Weak Drazin Inverses

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ABSTRACT

A new type of generalized inverse is defined which is a weakened form of the Drazin inverse. These new inverses are called (d)-inverses. Basic properties of (d)-inverses are developed. It is shown that (d)-inverses are often easier to compute than Drazin inverses and can frequently be used in place of the Drazin inverse when studying systems of differential equations with singular coefficients or when studying Markov chains.

1. INTRODUCTION

Many applications of the Moore-Penrose inverse $A^{\dagger}$ of an $m\times n$ matrix A have been developed over the last several years. However, in some applications one can get by with a weaker type of inverse. For example, if one wishes to solve $Ax=b$, and the equation is consistent, then $x=A^{-}b$ is a solution for any (1)-inverse A^{-}. There is a trade-off here. The Moore-Penrose inverse has many properties, such as being unique, that a (1)-inverse does not have. On the other hand, it is often easier to find a (1)-inverse than the Moore-Penrose inverse.

The Drazin inverse has recently been shown to have applications in the theory of differential equations [5, 8], control theory [4, 9], numerical analysis [14], and Markov chains [13]. Many of its basic properties have been developed [3, 6, 8, 10, 11, 13, 15–17], and the extension to infinite matrices has been begun [2]. It can, however, be difficult to compute the Drazin inverse. One way to lessen this latter problem is to look for a generalized inverse that would play much the same role for A^{D} as the (1)-inverses play for $A^{\dagger}$. One would not expect such an inverse to be unique. It should, at least in some cases of interest, be easier to compute. Furthermore, it should

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be usable as a replacement for A^D in many of the applications as well as having additional applications of its own.

We shall proceed as follows: First, a particular application will be used to motivate our definition of a "weak Drazin inverse". Its basic properties will then be developed. An algorithm for computing weak Drazin inverses will be given. Finally, applications of our work will be presented.

We assume the reader is familiar with the basic properties of the Drazin inverse [1, pp. 169–180]. Just as the Drazin inverse may be defined over an arbitrary field, the inverses discussed in this paper may be defined for square matrices over an arbitrary field.

The range and null space of a matrix A are denoted $R(A)$ and $N(A)$ respectively. $\mathbf{C}^{m\times n}$ denotes the space of $m\times n$ matrices over the complex numbers.

2. DEFINITIONS AND BASIC PROPERTIES

Consider the difference equation

$$Ax_{m+1}=x_m, \qquad m\geqslant 0, \quad A\in\mathbf{C}^{n\times n}. \tag{1}$$

All solutions of (1) are of the form $x_m=(A^D)^mA^DAq$, $q\in\mathbf{C}^n$ [8]. As observed in [8], it is the fact that the Drazin inverse solves (1) that helps explain its applications to differential equations. We shall define an inverse so that it solves (1) when (1) is consistent. Note that in (1), we have $x_m=A^lx_{m+l}$ for $l\geqslant 0$. Thus if our inverse is to always solve (1) it must send $R(A^k)$, $k=\operatorname{Index}A$, onto itself and have its restriction to $R(A^k)$ the same as the inverse of A restricted to $R(A^k)$. That is, it provides the unique solution to $Ax=b$, $x\in R(A^k)$, when $b\in R(A^k)$.

DEFINITION 1. Suppose that $A\in\mathbf{C}^{n\times n}$ and $k=\operatorname{Index}A$. Then B is a weak Drazin inverse, denoted A^d, if

(*d*) $BA^{k+1}=A^k$.

B is called a projective weak Drazin inverse of A if B satisfies (d) and

(*p*) $R(BA)=R(AA^D)$.

B is called a commuting weak Drazin inverse of A if B satisfies (d) and

(*c*) $AB=BA$.

B is called a minimal rank weak Drazin inverse of A if B satisfies (d) and

(m) $\mathrm{Rank}(B) = \mathrm{Rank}(A^D)$.

DEFINITION 2. An $(i_1, \dots, i_m)$-inverse of A is a matrix B satisfying the properties listed in the m-tuple. Here $i_l \in \{1,2,3,4,d,m,c,p\}$. The integers 1, 2, 3, 4 represent the usual defining relations of the Moore-Penrose inverse. Properties d, m, c, p are as in Definition 1.

We shall only be concerned with properties $\{1,2,m,d,c,p\}$. Note that they are all invariant under a simultaneous similarity transformation of A and B. Also note that one could define a right weak (d)-inverse by $A^{k+1}B = A^k$, and get a theory analogous to that developed here.

THEOREM 1. *Suppose that* $A \in \mathbf{C}^{n\times n}$, $k = \mathrm{Index}\,A$. *Suppose* $T \in \mathbf{C}^{n\times n}$ *is a nonsingular matrix such that*

$$TAT^{-1} = \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}, \qquad C \text{ nonsingular}, \quad N^k = 0. \tag{2}$$

Then B *is a* (d)*-inverse of* A *if and only if*

$$TBT^{-1} = \begin{bmatrix} C^{-1} & X \\ 0 & Y \end{bmatrix}, \qquad X, Y \text{ arbitrary}. \tag{3}$$

B *is an* (m,d)*-inverse for* A *if and only if*

$$TBT^{-1} = \begin{bmatrix} C^{-1} & X \\ 0 & 0 \end{bmatrix}, \qquad X \text{ arbitrary}. \tag{4}$$

B *is a* (p,d)*-inverse of* A *if and only if*

$$TBT^{-1} = \begin{bmatrix} C^{-1} & X \\ 0 & Y \end{bmatrix}, \qquad X \text{ arbitrary}, \quad YN = 0. \tag{5}$$

B *is a* (c,d)*-inverse of* A *if and only if*

$$TBT^{-1} = \begin{bmatrix} C^{-1} & 0 \\ 0 & Y \end{bmatrix}, \qquad YN = NY. \tag{6}$$

B is a $(1,d)$*-inverse of A if and only if*

$$TBT^{-1}=\begin{bmatrix} C^{-1} & X \\ 0 & N^{-} \end{bmatrix},\qquad XN=0,\quad N^{-}\ a\ (1)\text{-inverse of } N. \tag{7}$$

B is a $(2,d)$*-inverse of A if and only if*

$$TBT^{-1}=\begin{bmatrix} C^{-1} & X \\ 0 & Y \end{bmatrix},\qquad YNY=Y,\quad XNY=0. \tag{8}$$

If TAT^{-1} *is nilpotent, then* (3)–(8) *are to be interpreted as the* $(2,2)$*-block in the matrix. If A is invertible, then all reduce to* A^{-1}.

Proof. Let A be written as in (2). That each of (3)–(8) is the required type of inverse is a straightforward verification. Suppose then that B is a (d)-inverse of A. The case when A is nilpotent or invertible is trivial, so assume that A is neither nilpotent nor invertible. Since B leaves $R(A^k)$ invariant, we have

$$TBT^{-1}=\begin{bmatrix} Z & X \\ 0 & Y \end{bmatrix} \tag{9}$$

for some Z,X,Y. Substituting (9) into (d) gives only $ZC^{k+1}=C^k$. Hence $Z=C^{-1}$ and (3) follows. Equation (4) is clear. Assume now that B satisfies (3). If B is an (p,d)-inverse, then

$$R\left(\begin{bmatrix} I & XN \\ 0 & YN \end{bmatrix}\right)=R\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right).$$

Thus (5) follows. If B is a (c,d)-inverse of A, then

$$\begin{bmatrix} I & CX \\ 0 & NY \end{bmatrix}=\begin{bmatrix} I & XN \\ 0 & YN \end{bmatrix}.$$

But then $C^kX=XN^k=0$ and (6) follows. Similarly, (7) and (8) follow from (3) and the definition of properties $\{1,2\}$. ■

Note that for any $A\in\mathbf{C}^{n\times n}$ there always exists a T so that (2) holds. Also note that any number $l\geqslant \operatorname{Index} A$ can be used in place of k in (d).

Observe that, in general, $A^kA^dA\neq A^k$ and $A^{k+1}A^d\neq A^k$. Also A^dA and AA^d are not always projections. However, AA^d and A^dA are both the identity on $R(A^k)$.

COROLLARY 1. *A^D is the unique (p,c,d)-inverse of A. A^D is also a $(2,p,c,d)$-inverse and is the unique $(2,c,d)$-inverse of A by definition.*

Proof. Suppose B is a (p,c,d)-inverse of A. Then B has the form (3) with $Y=0$ by (4) and $X=0$ by (6). ∎

COROLLARY 2. *Suppose that* $\text{Index}A=1$. *Then*

(i) *B is a $(1,d)$-inverse of A if and only if B is a (d)-inverse, and*
(ii) *B is a $(2,d)$-inverse of A if and only if B is an (m,d)-inverse.*

Note that the Scroggs-Odell inverse [17] is a $(1,2,d)$-inverse of A.

COROLLARY 3. *Suppose that* $\text{Index}A\geqslant 2$. *Then there are no $(1,c,d)$-inverses or $(1,p,d)$-inverses.*

Proof. Suppose that $\text{Index}A\geqslant 2$ and B is a $(1,c,d)$-inverse of A. Then by (3), (6), (7) we have $X=0$, $NYN=N$, and $NY=YN$. But then $N^{k-1}=0$ which is a contradiction. If B is a $(1,p,d)$-inverse we have by (3), (5), (7) that $X=0$, $Y=0$, and $N0N=N$, which is a contradiction. ∎

Most of the (d)-inverses are not spectral in the sense of [13], since no assumptions have been placed on $N(A),N(A^d)$. However,

COROLLARY 4. *The operation of taking (m,d)-inverses has the spectral mapping property. That is, λ is a nonzero eigenvalue for A if and only if $1/\lambda$ is a nonzero eigenvalue for the (m,d)-inverse B. Furthermore, the eigenspaces for λ and $1/\lambda$ are the same. Either both A and B have a zero eigenvalue or both are invertible. The zero eigenspaces need not be the same.*

Note that if λ is a nonzero eigenvalue of A, then $1/\lambda$ is an eigenvalue of A^d.

COROLLARY 5. *If $B_1,\dots,B_r$ are (d)-inverses of A, then $B_1B_2\cdots B_r$ is a (d)-inverse of A^r. In particular, $(A^d)^m$ is a (d)-inverse of A^m.*

Corollary 5 is not true for (1)-inverses. For

$$B=\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}$$

is a $(1,2)$-inverse of

$$A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},$$

but $B^2=0$ and hence B^2 is not a (1)-inverse of $A^2=A$. This is not surprising, as $(A^\dagger)^2$ may not be even a (1)-inverse of A^2 [7].

THEOREM 2. *Suppose that* $A\in\mathbf{C}^{n\times n}$, $\operatorname{Index}A=k$. *Then*

(i) $\{A^D+Z(I-A^DA)|Z\in\mathbf{C}^{n\times n}\}$ *is the set of all* (d)*-inverses of* A,

(ii) $\{A^D+A^DAZ(I-A^DA)|Z\in\mathbf{C}^{n\times n}\}$ *is the set of all* (m,d)*-inverses of* A,

(iii) $\{A^D+Z(I-A^DA)|ZA=AZ\}$ *is the set of all* (c,d)*-inverses of* A, *and*

(iv) $\{A^D+(I-A^DA)[A(I-A^DA)]^-|(I-A^DA)[A(I-A^DA)]^-A(I-A^DA)=0\}$ *is the set of all* $(1,d)$*-inverses of* A.

Proof. (i)–(iv) follow from Theorem 1. We have omitted the (p,d) and $(2,d)$-inverses, since they are about as unappealing as (iv). ■

Just as it is possible to calculate $A^\dagger$ given an A^-, one may calculate A^D from any A^d.

COROLLARY 6. *If* $k=\operatorname{Index}A$, *then* $A^D=(A^d)^{l+1}A^l$ *for any* $l\geqslant k$.

3. BLOCK TRIANGULAR MATRICES

It is known [15] that if C,E are square, then

$$\begin{bmatrix} C & F \\ 0 & E \end{bmatrix}^D=\begin{bmatrix} C^D & X \\ 0 & E^D \end{bmatrix}, \tag{10}$$

$$X=(C^D)^2\left(\sum_{n=0}^{l-1}(C^D)^nFE^n\right)(I-EE^D)$$

$$+(I-C^DC)\left(\sum_{n=0}^{k-1}C^nF(E^D)^n\right)(E^D)^2-C^DFE^D,$$

$$l=\operatorname{Index}C,\qquad k=\operatorname{Index}E.$$

In this section we shall develop similar results for weak Drazin inverses. A useful special case of (10) is when C is invertible.

THEOREM 3. *Suppose that*

$$A \in \mathbf{C}^{n\times n} \quad \text{and} \quad A = \begin{bmatrix} C & D \\ 0 & E \end{bmatrix},$$

where C is invertible. Then all (d)-inverses of A are given by

$$A^d = \begin{bmatrix} C^{-1} & -C^{-1}DE^d + Z\left(I - E^{\tilde{d}}E\right) \\ 0 & E^d \end{bmatrix}, \tag{11}$$

E^d any (d)-inverse of E, $E^{\tilde{d}}$ an (m,d)-inverse of E, Z an arbitrary matrix of the correct size.

Proof. Suppose

$$A = \begin{bmatrix} C & D \\ 0 & E \end{bmatrix}$$

with C invertible. Let $k = \operatorname{Index} A = \operatorname{Index} E$. Then

$$A^k = \begin{bmatrix} C^k & \Theta \\ 0 & E^k \end{bmatrix},$$

where Θ is some matrix. Now the range of $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ is in $R(A^k)$. Hence A^D and any A^d agree on it. Thus

$$A^d = \begin{bmatrix} C^{-1} & X_1 \\ 0 & X_2 \end{bmatrix}. \tag{12}$$

Now suppose (12) is a (d)-inverse of A. Then $AA^dA^k = A^k$. Hence

$$\begin{bmatrix} C & D \\ 0 & E \end{bmatrix} \begin{bmatrix} C^{-1} & X_1 \\ 0 & X_2 \end{bmatrix} \begin{bmatrix} C^k & \Theta \\ 0 & E^k \end{bmatrix} = \begin{bmatrix} C^k & \Theta \\ 0 & E^k \end{bmatrix}. \tag{13}$$

Thus

$$\begin{bmatrix} I & CX_1 + DX_2 \\ 0 & EX_2 \end{bmatrix} \begin{bmatrix} C^k & \Theta \\ 0 & E^k \end{bmatrix} = \begin{bmatrix} C^k & \Theta \\ 0 & E^k \end{bmatrix},$$

or

$$\Theta+(CX_1+DX_2)E^k=\Theta, \tag{14}$$

$$EX_2E^k=E^k. \tag{15}$$

If $A^dA^{k+1}=A^k$ is to hold, one must have X_2 a (d)-inverse of E. Let $X_2=E^d$ for some (d)-inverse of E. Then (15) holds. Now (14) becomes $X_1E^k=-C^{-1}DE^dE^k$. Let E^d be a (m,d)-inverse of E. Then E^dE is a projection onto $R(E^k)$. Hence X_1 must be of the form $-C^{-1}DE^d+Z(I-E^dE)$, and (11) follows. To see that (11) defines a (d)-inverse of A is a direct computation. ■

It should be pointed out that while $BA^{k+1}=A^k$ implies $ABA^k=A^k$, the two conditions are not equivalent.

COROLLARY 7. *Suppose there exists an invertible T such that*

$$TAT^{-1}=\begin{bmatrix} C & X \\ 0 & N \end{bmatrix} \tag{16}$$

with C invertible and N nilpotent. Then

$$A^d=T^{-1}\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}T$$

is a (m,d)-inverse for A.

If one wanted A^D from (16) it would be given by the more complicated expression [15]

$$TA^DT^{-1}=\begin{bmatrix} C^{-1} & \tilde{X} \\ 0 & 0 \end{bmatrix}, \qquad \tilde{X}=C^{-2}\left(\sum_{n=0}^{k-1} C^{-k}XN^K\right).$$

4. COMPUTATION OF WEAK DRAZIN INVERSES

As observed in the preceeding section, if one has a block triangular matrix it is easier to compute a weak Drazin, than a Drazin inverse. However, in practice one frequently does not begin with a block triangular matrix. We shall now give two results which have analogues for the Drazin inverse, but work out much simpler for computing a weak Drazin inverse.

THEOREM 4. *Suppose that* $A \in \mathbf{C}^{n\times n}$ *and that* $p(x)=x^l(c_0+\cdots+c_rx^r)$, $c_0\neq 0$, *is the characteristic (or minimal) polynomial of* A. *Then*

$$A^d=-\frac{1}{c_0}\left(c_1I+\cdots+c_rA^{r-1}\right) \tag{17}$$

is a (c,d)*-inverse of* A. *If* (17) *is not invertible, then* $A^d+(I-A^dA)$ *is an invertible* (c,d)*-inverse of* A.

Proof. Since $p(A) = 0$, we have $(c_0I+\cdots+c_rA^r)A^l = 0$. Hence $(c_1I+\cdots+c_rA^{r-1})A^{l+1}=-c_0A^l$. Since $\operatorname{Index}A\leqslant l$, we have that (17) is a (d)-inverse. It is commuting, since it is a polynomial in A. Now let A be as in (2). Then since A^d is a (c,d)-inverse, it is in the form (6). But then $Y=-(1/c_0)(c_1I+c_2N+\cdots+c_rN^{r-1})$. If $c_1\neq 0$, then Y is invertible, since N is nilpotent, and we are done. Suppose that $c_1=0$. Then

$$A^d+(I-A^dA)=\begin{bmatrix} C^{-1} & 0\\ 0 & I-YN+Y\end{bmatrix},$$

and $I-YN+Y$ is invertible, since $Y-YN$ is nilpotent. That $A^d+(I-A^dA)$ is a (c,d)-inverse follows from the fact that A^d is a (c,d)-inverse. ■

Note that Theorem 4 requires no information on eigenvalues or their multiplicities to calculate a (c,d)-inverse. If A had rational entries, (17) would provide an exact answer if exact arithmetic were used.

Theorem 4 suggests that a variant of the Souriau-Frame algorithm could be used to compute (c,d)-inverses. In fact, the algorithm goes through almost unaltered.

THEOREM 5. *Suppose that* $A\in\mathbf{C}^{n\times n}$. *Let* $B_0=I$. *For* $j=1,2,\dots,n$, *let* $p_j=(1/j)\operatorname{Trace}(AB_{j-1})$ *and* $B_j=AB_{j-1}-p_jI$. *If* $p_s\neq 0$, *but* $p_{s+1}=p_{s+2}=\cdots=p_n=0$, *then*

$$A^d=\frac{1}{p_s}B_{s-1} \tag{18}$$

is a (c,d)*-inverse. In fact,* (17) *and* (18) *are the same matrix.*

Proof. Let $k = \operatorname{Index} A$. Observe that $B_j = A^j - p_1 A^{j-1} - p_2 A^{j-2} - \cdots - p_j I$. If r is the smallest integer such that $B_r = 0$, and s is the largest integer such that $p_s \neq 0$, then Index $A = r - s$ [16]. Since $B_r = 0$, we have $A^r = p_1 A^{r-1} - \cdots - p_{s-1} A^{r-s+1} - p_s A^{r-s} = 0$. Hence,

$$A^{r-s} = \frac{1}{p_s}\left(A^r - p_1 A^{r-1} - \cdots - p_{s-1} A^{r-s+1}\right)$$

$$= \frac{1}{p_s}\left(A^{s-1} - p_1 A^{s-2} - \cdots - p_{s-1} I\right) A^{r-s+1}.$$

That is,

$$A^k = \left(\frac{1}{p_s} B_{s-1}\right) A^{k+1}$$

as desired. ∎

5. APPLICATIONS

LEMMA 1. *Suppose $A, B \in \mathbf{C}^{n\times n}$ and $AB = BA$. Let A^d be any (d)-inverse of A. Then $A^d BAA^D = BA^d AA^D = BA^D AA^D = A^D BAA^D$.*

Proof. If $AB = BA$, then $A^D B = BA^D$. Also if A is given by (2), then

$$TBT^{-1} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix},$$

with $B_1 C = CB_1$. Lemma 1 now follows from Theorem 1. ∎

As an immediate consequence of Lemma 1, one may use a (d)-inverse in [4], [8] provided the appropriate projection is on the right. For example, we have

THEOREM 6. *Suppose that $A, B \in \mathbf{C}^{n\times n}$. Suppose that $A\dot{x} - Bx = 0$ has unique solutions for consistent initial conditions, that is, there is a scalar c such that $cA + B$ is invertible. Let $\hat{A} = (cA + B)^{-1}A$, $\hat{B} = (cA + B)^{-1}B$. Let $k = \operatorname{Index} \hat{A}$. If $A\dot{x} - Bx = 0$, $x(0) = q$, is consistent, then the solution is $x = e^{\hat{A}^d \hat{B} t} q$. If $\hat{A}^d$ is an (m,d)-inverse of $\hat{A}$, then all solutions of $A\dot{x} - Bx = 0$ are of the form $x = e^{\hat{A}^d \hat{B} t} \hat{A}\hat{A}^d q$, $q \in \mathbf{C}^n$, and the space of consistent initial conditions is $R(\hat{A}\hat{A}^d) = R(\hat{A}^d \hat{A})$.*

Note in Theorem 6 that $\hat{A}\hat{A}^d$ need not equal $\hat{A}^d\hat{A}$ even if $\hat{A}^d$ is an (m,d)-inverse of A.

THEOREM 7. *Let $A, B, \hat{A}, \hat{B}$, be as in Theorem 6. If*

$$Ax_{m+1} = Bx_m, \qquad m \geqslant 0, \tag{19}$$

is consistent for $x_0 = q$, then the solution is $x_m = (\hat{A}^d\hat{B})^m q = (\hat{A}^d)^m \hat{B}^m q$. If $\hat{A}^d$ is an (m,d)-inverse of $\hat{A}$, then all solutions of (19) are given by $x_m = (\hat{A}^d\hat{B})^m \hat{A}^d\hat{A}q$, and the space of consistent initial conditions is $R(\hat{A}^d\hat{A})$.

Weak Drazin inverses can also be used in the theory of Markov chains. For example, we have

THEOREM 8. *If T is the transition matrix of an m-state ergodic chain and if $A = I - T$, then the rows of $I - A^dA$ are all equal to the unique fixed probability vector w^* of T for any (d)-inverse of A.*

Proof. The rows of $I - AA^D$ are all equal to w^* [13, Theorem 2.3]. From (3) we have $AA^D = A^dA = A^{\#}A$, since Index$A = 1$ [13, Theorem 2.1]. Here $^{\#}$ denotes the group inverse of A. ∎

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