^i + \frac{1}{4}\gamma\cdot\left(\delta^{(2)}\tilde F^I\right)\big|_{\mathrm{bg}}h^x_I\epsilon^i\\
&+ \frac{1}{4}\gamma^{ab}\left[\left(\delta^{(2)}e^\mu_a\right)\big|_{\mathrm{bg}}e^\nu_b + e^\mu_a\left(\delta^{(2)}e^\nu_b\right)\big|_{\mathrm{bg}}\right]\mathcal{F}^I_{\mu\nu}h^x_I\epsilon^i. \end{aligned} \quad (4.3b)$$

For the supercovariant field strength, the covariant derivative on ϕ^x, and the spin connection ω^{ab}_μ, it holds that:

$$\begin{aligned}\left(\delta^{(2)}\tilde F^I_{\mu\nu}\right)\big|_{\mathrm{bg}} ={}& 2\partial_{[\mu}\left(\delta^{(2)}A^I_{\nu]}\right)\big|_{\mathrm{bg}}\\
&+ 2\left(\delta^{(1)}\bar\psi_{[\mu}\right)\big|_{\mathrm{bg}}\gamma_{\nu]}\left(\delta^{(1)}\lambda^x\right)\big|_{\mathrm{bg}}h^I_x\\
&+ i\sqrt{\frac{3}{2}}\left(\delta^{(1)}\bar\psi_\nu\right)\big|_{\mathrm{bg}}\left(\delta^{(1)}\psi_\mu\right)\big|_{\mathrm{bg}}h^I, \end{aligned} \quad (4.4a)$$

$$\left(\delta^{(2)}\widehat{\mathcal{D}}_\mu\phi^x\right)\big|_{\mathrm{bg}} = \partial_\mu\left(\delta^{(2)}\phi^x\right)\big|_{\mathrm{bg}} - i\left(\delta^{(1)}\bar\psi_\mu\right)\big|_{\mathrm{bg}}\left(\delta^{(1)}\lambda^x\right)\big|_{\mathrm{bg}}, \quad (4.4b)$$

$$\begin{aligned}\left(\delta^{(2)}\omega^{ab}_\mu\right)\big|_{\mathrm{bg}} ={}& \frac{1}{2}\left(\delta^{(2)}e_{c\mu}\right)\big|_{\mathrm{bg}}\left(\Omega^{abc} - \Omega^{bca} - \Omega^{cab}\right)\\
&+ \frac{1}{2}\left[\left(\delta^{(2)}\Omega^{abc}\right)\big|_{\mathrm{bg}} - \left(\delta^{(2)}\Omega^{bca}\right)\big|_{\mathrm{bg}}\right.\\
&\left.- \left(\delta^{(2)}\Omega^{cab}\right)\big|_{\mathrm{bg}}\right] + \left(\delta^{(2)}K^a{}_\mu{}^b\right)\big|_{\mathrm{bg}}, \end{aligned} \quad (4.4c)$$

$$\begin{aligned}\left(\delta^{(2)}\Omega^{abc}\right)\big|_{\mathrm{bg}} ={}& \left[\left(\delta^{(2)}e^{\mu a}\right)\big|_{\mathrm{bg}}e^{\nu b}\right.\\
&\left.+ e^{\mu a}\left(\delta^{(2)}e^{\nu b}\right)\big|_{\mathrm{bg}}\right]\left(\partial_\mu e^c_\nu - \partial_\nu e^c_\mu\right)\\
&+ e^{\mu a}e^{\nu b}\left[\partial_\mu\left(\delta^{(2)}e^c_\nu\right)\big|_{\mathrm{bg}} - \partial_\nu\left(\delta^{(2)}e^c_\mu\right)\big|_{\mathrm{bg}}\right], \end{aligned} \quad (4.4d)$$

$$\begin{aligned}\left(\delta^{(2)}K^a{}_\mu{}^b\right)\big|_{\mathrm{bg}} ={}& \left(\delta^{(1)}\bar\psi_\rho\right)\big|_{\mathrm{bg}}e^{\rho[a}\gamma^{b]}\left(\delta^{(1)}\psi_\mu\right)\big|_{\mathrm{bg}}\\
&+ \frac{1}{2}\left(\delta^{(1)}\bar\psi_\rho\right)\big|_{\mathrm{bg}}\gamma_\mu\left(\delta^{(1)}\psi_\nu\right)\big|_{\mathrm{bg}}e^{\rho a}e^{\nu b}. \end{aligned} \quad (4.4e)$$

4.4. Fourth order

Finally, at the fourth order, one achieves the following expressions:

$$\left(\delta^{(4)}e^a_\mu\right)\big|_{\mathrm{bg}} = \frac{1}{2}\bar\epsilon\gamma^a\left(\delta^{(3)}\psi_\mu\right)\big|_{\mathrm{bg}}, \quad (4.5a)$$

$$\left(\delta^{(4)}\phi^x\right)\big|_{\mathrm{bg}} = \frac{i}{2}\bar\epsilon\left(\delta^{(3)}\lambda^x\right)\big|_{\mathrm{bg}}, \quad (4.5b)$$

$$\begin{aligned}\left(\delta^{(4)}A^I_\mu\right)\big|_{\mathrm{bg}} ={}& -\frac{1}{2}\bar\epsilon\gamma_\mu\left(\delta^{(3)}\lambda^x\right)\big|_{\mathrm{bg}}h^I_x\\
&- \frac{i}{2}\sqrt{\frac{3}{2}}\bar\epsilon h^I\left(\delta^{(3)}\psi_\mu\right)\big|_{\mathrm{bg}}\\
&+ \frac{3i}{2}h^I_x\left(\delta^{(2)}\phi^x\right)\big|_{\mathrm{bg}}\bar\epsilon\left(\delta^{(1)}\psi_\mu\right)\big|_{\mathrm{bg}}\\
&- \frac{3}{2}\left(\delta^{(2)}e^a_\mu\right)\big|_{\mathrm{bg}}\bar\epsilon\gamma_a\left(\delta^{(1)}\lambda^x\right)\big|_{\mathrm{bg}}h^I_x\\
&- \frac{3}{2}\bar\epsilon\gamma_\mu\left(\delta^{(1)}\lambda^x\right)\big|_{\mathrm{bg}}\nabla_y h^I_x\left(\delta^{(2)}\phi^y\right)\big|_{\mathrm{bg}}. \end{aligned} \quad (4.5c)$$

Again, the supercovariant field strength, the covariant derivative on ϕ^x and the spin connection ω^{ab}_μ all vanish.

5. Wigging of BPS extremal black hole

Following the treatment of the $D = 5$ *attractor mechanism* given in [17,18] and [19], we consider the 1/2-BPS near-horizon conditions for extremal electric black hole (with near-horizon geometry $AdS_2 \times S^3$):

$$\partial_\mu h^I = 0 \quad \Longrightarrow \quad \partial_\mu \phi^x = 0, \qquad h_{Ix} F^I_{\mu\nu} = 0, \tag{5.1}$$

and we evaluate the results for purely bosonic background (computed in the previous section) onto such conditions (denoted by $|_{\mathrm{BPS}}$, and always understood on the r.h.s. of equations, throughout the following treatment).

5.1. First order

At the first order, the gravitino variation is non-zero, while the gaugino variation vanishes:

$$\left(\delta^{(1)}\psi^i_\mu\right)\big|_{\mathrm{BPS}} = D_\mu(\hat{\omega})\epsilon^i + \frac{i}{4\sqrt{6}} h_I \mathcal{F}^I_{\nu\rho}\left(\gamma_\mu{}^{\nu\rho} - 4\delta^\nu_\rho\gamma^\rho\right)\epsilon^i \neq 0, \tag{5.2a}$$

$$\left(\delta^{(1)}\lambda^{xi}\right)\big|_{\mathrm{BPS}} = 0. \tag{5.2b}$$

5.2. Second order

At the second order, one obtains:

$$\left(\delta^{(2)}e^a_\mu\right)\big|_{\mathrm{BPS}} = \frac{1}{2}\bar{\epsilon}\gamma^a\left(\delta^{(1)}\psi_\mu\right)\big|_{\mathrm{BPS}} \neq 0, \tag{5.3a}$$

$$\left(\delta^{(2)}\phi^x\right)\big|_{\mathrm{BPS}} = 0, \tag{5.3b}$$

$$\left(\delta^{(2)}A^I_\mu\right)\big|_{\mathrm{BPS}} = 0. \tag{5.3c}$$

5.3. Third order

At the third order, it holds that:

$$\begin{aligned}\left(\delta^{(3)}\psi_\mu\right)\big|_{\mathrm{BPS}} = {} & \left(\delta^{(2)}\mathcal{D}_\mu\right)\big|_{\mathrm{BPS}}\epsilon \\ & + \frac{i}{4\sqrt{6}} h_I \mathcal{F}^I_{bc}\left[\left(\delta^{(2)}e^a_\mu\right)\big|_{\mathrm{BPS}} e^\nu_b e^\rho_c \right. \\ & + e^a_\mu\left(\delta^{(2)}e^\nu_b\right)\big|_{\mathrm{BPS}} e^\rho_c + e^a_\mu e^\nu_b\left(\delta^{(2)}e^\rho_c\right)\big|_{\mathrm{BPS}}\left.\right] \\ & \times \left(\gamma_a{}^{bc} - 4\delta^b_a\gamma^c\right)\epsilon^i \\ & + \frac{i}{4\sqrt{6}} h_I\left(\delta^{(2)}\tilde{F}^I_{\nu\rho}\right)\big|_{\mathrm{BPS}}\left(\gamma_\mu{}^{\nu\rho} - 4\delta^\nu_\mu\gamma^\rho\right)\epsilon^i,\end{aligned} \tag{5.4a}$$

and

$$\left(\delta^{(3)}\lambda^{ix}\right)\big|_{\mathrm{BPS}} = 0. \tag{5.4b}$$

Concerning the supercovariant field strength, the covariant derivative on ϕ^x and the spin connection ω^{ab}_μ, the following expressions hold:

$$\left(\delta^{(2)}\tilde{F}^I_{\mu\nu}\right)\big|_{\mathrm{BPS}} = 2\partial_{[\mu\cdot}\left(\delta^{(2)}A^I_{\nu]}\right)\big|_{\mathrm{BPS}} + i\sqrt{\frac{3}{2}}\left(\delta^{(1)}\bar{\psi}_\nu\right)\big|_{\mathrm{BPS}}\left(\delta^{(1)}\psi_\mu\right)\big|_{\mathrm{BPS}} h^I, \tag{5.5a}$$

$$\left(\delta^{(2)}\widehat{\mathcal{D}}_\mu\phi^x\right)\big|_{\mathrm{BPS}} = \partial_\mu\left(\delta^{(2)}\phi^x\right)\big|_{\mathrm{BPS}} - i\left(\delta^{(1)}\bar{\psi}_\mu\right)\big|_{\mathrm{BPS}}\left(\delta^{(1)}\lambda^x\right)\big|_{\mathrm{BPS}} = 0, \tag{5.5b}$$

$$\begin{aligned}\left(\delta^{(2)}\omega^{ab}_\mu\right)\big|_{\mathrm{BPS}} = {} & \frac{1}{2}\left(\delta^{(2)}e_{c\mu}\right)\big|_{\mathrm{BPS}}\left(\Omega^{abc} - \Omega^{bca} - \Omega^{cab}\right) \\ & + \frac{1}{2}\left[\left(\delta^{(2)}\Omega^{abc}\right)\big|_{\mathrm{BPS}} - \left(\delta^{(2)}\Omega^{bca}\right)\big|_{\mathrm{BPS}} \right. \\ & \left. - \left(\delta^{(2)}\Omega^{cab}\right)\big|_{\mathrm{BPS}}\right] + \left(\delta^{(2)}K^a{}_\mu{}^b\right)\big|_{\mathrm{BPS}},\end{aligned} \tag{5.5c}$$

$$\begin{aligned}\left(\delta^{(2)}\Omega^{abc}\right)\big|_{\mathrm{BPS}} = {} & \left[\left(\delta^{(2)}e^{\mu a}\right)\big|_{\mathrm{BPS}} e^{\nu b} \right. \\ & \left. + e^{\mu a}\left(\delta^{(2)}e^{\nu b}\right)\big|_{\mathrm{BPS}}\right]\left(\partial_\mu e^c_\nu - \partial_\nu e^c_\mu\right) \\ & + e^{\mu a}e^{\nu b}\left[\partial_\mu\left(\delta^{(2)}e^c_\nu\right)\big|_{\mathrm{BPS}} - \partial_\nu\left(\delta^{(2)}e^c_\mu\right)\big|_{\mathrm{BPS}}\right],\end{aligned} \tag{5.5d}$$

$$\begin{aligned}\left(\delta^{(2)}K^a{}_\mu{}^b\right)\big|_{\mathrm{BPS}} = {} & \left(\delta^{(1)}\bar{\psi}_\rho\right)\big|_{\mathrm{BPS}} e^{\rho[a}\gamma^{b]}\left(\delta^{(1)}\psi_\mu\right)\big|_{\mathrm{BPS}} \\ & + \frac{1}{2}\left(\delta^{(1)}\bar{\psi}_\rho\right)\big|_{\mathrm{BPS}}\gamma_\mu\left(\delta^{(1)}\psi_\nu\right)\big|_{\mathrm{BPS}} e^{\rho a}e^{\nu b}.\end{aligned} \tag{5.5e}$$

5.4. Fourth order

Finally, at the fourth order, by using the identity [8]

$$h^I h_{Ix} = 0,$$

one achieves the following results:

$$\left(\delta^{(4)}e^a_\mu\right)\big|_{\mathrm{BPS}} = \frac{1}{2}\bar{\epsilon}\gamma^a\left(\delta^{(3)}\psi_\mu\right)\big|_{\mathrm{BPS}} \neq 0, \tag{5.6a}$$

$$\left(\delta^{(4)}\phi^x\right)\big|_{\mathrm{BPS}} = 0, \tag{5.6b}$$

$$\left(\delta^{(4)}A^I_\mu\right)\big|_{\mathrm{BPS}} = -\frac{i}{2}\sqrt{\frac{3}{2}}\bar{\epsilon}h^I\left(\delta^{(3)}\psi_\mu\right)\big|_{\mathrm{BPS}} \neq 0. \tag{5.6c}$$

Once again, the supercovariant field strength, the covariant derivative on ϕ^x and the spin connection ω^{ab}_μ all vanish.

6. Conclusion

The general structure of the fermionic wigging (3.1) along a 4-component anti-Killing spinor, as well as the results reported in Sections 5.2 and 5.4, do imply that the attractor values of the real scalar fields ϕ^x in the near-horizon $AdS_2 \times S^3$ geometry of the 1/2-BPS extremal (electric) black hole are *not* corrected by the fermionic wigging itself; an analogous result holds for extremal (magnetic) black string with a near horizon geometry $AdS_3 \times S^2$ (*cf. e.g.* [19] and [12]).

Thus, the attractor values of the scalar fields ϕ^x are still fixed purely in terms of the black hole (electric) charges:

$$\begin{aligned}\widehat{\phi}^x\big|_{\mathrm{BPS}} & = \left(e^\delta\phi\right)\big|_{\mathrm{BPS}} \\ & = \phi^x\big|_{BPS} + \left(\delta^{(1)}\phi^x\right)\big|_{\mathrm{BPS}} + \frac{1}{2!}\left(\delta^{(2)}\phi^x\right)\big|_{\mathrm{BPS}} \\ & \quad + \frac{1}{3!}\left(\delta^{(3)}\phi^x\right)\big|_{\mathrm{BPS}} + \frac{1}{4!}\left(\delta^{(4)}\phi^x\right)\big|_{\mathrm{BPS}} \\ & = \phi\big|_{\mathrm{BPS}},\end{aligned} \tag{6.1}$$

as it holds for the attractor mechanism on the purely bosonic background (*cf. e.g.* [17–19]). It should also be stressed that the result (6.1) does *not* depend on the specific data of the real special geometry of the manifold defined by the scalars of the vector multiplets.

We would like to stress once again that we adopted the approximation of computing the fermionic wig by performing a perturbation of the unwigged, purely bosonic BPS extremal black hole solution while keeping the radius of the event horizon unchanged.

The complete analysis of the fully-backreacted wigged black hole solution, including the study of its thermodynamical properties and the computation of its Bekenstein–Hawking entropy is left for future work. This study can also be generalized to the non-supersymmetric (non-BPS) case.[7]

It should also be remarked that in $D=4$, the attractor mechanism receives a priori *non-vanishing* corrections from bilinear terms in the anti-Killing spinor ϵ [7].

Further investigation of such an important difference concerning wig corrections to the attractor mechanism in $D=4$ and $D=5$ is currently in progress, and results will be reported elsewhere. Here, we confine ourselves to anticipate that the aforementioned *non-vanishing* wig corrections in $D=4$ can be related to the intrinsically *dyonic* nature of the four-dimensional *"large"* charge configurations, namely to the fact that charge configurations giving rise to a non-vanishing area of the horizon, and thus to a well-defined attractor mechanism for scalar dynamics, contain *both* electric and magnetic charges.

As further venues of research, we finally would like to mention that fermionic wigging techniques could also be applied to other asymptotically flat $D=5$ solutions, such as black rings [20,19] and "black Saturns" [21], as well to extended $\mathcal{N}>2$ supergravity theories in five dimensions.

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Appendix A. Notation and identities

We follow the notations in [8]. We adopt the Lorentzian $D=5$ metric signature $(-,+,+,+,+)$ and we consider symplectic–Majorana spinors satisfying

$$\bar{\lambda}^i = (\lambda_i)^\dagger \gamma^0 = \lambda^{iT}\mathcal{C}, \tag{A.1}$$

where the charge conjugation matrix $\mathcal{C}$ fulfills the condition

$$\mathcal{C}^T = -\mathcal{C} = \mathcal{C}^{-1}, \qquad \mathcal{C}^2 = -1, \tag{A.2}$$

and

$$\begin{aligned} &\mathcal{C}\gamma_\mu\mathcal{C}^{-1} = (\gamma_\mu)^T \quad \Longrightarrow \quad \mathcal{C}(\gamma_\mu)^T\mathcal{C} = -\gamma_\mu, \\ &\mathcal{C}(\gamma_{\mu\nu})^T\mathcal{C} = \gamma_{\mu\nu}, \end{aligned} \tag{A.3}$$

from which one obtains

$$(\mathcal{C}\gamma_\mu)^T = -\mathcal{C}\gamma_\mu, \qquad (\mathcal{C}\gamma_{\mu\nu})^T = \mathcal{C}\gamma_{\mu\nu}.$$

Notice that $\mathcal{C}$ and $\mathcal{C}\gamma_\mu$ are antisymmetric matrices, while $\mathcal{C}\gamma_{\mu\nu}$ is a symmetric one. Spinorial indices $i=1,2$ are raised and lowered as follows

$$V^i = \varepsilon^{ij}V_j, \qquad V_i = V^j\varepsilon_{ji},$$

with

$$\varepsilon_{12} = \varepsilon^{12} = 1.$$

From these relations, one can derive the following identities:

$$\bar{\lambda}^i\chi_i = \bar{\lambda}^i\chi^j\varepsilon_{ji} = -\bar{\chi}^i\lambda_i = \bar{\chi}_i\lambda^i, \tag{A.4}$$

$$\bar{\lambda}^i\gamma_\mu\chi_i = \bar{\lambda}^i\gamma_\mu\chi^j\varepsilon_{ji} = -\bar{\chi}^j\gamma_\mu\lambda_j = \bar{\chi}_i\gamma_\mu\lambda^i, \tag{A.5}$$

$$\bar{\lambda}^i\gamma_{\mu\nu}\chi_i = \bar{\lambda}^i\gamma_{\mu\nu}\chi^j\varepsilon_{ji} = \bar{\chi}^j\gamma_{\mu\nu}\lambda_j = -\bar{\chi}_i\gamma_{\mu\nu}\lambda^i, \tag{A.6}$$

yielding

$$\bar{\lambda}^i\lambda_i = 0, \tag{A.7}$$

$$\bar{\lambda}^i\gamma_\mu\lambda_i = 0, \tag{A.8}$$

$$\bar{\lambda}^i\gamma_{\mu\nu}\lambda_i \neq 0. \tag{A.9}$$

Appendix B. Third order

At third order in $\mathcal{N}=2$, $D=5$ supersymmetry iterated transformations, one finds[8]

$$\big(\delta^{(3)}e^a_\mu\big) = \frac{1}{2}\bar{\epsilon}\gamma^a\big(\delta^{(2)}\psi_\mu\big), \tag{B.1}$$

$$\begin{aligned}
\big(\delta^{(3)}\psi^i_\mu\big) = {} & \big(\delta^{(2)}\mathcal{D}_\mu\big)\epsilon^i - \frac{1}{6}\epsilon_j\bar{\lambda}^{ix}\gamma_\mu\big(\delta^{(2)}\lambda^j_x\big) \\
& + \frac{1}{12}\gamma_{\mu\nu}\epsilon_j\bar{\lambda}^{ix}\gamma^\nu\big(\delta^{(2)}\lambda^j_x\big) \\
& - \frac{1}{48}\gamma_{\mu\nu\rho}\epsilon_j\bar{\lambda}^{ix}\gamma^{\nu\rho}\big(\delta^{(2)}\lambda^j_x\big) \\
& + \frac{1}{12}\gamma^\nu\epsilon_j\bar{\lambda}^{ix}\gamma_{\mu\nu}\big(\delta^{(2)}\lambda^j_x\big) \\
& - \frac{1}{3}\big(\delta^{(1)}e^a_\mu\big)\epsilon_j\bar{\lambda}^{ix}\gamma_a\big(\delta^{(1)}\lambda^j_x\big) \\
& - \frac{1}{3}\epsilon_j\big(\delta^{(1)}\bar{\lambda}^{ix}\big)\gamma_\mu\big(\delta^{(1)}\lambda^j_x\big) \\
& + \frac{1}{6}\big(\delta^{(1)}e^a_\mu\big)\gamma_{ab}\epsilon_j\bar{\lambda}^{ix}\gamma^b\big(\delta^{(1)}\lambda^j_x\big) \\
& + \frac{1}{6}\gamma_{\mu\nu}\epsilon_j\big(\delta^{(1)}\bar{\lambda}^{ix}\big)\gamma^\nu\big(\delta^{(1)}\lambda^j_x\big) \\
& - \frac{1}{24}\big(\delta^{(1)}e^a_\mu\big)\gamma_{abc}\epsilon_j\bar{\lambda}^{ix}\gamma^{bc}\big(\delta^{(1)}\lambda^j_x\big) \\
& - \frac{1}{24}\gamma_{\mu\nu\rho}\epsilon_j\big(\delta^{(1)}\bar{\lambda}^{ix}\big)\gamma^{\nu\rho}\big(\delta^{(1)}\lambda^j_x\big) \\
& + \frac{1}{6}\big(\delta^{(1)}e^a_\mu\big)\gamma_{ab}\epsilon_j\bar{\lambda}^{ix}\gamma^b\big(\delta^{(1)}\lambda^j_x\big) \\
& + \frac{1}{6}\gamma_{\mu\nu}\epsilon_j\big(\delta^{(1)}\bar{\lambda}^{ix}\big)\gamma^\nu\big(\delta^{(1)}\lambda^j_x\big) \\
& - \frac{1}{6}\epsilon_j\bar{\lambda}^{ix}\gamma_a\lambda^j_x\big(\delta^{(2)}e^a_\mu\big) - \frac{1}{3}\big(\delta^{(1)}e^a_\mu\big)\epsilon_j\big(\delta^{(1)}\bar{\lambda}^{ix}\big)\gamma_a\lambda^j_x \\
& - \frac{1}{3}\epsilon_j\big(\delta^{(2)}\bar{\lambda}^{ix}\big)\gamma_\mu\lambda^j_x + \frac{1}{12}\gamma_{ab}\epsilon_j\bar{\lambda}^{ix}\gamma^b\lambda^j_x\big(\delta^{(2)}e^a_\mu\big) \\
& + \frac{1}{6}\gamma_{ab}\epsilon_j\big(\delta^{(1)}\bar{\lambda}^{ix}\big)\gamma^b\lambda^j_x\big(\delta^{(1)}e^a_\mu\big)
\end{aligned}$$

[7] Note that in this case the series (3.1) truncates at the 8th order.

[8] $\nabla_t\nabla_y h^I_x$ can be elaborated by exploiting Eq. (2.5). Furthermore, $\nabla_w\nabla_u T^{xyz} = 12\widetilde{E}^{xyz}{}_{wu}$, where the rank-5 completely symmetric tensor $\widetilde{E}^{xyz}{}_{wu}$ is the real special geometry analogue [13] of the so-called *E-tensor* of special Kähler geometry [16]; by using the last of (2.2a), a similar result holds for $\nabla_u\nabla_t\Gamma^x_{yz}$.

$$
\begin{aligned}
&+\frac{1}{12}\gamma_{\mu\nu}\epsilon_j\big(\delta^{(2)}\bar\lambda^{ix}\big)\gamma^\nu\lambda_x^j\\
&-\frac{1}{48}\gamma_{abc}\epsilon_j\bar\lambda^{ix}\gamma^{bc}\lambda_x^j\big(\delta^{(2)}e_\mu^a\big)\\
&-\frac{1}{24}\gamma_{abc}\epsilon_j\big(\delta^{(1)}\bar\lambda^{ix}\big)\gamma^{bc}\lambda_x^j\big(\delta^{(1)}e_\mu^a\big)\\
&-\frac{1}{48}\gamma_{\mu\nu\rho}\epsilon_j\big(\delta^{(2)}\bar\lambda^{ix}\big)\gamma^{\nu\rho}\lambda_x^j\\
&+\frac{1}{12}\gamma^\nu\epsilon_j\big(\delta^{(2)}\bar\lambda^{ix}\big)\gamma_{\mu\nu}\lambda_x^j\\
&+\frac{1}{12}\gamma^b\epsilon_j\bar\lambda^{ix}\gamma_{ab}\lambda_x^j\big(\delta^{(2)}e_\mu^a\big)\\
&+\frac{1}{6}\gamma^b\epsilon_j\big(\delta^{(1)}\bar\lambda^{ix}\big)\gamma_{ab}\lambda_x^j\big(\delta^{(1)}e_\mu^a\big)\\
&+\frac{i}{4\sqrt6}h_I\tilde F^I_{\nu\rho}\big[\big(\delta^{(2)}e_\mu^a\big)\epsilon_b^\nu e_c^\rho+e_\mu^a\big(\delta^{(2)}\epsilon_b^\nu\big)e_c^\rho\\
&+e_\mu^a\epsilon_b^\nu\big(\delta^{(2)}e_c^\rho\big)\\
&+2\big(\delta^{(1)}e_\mu^a\big)\big(\delta^{(1)}\epsilon_b^\nu\big)e_c^\rho+2\big(\delta^{(1)}e_\mu^a\big)\epsilon_b^\nu\big(\delta^{(1)}e_c^\rho\big)\\
&+2e_\mu^a\big(\delta^{(1)}\epsilon_b^\nu\big)\big(\delta^{(1)}e_c^\rho\big)\big]\big(\gamma_a{}^{bc}-4\delta_a^b\gamma^c\big)\epsilon^i\\
&+\frac{i}{6}h_{Iz}\big(\delta^{(1)}\phi^z\big)\tilde F^I_{\nu\rho}\big[\big(\delta^{(1)}e_\mu^a\big)e_b^\nu e_c^\rho+e_\mu^a\big(\delta^{(1)}e_b^\nu\big)e_c^\rho\\
&+e_\mu^a e_b^\nu\big(\delta^{(1)}e_c^\rho\big)\big]\big(\gamma_a{}^{bc}-4\delta_a^b\gamma^c\big)\epsilon^i\\
&+\frac{i}{12}h_{Iz}\big(\delta^{(2)}\phi^z\big)\tilde F^I_{\nu\rho}\big(\gamma_\mu{}^{\nu\rho}-4\delta_\mu^\nu\gamma^\rho\big)\epsilon^i\\
&+\frac{i}{12}\nabla_y h_{Iz}\big(\delta^{(1)}\phi^z\big)\big(\delta^{(1)}\phi^y\big)\tilde F^I_{\nu\rho}\big(\gamma_\mu{}^{\nu\rho}-4\delta_\mu^\nu\gamma^\rho\big)\epsilon^i\\
&+\frac{i}{2\sqrt6}h_I\big(\delta^{(1)}\tilde F^I_{\nu\rho}\big)\big[\big(\delta^{(1)}e_\mu^a\big)e_b^\nu e_c^\rho+e_\mu^a\big(\delta^{(1)}e_b^\nu\big)e_c^\rho\\
&+e_\mu^a e_b^\nu\big(\delta^{(1)}e_c^\rho\big)\big]\big(\gamma_a{}^{bc}-4\delta_a^b\gamma^c\big)\epsilon^i\\
&+\frac{i}{6}h_{Iz}\big(\delta^{(1)}\phi^z\big)\big(\delta^{(1)}\tilde F^I_{\nu\rho}\big)\big(\gamma_\mu{}^{\nu\rho}-4\delta_\mu^\nu\gamma^\rho\big)\epsilon^i\\
&+\frac{i}{4\sqrt6}h_I\big(\delta^{(2)}\tilde F^I_{\nu\rho}\big)\big(\gamma_\mu{}^{\nu\rho}-4\delta_\mu^\nu\gamma^\rho\big)\epsilon^i,
\end{aligned}
\qquad (B.2)
$$

$$
\big(\delta^{(3)}\phi^x\big)=\frac{i}{2}\bar\epsilon\big(\delta^{(2)}\lambda^x\big),\qquad (B.3)
$$

$$
\begin{aligned}
\big(\delta^{(3)}A_\mu^I\big)=&-\frac{1}{2}\bar\epsilon\gamma_\mu\big(\delta^{(2)}\lambda^x\big)h_x^I-\frac{1}{2}\big(\delta^{(2)}e_\mu^a\big)\bar\epsilon\gamma_a\lambda^x h_x^I\\
&-\frac{i}{2}\sqrt{\frac{3}{2}}\bar\epsilon h^I\big(\delta^{(2)}\psi_\mu\big)+\frac{i}{2}h_x^I\big(\delta^{(2)}\phi^x\big)\bar\epsilon\psi_\mu\\
&-\frac{1}{2}\bar\epsilon\gamma_\mu\lambda^x\nabla_y h_x^I\big(\delta^{(2)}\phi^y\big)\\
&+ih_x^I\big(\delta^{(1)}\phi^x\big)\bar\epsilon\big(\delta^{(1)}\psi_\mu\big)\\
&+\frac{i}{2}\nabla_y h_x^I\big(\delta^{(1)}\phi^y\big)\big(\delta^{(1)}\phi^x\big)\bar\epsilon\psi_\mu\\
&-\big(\delta^{(1)}e_\mu^a\big)\bar\epsilon\gamma_a\big(\delta^{(1)}\lambda^x\big)h_x^I\\
&-\big(\delta^{(1)}e_\mu^a\big)\bar\epsilon\gamma_a\lambda^x\nabla_y h_x^I\big(\delta^{(1)}\phi^y\big)\\
&-\bar\epsilon\gamma_\mu\big(\delta^{(1)}\lambda^x\big)\nabla_y h_x^I\big(\delta^{(1)}\phi^y\big)\\
&-\frac{1}{2}\bar\epsilon\gamma_\mu\lambda^x\nabla_t\nabla_y h_x^I\big(\delta^{(1)}\phi^y\big)\big(\delta^{(1)}\phi^t\big),
\end{aligned}
\qquad (B.4)
$$

$$
\begin{aligned}
\big(\delta^{(3)}\lambda^{ix}\big)=&-\big(\delta^{(1)}e_a^\mu\big)\gamma^a\big(\delta^{(1)}\widehat{\mathcal D}_\mu\phi^x\big)\epsilon^i-\frac{i}{2}\big(\delta^{(2)}e_a^\mu\big)\gamma^a\widehat{\mathcal D}_\mu\phi^x\epsilon^i\\
&-\frac{i}{2}\gamma^\mu\big(\delta^{(2)}\widehat{\mathcal D}_\mu\phi^x\big)\epsilon^i\\
&-\frac{1}{4\sqrt6}T^{xyz}\gamma^\mu\epsilon_j\bar\lambda_y^i\gamma_\mu\big(\delta^{(2)}\lambda_z^j\big)\\
&+\frac{1}{4}\sqrt{\frac{3}{2}}T^{xyz}\epsilon_j\bar\lambda_y^i\big(\delta^{(2)}\lambda_z^j\big)\\
&-\frac{1}{8\sqrt6}T^{xyz}\gamma^{\mu\nu}\epsilon_j\bar\lambda_y^i\gamma_{\mu\nu}\big(\delta^{(2)}\lambda_z^j\big)\\
&-2\big(\delta^{(2)}\phi^y\big)\Gamma^x_{yz}\big(\delta^{(1)}\lambda^{zi}\big)\\
&+\frac{1}{2}\sqrt{\frac{3}{2}}T^{xyz}\epsilon_j\big(\delta^{(1)}\bar\lambda_y^i\big)\big(\delta^{(1)}\lambda_z^j\big)\\
&-\frac{1}{4\sqrt6}T^{xyz}\gamma^{\mu\nu}\epsilon_j\big(\delta^{(1)}\bar\lambda_y^i\big)\gamma_{\mu\nu}\big(\delta^{(1)}\lambda_z^j\big)\\
&-\frac{1}{2\sqrt6}T^{xyz}\gamma^\mu\epsilon_j\big(\delta^{(1)}\bar\lambda_y^i\big)\gamma_\mu\big(\delta^{(1)}\lambda_z^j\big)\\
&-\big(\delta^{(1)}\phi^y\big)\Gamma^x_{yz}\big(\delta^{(2)}\lambda^{zi}\big)-\big(\delta^{(3)}\phi^y\big)\Gamma^x_{yz}\lambda^{zi}\\
&+\frac{1}{4}\sqrt{\frac{3}{2}}T^{xyz}\epsilon_j\big(\delta^{(2)}\bar\lambda_y^i\big)\lambda_z^j\\
&-\frac{1}{8\sqrt6}T^{xyz}\gamma^{\mu\nu}\epsilon_j\big(\delta^{(2)}\bar\lambda_y^i\big)\gamma_{\mu\nu}\lambda_z^j\\
&+\frac{1}{2}\sqrt{\frac{3}{2}}\nabla_u T^{xyz}\big(\delta^{(1)}\phi^u\big)\epsilon_j\bar\lambda_y^i\big(\delta^{(1)}\lambda_z^i\big)\\
&-\frac{1}{4\sqrt6}\nabla_u T^{xyz}\big(\delta^{(1)}\phi^u\big)\gamma^{\mu\nu}\epsilon_j\big(\delta^{(1)}\bar\lambda_y^i\big)\gamma_{\mu\nu}\lambda_z^j\\
&-\frac{1}{2\sqrt6}\nabla_u T^{xyz}\big(\delta^{(1)}\phi^u\big)\gamma^\mu\epsilon_j\bar\lambda_y^i\gamma_\mu\big(\delta^{(1)}\lambda_z^j\big)\\
&+\frac{1}{2}\sqrt{\frac{3}{2}}\nabla_u T^{xyz}\big(\delta^{(1)}\phi^u\big)\epsilon_j\big(\delta^{(1)}\bar\lambda_y^i\big)\lambda_z^j\\
&-\frac{1}{2\sqrt6}\nabla_u T^{xyz}\big(\delta^{(1)}\phi^u\big)\gamma^\mu\epsilon_j\big(\delta^{(1)}\bar\lambda_y^i\big)\gamma_\mu\lambda_z^j\\
&-2\big(\delta^{(1)}\phi^y\big)\nabla_t\Gamma^x_{yz}\big(\delta^{(1)}\phi^t\big)\big(\delta^{(1)}\lambda^{zi}\big)\\
&-2\big(\delta^{(2)}\phi^y\big)\nabla_t\Gamma^x_{yz}\big(\delta^{(1)}\phi^t\big)\lambda^{zi}\\
&+\frac{1}{4}\sqrt{\frac{3}{2}}\nabla_u T^{xyz}\big(\delta^{(2)}\phi^u\big)\epsilon_j\bar\lambda_y^i\lambda_z^j\\
&+\frac{1}{4}\sqrt{\frac{3}{2}}\nabla_w\nabla_u T^{xyz}\big(\delta^{(1)}\phi^u\big)\big(\delta^{(1)}\phi^w\big)\epsilon_j\bar\lambda_y^i\lambda_z^j\\
&-\frac{1}{8\sqrt6}\nabla_w\nabla_u T^{xyz}\big(\delta^{(1)}\phi^u\big)\big(\delta^{(1)}\phi^w\big)\gamma^{\mu\nu}\epsilon_j\bar\lambda_y^i\gamma_{\mu\nu}\lambda_z^j\\
&-\frac{1}{4\sqrt6}\nabla_w\nabla_u T^{xyz}\big(\delta^{(1)}\phi^u\big)\big(\delta^{(1)}\phi^w\big)\gamma^\mu\epsilon_j\bar\lambda_y^i\gamma_\mu\lambda_z^j\\
&-2\big(\delta^{(1)}\phi^y\big)\nabla_u\nabla_t\Gamma^x_{yz}\big(\delta^{(1)}\phi^t\big)\big(\delta^{(1)}\phi^u\big)\lambda^{zi}\\
&+\frac{1}{4}\gamma\cdot\tilde F^I\nabla_t h_I^x\big(\delta^{(2)}\phi^t\big)\epsilon^i\\
&+\frac{1}{2}\gamma\cdot\big(\delta^{(1)}\tilde F^I\big)\nabla_t h_I^x\big(\delta^{(1)}\phi^t\big)\epsilon^i\\
&+\frac{1}{4}\gamma\cdot\big(\delta^{(2)}\tilde F^I\big)h_I^x\epsilon^i\\
&+\frac{1}{4}\gamma\cdot\tilde F^I\nabla_u\nabla_t h_I^x\big(\delta^{(1)}\phi^t\big)\big(\delta^{(1)}\phi^u\big)\epsilon^i\\
&+\frac{1}{4}\gamma^{ab}\big[\big(\delta^{(2)}e_a^\mu\big)e_b^\nu+2\big(\delta^{(1)}e_a^\mu\big)\big(\delta^{(1)}e_b^\nu\big)\\
&+e_a^\mu\big(\delta^{(2)}e_b^\nu\big)\big]\tilde F_{\mu\nu}h_I^x\\
&+\gamma^{ab}\big(\delta^{(1)}e_a^\mu\big)e_b^\nu\big(\delta^{(1)}\tilde F_{\mu\nu}\big)h_I^x
\end{aligned}
$$

$$+\gamma^{ab}\big(\delta^{(1)}e_a^{\mu}\big)e_b^{\nu}\tilde{F}_{\mu\nu}\nabla_t h_I^x\big(\delta^{(1)}\phi^t\big)$$
$$-\big(\delta^{(1)}\phi^y\big)\nabla_t\Gamma^x_{yz}\big(\delta^{(2)}\phi^t\big)\lambda^{zi}$$
$$-\frac{1}{4\sqrt{6}}\nabla_u T^{xyz}\big(\delta^{(2)}\phi^u\big)\gamma^{\mu}\epsilon_j\bar{\lambda}_y^i\gamma_\mu\lambda_z^j$$
$$-\frac{1}{8\sqrt{6}}\nabla_u T^{xyz}\big(\delta^{(2)}\phi^u\big)\gamma^{\mu\nu}\epsilon_j\bar{\lambda}_y^i\gamma_{\mu\nu}\lambda_z^j, \tag{B.5}$$

with

$$\big(\delta^{(2)}\tilde{F}^I_{\mu\nu}\big)=\big(\delta^{(2)}\mathcal{F}^I_{\mu\nu}\big)+2\big(\delta^{(1)}\bar{\psi}_{[\mu}\big)\gamma_{\nu]}\big(\delta^{(1)}\lambda^x\big)h_x^I$$
$$+2\big(\delta^{(1)}\bar{\psi}_{[\mu}\big)\gamma_{\nu]}\lambda^x\nabla_y h_x^I\big(\delta^{(1)}\phi^y\big)$$
$$+2\bar{\psi}_{[\mu}\gamma_{\nu]}\big(\delta^{(1)}\lambda^x\big)\nabla_y h_x^I\big(\delta^{(1)}\phi^y\big)$$
$$+\big(\delta^{(2)}\bar{\psi}_{[\mu}\big)\gamma_{.\nu]}\lambda^x h_x^I$$
$$+\bar{\psi}_{[\mu}\gamma_{\nu]}\big(\delta^{(2)}\lambda^x\big)h_x^I$$
$$+\bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^x\nabla_z\nabla_y h_x^I\big(\delta^{(1)}\phi^y\big)\big(\delta^{(1)}\phi^z\big)$$
$$+\bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^x\nabla_y h_x^I\big(\delta^{(2)}\phi^y\big)+\frac{i}{2}\sqrt{\frac{3}{2}}\big(\delta^{(2)}\bar{\psi}_\mu\big)\psi_\nu h^I$$
$$+\frac{i}{2}\sqrt{\frac{3}{2}}\bar{\psi}_\mu\big(\delta^{(2)}\psi_\nu\big)h^I+i\sqrt{\frac{3}{2}}\big(\delta^{(1)}\bar{\psi}_\mu\big)\big(\delta^{(1)}\psi_\nu\big)h^I$$
$$-i\bar{\psi}_\mu\big(\delta^{(1)}\psi_\nu\big)h_x^I\big(\delta^{(1)}\phi^x\big)-\frac{i}{2}\bar{\psi}_\mu\psi_\nu h_x^I\big(\delta^{(2)}\phi^x\big)$$
$$-i\big(\delta^{(1)}\bar{\psi}_\mu\big)\psi_\nu h_x^I\big(\delta^{(1)}\phi^x\big)$$
$$-\frac{i}{2}\bar{\psi}_\mu\psi_\nu\nabla_y h_x^I\big(\delta^{(1)}\phi^x\big)\big(\delta^{(1)}\phi^y\big)$$
$$+2\big(\delta^{(1)}\bar{\psi}_{[\mu}\big)\big(\delta^{(1)}e^a_{\nu]}\big)\gamma_a\lambda^x h_x^I$$
$$+\bar{\psi}_{[\mu}\big(\delta^{(2)}e^a_{\nu]}\big)\gamma_a\lambda^x h_x^I$$
$$+2\bar{\psi}_{[\mu}\big(\delta^{(1)}e^a_{\nu]}\big)\gamma_a\big(\delta^{(1)}\lambda^x\big)h_x^I$$
$$+2\bar{\psi}_{[\mu}\big(\delta^{(1)}e^a_{\nu]}\big)\gamma_a\lambda^x\nabla_t h_x^I\big(\delta^{(1)}\phi^t\big), \tag{B.6}$$

$$\big(\delta^{(2)}\mathcal{D}_\mu\big)=\frac{1}{4}\big(\delta^{(2)}\omega_\mu^{ab}\big)\gamma_{ab}, \tag{B.7}$$

$$\big(\delta^{(2)}\omega_\mu^{ab}\big)=\frac{1}{2}\big(\delta^{(2)}e_{c\mu}\big)\big(\Omega^{abc}-\Omega^{bca}-\Omega^{cab}\big)$$
$$+\big(\delta^{(1)}e_{c\mu}\big)\big[\big(\delta^{(1)}\Omega^{abc}\big)-\big(\delta^{(1)}\Omega^{bca}\big)-\big(\delta^{(1)}\Omega^{cab}\big)\big]$$
$$+\frac{1}{2}e_{c\mu}\big[\big(\delta^{(2)}\Omega^{abc}\big)-\big(\delta^{(2)}\Omega^{bca}\big)-\big(\delta^{(2)}\Omega^{cab}\big)\big]$$
$$+\big(\delta^{(2)}K^a{}_\mu{}^b\big), \tag{B.8}$$

$$\big(\delta^{(2)}\Omega^{abc}\big)=\big[\big(\delta^{(2)}e^{\mu a}\big)e^{\nu b}+2\big(\delta^{(1)}e^{\mu a}\big)\big(\delta^{(1)}e^{\nu b}\big)$$
$$+e^{\mu a}\big(\delta^{(2)}e^{\nu b}\big)\big]\big(\partial_\mu e_\nu^c-\partial_\nu e_\mu^c\big)$$
$$+2\big[\big(\delta^{(1)}e^{\mu a}\big)e^{\nu b}+e^{\mu a}\big(\delta^{(1)}e^{\nu b}\big)\big]$$
$$\times\big[\partial_\mu\big(\delta^{(1)}e_\nu^c\big)-\partial_\nu\big(\delta^{(1)}e_\mu^c\big)\big]$$
$$+e^{\mu a}e^{\nu b}\big[\partial_\mu\big(\delta^{(2)}e_\nu^c\big)-\partial_\nu\big(\delta^{(2)}e_\mu^c\big)\big], \tag{B.9}$$

$$\big(\delta^{(2)}K^a{}_\mu{}^b\big)=\frac{1}{2}\bigg[\big(\delta^{(2)}\bar{\psi}_\rho\big)e^{\rho[a}\gamma^{b]}\psi_\mu$$
$$+2\big(\delta^{(1)}\bar{\psi}_\rho\big)\big(\delta^{(1)}e^{\rho[a}\big)\gamma^{b]}\psi_\mu$$
$$+2\big(\delta^{(1)}\bar{\psi}_\rho\big)e^{\rho[a}\gamma^{b]}\big(\delta^{(1)}\psi_\mu\big)$$
$$+\bar{\psi}_\rho\big(\delta^{(2)}e^{\rho[a}\big)\gamma^{b]}\psi_\mu$$
$$+2\bar{\psi}_\rho\big(\delta^{(1)}e^{\rho[a}\big)\gamma^{b]}\big(\delta^{(1)}\psi_\mu\big)+\bar{\psi}^{[a}\gamma^{b]}\big(\delta^{(2)}\psi_\mu\big)$$
$$+\frac{1}{2}\big(\delta^{(2)}\bar{\psi}_\rho\big)e^{\rho a}\gamma_\mu\psi^b$$
$$+\big(\delta^{(1)}\bar{\psi}_\rho\big)\gamma_\mu\big(\delta^{(1)}\psi_\nu\big)e^{\rho a}e^{\nu b}$$
$$+\big(\delta^{(1)}\bar{\psi}_\rho\big)\gamma_\mu\psi^b\big(\delta^{(1)}e^{\rho a}\big)$$
$$+\big(\delta^{(1)}\bar{\psi}_\rho\big)\gamma_c\psi^b e^{\rho a}\big(\delta^{(1)}e_\mu^c\big)$$
$$+\big(\delta^{(1)}\bar{\psi}_\rho\big)\gamma_\mu\psi_\nu e^{\rho a}\big(\delta^{(1)}e^{\nu b}\big)$$
$$+\frac{1}{2}\bar{\psi}^a\gamma_\mu\big(\delta^{(2)}\psi_\nu\big)e^{\nu b}$$
$$+\bar{\psi}_\rho\gamma_\mu\big(\delta^{(1)}\psi_\nu\big)\big(\delta^{(1)}e^{\rho a}\big)e^{\nu b}$$
$$+\bar{\psi}^a\gamma_c\big(\delta^{(1)}\psi_\nu\big)\big(\delta^{(1)}e_\mu^c\big)e^{\nu b}$$
$$+\bar{\psi}^a\gamma_\mu\big(\delta^{(1)}\psi_\nu\big)\big(\delta^{(1)}e^{\nu b}\big)+\frac{1}{2}\bar{\psi}_\rho\gamma_\mu\psi^b\big(\delta^{(2)}e^{\rho a}\big)$$
$$+\bar{\psi}_\rho\gamma_c\psi^b\big(\delta^{(1)}e^{\rho a}\big)\big(\delta^{(1)}e_\mu^c\big)$$
$$+\bar{\psi}_\rho\gamma_\mu\psi_\nu\big(\delta^{(1)}e^{\rho a}\big)\big(\delta^{(1)}e^{\nu b}\big)$$
$$+\frac{1}{2}\bar{\psi}^a\gamma_c\psi^b\big(\delta^{(2)}e_\mu^c\big)+\bar{\psi}^a\gamma_c\psi_\nu\big(\delta^{(1)}e_\mu^c\big)\big(\delta^{(1)}e^{\nu b}\big)$$
$$+\frac{1}{2}\bar{\psi}^a\gamma_\mu\psi_\nu\big(\delta^{(2)}e^{\nu b}\big)\bigg], \tag{B.10}$$

$$\big(\delta^{(2)}\widehat{\mathcal{D}}_\mu\phi^x\big)=\partial_\mu\big(\delta^{(2)}\phi^x\big)-\frac{i}{2}\big(\delta^{(2)}\bar{\psi}_\mu\big)\lambda^x-i\big(\delta^{(1)}\bar{\psi}_\mu\big)\big(\delta^{(1)}\lambda^x\big)$$
$$-\frac{i}{2}\bar{\psi}_\mu\big(\delta^{(2)}\lambda^x\big). \tag{B.11}$$

Appendix C. Fourth order

Finally, at the fourth order we find[9]

$$\big(\delta^{(4)}e_\mu^a\big)=\frac{1}{2}\bar{\epsilon}\gamma^a\big(\delta^{(3)}\psi_\mu\big), \tag{C.1}$$

$$\big(\delta^{(4)}\psi_\mu^i\big)=\big(\delta^{(3)}\mathcal{D}_\mu\big)\epsilon^i-\frac{1}{6}\epsilon_j\bar{\lambda}^{ix}\gamma_\mu\big(\delta^{(3)}\lambda_x^j\big)$$
$$+\frac{1}{12}\gamma_{\mu\nu}\epsilon_j\bar{\lambda}^{ix}\gamma^\nu\big(\delta^{(3)}\lambda_x^j\big)$$
$$-\frac{1}{48}\gamma_{\mu\nu\rho}\epsilon_j\bar{\lambda}^{ix}\gamma^{\nu\rho}\big(\delta^{(3)}\lambda_x^j\big)$$
$$+\frac{1}{12}\gamma^\nu\epsilon_j\bar{\lambda}^{ix}\gamma_{\mu\nu}\big(\delta^{(3)}\lambda_x^j\big)$$
$$-\frac{1}{6}\epsilon_j\bar{\lambda}^{ix}\gamma_a\lambda_x^j\big(\delta^{(3)}e_\mu^a\big)$$
$$-\frac{1}{3}\epsilon_j\big(\delta^{(3)}\bar{\lambda}^{ix}\big)\gamma_\mu\lambda_x^j+\frac{1}{12}\gamma_{ab}\epsilon_j\bar{\lambda}^{ix}\gamma^b\lambda_x^j\big(\delta^{(3)}e_\mu^a\big)$$
$$+\frac{1}{12}\gamma_{\mu\nu}\epsilon_j\big(\delta^{(3)}\bar{\lambda}^{ix}\big)\gamma^\nu\lambda_x^j$$
$$-\frac{1}{48}\gamma_{abc}\epsilon_j\bar{\lambda}^{ix}\gamma^{bc}\lambda_x^j\big(\delta^{(3)}e_\mu^a\big)$$
$$-\frac{1}{48}\gamma_{\mu\nu\rho}\epsilon_j\big(\delta^{(3)}\bar{\lambda}^{ix}\big)\gamma^{\nu\rho}\lambda_x^j$$
$$+\frac{1}{12}\gamma^\nu\epsilon_j\big(\delta^{(3)}\bar{\lambda}^{ix}\big)\gamma_{\mu\nu}\lambda_x^j$$
$$+\frac{1}{12}\gamma^b\epsilon_j\bar{\lambda}^{ix}\gamma_{ab}\lambda_x^j\big(\delta^{(3)}e_\mu^a\big)$$
$$+\frac{i}{4\sqrt{6}}h_I\tilde{F}^I_{\nu\rho}\big[\big(\delta^{(3)}e_\mu^a\big)e_b^\nu e_c^\rho$$
$$+e_\mu^a\big(\delta^{(3)}e_b^\nu\big)e_c^\rho+e_\mu^a e_b^\nu\big(\delta^{(3)}e_c^\rho\big)$$

[9] Note that $\nabla_w\nabla_t\nabla_u T^{xyz}=12\nabla_w\widetilde{E}^{xyz}{}_{tu}$ [13]; similarly, $\nabla_t\nabla_z\nabla_y h_x^I$ can be related to $\widetilde{E}$-*tensor* (*cf.* footnote 8).

$$+ 3e_\mu^a(\delta^{(2)}e_b^\nu)(\delta^{(1)}e_c^\rho) + 3e_\mu^a(\delta^{(1)}e_b^\nu)(\delta^{(2)}e_c^\rho)$$
$$+ 3(\delta^{(2)}e_\mu^a)e_b^\nu(\delta^{(1)}e_c^\rho)$$
$$+ 3(\delta^{(2)}e_\mu^a)(\delta^{(1)}e_b^\nu)e_c^\rho + 3(\delta^{(1)}e_\mu^a)(\delta^{(2)}e_b^\nu)e_c^\rho$$
$$+ 3(\delta^{(1)}e_\mu^a)e_b^\nu(\delta^{(2)}e_c^\rho)$$
$$+ 6(\delta^{(1)}e_\mu^a)(\delta^{(1)}e_b^\nu)(\delta^{(1)}e_c^\rho)\big](\gamma_a{}^{bc} - 4\delta_a^b\gamma^c)\epsilon^i$$
$$+ \frac{i}{4}\nabla_t h_{Ix}\tilde{F}^I_{\nu\rho}\big[(\delta^{(1)}e_\mu^a)e_b^\nu e_c^\rho + e_\mu^a(\delta^{(1)}e_b^\nu)e_c^\rho$$
$$+ e_\mu^a e_b^\nu(\delta^{(1)}e_c^\rho)\big](\delta^{(1)}\phi^x)(\delta^{(1)}\phi^t)(\gamma_a{}^{bc} - 4\delta_a^b\gamma^c)\epsilon^i$$
$$+ \frac{i}{2}h_{Ix}(\delta^{(1)}\tilde{F}^I_{\nu\rho})\big[(\delta^{(1)}e_\mu^a)e_b^\nu e_c^\rho + e_\mu^a(\delta^{(1)}e_b^\nu)e_c^\rho$$
$$+ e_\mu^a e_b^\nu(\delta^{(1)}e_c^\rho)\big](\delta^{(1)}\phi^x)(\gamma_a{}^{bc} - 4\delta_a^b\gamma^c)\epsilon^i$$
$$+ \frac{i}{4}h_{Ix}\tilde{F}^I_{\nu\rho}\big[(\delta^{(2)}e_\mu^a)e_b^\nu e_c^\rho + e_\mu^a(\delta^{(2)}e_b^\nu)e_c^\rho$$
$$+ e_\mu^a e_b^\nu(\delta^{(2)}e_c^\rho)$$
$$+ 2e_\mu^a(\delta^{(1)}e_b^\nu)(\delta^{(1)}e_c^\rho) + 2(\delta^{(1)}e_\mu^a)e_b^\nu(\delta^{(1)}e_c^\rho)$$
$$+ 2(\delta^{(1)}e_\mu^a)(\delta^{(1)}e_b^\nu)e_c^\rho\big](\delta^{(1)}\phi^x)(\gamma_a{}^{bc} - 4\delta_a^b\gamma^c)\epsilon^i$$
$$+ \frac{i}{4}h_{Ix}\tilde{F}^I_{\nu\rho}\big[(\delta^{(1)}e_\mu^a)e_b^\nu e_c^\rho + e_\mu^a(\delta^{(1)}e_b^\nu)e_c^\rho$$
$$+ e_\mu^a e_b^\nu(\delta^{(1)}e_c^\rho)\big](\delta^{(2)}\phi^x)(\gamma_a{}^{bc} - 4\delta_a^b\gamma^c)\epsilon^i$$
$$+ \frac{i}{4}\sqrt{\frac{3}{2}}h_I(\delta^{(2)}\tilde{F}^I_{\nu\rho})\big[(\delta^{(1)}e_\mu^a)e_b^\nu e_c^\rho + e_\mu^a(\delta^{(1)}e_b^\nu)e_c^\rho$$
$$+ e_\mu^a e_b^\nu(\delta^{(1)}e_c^\rho)\big](\gamma_a{}^{bc} - 4\delta_a^b\gamma^c)\epsilon^i$$
$$+ \frac{i}{4}\sqrt{\frac{3}{2}}h_I(\delta^{(1)}\tilde{F}^I_{\nu\rho})\big[(\delta^{(2)}e_\mu^a)e_b^\nu e_c^\rho$$
$$+ e_\mu^a(\delta^{(2)}e_b^\nu)e_c^\rho + e_\mu^a e_b^\nu(\delta^{(2)}e_c^\rho)$$
$$+ 2e_\mu^a(\delta^{(1)}e_b^\nu)(\delta^{(1)}e_c^\rho) + 2(\delta^{(1)}e_\mu^a)e_b^\nu(\delta^{(1)}e_c^\rho)$$
$$+ 2(\delta^{(1)}e_\mu^a)(\delta^{(1)}e_b^\nu)e_c^\rho\big](\gamma_a{}^{bc} - 4\delta_a^b\gamma^c)\epsilon^i$$
$$+ \frac{i}{12}h_{Iz}(\delta^{(3)}\phi^z)\tilde{F}^I_{\nu\rho}(\gamma_\mu{}^{\nu\rho} - 4\delta_\mu^\nu\gamma^\rho)\epsilon^i$$
$$+ \frac{i}{4\sqrt{6}}h_I(\delta^{(3)}\tilde{F}^I_{\nu\rho})(\gamma_\mu{}^{\nu\rho} - 4\delta_\mu^\nu\gamma^\rho)\epsilon^i$$
$$- \frac{1}{2}(\delta^{(1)}e_\mu^a)\epsilon_j\bar{\lambda}^{ix}\gamma_a(\delta^{(2)}\lambda_x^j)$$
$$- \frac{1}{2}\epsilon_j(\delta^{(1)}\bar{\lambda}^{ix})\gamma_\mu(\delta^{(2)}\lambda_x^j)$$
$$+ \frac{1}{4}(\delta^{(1)}e_\mu^a)\gamma_{ab}\epsilon_j\bar{\lambda}^{ix}\gamma^b(\delta^{(2)}\lambda_x^j)$$
$$+ \frac{1}{4}\gamma_{\mu\nu}\epsilon_j(\delta^{(1)}\bar{\lambda}^{ix})\gamma^\nu(\delta^{(2)}\lambda_x^j)$$
$$- \frac{1}{16}(\delta^{(1)}e_\mu^a)\gamma_{abc}\epsilon_j\bar{\lambda}^{ix}\gamma^{bc}(\delta^{(2)}\lambda_x^j)$$
$$- \frac{1}{16}\gamma_{\mu\nu\rho}\epsilon_j(\delta^{(1)}\bar{\lambda}^{ix})\gamma^{\nu\rho}(\delta^{(2)}\lambda_x^j)$$
$$+ \frac{1}{4}\gamma^b\epsilon_j\bar{\lambda}^{ix}\gamma_{ab}(\delta^{(2)}\lambda_x^j)(\delta^{(1)}e_\mu^a)$$
$$+ \frac{1}{4}\gamma^\nu\epsilon_j(\delta^{(1)}\bar{\lambda}^{ix})\gamma_{\mu\nu}(\delta^{(2)}\lambda_x^j)$$
$$- \frac{1}{2}\epsilon_j\bar{\lambda}^{ix}\gamma_a(\delta^{(1)}\lambda_x^j)(\delta^{(2)}e_\mu^a)$$
$$- \epsilon_j(\delta^{(1)}\bar{\lambda}^{ix})\gamma_a(\delta^{(1)}\lambda_x^j)(\delta^{(1)}e_\mu^a)$$
$$- \frac{1}{2}\epsilon_j(\delta^{(2)}\bar{\lambda}^{ix})\gamma_\mu(\delta^{(1)}\lambda_x^j)$$
$$+ \frac{1}{4}(\delta^{(2)}e_\mu^a)\gamma_{ab}\epsilon_j\bar{\lambda}^{ix}\gamma^b(\delta^{(1)}\lambda_x^j)$$
$$+ \frac{1}{2}(\delta^{(1)}e_\mu^a)\gamma_{ab}\epsilon_j(\delta^{(1)}\bar{\lambda}^{ix})\gamma^b(\delta^{(1)}\lambda_x^j)$$
$$+ \frac{1}{4}\gamma_{\mu\nu}\epsilon_j(\delta^{(2)}\bar{\lambda}^{ix})\gamma^\nu(\delta^{(1)}\lambda_x^j)$$
$$- \frac{1}{16}(\delta^{(2)}e_\mu^a)\gamma_{abc}\epsilon_j\bar{\lambda}^{ix}\gamma^{bc}(\delta^{(1)}\lambda_x^j)$$
$$- \frac{1}{8}(\delta^{(1)}e_\mu^a)\gamma_{abc}\epsilon_j(\delta^{(1)}\bar{\lambda}^{ix})\gamma^{bc}(\delta^{(1)}\lambda_x^j)$$
$$- \frac{1}{16}\gamma_{\mu\nu\rho}\epsilon_j(\delta^{(2)}\bar{\lambda}^{ix})\gamma^{\nu\rho}(\delta^{(1)}\lambda_x^j)$$
$$+ \frac{1}{4}\gamma^b\epsilon_j\bar{\lambda}^{ix}\gamma_{ab}(\delta^{(1)}\lambda_x^j)(\delta^{(2)}e_\mu^a)$$
$$+ \frac{1}{2}\gamma^b\epsilon_j(\delta^{(1)}\bar{\lambda}^{ix})\gamma_{ab}(\delta^{(1)}\lambda_x^j)(\delta^{(1)}e_\mu^a)$$
$$+ \frac{1}{4}\gamma^\nu\epsilon_j(\delta^{(2)}\bar{\lambda}^{ix})\gamma_{\mu\nu}(\delta^{(1)}\lambda_x^j)$$
$$- \frac{1}{2}(\delta^{(2)}e_\mu^a)\epsilon_j(\delta^{(1)}\bar{\lambda}^{ix})\gamma_a\lambda_x^j$$
$$- \frac{1}{2}(\delta^{(1)}e_\mu^a)\epsilon_j(\delta^{(2)}\bar{\lambda}^{ix})\gamma_a\lambda_x^j$$
$$+ \frac{1}{4}(\delta^{(2)}e_\mu^a)\gamma_{ab}\epsilon_j(\delta^{(1)}\bar{\lambda}^{ix})\gamma^b\lambda_x^j$$
$$+ \frac{1}{4}(\delta^{(1)}e_\mu^a)\epsilon_j\gamma_{ab}(\delta^{(2)}\bar{\lambda}^{ix})\gamma^b\lambda_x^j$$
$$- \frac{1}{16}(\delta^{(2)}e_\mu^a)\gamma_{abc}\epsilon_j(\delta^{(1)}\bar{\lambda}^{ix})\gamma^{bc}\lambda_x^j$$
$$- \frac{1}{16}(\delta^{(1)}e_\mu^a)\gamma_{abc}\epsilon_j(\delta^{(2)}\bar{\lambda}^{ix})\gamma^{bc}\lambda_x^j$$
$$+ \frac{1}{4}(\delta^{(2)}e_\mu^a)\gamma^b\epsilon_j(\delta^{(1)}\bar{\lambda}^{ix})\gamma_{ab}\lambda_x^j$$
$$+ \frac{1}{4}(\delta^{(1)}e_\mu^a)\gamma^b\epsilon_j(\delta^{(2)}\bar{\lambda}^{ix})\gamma_{ab}\lambda_x^j$$
$$+ \frac{i}{4}\big[h_{Iz}(\delta^{(2)}\phi^z) + \nabla_y h_{Iz}(\delta^{(1)}\phi^z)(\delta^{(1)}\phi^y)\big]$$
$$\times \tilde{F}^I_{\nu\rho}(\delta^{(1)}e_\mu^a)(\gamma_a{}^{\nu\rho} - 4\delta_a^\nu\gamma^\rho)\epsilon^i$$
$$+ \frac{i}{12}\big[\nabla_y h_{Iz}(\delta^{(1)}\phi^z)(\delta^{(2)}\phi^y)$$
$$+ 2\nabla_y h_{Iz}(\delta^{(2)}\phi^z)(\delta^{(1)}\phi^y)$$
$$+ \nabla_t\nabla_y h_{Iz}(\delta^{(1)}\phi^z)(\delta^{(1)}\phi^y)(\delta^{(1)}\phi^t)\big]$$
$$\times \tilde{F}^I_{\nu\rho}(\gamma_\mu{}^{\nu\rho} - 4\delta_\mu^\nu\gamma^\rho)$$
$$+ \frac{i}{4}\big[h_{Iz}(\delta^{(2)}\phi^z) + \nabla_y h_{Iz}(\delta^{(1)}\phi^z)(\delta^{(1)}\phi^y)\big]$$
$$\times (\delta^{(1)}\tilde{F}^I_{\nu\rho})(\gamma_\mu{}^{\nu\rho} - 4\delta_\mu^\nu\gamma^\rho)\epsilon^i$$
$$+ \frac{i}{4}h_{Iz}(\delta^{(1)}\phi^z)(\delta^{(2)}\tilde{F}^I_{\nu\rho})(\gamma_\mu{}^{\nu\rho} - 4\delta_\mu^\nu\gamma^\rho)\epsilon^i, \qquad (C.2)$$

$$(\delta^{(4)}\phi^x) = \frac{i}{2}\bar{\epsilon}(\delta^{(3)}\lambda^x), \qquad (C.3)$$

$$(\delta^{(4)}A_\mu^I) = -\frac{1}{2}\bar{\epsilon}\gamma_\mu(\delta^{(3)}\lambda^x)h_x^I - \frac{1}{2}(\delta^{(3)}e_\mu^a)\bar{\epsilon}\gamma_a\lambda^x h_x^I$$
$$- \frac{i}{2}\sqrt{\frac{3}{2}}\bar{\epsilon}h^I(\delta^{(3)}\psi_\mu) + \frac{i}{2}h_x^I(\delta^{(3)}\phi^x)\bar{\epsilon}\psi_\mu$$

$$
\begin{aligned}
&- \frac{1}{2}\bar{\epsilon}\gamma_\mu \lambda^x \nabla_y h^I_x\big(\delta^{(3)}\phi^y\big) \\
&+ \frac{3i}{2} h^I_x\big(\delta^{(1)}\phi^x\big)\bar{\epsilon}\big(\delta^{(2)}\psi_\mu\big) + \frac{3i}{2} h^I_x\big(\delta^{(2)}\phi^x\big)\bar{\epsilon}\big(\delta^{(1)}\psi_\mu\big) \\
&+ \frac{3i}{2}\nabla_y h^I_x\big(\delta^{(1)}\phi^x\big)\big(\delta^{(1)}\phi^y\big)\bar{\epsilon}\big(\delta^{(1)}\psi_\mu\big) \\
&+ \frac{i}{2}\nabla_y h^I_x\big(\delta^{(2)}\phi^y\big)\big(\delta^{(1)}\phi^x\big)\bar{\epsilon}\psi_\mu \\
&+ i\nabla_y h^I_x\big(\delta^{(2)}\phi^x\big)\big(\delta^{(1)}\phi^y\big)\bar{\epsilon}\psi_\mu \\
&+ \frac{i}{2}\nabla_z\nabla_y h^I_x\big(\delta^{(1)}\phi^x\big)\big(\delta^{(1)}\phi^y\big)\big(\delta^{(1)}\phi^z\big)\bar{\epsilon}\psi_\mu \\
&- \frac{3}{2}\big(\delta^{(1)}e^a_\mu\big)\bar{\epsilon}\gamma_a\big(\delta^{(2)}\lambda^x\big)h^I_x \\
&- \frac{3}{2}\big(\delta^{(2)}e^a_\mu\big)\bar{\epsilon}\gamma_a\big(\delta^{(1)}\lambda^x\big)h^I_x \\
&- \frac{3}{2}\bar{\epsilon}\gamma_\mu\big(\delta^{(2)}\lambda^x\big)\nabla_y h^I_x\big(\delta^{(1)}\phi^y\big) \\
&- 3\big(\delta^{(1)}e^a_\mu\big)\bar{\epsilon}\gamma_a\big(\delta^{(1)}\lambda^x\big)\nabla_y h^I_x\big(\delta^{(1)}\phi^y\big) \\
&- \frac{3}{2}\big(\delta^{(2)}e^a_\mu\big)\bar{\epsilon}\gamma_a\lambda^x\nabla_y h^I_x\big(\delta^{(1)}\phi^y\big) \\
&- \frac{3}{2}\bar{\epsilon}\gamma_\mu\big(\delta^{(1)}\lambda^x\big)\big[\nabla_y h^I_x\big(\delta^{(2)}\phi^y\big) \\
&+ \nabla_z\nabla_y h^I_x\big(\delta^{(1)}\phi^y\big)\big(\delta^{(1)}\phi^z\big)\big] \\
&- \frac{3}{2}\big(\delta^{(1)}e^a_\mu\big)\bar{\epsilon}\gamma_a\lambda^x\big[\nabla_y h^I_x\big(\delta^{(2)}\phi^y\big) \\
&+ \nabla_z\nabla_y h^I_x\big(\delta^{(1)}\phi^y\big)\big(\delta^{(1)}\phi^z\big)\big] \\
&- \frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^x\big[\nabla_z\nabla_y h^I_x\big(\delta^{(1)}\phi^y\big)\big(\delta^{(2)}\phi^z\big) \\
&+ 2\nabla_z\nabla_y h^I_x\big(\delta^{(2)}\phi^y\big)\big(\delta^{(1)}\phi^z\big) \\
&+ \nabla_t\nabla_z\nabla_y h^I_x\big(\delta^{(1)}\phi^y\big)\big(\delta^{(1)}\phi^z\big)\big(\delta^{(1)}\phi^t\big)\big],
\end{aligned}
\tag{C.4}
$$

$$
\begin{aligned}
\big(\delta^{(4)}\lambda^{ix}\big) = &-\frac{i}{2}\gamma^\mu\big(\delta^{(3)}\widehat{\mathcal{D}}_\mu\phi^x\big)\epsilon^i - \frac{3i}{2}\gamma^a\big(\delta^{(2)}e^\mu_a\big)\big(\delta^{(1)}\widehat{\mathcal{D}}_\mu\phi^x\big)\epsilon^i \\
&- \frac{3i}{2}\gamma^a\big(\delta^{(1)}e^\mu_a\big)\big(\delta^{(2)}\widehat{\mathcal{D}}_\mu\phi^x\big)\epsilon^i \\
&- \frac{i}{2}\gamma^a\big(\delta^{(3)}e^\mu_a\big)\widehat{\mathcal{D}}_\mu\phi^x - \frac{1}{4\sqrt{6}}T^{xyz}\gamma^\mu\epsilon_j\bar{\lambda}^i_y\gamma_\mu\big(\delta^{(3)}\lambda^j_z\big) \\
&+ \frac{1}{4}\sqrt{\frac{3}{2}}T^{xyz}\epsilon_j\bar{\lambda}^i_y\big(\delta^{(3)}\lambda^j_z\big) \\
&- \frac{1}{8\sqrt{6}}T^{xyz}\gamma^{\mu\nu}\epsilon_j\bar{\lambda}^i_y\gamma_{\mu\nu}\big(\delta^{(3)}\lambda^j_z\big) \\
&+ \frac{3}{4}\sqrt{\frac{3}{2}}T^{xyz}\epsilon_j\big(\delta^{(1)}\bar{\lambda}^i_y\big)\big(\delta^{(2)}\lambda^j_z\big) \\
&- \frac{1}{8}\sqrt{\frac{3}{2}}T^{xyz}\gamma^{\mu\nu}\epsilon_j\big(\delta^{(1)}\bar{\lambda}^i_y\big)\gamma_{\mu\nu}\big(\delta^{(2)}\lambda^j_z\big) \\
&- \frac{1}{4}\sqrt{\frac{3}{2}}T^{xyz}\gamma^\mu\epsilon_j\big(\delta^{(1)}\bar{\lambda}^i_y\big)\gamma_\mu\big(\delta^{(2)}\lambda^j_z\big) \\
&- 3\big(\delta^{(2)}\phi^y\big)\varGamma^x_{yz}\big(\delta^{(2)}\lambda^{zi}\big) - 3\big(\delta^{(3)}\phi^y\big)\varGamma^x_{yz}\big(\delta^{(1)}\lambda^{zi}\big) \\
&+ \frac{3}{4}\sqrt{\frac{3}{2}}T^{xyz}\epsilon_j\big(\delta^{(2)}\bar{\lambda}^i_y\big)\big(\delta^{(1)}\lambda^j_z\big) \\
&- \frac{1}{8}\sqrt{\frac{3}{2}}T^{xyz}\gamma^{\mu\nu}\epsilon_j\big(\delta^{(2)}\bar{\lambda}^i_y\big)\gamma_{\mu\nu}\big(\delta^{(1)}\lambda^j_z\big) \\
&- \frac{1}{4}\sqrt{\frac{3}{2}}T^{xyz}\gamma^\mu\epsilon_j\big(\delta^{(2)}\bar{\lambda}^i_y\big)\gamma_\mu\big(\delta^{(1)}\lambda^j_z\big) \\
&- \big(\delta^{(1)}\phi^y\big)\varGamma^x_{yz}\big(\delta^{(3)}\lambda^{zi}\big) - \big(\delta^{(4)}\phi^y\big)\varGamma^x_{yz}\lambda^{zi} \\
&+ \frac{1}{4}\sqrt{\frac{3}{2}}T^{xyz}\epsilon_j\big(\delta^{(3)}\bar{\lambda}^i_y\big)\lambda^j_z \\
&- \frac{1}{8\sqrt{6}}T^{xyz}\gamma^{\mu\nu}\epsilon_j\big(\delta^{(3)}\bar{\lambda}^i_y\big)\gamma_{\mu\nu}\lambda^j_z \\
&- \frac{1}{4\sqrt{6}}T^{xyz}\gamma^\mu\epsilon_j\big(\delta^{(3)}\bar{\lambda}^i_y\big)\gamma_\mu\lambda^j_z \\
&+ \frac{3}{4}\sqrt{\frac{3}{2}}\nabla_t T^{xyz}\big(\delta^{(1)}\phi^t\big)\epsilon_j\big(\delta^{(2)}\bar{\lambda}^i_y\big)\lambda^j_z \\
&- \frac{1}{8}\sqrt{\frac{3}{2}}\nabla_t T^{xyz}\big(\delta^{(1)}\phi^t\big)\gamma^{\mu\nu}\epsilon_j\bar{\lambda}^i_y\gamma_{\mu\nu}\big(\delta^{(2)}\lambda^j_z\big) \\
&- \frac{1}{4}\sqrt{\frac{3}{2}}\nabla_t T^{xyz}\big(\delta^{(1)}\phi^t\big)\gamma^\mu\epsilon_j\bar{\lambda}^i_y\gamma_\mu\big(\delta^{(2)}\lambda^j_z\big) \\
&+ \frac{3}{2}\sqrt{\frac{3}{2}}\nabla_t T^{xyz}\big(\delta^{(1)}\phi^t\big)\epsilon_j\big(\delta^{(1)}\bar{\lambda}^i_y\big)\big(\delta^{(1)}\lambda^j_z\big) \\
&- \frac{1}{4}\sqrt{\frac{3}{2}}\nabla_t T^{xyz}\big(\delta^{(1)}\phi^t\big)\gamma^{\mu\nu}\epsilon_j\big(\delta^{(1)}\bar{\lambda}^i_y\big)\gamma_{\mu\nu}\big(\delta^{(1)}\lambda^j_z\big) \\
&- \frac{1}{2}\sqrt{\frac{3}{2}}\nabla_t T^{xyz}\big(\delta^{(1)}\phi^t\big)\gamma^\mu\epsilon_j\big(\delta^{(1)}\bar{\lambda}^i_y\big)\gamma_\mu\big(\delta^{(1)}\lambda^j_z\big) \\
&+ \frac{3}{4}\sqrt{\frac{3}{2}}\nabla_t T^{xyz}\big(\delta^{(1)}\phi^t\big)\epsilon_j\big(\delta^{(2)}\bar{\lambda}^i_y\big)\lambda^j_z \\
&- \frac{1}{8}\sqrt{\frac{3}{2}}\nabla_t T^{xyz}\big(\delta^{(1)}\phi^t\big)\gamma^{\mu\nu}\epsilon_j\big(\delta^{(2)}\bar{\lambda}^i_y\big)\gamma_{\mu\nu}\lambda^j_z \\
&- \frac{1}{4}\sqrt{\frac{3}{2}}\nabla_t T^{xyz}\big(\delta^{(1)}\phi^t\big)\gamma^\mu\epsilon_j\big(\delta^{(2)}\bar{\lambda}^i_y\big)\gamma_\mu\lambda^j_z \\
&- 6\big(\delta^{(2)}\phi^y\big)\nabla_t\varGamma^x_{yz}\big(\delta^{(1)}\phi^t\big)\big(\delta^{(1)}\lambda^{zi}\big) \\
&- 3\big(\delta^{(1)}\phi^y\big)\nabla_t\varGamma^x_{yz}\big(\delta^{(1)}\phi^t\big)\big(\delta^{(2)}\lambda^{zi}\big) \\
&- 3\big(\delta^{(3)}\phi^y\big)\nabla_t\varGamma^x_{yz}\big(\delta^{(1)}\phi^t\big)\lambda^{zi} \\
&+ \frac{3}{4}\sqrt{\frac{3}{2}}\epsilon_j\bar{\lambda}^i_y\big(\delta^{(1)}\lambda^j_z\big)\big[\nabla_t T^{xyz}\big(\delta^{(2)}\phi^t\big) \\
&+ \nabla_u\nabla_t T^{xyz}\big(\delta^{(1)}\phi^t\big)\big(\delta^{(1)}\phi^u\big)\big] \\
&- \frac{1}{8}\sqrt{\frac{3}{2}}\gamma^{\mu\nu}\epsilon_j\bar{\lambda}^i_y\gamma_{\mu\nu}\big(\delta^{(1)}\lambda^j_z\big)\big[\nabla_t T^{xyz}\big(\delta^{(2)}\phi^t\big) \\
&+ \nabla_u\nabla_t T^{xyz}\big(\delta^{(1)}\phi^t\big)\big(\delta^{(1)}\phi^u\big)\big] \\
&- \frac{1}{4}\sqrt{\frac{3}{2}}\gamma^\mu\epsilon_j\bar{\lambda}^i_y\gamma_\mu\big(\delta^{(1)}\lambda^j_z\big)\big[\nabla_t T^{xyz}\big(\delta^{(2)}\phi^t\big) \\
&+ \nabla_u\nabla_t T^{xyz}\big(\delta^{(1)}\phi^t\big)\big(\delta^{(1)}\phi^u\big)\big] \\
&+ \frac{3}{4}\sqrt{\frac{3}{2}}\epsilon_j\big(\delta^{(1)}\bar{\lambda}^i_y\big)\lambda^j_z\big[\nabla_t T^{xyz}\big(\delta^{(2)}\phi^t\big) \\
&+ \nabla_u\nabla_t T^{xyz}\big(\delta^{(1)}\phi^t\big)\big(\delta^{(1)}\phi^u\big)\big] \\
&- \frac{1}{8}\sqrt{\frac{3}{2}}\gamma^{\mu\nu}\epsilon_j\big(\delta^{(1)}\bar{\lambda}^i_y\big)\gamma_{\mu\nu}\lambda^j_z\big[\nabla_t T^{xyz}\big(\delta^{(2)}\phi^t\big) \\
&+ \nabla_u\nabla_t T^{xyz}\big(\delta^{(1)}\phi^t\big)\big(\delta^{(1)}\phi^u\big)\big] \\
&- \frac{1}{4}\sqrt{\frac{3}{2}}\gamma^\mu\epsilon_j\big(\delta^{(1)}\bar{\lambda}^i_y\big)\gamma_\mu\lambda^j_z\big[\nabla_t T^{xyz}\big(\delta^{(2)}\phi^t\big) \\
&+ \nabla_u\nabla_t T^{xyz}\big(\delta^{(1)}\phi^t\big)\big(\delta^{(1)}\phi^u\big)\big] \\
&- 3\big(\delta^{(1)}\phi^y\big)\big[\nabla_t\varGamma^x_{yz}\big(\delta^{(2)}\phi^t\big) \\
&+ \nabla_u\nabla_t\varGamma^x_{yz}\big(\delta^{(1)}\phi^t\big)\big(\delta^{(1)}\phi^u\big)\big]\big(\delta^{(1)}\lambda^{zi}\big)
\end{aligned}
$$

$$
\begin{aligned}
&- 3\big(\delta^{(2)}\phi^y\big)\big[\nabla_t \Gamma^x_{yz}\big(\delta^{(2)}\phi^t\big) \\
&+ \nabla_u\nabla_t \Gamma^x_{yz}\big(\delta^{(1)}\phi^t\big)\big(\delta^{(1)}\phi^u\big)\big]\lambda^{zi} \\
&+ \frac{1}{4}\sqrt{\frac{3}{2}}\big[\nabla_u T^{xyz}\big(\delta^{(3)}\phi^u\big) \\
&+ \nabla_t\nabla_u T^{xyz}\big(\delta^{(1)}\phi^u\big)\big(\delta^{(2)}\phi^t\big) \\
&+ 2\nabla_t\nabla_u T^{xyz}\big(\delta^{(2)}\phi^u\big)\big(\delta^{(1)}\phi^t\big) \\
&+ \nabla_w\nabla_t\nabla_u T^{xyz}\big(\delta^{(1)}\phi^u\big)\big(\delta^{(1)}\phi^t\big)\big(\delta^{(1)}\phi^w\big)\big]\epsilon_j\bar{\lambda}^i_y\lambda^j_z \\
&- \frac{1}{8\sqrt{6}}\gamma^{\mu\nu}\epsilon_j\bar{\lambda}^i_y\gamma_{\mu\nu}\lambda^j_z\big[\nabla_u T^{xyz}\big(\delta^{(3)}\phi^u\big) \\
&+ \nabla_t\nabla_u T^{xyz}\big(\delta^{(1)}\phi^u\big)\big(\delta^{(2)}\phi^t\big) \\
&+ 2\nabla_t\nabla_u T^{xyz}\big(\delta^{(2)}\phi^u\big)\big(\delta^{(1)}\phi^t\big) \\
&+ \nabla_w\nabla_t\nabla_u T^{xyz}\big(\delta^{(1)}\phi^u\big)\big(\delta^{(1)}\phi^t\big)\big(\delta^{(1)}\phi^w\big)\big] \\
&- \frac{1}{4\sqrt{6}}\gamma^{\mu}\epsilon_j\bar{\lambda}^i_y\gamma_{\mu}\lambda^j_z\big[\nabla_u T^{xyz}\big(\delta^{(3)}\phi^u\big) \\
&+ \nabla_t\nabla_u T^{xyz}\big(\delta^{(1)}\phi^u\big)\big(\delta^{(2)}\phi^t\big) \\
&+ 2\nabla_t\nabla_u T^{xyz}\big(\delta^{(2)}\phi^u\big)\big(\delta^{(1)}\phi^t\big) \\
&+ \nabla_w\nabla_t\nabla_u T^{xyz}\big(\delta^{(1)}\phi^u\big)\big(\delta^{(1)}\phi^t\big)\big(\delta^{(1)}\phi^w\big)\big] \\
&- \big(\delta^{(1)}\phi^y\big)\big[\nabla_u \Gamma^x_{yz}\big(\delta^{(3)}\phi^u\big) \\
&+ \nabla_u\nabla_t \Gamma^x_{yz}\big(\delta^{(1)}\phi^t\big)\big(\delta^{(2)}\phi^u\big) \\
&+ 2\nabla_u\nabla_t \Gamma^x_{yz}\big(\delta^{(2)}\phi^t\big)\big(\delta^{(1)}\phi^u\big) \\
&+ \nabla_w\nabla_u\nabla_t \Gamma^x_{yz}\big(\delta^{(1)}\phi^t\big)\big(\delta^{(1)}\phi^u\big)\big(\delta^{(1)}\phi^w\big)\big]\lambda^{zi} \\
&+ \frac{1}{4}\gamma\cdot\tilde{F}^I\big[\nabla_t h^x_I\big(\delta^{(3)}\phi^t\big) \\
&+ \nabla_u\nabla_t h^x_I\big(\delta^{(1)}\phi^t\big)\big(\delta^{(2)}\phi^u\big) \\
&+ 2\nabla_u\nabla_t h^x_I\big(\delta^{(2)}\phi^t\big)\big(\delta^{(1)}\phi^u\big)\epsilon^i \\
&+ \nabla_w\nabla_u\nabla_t h^x_I\big(\delta^{(1)}\phi^t\big)\big(\delta^{(1)}\phi^u\big)\big(\delta^{(1)}\phi^w\big)\big]\epsilon^i \\
&+ \frac{3}{4}\gamma\cdot\big(\delta^{(1)}\tilde{F}^I\big)\big[\nabla_t h^x_I\big(\delta^{(2)}\phi^t\big) \\
&+ \nabla_u\nabla_t h^x_I\big(\delta^{(1)}\phi^t\big)\big(\delta^{(1)}\phi^u\big)\big] \\
&+ \frac{3}{4}\gamma\cdot\big(\delta^{(2)}\tilde{F}^I\big)\nabla_t h^x_I\big(\delta^{(1)}\phi^t\big) + \frac{1}{4}\gamma\cdot\big(\delta^{(3)}\tilde{F}^I\big)h^x_I \\
&+ \frac{1}{2}\gamma^{ab}\big[\big(\delta^{(3)}e^\mu_a\big)e^\nu_b + 3\big(\delta^{(2)}e^\mu_a\big)\big(\delta^{(1)}e^\nu_b\big)\big]\tilde{F}^I_{\mu\nu}h^x_I \\
&+ \frac{3}{2}\gamma^{ab}\big[\big(\delta^{(2)}e^\mu_a\big)e^\nu_b + \big(\delta^{(1)}e^\mu_a\big)\big(\delta^{(1)}e^\nu_b\big)\big]\big(\delta^{(1)}\tilde{F}^I_{\mu\nu}\big)h^x_I \\
&+ \frac{3}{2}\gamma^{ab}\big[\big(\delta^{(2)}e^\mu_a\big)e^\nu_b + \big(\delta^{(1)}e^\mu_a\big)\big(\delta^{(1)}e^\nu_b\big)\big] \\
&\times \tilde{F}^I_{\mu\nu}\nabla_t h^x_I\big(\delta^{(1)}\phi^t\big) \\
&+ \frac{3}{2}\gamma^{ab}\big(\delta^{(1)}e^\mu_a\big)e^\nu_b\big(\delta^{(2)}\tilde{F}^I_{\mu\nu}\big)h^x_I \\
&+ 3\gamma^{ab}\big(\delta^{(1)}e^\mu_a\big)e^\nu_b\big(\delta^{(1)}\tilde{F}^I_{\mu\nu}\big)\nabla_t h^x_I\big(\delta^{(1)}\phi^t\big) \\
&+ \frac{3}{2}\gamma^{ab}\big(\delta^{(1)}e^\mu_a\big)e^\nu_b\tilde{F}^I_{\mu\nu}\nabla_t\nabla_u h^x_I\big(\delta^{(1)}\phi^t\big)\big(\delta^{(1)}\phi^u\big) \\
&+ \frac{3}{2}\gamma^{ab}\big(\delta^{(1)}e^\mu_a\big)e^\nu_b\tilde{F}^I_{\mu\nu}\nabla_t h^x_I\big(\delta^{(1)}\phi^t\big),
\end{aligned}
\tag{C.5}
$$

with

$$
\begin{aligned}
\big(\delta^{(3)}\tilde{F}^I_{\mu\nu}\big) = {}& \big(\delta^{(3)}\mathcal{F}^I_{\mu\nu}\big) + 3\big(\delta^{(1)}\bar{\psi}_{[\mu}\big)\gamma_{\nu]}\big(\delta^{(2)}\lambda^x\big)h^I_x \\
&+ 3\big(\delta^{(2)}\bar{\psi}_{[\mu}\big)\gamma_{\nu]}\big(\delta^{(1)}\lambda^x\big)h^I_x \\
&+ \bar{\psi}_{[\mu}\gamma_{\nu]}\big(\delta^{(3)}\lambda^x\big)h^I_x + \big(\delta^{(3)}\bar{\psi}_{[\mu}\big)\gamma_{\nu]}\lambda^x h^I_x \\
&+ \frac{i}{2}\sqrt{\frac{3}{2}}\bar{\psi}_\mu\big(\delta^{(3)}\psi_\nu\big)h^I + \frac{3i}{2}\sqrt{\frac{3}{2}}\big(\delta^{(1)}\bar{\psi}_\mu\big)\big(\delta^{(2)}\psi_\nu\big)h^I \\
&+ \frac{3i}{2}\sqrt{\frac{3}{2}}\big(\delta^{(2)}\bar{\psi}_\mu\big)\big(\delta^{(1)}\psi_\nu\big)h^I \\
&+ \frac{i}{2}\sqrt{\frac{3}{2}}\big(\delta^{(3)}\bar{\psi}_\mu\big)\psi_\nu h^I - \frac{3i}{2}\bar{\psi}_\mu\big(\delta^{(2)}\psi_\nu\big)h^I_x\big(\delta^{(1)}\phi^x\big) \\
&- \frac{3i}{2}\bar{\psi}_\mu\big(\delta^{(1)}\psi_\nu\big)h^I_x\big(\delta^{(2)}\phi^x\big) \\
&- 3i\big(\delta^{(1)}\bar{\psi}_\mu\big)\big(\delta^{(1)}\psi_\nu\big)h^I_x\big(\delta^{(1)}\phi^x\big) \\
&- \frac{3i}{2}\bar{\psi}_\mu\big(\delta^{(1)}\psi_\nu\big)\nabla_y h^I_x\big(\delta^{(1)}\phi^x\big)\big(\delta^{(1)}\phi^y\big) \\
&- \frac{i}{2}\bar{\psi}_\mu\psi_\nu h^I_x\big(\delta^{(3)}\phi^x\big) - \frac{3i}{2}\big(\delta^{(1)}\bar{\psi}_\mu\big)\psi_\nu h^I_x\big(\delta^{(2)}\phi^x\big) \\
&- \frac{3i}{2}\big(\delta^{(2)}\bar{\psi}_\mu\big)\psi_\nu h^I_x\big(\delta^{(1)}\phi^x\big) \\
&- \frac{i}{2}\bar{\psi}_\mu\psi_\nu\nabla_y h^I_x\big(\delta^{(1)}\phi^x\big)\big(\delta^{(2)}\phi^y\big) \\
&- i\bar{\psi}_\mu\psi_\nu\nabla_y h^I_x\big(\delta^{(2)}\phi^x\big)\big(\delta^{(1)}\phi^y\big) \\
&- \frac{3i}{2}\big(\delta^{(1)}\bar{\psi}_\mu\big)\psi_\nu\nabla_y h^I_x\big(\delta^{(1)}\phi^x\big)\big(\delta^{(1)}\phi^y\big) \\
&- \frac{i}{2}\bar{\psi}_\mu\psi_\nu\nabla_z\nabla_y h^I_x\big(\delta^{(1)}\phi^x\big)\big(\delta^{(1)}\phi^y\big)\big(\delta^{(1)}\phi^z\big) \\
&+ 3\bar{\psi}_{[\mu}\gamma_{\nu]}\big(\delta^{(1)}\lambda^x\big)\nabla_y h^I_x\big(\delta^{(2)}\phi^y\big) \\
&+ 3\bar{\psi}_{[\mu}\gamma_{\nu]}\big(\delta^{(2)}\lambda^x\big)\nabla_y h^I_x\big(\delta^{(1)}\phi^y\big) \\
&+ 6\big(\delta^{(1)}\bar{\psi}_{[\mu}\big)\gamma_{\nu]}\big(\delta^{(1)}\lambda^x\big)\nabla_y h^I_x\big(\delta^{(1)}\phi^y\big) \\
&+ \bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^x\nabla_y h^I_x\big(\delta^{(3)}\phi^y\big) \\
&+ 3\big(\delta^{(1)}\bar{\psi}_{[\mu}\big)\gamma_{\nu]}\lambda^x\nabla_y h^I_x\big(\delta^{(2)}\phi^y\big) \\
&+ 3\big(\delta^{(2)}\bar{\psi}_{[\mu}\big)\gamma_{\nu]}\lambda^x\nabla_y h^I_x\big(\delta^{(1)}\phi^y\big) \\
&+ 3\bar{\psi}_{[\mu}\gamma_{\nu]}\big(\delta^{(1)}\lambda^x\big)\nabla_z\nabla_y h^I_x\big(\delta^{(1)}\phi^y\big)\big(\delta^{(1)}\phi^z\big) \\
&+ \bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^x\nabla_z\nabla_y h^I_x\big(\delta^{(1)}\phi^y\big)\big(\delta^{(2)}\phi^z\big) \\
&+ 2\bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^x\nabla_z\nabla_y h^I_x\big(\delta^{(2)}\phi^y\big)\big(\delta^{(1)}\phi^z\big) \\
&+ 3\big(\delta^{(1)}\bar{\psi}_{[\mu}\big)\gamma_{\nu]}\lambda^x\nabla_z\nabla_y h^I_x\big(\delta^{(1)}\phi^y\big)\big(\delta^{(1)}\phi^z\big) \\
&+ \bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^x\nabla_w\nabla_z\nabla_y h^I_x\big(\delta^{(1)}\phi^y\big)\big(\delta^{(1)}\phi^z\big)\big(\delta^{(1)}\phi^w\big) \\
&+ 3\big(\delta^{(2)}\bar{\psi}_{[\mu}\big)\big(\delta^{(1)}e^a_{\nu]}\big)\gamma_a\lambda^x h^I_x \\
&+ 3\big(\delta^{(1)}\bar{\psi}_{[\mu}\big)\big(\delta^{(2)}e^a_{\nu]}\big)\gamma_a\lambda^x h^I_x \\
&+ 6\big(\delta^{(1)}\bar{\psi}_{[\mu}\big)\big(\delta^{(1)}e^a_{\nu]}\big)\gamma_a\big(\delta^{(1)}\lambda^x\big)h^I_x \\
&+ 6\big(\delta^{(1)}\bar{\psi}_{[\mu}\big)\big(\delta^{(1)}e^a_{\nu]}\big)\gamma_a\lambda^x\nabla_t h^I_x\big(\delta^{(1)}\phi^t\big) \\
&+ \bar{\psi}_{[\mu}\big(\delta^{(3)}e^a_{\nu]}\big)\gamma_a\lambda^x h^I_x + 3\bar{\psi}_{[\mu}\big(\delta^{(2)}e^a_{\nu]}\big)\gamma_a\big(\delta^{(1)}\lambda^x\big)h^I_x \\
&+ 3\bar{\psi}_{[\mu}\big(\delta^{(2)}e^a_{\nu]}\big)\gamma_a\lambda^x\nabla_t h^I_x\big(\delta^{(1)}\phi^t\big) \\
&+ 3\bar{\psi}_{[\mu}\big(\delta^{(1)}e^a_{\nu]}\big)\gamma_a\big(\delta^{(2)}\lambda^x\big)h^I_x \\
&+ 6\bar{\psi}_{[\mu}\big(\delta^{(1)}e^a_{\nu]}\big)\gamma_a\big(\delta^{(1)}\lambda^x\big)\nabla_t h^I_x\big(\delta^{(1)}\phi^t\big) \\
&+ 3\bar{\psi}_{[\mu}\big(\delta^{(1)}e^a_{\nu]}\big)\gamma_a\lambda^x\nabla_u\nabla_t h^I_x\big(\delta^{(1)}\phi^t\big)\big(\delta^{(1)}\phi^u\big) \\
&+ 3\bar{\psi}_{[\mu}\big(\delta^{(1)}e^a_{\nu]}\big)\gamma_a\lambda^x\nabla_t h^I_x\big(\delta^{(2)}\phi^t\big),
\end{aligned}
\tag{C.6}
$$

$$\big(\delta^{(3)}\mathcal{D}_\mu\big)=\frac{1}{4}\big(\delta^{(3)}\omega_\mu^{ab}\big)\gamma_{ab}, \tag{C.7}$$

$$\begin{aligned}\big(\delta^{(3)}\omega_\mu^{ab}\big)=&\frac{1}{2}\big(\delta^{(3)}e_{c\mu}\big)\big(\Omega^{abc}-\Omega^{bca}-\Omega^{cab}\big)\\&+2\big(\delta^{(2)}e_{c\mu}\big)\big[\big(\delta^{(1)}\Omega^{abc}\big)-\big(\delta^{(1)}\Omega^{bca}\big)-\big(\delta^{(1)}\Omega^{cab}\big)\big]\\&+2\big(\delta^{(1)}e_{c\mu}\big)\big[\big(\delta^{(2)}\Omega^{abc}\big)-\big(\delta^{(2)}\Omega^{bca}\big)-\big(\delta^{(2)}\Omega^{cab}\big)\big]\\&+\frac{1}{2}e_{c\mu}\big[\big(\delta^{(3)}\Omega^{abc}\big)-\big(\delta^{(3)}\Omega^{bca}\big)-\big(\delta^{(3)}\Omega^{cab}\big)\big]\\&+\big(\delta^{(3)}K^a{}_\mu{}^b\big),\end{aligned} \tag{C.8}$$

$$\begin{aligned}\big(\delta^{(3)}\Omega^{abc}\big)=&\big[\big(\delta^{(3)}e^{\mu a}\big)e^{\nu b}+3\big(\delta^{(2)}e^{\mu a}\big)\big(\delta^{(1)}e^{\nu b}\big)\\&+3\big(\delta^{(1)}e^{\mu a}\big)\big(\delta^{(2)}e^{\nu b}\big)\\&+e^{\mu a}\big(\delta^{(3)}e^{\nu b}\big)\big]\big(\partial_\mu e_\nu^c-\partial_\nu e_\mu^c\big)\\&+3\big[\big(\delta^{(1)}e^{\mu a}\big)e^{\nu b}+e^{\mu a}\big(\delta^{(1)}e^{\nu b}\big)\big]\\&\times\big[\partial_\mu\big(\delta^{(2)}e_\nu^c\big)-\partial_\nu\big(\delta^{(2)}e_\mu^c\big)\big]\\&+3\big[\big(\delta^{(2)}e^{\mu a}\big)e^{\nu b}+2\big(\delta^{(1)}e^{\mu a}\big)\big(\delta^{(1)}e^{\nu b}\big)\\&+e^{\mu a}\big(\delta^{(2)}e^{\nu b}\big)\big]\big[\partial_\mu\big(\delta^{(1)}e_\nu^c\big)-\partial_\nu\big(\delta^{(1)}e_\mu^c\big)\big]\\&+e^{\mu a}e^{\nu b}\big[\partial_\mu\big(\delta^{(3)}e_\nu^c\big)-\partial_\nu\big(\delta^{(3)}e_\mu^c\big)\big],\end{aligned} \tag{C.9}$$

$$\begin{aligned}\big(\delta^{(3)}K^a{}_\mu{}^b\big)=&\frac{3}{2}\big(\delta^{(1)}\bar\psi_\rho\big)\big(\delta^{(1)}e^{\rho a}\big)\gamma_c\big(\delta^{(1)}e_\mu^c\big)\psi^b\\&+\frac{3}{4}\big(\delta^{(2)}\bar\psi_\rho\big)e^{\rho a}\gamma_c\big(\delta^{(1)}e_\mu^c\big)\psi^b\\&+\frac{3}{4}\big(\delta^{(2)}\bar\psi_\rho\big)\big(\delta^{(1)}e^{\rho a}\big)\gamma_\mu\psi^b\\&+\frac{3}{4}\big(\delta^{(1)}\bar\psi_\rho\big)e^{\rho a}\gamma_c\big(\delta^{(2)}e_\mu^c\big)\psi^b\\&+\frac{3}{4}\bar\psi_\rho\big(\delta^{(1)}e^{\rho a}\big)\gamma_c\big(\delta^{(2)}e_\mu^c\big)\psi^b\\&+\frac{3}{4}\big(\delta^{(1)}\bar\psi_\rho\big)\big(\delta^{(2)}e^{\rho a}\big)\gamma_\mu\psi^b\\&+\frac{3}{4}\bar\psi_\rho\big(\delta^{(2)}e^{\rho a}\big)\gamma_c\big(\delta^{(1)}e_\mu^c\big)\psi^b\\&+\frac{1}{4}\big(\delta^{(3)}\bar\psi_\rho\big)e^{\rho a}\gamma_\mu\psi^b\\&+\frac{1}{4}\bar\psi^a\gamma_c\big(\delta^{(3)}e_\mu^c\big)\psi^b+\frac{1}{4}\bar\psi_\rho\big(\delta^{(3)}e^{\rho a}\big)\gamma_\mu\psi^b\\&+\frac{1}{2}\bar\psi^{[a}\gamma^{b]}\big(\delta^{(3)}\psi_\mu\big)\\&+\frac{1}{4}\bar\psi^a\gamma_\mu\big(\delta^{(3)}\psi_\sigma\big)e^{\sigma b}\\&+\frac{3}{2}\big(\delta^{(1)}\bar\psi_\rho\big)e^{\rho[a}\gamma^{b]}\big(\delta^{(2)}\psi_\mu\big)\\&+\frac{3}{4}\big(\delta^{(1)}\bar\psi_\rho\big)e^{\rho a}\gamma_\mu\big(\delta^{(2)}\psi_\sigma\big)e^{\sigma b}\\&+\frac{3}{4}\bar\psi^a\gamma_c\big(\delta^{(1)}e_\mu^c\big)\big(\delta^{(2)}\psi_\sigma\big)e^{\sigma b}\\&+\frac{3}{2}\bar\psi_\rho\big(\delta^{(1)}e^{\rho[a}\big)\gamma^{b]}\big(\delta^{(2)}\psi_\mu\big)\\&+\frac{3}{4}\bar\psi_\rho\big(\delta^{(1)}e^{\rho a}\big)\gamma_\mu\big(\delta^{(2)}\psi_\sigma\big)e^{\sigma b}\\&+\frac{3}{4}\bar\psi^a\gamma_\mu\big(\delta^{(2)}\psi_\sigma\big)\big(\delta^{(1)}e^{\sigma b}\big)\\&+\frac{3}{2}\big(\delta^{(1)}\bar\psi_\rho\big)e^{\rho a}\gamma_c\big(\delta^{(1)}e_\mu^c\big)\big(\delta^{(1)}\psi_\sigma\big)e^{\sigma b}\\&+3\big(\delta^{(1)}\bar\psi_\rho\big)\big(\delta^{(1)}e^{\rho[a}\big)\gamma^{b]}\big(\delta^{(1)}\psi_\mu\big)\\&+\frac{3}{2}\big(\delta^{(1)}\bar\psi_\rho\big)\big(\delta^{(1)}e^{\rho a}\big)\gamma_\mu\big(\delta^{(1)}\psi_\sigma\big)e^{\sigma b}\\&+\frac{3}{2}\bar\psi_\rho\big(\delta^{(1)}e^{\rho a}\big)\gamma_c\big(\delta^{(1)}e_\mu^c\big)\big(\delta^{(1)}\psi_\sigma\big)e^{\sigma b}\\&+\frac{3}{2}\big(\delta^{(1)}\bar\psi_\rho\big)e^{\rho a}\gamma_\mu\big(\delta^{(1)}\psi_\sigma\big)\big(\delta^{(1)}e^{\sigma b}\big)\\&+\frac{3}{2}\bar\psi^a\gamma_c\big(\delta^{(1)}e_\mu^c\big)\big(\delta^{(1)}\psi_\sigma\big)\big(\delta^{(1)}e^{\sigma b}\big)\\&+\frac{3}{2}\bar\psi_\rho\big(\delta^{(1)}e^{\rho a}\big)\gamma_\mu\big(\delta^{(1)}\psi_\sigma\big)\big(\delta^{(1)}e^{\sigma b}\big)\\&+\frac{3}{2}\big(\delta^{(2)}\bar\psi_\rho\big)e^{\rho[a}\gamma^{b]}\big(\delta^{(1)}\psi_\mu\big)\\&+\frac{3}{4}\big(\delta^{(2)}\bar\psi_\rho\big)e^{\rho a}\gamma_\mu\big(\delta^{(1)}\psi_\sigma\big)e^{\sigma b}\\&+\frac{3}{4}\bar\psi^a\gamma_c\big(\delta^{(2)}e_\mu^c\big)\big(\delta^{(1)}\psi_\sigma\big)e^{\sigma b}\\&+\frac{3}{2}\bar\psi_\rho\big(\delta^{(2)}e^{\rho[a}\big)\gamma^{b]}\big(\delta^{(1)}\psi_\mu\big)\\&+\frac{3}{4}\bar\psi_\rho\big(\delta^{(2)}e^{\rho a}\big)\gamma_\mu\big(\delta^{(1)}\psi_\sigma\big)e^{\sigma b}\\&+\frac{3}{4}\bar\psi^a\gamma_\mu\big(\delta^{(1)}\psi_\sigma\big)\big(\delta^{(2)}e^{\sigma b}\big)\\&+\frac{3}{2}\big(\delta^{(1)}\bar\psi_\rho\big)e^{\rho a}\gamma_c\big(\delta^{(1)}e_\mu^c\big)\psi_\sigma\big(\delta^{(1)}e^{\sigma b}\big)\\&+\frac{3}{2}\big(\delta^{(1)}\bar\psi_\rho\big)\big(\delta^{(1)}e^{\rho a}\big)\gamma_\mu\psi_\sigma\big(\delta^{(1)}e^{\sigma b}\big)\\&+\frac{3}{2}\bar\psi_\rho\big(\delta^{(1)}e^{\rho a}\big)\gamma_c\big(\delta^{(1)}e_\mu^c\big)\psi_\sigma\big(\delta^{(1)}e^{\sigma b}\big)\\&+\frac{3}{2}\big(\delta^{(2)}\bar\psi_\rho\big)\big(\delta^{(1)}e^{\rho[a}\big)\gamma^{b]}\psi_\mu\\&+\frac{3}{4}\big(\delta^{(2)}\bar\psi_\rho\big)e^{\rho a}\gamma_\mu\psi_\sigma\big(\delta^{(1)}e^{\sigma b}\big)\\&+\frac{3}{4}\bar\psi^a\gamma_c\big(\delta^{(2)}e_\mu^c\big)\psi_\sigma\big(\delta^{(1)}e^{\sigma b}\big)\\&+\frac{3}{2}\big(\delta^{(1)}\bar\psi_\rho\big)\big(\delta^{(2)}e^{\rho[a}\big)\gamma^{b]}\psi_\mu\\&+\frac{3}{4}\bar\psi_\rho\big(\delta^{(2)}e^{\rho a}\big)\gamma_\mu\psi_\sigma\big(\delta^{(1)}e^{\sigma b}\big)\\&+\frac{3}{4}\big(\delta^{(1)}\bar\psi_\rho\big)e^{\rho a}\gamma_\mu\psi_\sigma\big(\delta^{(2)}e^{\sigma b}\big)\\&+\frac{3}{4}\bar\psi^a\gamma_c\big(\delta^{(1)}e_\mu^c\big)\psi_\sigma\big(\delta^{(2)}e^{\sigma b}\big)\\&+\frac{3}{4}\bar\psi_\rho\big(\delta^{(1)}e^{\rho a}\big)\gamma_\mu\psi_\sigma\big(\delta^{(2)}e^{\sigma b}\big)\\&+\frac{1}{2}\big(\delta^{(3)}\bar\psi_\rho\big)e^{\rho[a}\gamma^{b]}\psi_\mu\\&+\frac{1}{2}\bar\psi_\rho\big(\delta^{(3)}e^{\rho[a}\big)\gamma^{b]}\psi_\mu\\&+\frac{1}{4}\bar\psi^a\gamma_\mu\psi_\sigma\big(\delta^{(3)}e^{\sigma b}\big),\end{aligned} \tag{C.10}$$

$$\begin{aligned}\big(\delta^{(3)}\widehat{\mathcal{D}}_\mu\phi^x\big)=&\partial_\mu\big(\delta^{(3)}\phi^x\big)-\frac{i}{2}\big(\delta^{(3)}\bar\psi_\mu\big)\lambda^x-\frac{3i}{2}\big(\delta^{(2)}\bar\psi_\mu\big)\big(\delta^{(1)}\lambda^x\big)\\&-\frac{3i}{2}\big(\delta^{(1)}\bar\psi_\mu\big)\big(\delta^{(2)}\lambda^x\big)-\frac{i}{2}\bar\psi_\mu\big(\delta^{(3)}\lambda^x\big).\end{aligned} \tag{C.11}$$

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