MATHEMATICS

CHARACTERIZING HILBERT SPACE BY SMOOTHNESS *)

BY

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1. *Introduction.*

While discussing the smoothenss properties of certain Banach (or B-) spaces, in [1], BONIC and FRAMPTON state: "It is likely that if a B-space and its dual are C^2-smooth, then it is a Hilbert space". The purpose of this note is to prove this statement in the following form.

Theorem 1. *If $\mathscr{X}$ is a B-space and $\mathscr{X}^*$ its dual and if the norms in $\mathscr{X}$ and $\mathscr{X}^*$ are twice Fréchet (or F-) differentiable at every point except possibly the origin, then $\mathscr{X}$ is isomorphic to a Hilbert space in the sense that $\mathscr{X}$ may be provided with an inner-product in such a way that the resulting (inner-product) norm is equivalent to the given norm of $\mathscr{X}$.*

Even though there are several characterizations of the Hilbert space (e.g., [3], [6], [8], [5]), they are mostly based on geometric considerations. The corresponding characterizations based on smoothness properties are of interest in certain analytical work (cf. [1]). Such a method is considered here. The terminology employed here essentially follows [2], and some results of the latter will be used freely.

2. Proof. The proof obtains from the following seven steps starting with the real case. Then the complex case will be obtained using ([3], p. 335).

(1) If $\mathscr{X}$ is a B-space and the norm of $\mathscr{X}^*$ is F-differentiable then $\mathscr{X}$ is reflexive. If also the norm of $\mathscr{X}$ is F-differentiable then $\mathscr{X}$ and $\mathscr{X}^*$ are homeomorphic (under a 'spherical image map', [2], p. 306).

For, by ([2], Cor. 3.18) the norm in $\mathscr{X}^*$ is F-differentiable iff the unit sphere of $\mathscr{X}$ is weakly uniformly rotund (and hence it is k-rotund, [2], p. 309). This implies in turn (cf. [2], Thm. 5.4 (i)) that the unit ball of $\mathscr{X}$ is weakly compact since $\mathscr{X}$ is complete. It follows that $\mathscr{X}$ is reflexive ([4], p. 38). The last statement is a consequence of ([2], Cor. 3.18 and Thm. 4.18). Moreover, by a well-known result of ŠMULIAN (cf. also [2], p. 289), $\mathscr{X}$ and $\mathscr{X}^*$ are simultaneously rotund and smooth.

(2) If $\mathscr{X}$ is a B-space whose norm is twice F-differentiable and if T_{x_0}

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is the second derivative at $x_0(\neq 0)$, then the map $T_{x_0}: \mathscr{X}\times\mathscr{X}\to$ reals, is a positive definite symmetric bounded bilinear functional.

Indeed, if $\varphi(t)=\|x_0+tx\|$, for x_0, x in the unit sphere $S=\{x: \|x\|=1\}$ of $\mathscr{X}$, then the convex function $\varphi(\cdot)$ is twice differentiable so that $\varphi''(t)\geqslant 0$ and $\varphi''(0)=T_{x_0}(x, x)\geqslant 0$. The result is now an immediate consequence of ([10], Thm. 5.1; or [4], Thm. 26.3.5). (Here, positive = non negative.)

(3) For all x_0, x in the unit sphere S, one has $T_{x_0}(x_0, x)=T_{x_0}(x, x_0)=0$.

By symmetry, consider $T_{x_0}(x_0, x)$. Let $G(x_0; \cdot)$ be the weak derivative of the norm in $\mathscr{X}$ at $x_0\in S$. Then

$$G(x_0; x)=\lim_{t\to 0}\frac{\|x_0+tx\|-1}{t},$$

by definition, and clearly $G(\beta x_0; \cdot)=G(x_0; \cdot)$ for all $\beta>0$. Since

$$T_{x_0}(h_1, h_2)=\lim_{t\to 0}\frac{G(x_0+th_1; h_2)-G(x_0; h_2)}{t},\quad h_1, h_2\in S,$$

it follows from the preceding comment, on setting $h_1=x_0$, that

$$G((1+t)x_0; h_2)-G(x_0; h_2)=0$$

for $|t|<1$, and all $h_2\in S$. Hence $T_{x_0}(x_0, x)=0$.

(4) For each $x_0\in S$, $G(x_0; \cdot)\in S^*$, the unit sphere of $\mathscr{X}^*$. This is immediate from $G(x_0; x_0)=1$ and $|G(x_0, x)|\leqslant 1$ for all $x\in S$.

(5) Under the hypothesis of the theorem, if $0\neq x\in\mathscr{X}_1=\mathscr{X}\ominus x_0$ (direct difference) then $T_{x_0}(x, x)>0$.

Suppose this is not true. Then $T_{x_0}(x, x)=0$ for some $0\neq x\in\mathscr{X}_1$. Then by step (2), via Schwarz inequality, it follows that $T_{x_0}(x, y)=0$ for all $y\in\mathscr{X}$ so that $T_{x_0}x=0$, i.e., $T_{x_0}: \mathscr{X}_1\to\mathscr{X}_1{}^*$ is not $1-1$ and has no inverse on $\mathscr{X}_1$. Consider the map $T: \mathscr{X}\to\mathscr{X}^*$ defined by $Tx=\|x\|\nu_{x/\|x\|}$ for $x\neq 0$, and $=0$ for $x=0$, where $\nu_{(\cdot)}$ is the spherical image map of S (cf. [2]). Then under the hypothesis of the theorem T and T^{-1} are single-valued continuous onto maps ([2], Thm. 4.18). In fact on S and S^*, one can identify Tx as $G(x; \cdot)$ and $T^{-1}x^*$ as $\tilde{G}(x^*; \cdot)$ where G and $\tilde{G}$ are the first (strong or F-) derivatives of the norms in $\mathscr{X}$ and $\mathscr{X}^*$ at x and x^* respectively. If $y_t=(x_0+ty)/\|x_0+ty\|$, $x_0, y\in S$, then with the hypothesis of twice F-differentiability of norms in $\mathscr{X}$ it follows that

$$\lim_{t\to 0}\frac{(Ty_t-Tx_0)}{t}(y)$$

exists and is continuous and hence T is F-differentiable, (cf. [10], Th. 3.3). It also follows easily from the definition of T on S, that the above limit equals $T_{x_0}(y, y)$. Thus $T'(x_0)=T_{x_0}$ where prime denotes the F-derivative of T at x_0. Since $\mathscr{X}^*$ satisfies the same hypothesis, $(T^{-1})'(x_0{}^*)$ exists and $=T_{x_0^*}: \mathscr{X}^*\to\mathscr{X}$. Now consider the identity $x_0=T^{-1}(Tx_0)$ where $x_0, x\in S$ are given at the beginning of this step. Now using the 'chain rule' for the F-derivatives (cf. [10], p. 41) one obtains

$$I=(T^{-1})'(y)T'(x_0),\ y=Tx_0.$$

Here I is the identity operator on $\mathscr{X}$. If this operator equation is applied to the given element $x \in S$, one gets $x=0$, (recall that $T_{x_0}x=0$) which is a contradiction since $x \neq 0$, $(\neq x_0)$. This proves that the supposition is false and $T_{x_0}(x, x)>0$, for $0 \neq x \in \mathscr{X}_1$.

(6) If $x, y \in \mathscr{X}_1$ and $[x, y]=T_{x_0}(x, y)$, then $[\cdot,\cdot]$ is an inner-product and $|||x|||=[x, x]^{\frac{1}{2}}$ gives an equivalent norm to the given norm $\|\cdot\|$ of $\mathscr{X}$.

In view of the preceding steps only the equivalence of norms need be shown and this is similar to ([6], p. 52). Briefly,

$$|||x|||^2=T_{x_0}(x, x) \leqslant \|T_{x_0}\|\ \|x\|^2=C_2{}^2\|x\|^2,$$

being true with $C_2{}^2=\|T_{x_0}\|$, suppose there was no $C_1>0$ such that $|||x||| \geqslant C_1\|x\|$. Then there exist $x_n \in \mathscr{X}_1$ with $|||x_n|||=1$ and $\|x_n\| \geqslant n$. If $y \in \mathscr{X}_1$ and $x_y{}^*(x_n)=[x_n, y]$ so that $|x_y{}^*(x_n)| \leqslant |||y|||<\infty$, $n \geqslant 1$, it follows from the uniform boundedness that $|x_y{}^*(x_n)| \leqslant M \cdot \|x_y{}^*\|$, for some constant $0<M<\infty$. This implies $\|x_n\| \leqslant M$ and contradicts the choice $\|x_n\| \geqslant n$. Hence there is a $C_1>0$ such that $C_1\|x\| \leqslant |||x||| \leqslant C_2\|x\|$.

(7) The theorem holds as stated. For, since $\mathscr{X}=\mathscr{X}_1 \oplus x_0$, if $x, y \in \mathscr{X}$, let $x=x_1+\lambda_1 x_0$, $y=y_1+\lambda_2 x_0$ be the unique direct sum representations for $x_1, y_1 \in \mathscr{X}_1$, and λ_1, λ_2 real. Define $(x, y)=[x_1, y_1]+\lambda_1\lambda_2\|x_0\|^2$. Then $(\cdot,\cdot)$ is an inner product and its norm is clearly equivalent to $\|\cdot\|$ of $\mathscr{X}$. Thus the theorem holds if $\mathscr{X}$ is real. If $\mathscr{X}$ is complex, then the result can be deduced from the real case by ([3], Thm. 7.2). This completes the proof of the theorem.

3. Complements. As a consequence of the theorem one has:

Corollary. *If Φ, Ψ are Young's complementary functions, both satisfying a growth condition (the Δ_2-condition of* [7], *or see* [9]), *and if L^Φ, L^Ψ are Orlicz spaces on a measure space (Ω, Σ, μ), then L^Φ is isomorphic to a Hilbert space whenever Φ, Ψ are twice continuously differentiable on the line.*

The result follows from the theorem if it is noted that twice continuous differentiability of Φ and the growth condition imply the norm in L^Φ is twice F-differentiable, which results from a second order implicit differentiation, analogous to the first order case of ([7], pp. 188–189; [9], p. 675). It may be remarked that without a growth condition while L^Φ and L^Ψ are rotund since Φ, Ψ are strictly convex ([9], Thm. 4), the F-differentiation of the norm does not seem to hold at *all* elements of L^Φ (since $(L^\Phi)^* \neq L^\Psi$).

Remarks. 1. By step (1) of the proof of theorem, if the norm of $\mathscr{X}^*$ is F-differentiable and $\mathscr{X}$ is complete then $\mathscr{X}$ is reflexive. Thus the additional assumption (made in [9], Thm. 6) that $\mathscr{X}$ be reflexive is redundant.

2. The following remark, due to Mr. J. J. Uhl, Jr., has some interest here. It follows easily from two applications of Hölder and inverse Hölder inequalities: If Φ, Ψ are continuous Young's complementary functions, then the following three statements are equivalent. (i) $L^{\Phi} \supset L^{\Psi}$, (ii) $L^2 \supset L^{\Psi}$, (iii) $L^{\Phi} \supset L^2$.

3. The norm functional of a complex B-space, even if it is F-differentiable, is not necessarily analytic (cf. [4], pp. 766–769). This follows from the above theorem applied to, for instance, the complex L^p spaces with $p > 1$.

4. I was informed by Professor F. Browder that the result of Theorem 1 was also established in an unpublished manuscript, which is not yet available to me, by Bonic and Reis. Also after this paper was prepared, an announcement without proof of a similar result has appeared in the Bulletin of the Amer. Math. Soc., (1966), p. 521, by Sundaresan. He states that it is a consequence of certain other results. But, I think, the above simple and direct proof may be of independent interest.

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