JOURNAL OF ALGEBRA 25, 222–225 (1973)

A Generalization of a Theorem of Z. Janko and J. G. Thompson

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Received June 14, 1971

Z. Janko and J. G. Thompson [6] have shown that if G is a finite non-abelian simple group having a Sylow 2 subgroup T which has no normal, elementary abelian subgroups of order 8 and whenever t is an involution in T satisfying $[T: C_T(t)] \leqslant 2$, then $C_G(t)$ is solvable, then the simple group G is isomorphic to one of the following simple groups: $L_2(q)$, A_7, M_{11}, $L_3(3)$, $U_3(3)$ or $U_3(4)$. In this paper, we replace the condition of solvability of $C_G(t)$ by that $C_G(t)$ is 2-constrained and show that the simple group G is isomorphic to one of the simple groups $L_2(q)$, A_7, M_{11}, $L_3(3)$, $U_3(3)$, $U_3(4)$ or to one of the Janko simple groups of order 604,800 or 50,232,960.

We will use the following unpublished results besides the ones quoted in [6]. The terminology is the same as in [6].

J. G. Thompson. Let G be a non abelian finite simple group, T a Sylow 2 subgroup of G and suppose that T has no normal elementary abelian subgroups of order $\geqslant 8$, $Z(T)$ is noncyclic, and $N_G(T) = T \cdot C_G(T)$, then $T \simeq T_1 \times T_2$ where T_i is dihedral or semidihedral and $|T_i| \geqslant 16$ for $i = 1, 2$.

Lyons. Let G be a finite, nonabelian simple group with a Sylow 2 subgroup which is isomorphic to a Sylow 2 subgroup of $U_3(4)$. Then G is isomorphic to $U_3(4)$.

Throughout the paper, we assume that G is a finite, nonabelian, simple group with a Sylow 2 subgroup T which has no normal, elementary abelian subgroups of order 8 and whenever t is an involution in T satisfying $[T: C_T(t)] \leqslant 2$ then $C_G(t)$ is 2-constrained.

LEMMA 1. *Suppose* $|Z| = 2$, $|U(T)| = 1$ *and* $i \not\sim z$ *then* $C_G(i) = C_G(W) \cdot O_{2'}(C_G(i))$.

Proof. Let $C = C_G(i)$, $T_0 = C_T(i)$. Since $i \not\sim z$ it follows that T_0 is an S_2 subgroup of C and $W = \Omega_1(Z(T_0))$. Let $H_0 = T_0 \cap O_{2',2}(C)$, $C_0 = N_C(H_0)$ and let W^* be the normal closure of W in C_0. Assume that $W^* = W$. Then

$W \lhd C_0$ and so $W \subseteq Z(C_0)$. Since $C = C_0 \cdot O_{2'}(C)$ we get $C = C(W) \cdot O_{2'}(C)$ and the lemma is proved. Hence we may assume $W^* \supset W$. Let

$$D = O_2(C_0 \bmod C_{C_0}(W^*))$$

and so $D \cap T_0$ is an S_2 subgroup of D. This implies that

$$D = (D \cap T_0) \cdot C_{C_0}(W^*)$$

and so $W \subseteq Z(D)$. Thus $W^* \subseteq Z(D)$ and so we get $D = C_{C_0}(W^*)$ i.e. we have proved $O_2(A_{C_0}(W^*)) = 1$. Since $C_{C_0}(W^*)$ does not contain an S_2 subgroup of C_0, we get $A_{C_0}(W^*)$ has even order. Now we will show that $A_{C_0}(W^*)$ is solvable and then the same proof as in Lemma 3.1 [6] goes through completing the proof of this lemma. Assume $A_{C_0}(W^*)$ is not solvable. By (1.1) of [6] we have either $|W^*| = 8$ or $|W^*| = 16$. Case (i) $|W^*| = 8$. So $A_{C_0}(W^*)$ is isomorphic to a subgroup of $L_2(7)$. But since $A_{C_0}(W^*)$ is non solvable, so $A_{C_0}(W^*) \cong L_2(7)$. In this case the 7 element, which acts non trivially on W^*, conjugates i to z i.e. $i \sim z$ a contradiction. Hence we can assume $|W^*| = 16$. So $A_{C_0}(W^*)$ is isomorphic to a subgroup of $GL(4, 2)$. If $5 / |(A_{C_0}(W^*)|$ then the 5 element say α acts non trivially on W^*. Consider the cycle decomposition of α acting on W^*. Since α fixes i, α is either a 5 cycle or the product of two 5 cycles. So α fixes either 6 or 11 elements of W^*, which can't happen as $C_{W^*}(\alpha)$ is a subgroup of W^*. Now $A_{C_0}(W^*)$ is isomorphic to a non solvable subgroup of $GL(4, 2)$ and $|A_{C_0}(W^*)| \mid 2^6 \cdot 3^2 \cdot 7$. So $A_{C_0}(W^*) \cong L_2(7)$. Let $K = C_{C_0}(W^*)$ and $H = T_0 \cap K$. Suppose α, $\beta \in A_{C_0}(W^*)$ such that $|\alpha| = 7$, $|\beta| = 3$ and β normalizers $\langle\alpha\rangle$. Under the action of α, W^* splits as $L \times \langle i \rangle$, where α permutes the involutions in L. Since β normalizes $\langle\alpha\rangle$, β normalizes L. Claim $L \lhd C_0$. Suppose not, i.e. $1 < [C_0 \colon N_{C_0}(L)] \leqslant 8$. Since the Dihedral subgroup of order 21 is a maximal subgroup of $L_2(7)$ we get $[C_0 \colon N_{C_0}(L)] = 8$. Hence $[T_0 \colon N_{T_0}(L)] = 8$. So $\bigcap_{c \in C_0} L^c = \bigcap_{t \in T_0} L^t$. Since $\langle i, z \rangle \subseteq Z(T_0)$. We have $|\bigcap_{t \in T_0} L^t| \geqslant 2$. So $|\bigcap_{c \in C_0} L^c| = 2$ or 4 but then α centralizes $\bigcap_{c \in C_0} L^c$ which cannot happen. Hence $L \lhd C_0$. Let $t \in T - T_0$ and let $W_1 = W^{*t} \cdot W^* \cap W_1 \lhd T$ and $W \subseteq W^* \cap W_1 \Rightarrow W = W^* \cap W_1$. Now W^*/W, W_1/W are normal subgroups fo T_0/W so $W^* \cdot W_1/W$ is an elementary abelian group so $\phi(W^* \cdot W_1) \subseteq W = \langle i, z \rangle$ but $\phi(W^* \cdot W_1)$ can't be of order 2, for otherwise $W^* \cdot W_1$ is generated by 5 elements, contradicting the result of McWilliams [7]. Hence $\phi(W^* \cdot W_1) = \langle i, z \rangle$. Also $L \lhd T_0$, so $L \lhd W^* \cdot W_1$ but i is a central involution in T_0. Therefore $W^* \cdot W_1/L$ is an elementary abelian group, so $\phi(W^* \cdot W_1) \subseteq L$ i.e. $\langle i, z \rangle = \phi(W^* \cdot W_1) \subseteq L$ i.e. $i \in L$, which is a contradiction. Hence $A_{C_0}(W^*)$ is solvable. Now the same proof as in [6] goes through.

LEMMA 2. *Suppose* $|Z| = 2$; $|U(T)| = 1$ *and* $i \not\sim_G z$. *If* V *is any subgroup of* $C(i)$ *which contains* W *then* V *centralizes* W.

Proof. Same as in [6].

LEMMA 3. *Suppose* $|Z| = 2$, $|U(T)| = 1$ *and* $\bar{W} \ntriangleleft \bar{N}$ *then* G *is isomorphic to* $U_3(3)$ *or to one of the Janko simple groups of order* 604,800 *or* 50,232,960.

Proof. Since $W \in U(T)$ and N is 2 constrained then $W \subseteq H$. Now the same argument as in [6] gives that H is of symplectic type. Since N is 2 constrained, $\bar{N}/\bar{H}$ acts faithfully on $\bar{H}$.

By [5] either $\bar{H}$ is 2-group of maximal class of $\bar{H} = H_1 * H_2$ where H_1 is extra special 2 group of order $2^{2m+1}/m \geqslant 1$ and H_2 is cyclic or of maximal class. Since the automorphism group of a 2-group of maximal class is solvable, then by [6] we can assume $\bar{H}$ is of the kind $H_1 * H_2$. If $m \geqslant 3$ then $|\bar{H}/\phi(\bar{H})| \geqslant 2^6$ contradicting MacWilliams [7]. So $m \leqslant 2$. If $m = 1$ then $\bar{N}/\bar{H}$ is solvable and the same proof as in [6] goes through. Furthermore if $m = 2$ and $|H_2| > 2$ then $|\bar{H}/\phi(\bar{H})| \geqslant 2^5$ contradicting McWilliams [7]. Hence $\bar{H} = H_1$ is extra special 2-group. By [7] we have $|\bar{H}| = 32$. So $\bar{H} \cong Q_8 * Q_8$ or $Q_8 * D_8$. In case $\bar{H} = Q_8 * Q_8$, $\bar{N}/\bar{H}$ is solvable and the same proof as in [6] goes through. Now we have $\bar{H} \cong Q_8 * D_8$. By Huppert [5], $\bar{N}/\bar{H}$ is isomorphic to a subgroup of $\mathscr{S}_5$ the symmetric group on 5 letters. If $\bar{N}/\bar{H}$ is non solvable then $\bar{N}/\bar{H} \cong \mathfrak{A}_5$ or $\mathscr{S}_5$. If $\bar{N}/\bar{H} \cong \mathfrak{A}_5$ then a Sylow 2 subgroup of $\bar{N}$ is isomorphic to a Sylow 2 subgroup of the Janko simple group of order 604,800, see Lemma 2.6[3]. But by Gorenstein and Harada [1]; G is isomorphic to one of the Janko simple groups of order 604,800 or 50,232,960. In the other case $\bar{N}/\bar{H} \cong \mathscr{S}_5$. By the Gorenstein and Harada Lemma 5.6, [3], there are no simple groups G with this property. Hence the result is proved by [6].

LEMMA 4. *Assume* $|T'| \neq 1$ *and* $|Z| = 4$, *then there is precisely one class of involutions in G and if z is an involution in G then* $C_G(z)/O_{2'}(C_G(z))$ *is isomorphic to the centralizer of an involution in* $U_3(4)$.

Proof. If $N_G(T) \neq T \cdot C_G(T)$, then same proof as in [6] proves the result. So we can assume $N_G(T) = T \cdot C_G(T)$. We will show in this case that there are no simple groups with these properties. We will show that $C_G(z)$ is solvable for every involution $z \in Z$. Then the same proof as in [6] completes the proof. Let $C = C_G(z)$. Let H be a sylow 2 subgroup of C. Then $C_T(H) \subseteq H$. By Thompson's result stated before $T \cong T_1 \times T_2$ where T_i is dihedral or semidihedral and $|T_i| \geqslant 16$. Hence $[T: H] \leqslant 4$. Assume C is not solvable since $C/O_{2',2}(C)$ acts faithfully on $H/\phi(H)$ and $|H/\phi(H)| \leqslant 2^4$, therefore $C/O_{2',2}(C)$ is isomorphic to a subgroup of $GL(4, 2)$ and so $C/O_{2'2}(C) \simeq \mathfrak{A}_5$. But $N(T) \neq T \cdot C_G(T)$ in $\mathfrak{A}_5$, a contradiction. Hence C is solvable.

MAIN THEOREM. *Let G be a nonabelian, finite simple group and T a sylow 2 subgroup of G. We assume that T has no normal elementary subgroups of order 8 and whenever t is an involution in T such that $[T: C_T(t)] \leqslant 2$ then $C_G(t)$ is 2-constrained. Then G is isomorphic to one of the simple groups $L_2(q)$, $\mathfrak{A}_7$ M_{11}, $L_3(3)$, $U_3(3)$, $U_3(4)$ or to one of the Janko simple groups of order* 604,800 *or* 50,232,960.

Proof. The same argument as in [6] shows that either G is isomorphic to one of the groups named in the theorem or T is isomorphic to a sylow 2 subgroup of $\hat{\mathfrak{A}}_{11}$. In the latter case by Gorenstein and Harada [2], $C_G(z) \simeq \hat{\mathfrak{A}}_{11}$ which is not 2-constrained which can not happen and proves the result.

ACKNOWLEDGMENTS

The author wishes to express his gratitude to Professor Z. Janko for suggesting this problem and for his help and encouragement and Professor S. K. Wong for helpful discussions in the preparation of this paper.

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