JOURNAL OF COMBINATORIAL THEORY (A) 12, 21–30 (1972)

Tensor Representation of Finite Groups

OLIVER ABERTH

Mathematics Department, Texas A & M University, College Station, Texas 77843

Communicated by H. J. Ryser

Received August 24, 1969

A cubic array of 0's and 1's, a 3-dimensional analog of the 2-dimensional array of 0's and 1's defined by a permutation matrix, can be employed to represent a finite group. The elements of this array equal the components of a certain third order tensor, and some of the standard group representation results may be interpreted as diagonalization assertions about this tensor and a similar one defined for group classes. Employing the tensor terminology introduced, a proof that the degrees of the irreducible representations of a finite group divide the group order is given which is somewhat different from the usual proofs.

INTRODUCTION

The orthogonality relations for group representations and group characters are of fundamental importance in the study of finite groups. In this paper we show how these relations can be interpreted in a new way, which hopefully may provide fresh insight into them. For any finite group, two third-order tensors are defined, one whose components may be obtained from the group multiplication table, and another whose components may be obtained from the class multiplication table. The orthogonality relations then translate into diagonalization assertions for these two tensors. As an example of this new point of view, we provide a proof that the degree of an irreducible representation of a finite group divides the order of the group, in a way related to, but somewhat different from, the usual proof by algebraic integer theory.

As a preliminary, we give here a definition of the term "tensor" as we employ it in this paper, and a brief description of some elementary tensor concepts that are needed. (A more detailed exposition may be found in Mal'cev [3].)

1. *Tensor definition.* Let V be a vector space of dimension n over the complex number field. A tensor of order $(r + s)$ with r covariant and s

contravariant indices is a correspondence which associates with each basis of V n^{r+s} complex numbers (components) $A^{j_1 \cdots j_s}_{i_1 \cdots i_r}$, $1 \leqslant i_k, j_k \leqslant n$, in such a way that a certain relationship holds between the components associated with any two bases of V. Let $P_j{}^i$ denote a non-singular n-square matrix with inverse matrix $Q_j{}^i$ (upper indices denote row, lower indices denote column). Then, if e_j, e_j', $1 \leqslant j \leqslant n$, are any two bases such that

$$e_j' = \sum_{t=1}^{n} P_j{}^t e_t ,$$

the components $A'^{j_1 \cdots j_s}_{i_1 \cdots i_r}$ associated with e_j' and the components $A^{j_1 \cdots j_s}_{i_1 \cdots i_r}$ associated with e_j satisfy the equation

$$A'^{j_1 \cdots j_s}_{i_1 \cdots i_r} = \sum A^{u_1 \cdots u_s}_{t_1 \cdots t_r} P^{t_1}_{i_1} \cdots P^{t_r}_{i_r} Q^{j_1}_{u_1} \cdots Q^{j_s}_{u_s} ,$$

where the summation is over all the indices $t_1, \ldots, t_r, u_1, \ldots, u_s$. The right side of the above equation is called the

"transform of $A^{j_1 \cdots j_s}_{i_1 \cdots i_r}$ under $P_j{}^i$."

2. *Multiplication of tensors.* Given two tensors, a new tensor may be formed by multiplication. For instance, if A^{pq}_{ijk} and B^{gh} are tensors, we may define the tensor C^{pqgk}_{ijk} by the equation

$$C^{pqgh}_{ijk} = A^{pq}_{ijk} B^{gh}.$$

This gives the correspondence of n^7 components for each basis, with the necessary relation connecting the components for two bases satisfied.

3. *Contraction of a tensor.* From any tensor with both contravariant and covariant indices, a new tensor with one less covariant and one less contravariant indices may be formed by choosing any contravariant index and any covariant index, setting them equal, and summing over this pair of indices. Thus from the tensor A^{pq}_{ijk} we may form the tensor D^q_{ik} by the rule

$$D^q_{ik} = \sum_{t=1}^{n} A^{tq}_{itk} .$$

This defines the correspondence of n^3 components for each basis, and the necessary relation which connects the components for any two bases is again satisfied.

4. *Tensor equations.* If two tensors of the same type have identical components for one basis, then they have identical components for every basis and the tensors are identical.

AN ALGEBRA TENSOR

Let G_{ij}^k $(1 \leqslant i, j, k \leqslant n)$ be any n^3 complex numbers. These numbers can be used to define a multiplication between any two ordered n-tuples of complex numbers x^i, y^i $(1 \leqslant i \leqslant n)$ by the rule $x^i \cdot y^i = z^i$,

$$z^i = \sum_{t,u=1}^{n} G_{tu}^i x^t y^u.$$

If scalar multiplication of an n-tuple and addition of two n-tuples are taken as the usual vector operations, then G_{ij}^k defines what is commonly called an algebra of order n over the complex number field. The algebra is not necessarily associative.

If e_j $(1 \leqslant j \leqslant n)$ equals the n-tuple with all components 0 except for the j-th, which equals 1, then we have

$$e_i \cdot e_j = \sum_{k=1}^{n} G_{ij}^k e_k .$$

Thus the elements e_j, which we call the basis elements associated with G_{ij}^k, define the algebra by their multiplication table. If new basis elements e_j' are defined by $e_j' = \sum_{t=1}^{n} P_j{}^t e_t$, $P_j{}^i$ a non-singular matrix with inverse $Q_j{}^i$, then

$$e_i' \cdot e_j' = \sum_{k=1}^{n} G_{ij}'^k e_k',$$

with

$$G_{ij}'^k = \sum_{t,u,v=1}^{n} G_{tu}^v P_i{}^t P_j{}^u Q_v{}^k, \tag{1}$$

and it is clear that the array G_{ij}^k transforms just like the components of a third-order tensor with two covariant and one contravariant indices. Accordingly, let us call G_{ij}^k an algebra tensor.

Now, just as the components of a matrix are visualized in a planar rectangular array as an aid in following the matrix operations, we can visualize the numbers G_{ij}^k as a spatial cubic array. The n-tuple obtained from G_{ij}^{k} by fixing two of the three indices is a row, normal, or column according as the free index is i, j, or k, respectively (see Figure 1), and the components G_{ii}^{i} , $i = 1, 2, \ldots, n$, form the main diagonal.

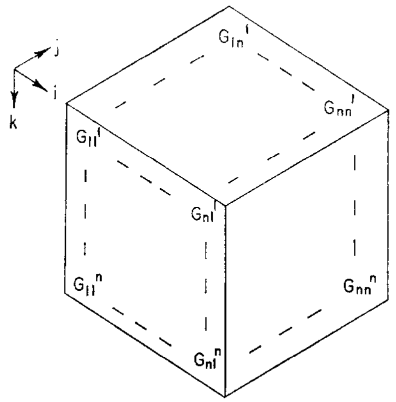

FIGURE 1

QUASIGROUP AND GROUP ALGEBRAS

Suppose now that G_{ij}^{k} is such that every row, column, and normal contains exactly one component equal to 1 with all other components 0. The array of components is then a three-dimensional analog of a permutation matrix array. The structure of the columns implies that the product of any two basis elements is again a basis element, and the structure of the rows and normals implies that the equations $x \cdot e_j = e_k$, $e_i \cdot y = e_k$ always have unique basis element solutions for x, y. Thus, under multiplication the basis elements have all the properties required to be a quasigroup. If, in addition, they satisfy the associative law, they define a group, and the algebra is a group algebra.

G_{ij}^{k} defines an associative algebra, $x^i \cdot (y^i \cdot z^i) = (x^i \cdot y^i) \cdot z^i$, if and only if

$$\sum_{t=1}^{n} G_{ht}^{k} G_{ij}^{t} = \sum_{t=1}^{n} G_{hi}^{t} G_{tj}^{k}, \tag{2}$$

and a commutative algebra, $x^i \cdot y^i = y^i \cdot x^i$, if and only if

$$G_{ij}^{k} = G_{ji}^{k}. \tag{3}$$

Note that each side of (2) may be interpreted as defining a tensor with three covariant and one contravariant indices, obtained from G_{ij}^{k} by the operations of multiplication and contraction. Since these two tensors have identical components for the basis e_j, they have identical components for any basis, and (2) holds when G_{ij}^k is replaced by $G_{ij}'^k$. Similarly (3) holds when G_{ij}^k is replaced by $G_{ij}'^k$.

We have now that the tensor G_{ij}^k defines a group (for its basis elements) if it satisfies (2) and its components have the "permutation" property mentioned. If (3) also holds, the group is Abelian. The sections of the array obtained by holding the index i fixed give the right regular representations of the group elements. That is, if the matrices $R_{(i)}$, $1 \leqslant i \leqslant n$, are defined by setting the component of the k-th row and j-th column equal to G_{ij}^k, then, by (2), $R_{(i)}R_{(j)} = R_{(k)}$ if $e_i \cdot e_j = e_k$. Similarly, the left regular matrix representations $L_{(j)}$ are the sections obtained by holding the index j fixed. Note that (2) is equivalent to the condition that all the matrices $R_{(i)}$ commute with all the matrices $L_{(j)}$; (3) is equivalent to the condition that $R_{(i)} = L_{(i)}$ for all i. A multiplication table for the group elements may be obtained by forming

$$B_{ij} = \sum_{k=1}^{n} G_{ij}^k v^k ,$$

v^k an n-tuple whose k-th component is the integer k. If $e_i \cdot e_j = e_k$, then the entry B_{ij} is equal to k.

When $P_j{}^i$ is a permutation matrix, then the transform of a group algebra tensor defined by (1) also has the property that every row, normal, and column has a single component equal to 1 with all others 0. The groups defined by G_{ij}^k and $G_{ij}'^k$ are isomorphic since each basis element e_j' equals some element e_k. If the group G defined by e_k has a normal subgroup H of order m, then we may choose $P_j{}^i$ so that e_j' for $(r-1)\,m < j \leqslant rm$, $r = 1, 2, \ldots, n/m$, belong to the same coset of H. Then $G_{ij}'^k$ will have all the components which equal 1 lying in cubic subarrays of size m on a side, and the spatial arrangement of these smaller cubes can serve to define a group algebra tensor of order n/m for the quotient group G/H.

The components of a group algebra tensor may be displayed on a printed page by giving the matrix sections in order, but then some of the relationships among the non-zero components become hard to follow. For low-order groups, $n = 8$ say, with 5 groups to choose from, the group algebra tensor may be represented by using small spheres or other objects to designate the 1 components and fixing these by some means at their appropriate positions in a cubic array. These models can be surprisingly attractive, with many interesting perspectives, especially when normal subgroups are made evident as described above.

Diagonalization of the Group Algebra Tensor

We apply now the theory of group representations to construct for a group algebra tensor G_{ij}^k a matrix P_j^i so that $G_{ij}'^k$, the transform of G_{ij}^k under P_j^i, assumes a simple form. Let the r irreducible unitary representations of the group elements e_j be numbered in some arbitrary manner with $m(k)$ the degree of the k-th representation. Then $\sum_{k=1}^r m^2(k) = n$, and if $A_i^h(e_j\,; k)$ designates the (h, i) component of the k-th representation for the element e_j, we have (cf. [5, pages 355–359])

$$\sum_{s=1}^{m(k)} A_s^h(e_j\,; k)\, A_f^s(e_t\,; k) = A_f^h(e_j \cdot e_t\,; k), \tag{4}$$

and

$$\sum_{j=1}^{n} A_i^h(e_j\,; k)\, \overline{A_f^d(e_j\,; u)} = \frac{n}{m(k)}\, \delta_d^h \delta_f^i \delta_u^k, \tag{5}$$

where $\delta_b^a = 1$ if the indices a and b are equal, otherwise $\delta_b^a = 0$.

The matrix components of the first representation, multiplied by a constant, are used to form the first $m^2(1)$ columns of P_j^i, then those of the second representation to form the next $m^2(2)$ columns, and so on. Specifically, for any integer j satisfying $1 \leqslant j \leqslant n$, let us denote by j_1, j_2, j_3 the three unique positive integers such that

$$0 \leqslant \sum_{t=1}^{j_1} m^2(t) - j < m^2(j_1),$$

$$1 \leqslant j_2\,, j_3 \leqslant m(j_1),$$

and

$$j = \sum_{t=1}^{j_1} m^2(t) - m^2(j_1) + (j_2 - 1)\, m(j_1) + j_3\,.$$

Conversely, let $\langle j_1, j_2, j_3 \rangle$ denote the integer j defined by j_1, j_2, j_3. Then the components P_j^i are given by the equation

$$P_j^i = \frac{m(j_1)}{n} A_{j_3}^{j_2}(e_i\,; j_1),$$

and by (5) the components of the inverse matrix are given by

$$Q_j^i = \overline{A_{i_3}^{i_2}(e_j; i_1)}.$$

Transforming G_{ij}^k under $P_j^{\,i}$, we obtain from (1)

$$G_{ij}'^{k} = \sum_{\substack{t,u=1\\ e_v=e_t\cdot e_u}}^{n} P_i^{\,t} P_j^{\,u} Q_v^{\,k}$$

$$= \sum_{t,u=1}^{n} \sum_{s=1}^{m(k_1)} P_i^{\,t} P_j^{\,u} Q_t^{\langle k_1,k_2,s\rangle} Q_u^{\langle k_1,s,k_3\rangle} \qquad \text{by} \quad (4)$$

$$= \sum_{s=1}^{m(k_1)} \delta_{i_1}^{k_1}\delta_{i_2}^{k_2}\delta_{i_3}^{s}\delta_{j_1}^{k_1}\delta_{j_2}^{s}\delta_{j_3}^{k_3}$$

$$= \delta_{i_1}^{k_1}\delta_{i_2}^{k_2}\delta_{i_3}^{j_2}\delta_{j_1}^{k_1}\delta_{j_3}^{k_3} .$$

Thus the components $G_{ij}'^{k}$ equal 0 unless $i_1 = j_1 = k_1$, and then they equal $\delta_{i_2}^{k_2}\delta_{i_3}^{j_2}\delta_{j_3}^{k_3}$. If we set $e_j' = f_{j_2,j_3}^{(j_1)}$, then we have

$$f_{i_2,i_3}^{(i_1)} \cdot f_{j_2,j_3}^{(j_1)} = \delta_{i_1}^{j_1}\delta_{i_3}^{j_2} f_{i_2,j_3}^{(j_1)} ,$$

and it is clear that the elements e_j' with j_1 fixed are a basis for a matrix algebra of order $m^2(j_1)$. Figure 2 depicts a typical array $G_{ij}'^{k}$, that of the dihedral group of order 10 with matrix algebras of orders 4, 4, 1, and 1, respectively. The non-zero components are located in cubic subarrays of size $m^2(j_1)$ on a side, arranged in diagonal position. If G_{ij}^k defines an Abelian group, then since $G_{ij}'^{k}$ also satisfies (3), and this requires that the matrix algebras are commutative and hence of order 1, the only non-zero components of $G_{ij}'^{k}$ are those of the main diagonal, which equal 1.

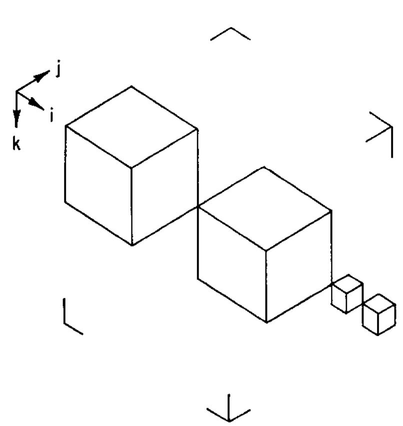

FIGURE 2

The Class Tensor

If there are r classes for the group elements e_j defined by a group algebra tensor, and c_j is the sum of the elements belonging to the j-th class, then we may write

$$c_i \cdot c_j = \sum_{k=1}^{r} C_{ij}^k c_k ,$$

where the components of the array C_{ij}^k are non-negative integers and satisfy equations (2) and (3). Setting $\chi_j{}^i$, the character of members of the j-th class for i-th irreducible representation, equal to $\sum_{t=1}^{m(i)} A_t{}^t(e_u ; i)$, e_u in the j-th class, then as a consequence of (4) and (5) the following character relations may be obtained [5, pages 359–365]:

$$\sum_{t=1}^{r} n_t \chi_t{}^i \overline{\chi_t{}^k} = n \delta_k{}^i, \tag{6}$$

$$n_t \chi_t{}^k n_u \chi_u{}^k = m(k) \sum_{v=1}^{r} C_{tu}^v n_v \chi_v{}^k. \tag{7}$$

n_j equals the number of group elements belonging to the j-th class, and $m(k)$ is again the degree of the k-th irreducible representation. If we view C_{ij}^k as a tensor, then the above equations may be interpreted as defining a matrix $R_j{}^i$ such that the transform of C_{ij}^k under $R_j{}^i$ is in diagonal form with the components along the main diagonal equal to 1 and all other components 0. The components of $R_j{}^i$ are given by

$$R_j{}^i = \chi_i{}^j \frac{m(j)}{n}, \tag{8}$$

and by (6) the components of the inverse matrix $S_j{}^i$ satisfy the equation

$$S_j{}^i = \overline{\chi_j{}^i} \frac{n_j}{m(i)}. \tag{9}$$

After taking the conjugate of equation (7), it may be written as

$$S_t{}^k S_u{}^k = \sum_{v=1}^{r} C_{tu}^v S_v{}^k. \tag{10}$$

Multiplying (10) by $R_i{}^t R_j{}^u$ and summing over t and u, we obtain for

$$C_{ij}'^k = \sum_{t,u,v=1}^{r} C_{tu}^v R_i{}^t R_j{}^u S_v{}^k$$

the result that the components with indices satisfying $i = j = k$ are equal to 1 and all other components equal 0. Thus C'^{k}_{ij} has the diagonal form claimed.

PROOF THAT THE DEGREES $m(i)$ DIVIDE THE GROUP ORDER n

Briefly, in this proof we use tensors to define a certain r-square matrix with integer components which has the rational numbers $(n/m(i))^2$, $1 \leqslant i \leqslant r$, as eigenvalues. These, being the roots of a monic polynomial with integer coefficients, must then be integers and so $m(i)$ divides n. (Related matrix proofs are given in [4] and [7].)

First, we introduce the second-order tensor H^{ij} which is defined as follows. For the basis associated with C^{k}_{ij}, H^{ij} equals n/n_i if $i = j$ and 0 if $i \neq j$. For all other bases the tensor's components are obtained by the usual transformation relations. Thus, for the basis associated with the diagonalized C'^{k}_{ij}, we have

$$H'^{ij} = \sum_{t,u=1}^{r} H^{tu} S_t{}^i S_u{}^j \tag{11}$$

$$= \sum_{t=1}^{r} \frac{nn_t}{m(i)\,m(j)} \overline{\chi_t{}^i \chi_t{}^j} \qquad \text{by} \quad (9).$$

Now we employ the elementary result that the j-th characters $\chi_t{}^j$, $1 \leqslant t \leqslant n$, equal the conjugate of some other characters, the k-th say, $\chi_t{}^j = \overline{\chi_t{}^k}$, with the j-th and k-th irreducible representations of identical degree, $m(j) = m(k)$. Employing equation (6), we have then that H'^{ij} equals 0 unless the i-th and j-th characters are conjugate and then H'^{ij} equals $(n/m(i))^2$.

Next, we define the second-order tensor $F_j{}^i$,

$$F_j{}^i = \sum_{t,u,v=1}^{r} C^{i}_{tj} C^{v}_{uv} H^{tu}, \tag{12}$$

formed by repeated multiplications and contractions from the tensors C^{k}_{ij}, H^{ij}. The transform of $F_j{}^i$ under $R_j{}^i$ may be determined in two ways:

$$F'^{i}_{j} = \sum_{t,u=1}^{r} F_u{}^t S_t{}^i R_j{}^u, \tag{13}$$

or

$$F'^{i}_{j} = \sum_{t,u,v=1}^{r} C'^{i}_{tj} C'^{v}_{uv} H'^{tu}. \tag{14}$$

If we let F, F' be matrices whose (i, j) components are $F_j^i, F_j'^i$, respectively, then on inserting the value of H^{tu} in (12) to obtain

$$F_j^i = \sum_{t,v=1}^{r} C_{tj}^i C_{tv}^v \frac{n}{n_t}$$

we see that the components of F are integers, since n_t divides n. Equation (13) may be written in matrix notation as

$$F' = R^{-1}FR,$$

and so F and F' are equivalent matrices. Finally, using the known values for $C_{ij}'^k$, H'^{ij} in (14), we see that the off-diagonal elements of F' are all zero, and that the i-th diagonal element, obtained by setting $j = i$, is equal to

$$\sum_{u,v=1}^{r} C_{uv}'^v H'^{iu} = \sum_{u=1}^{r} H'^{iu} = \left(\frac{n}{m(i)}\right)^2.$$

Thus the matrix F with integer components has the rational numbers $(n/m(i))^2$ as eigenvalues, and the proof is complete.

References

1. O. Aberth, The transformation of tensors into diagonal form, *SIAM J. Appl. Math.* **15** (1967), 1247–1252.
2. M. Hall, Jr., "The Theory of Groups," Macmillan, New York, 1959.
3. A. I. Mal'cev, "Foundations of Linear Algebra," Freeman, San Francisco, 1963.
4. F. D. Murnaghan, The dimensions of the irreducible representations of a finite group, *Proc. Nat. Acad. Sci. U. S. A.* **37** (1951), 441–442.
5. V. I. Smirnov, "Linear Algebra and Group Theory," McGraw-Hill, New York, 1961.
6. J. L. Synge and A. Schild, "Tensor Calculus," University of Toronto Press, Toronto, 1949.
7. H. Zassenhaus, An equation for the degrees of the absolutely irreducible representations of a group of finite order, *Canad. J. Math.* **2** (1950), 166–167.