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Quick cut-elimination for strictly positive cuts

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In this paper we show that the intuitionistic theory $\widehat{ID}^i_{<\omega}(SP)$ for finitely many iterations of strictly positive operators is a conservative extension of Heyting arithmetic. The proof is inspired by the quick cut-elimination due to G. Mints. This technique is also applied to fragments of Heyting arithmetic.

1. Introduction

Let us consider in this paper the fixed point predicate $I(x)$ for a positive formula $\Phi(X, x)$:

$$(FP)^{\Phi} \quad \forall x[I(x) \leftrightarrow \Phi(I, x)]. \tag{1}$$

Buchholz [3] showed that an intuitionistic fixed point theory $\widehat{ID}^i(\mathcal{M})$ is conservative over Heyting arithmetic HA with respect to almost negative formulas, (in which $\vee$ does not occur and $\exists$ occurs in front of atomic formulas only). The theory $\widehat{ID}^i(\mathcal{M})$ has the axioms (1) $(FP)^{\Phi}$ for fixed points for *monotone formulas* $\Phi(X, x)$, which are generated from arithmetic atomic formulas and $X(t)$ by means of (first order) monotonic connectives $\vee, \wedge, \exists, \forall$. Namely neither $\to$ nor $\neg$ occur in monotone formulas. The proof is based on a recursive realizability interpretation.

After seeing the result of Buchholz, we [1] showed that an intuitionistic fixed point (second order) theory is conservative over HA for all arithmetic formulas. In the theory the operator Φ for fixed points is generated from $X(t)$ and any second order formulas by means of first order monotonic connectives and second order existential quantifiers $\exists f (\in \omega \to \omega)$. Moreover the same holds for the finite iterations of these operations. The proof is based on Goodman's theorem [5].

Next, Rüede and Strahm [8] extend significantly the results in [3] and [1]. They showed that the intuitionistic fixed point theory $\widehat{ID}^i_{<\omega}(SP)$ for finitely many iterations of *strictly positive* operators is conservative over HA with respect to negative and Π^0_2-formulas.

In this paper we show a full result. Let L be a language obtained from the language L_{HA} of HA by adding unary predicate symbols $P, \ldots,$ and HA_L, the Heyting arithmetic in the expanded language L. In other words, the induction axioms are available for any L-formulas in HA_L. $\widehat{ID}^i(SP_L)$ denotes the intuitionistic fixed point theory for strictly positive operators in the language L.

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Theorem 1. *$\widehat{ID}^i(SP_L)$ is a conservative extension of* HA_L.

Let $\widehat{ID}^i_n(SP)$ denote the intuitionistic fixed point theory for n-fold iterations of strictly positive operators. $\widehat{ID}^i_0(SP)$ is another name for HA. Theorem 1 yields the following Corollary 2.

Corollary 2. *For each $n < \omega$, $\widehat{ID}^i_{n+1}(SP)$ is a conservative extension of $\widehat{ID}^i_n(SP)$, and $\widehat{ID}^i_{<\omega}(SP)$ is a conservative extension of* HA.

Our proof is based on a quick cut-elimination of strictly positive cuts with arbitrary antecedents, cf. Theorem 8. The proof is inspired by G. Mints' quick cut-elimination of monotone cuts in [7], and was found in an attempt to clarify ideas in [2].

Let us explain an idea of our proof more closely. The story is essentially the same as in [2]. First the finitary derivations in $\widehat{ID}^i(SP_L)$ are embedded to infinitary derivations, and eliminate cuts partially. This results in an infinitary derivation of depth less than ε_0, in which there occur cut inferences with cut formulas $I^{\Phi}(t)$ for fixed points only. Now the constraint on operator Φ admits us to eliminate strictly positive cut formulas quickly. In this way we will get an infinitary derivation of a depth less than ε_0, in which there occur no fixed point formulas.

By formalizing the arguments we see that the end formula is true in HA_L.

In Section 5 we show that monotone cuts with negative antecedents can be eliminated more quickly. In the final Section 6 these techniques are applied to fragments of Heyting arithmetic.

2. An intuitionistic theory $\widehat{ID}^i(SP_L)$

L_{HA} denotes the language of Heyting arithmetic. Logical connectives are $\vee, \wedge, \to, \exists, \forall$. $\neg A :\equiv (A \to \bot)$. Let L be a language obtained from the language L_{HA} by adding unary predicate symbols $P, \ldots$. Let I be a fresh unary predicate symbol not in L, and let $L(I)$ denote $L \cup \{I\}$.

Let SP_L be the class of $L(I)$-formulas such that $A \in SP_L$ iff I occurs only strictly positive in A. The class SP_L is defined inductively.

Definition 3. Define inductively a class of formulas SP_L in $L(I)$ as follows.

1. Any atomic formula in L belongs to SP_L.
2. Any atomic formula $I(t)$ belongs to the class SP_L.
3. If $R, S \in SP_L$, then $R \vee S$, $R \wedge S$, $\exists x R$, $\forall x R \in SP_L$.
4. If $L \in L$ and $R \in SP_L$, then $L \to R \in SP_L$.

Let $\widehat{ID}^i(SP_L)$ denote the following extension of HA_L. Its language is obtained from L by adding a unary set constant I for a $\Phi \equiv \Phi(I, x) \in SP_L$, in which only a fixed variable x occurs freely. Its axioms are those of HA_L in the expanded language (i.e., the induction axioms are available for any formulas in the expanded language $L(I)$) plus the axiom $(FP)^{\Phi}$.

3. Infinitary derivations

Given an $\widehat{ID}^i(SP_L)$-derivation D_0 of an L-sentence C_0, let us first transfer it to an infinitary derivation in an infinitary calculus $\widehat{ID}^{i\infty}(SP_L)$.

Let N denote a number which is big enough so that any formula occurring in D_0 has logical complexity (which is defined by the number of occurrences of logical connectives) smaller than N. In what follows, any formula occurring in infinitary derivations we are concerned with has logical complexity less than N.

The derived objects in the calculus $\widehat{ID}^{i\infty}(SP_L)$ are *sequents* $\Gamma \Rightarrow A$, where A is a *sentence* (in the language of $\widehat{ID}^i(SP_L)$) and Γ denotes a finite set of *sentences*, where each closed term t is identified with its value $\bar{n}$, the nth numeral.

$\bot$ stands ambiguously for false equations $t = s$ with closed terms t, s having different values. $\top$ stands ambiguously for true equations $t = s$ with closed terms t, s having equal values.

The *initial sequents* are

$$\Gamma, P(t) \Rightarrow P(t); \qquad \Gamma, \bot \Rightarrow A; \qquad \Gamma \Rightarrow \top$$

for predicate symbols $P \in (L(I) \setminus L_{HA})$.

The *inference rules* are $(L\vee)$, $(R\vee)$, $(L\wedge)$, $(R\wedge)$, $(L \to)$, $(R \to)$, $(L\exists)$, $(R\exists)$, $(L\forall)$, $(R\forall)$, (LI), (RI) and (cut). These are the standard ones.

1.
$$\frac{\Gamma, I(t), \Phi(I,t) \Rightarrow C}{\Gamma, I(t) \Rightarrow C}\ (LI)\ ; \quad \frac{\Gamma \Rightarrow \Phi(I,t)}{\Gamma \Rightarrow I(t)}\ (RI)$$

2.
$$\frac{\Gamma, A_0 \vee A_1, A_0 \Rightarrow C \quad \Gamma, A_0 \vee A_1, A_1 \Rightarrow C}{\Gamma, A_0 \vee A_1 \Rightarrow C}\ (L\vee)\ ; \quad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1}\ (R\vee)\ (i = 0, 1)$$

3.
$$\frac{\Gamma, A_0 \wedge A_1, A_i \Rightarrow C}{\Gamma, A_0 \wedge A_1 \Rightarrow C}\ (L\wedge)\ (i=0,1)\ ; \quad \frac{\Gamma \Rightarrow A_0 \quad \Gamma \Rightarrow A_1}{\Gamma \Rightarrow A_0 \wedge A_1}\ (R\wedge)$$

4.
$$\frac{\Gamma, A \to B \Rightarrow A \quad \Gamma, A \to B, B \Rightarrow C}{\Gamma, A \to B \Rightarrow C}\ (L\to)\ ; \quad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\ (R\to)$$

5.
$$\frac{\cdots \quad \Gamma, \exists x B(x), B(\bar{n}) \Rightarrow C \quad \cdots (n \in \omega)}{\Gamma, \exists x B(x) \Rightarrow C}\ (L\exists)\ ; \quad \frac{\Gamma \Rightarrow B(\bar{n})}{\Gamma \Rightarrow \exists x B(x)}\ (R\exists)$$

6.
$$\frac{\Gamma, \forall x B(x), B(\bar{n}) \Rightarrow C}{\Gamma, \forall x B(x) \Rightarrow C}\ (L\forall)\ ; \quad \frac{\cdots \quad \Gamma \Rightarrow B(\bar{n}) \quad \cdots (n \in \omega)}{\Gamma \Rightarrow \forall x B(x)}\ (R\forall)$$

7.
$$\frac{\Gamma \Rightarrow A \quad \Gamma, A \Rightarrow C}{\Gamma \Rightarrow C}\ (cut).$$

The *depth* of an infinitary derivation is defined to be the depth of the well founded tree.

As usual we see the following proposition. Recall that N is an upper bound of logical complexities of formulas occurring in the given finite derivation D_0 of L-sentence C_0.

Proposition 4. 1. *There exists an infinitary derivation D_1 of C_0 such that its depth is less than ω^2 and the logical complexity of any sentence, in particular cut formulas occurring in D_1, is less than N.*

2. *By partial cut-elimination, there exists an infinitary derivation D_2 of C_0 and an ordinal $\alpha_0 < \varepsilon_0$ such that the depth of the derivation D_2 is less than α_0 and any cut formula occurring in D_2 is an atomic formula $I(t)$, (and the logical complexity of any formula occurring in it is less than N).*

Definition 5. The rank $rk(A)$ of a sentence A is defined by

$$rk(A) := \begin{cases} 0 & \text{if } A \in L \\ 1 & \text{if } A \in (SP_L \setminus L) \\ 2 & \text{otherwise.} \end{cases}$$

Let us call a cut inference *L-cut* [SP_L-*cut*] if its cut formula is of rank 0 [of rank 1], resp.

Let $\vdash^{\alpha}_{r} \Gamma \Rightarrow C$ mean that there exists an infinitary derivation of $\Gamma \Rightarrow C$ such that its depth is at most α, and all its cut formulas have rank less than r, (and the logical complexity of any formula occurring in it is less than N).

The following lemmas are seen as usual.

Lemma 6 (*Weakening Lemma*). *If $\vdash^{\alpha}_{1} \Gamma \Rightarrow A$ and $\beta \geq \alpha$, then $\vdash^{\beta}_{1} \Delta, \Gamma \Rightarrow A$.*

Lemma 7 (*Inversion Lemma*). *Assume $\vdash^{\alpha}_{1} \Gamma \Rightarrow A$.*

1. *If $A \equiv \bot$, then $\vdash^{\alpha}_{1} \Gamma \Rightarrow C$ for any C.*
2. $\vdash^{\alpha}_{1} (\Gamma \setminus \{\top\}) \Rightarrow A$.

Let $3_2(\beta) := 3^{3^{\beta}}$.

Theorem 8. *Suppose that $\vdash^{\beta}_{2} \Gamma \Rightarrow C$. Then $\vdash^{3_2(\beta)}_{1} \Gamma \Rightarrow C$.*

Assuming Theorem 8, we can show Theorem 1 as follows. Suppose an L-sentence C_0 is provable in $\widehat{ID}^{i}(SP_L)$. By Proposition 4 we have $\vdash^{\alpha_0}_{2} \Rightarrow C_0$ for a big enough number N and an $\alpha_0 < \varepsilon_0$. Then Theorem 8 yields $\vdash^{\beta_0}_{1} \Rightarrow C_0$ for $\beta_0 = 3_2(\alpha_0) < \varepsilon_0$.

Let $\mathrm{Tr}_N(x)$ denote a partial truth definition for formulas of logical complexity less than N. By transfinite induction up to β_0 we see $\mathrm{Tr}_N(C_0)$. Note that any sentence occurring in the witnessing derivation for $\vdash^{\beta_0}_{1} \Rightarrow C_0$ has logical complexity less than N, and it is an L-sentence. Specifically there occurs no fixed point formula $I(t)$ in it. Now since everything up to this point is formalizable in HA_L, we have $\mathrm{Tr}_N(C_0)$, and hence C_0 in HA_L. This shows Theorem 1.

Additional information equipped with infinitary derivations together with the repetition rule (*Rep*)

$$\frac{\Gamma \Rightarrow C}{\Gamma \Rightarrow C}\ (Rep)$$

is helpful when we formalize our proof as in [6]. In this paper let us suppress these.

A proof of Theorem 8 is given in the next section.

4. Quick cut-elimination of strictly positive cuts with arbitrary antecedents

In this section we show that strictly positive cuts can be eliminated quickly even if antecedents of cut inferences and endsequents are arbitrary formulas. The only constraint is that any cut formula has to be strictly positive.

Let $\alpha\#\beta$ denote the *natural sum* or commutative sum, $\alpha\#\beta = \beta\#\alpha$, and $\alpha \times \beta$ the *natural product*.

Theorem 8 follows from the following Lemma 9.

As in Lemma 3.2, [7] the elimination procedure is fairly standard, leaving the resulted cut inferences of rank 0, but it has to perform in parallel.

$\boldsymbol{A}$ denotes a finite list $A_k, \dots, A_2, A_1$ ($k \geq 0$) of *SP*-formulas, and $\boldsymbol{\alpha} = \alpha_k, \dots, \alpha_2, \alpha_1$ a list of ordinals. Then $\vdash_1^{\boldsymbol{\alpha}} \Gamma \Rightarrow \boldsymbol{A}$ designates that $\vdash_1^{\alpha_i} \Gamma \Rightarrow A_i$ for each $i \in \{1, \dots, k\}$.

$$\sum \boldsymbol{\alpha} := 1\#\alpha_1\#\cdots\#\alpha_k.$$

$\boldsymbol{A}_1$ denotes the list $A_k, \dots, A_2$, in which A_1 is deleted. Likewise $\boldsymbol{\alpha}_1$ denotes the list $\alpha_k, \dots, \alpha_2$.

Lemma 9. *Suppose* $\vdash_1^{\boldsymbol{\alpha}} \Gamma \Rightarrow \boldsymbol{A}$ *and* $\vdash_2^{\beta} \Delta, \boldsymbol{A} \Rightarrow C$ *with* $rk(A_i) \leq 1$ *for* $i = 1, \dots, k$. *Then*

$$\vdash_1^{(\sum \boldsymbol{\alpha})\times 3_2(\beta)} \Delta, \Gamma \Rightarrow C \tag{2}$$

Note that the case $k = 0$ in Lemma 9 is nothing but Theorem 8.

We prove Lemma 9 by the main induction on β with subsidiary induction on $\sum \boldsymbol{\alpha} + k$, where k is the length of the list $\boldsymbol{A}$.

1. The case when one of $\Gamma \Rightarrow A_i$, and $\Delta, \boldsymbol{A} \Rightarrow C$ is an initial sequent.

   First consider the case when $\Delta, \boldsymbol{A} \Rightarrow C$ is an initial sequent.

   If $\Delta, \boldsymbol{A} \Rightarrow C$ is an initial sequent such that one of the cases $C \equiv \top, \bot \in \Delta$ or $C \in \Delta$ occurs, then $\Delta \Rightarrow C$, and hence $\Delta, \Gamma \Rightarrow C$ is still the same kind of initial sequent.

   If $\Delta, \boldsymbol{A} \Rightarrow C$ is an initial sequent with the principal formula $\boldsymbol{A} \ni A_i \equiv C$, then $\vdash_1^{(\sum \boldsymbol{\alpha})\times 3_2(\beta)} \Delta, \Gamma \Rightarrow A_i (\equiv C)$ follows by weakening from the premise $\vdash_1^{\alpha_i} \Gamma \Rightarrow A_i$ and $(\sum \boldsymbol{\alpha}) \times 3_2(\beta) \geq \alpha_i$.

   If $A_i \equiv \bot$, then Inversion Lemma 7.1 with a weakening yields $\vdash_1^{\alpha_i} \Delta, \Gamma \Rightarrow C$.

   Next assume $\Gamma \Rightarrow A_i$ is an initial sequent for an i. This implies $k > 0$. For simplicity assume $i = 1$.

   If $A_1 \in \Gamma$, then by SIH (=Subsidiary Induction Hypothesis) we have $\vdash_1^{(\sum \boldsymbol{\alpha}_1)\times 3_2(\beta)} \Delta, A_1, \Gamma \Rightarrow C$ with $A_1 \in \Gamma$ and $(\sum \boldsymbol{\alpha}_1) \times 3_2(\beta) \leq (\sum \boldsymbol{\alpha}) \times 3_2(\beta)$.

   If $\bot \in \Gamma$, then $\Delta, \Gamma \Rightarrow C$ is an initial sequent.

   If $A_1 \equiv \top$, then Inversion Lemma 7.2 yields $\vdash_2^{\beta} \Delta, \boldsymbol{A}_1 \Rightarrow C$, and by SIH $\vdash_1^{(\sum \boldsymbol{\alpha}_1)\times 3_2(\beta)} \Delta, \Gamma \Rightarrow C$.

   In what follows assume that none of $\Gamma \Rightarrow A_i$, and $\Delta, \boldsymbol{A} \Rightarrow C$ is an initial sequent.
2. Consider the case when $\Delta, \boldsymbol{A} \Rightarrow C$ is a lowersequent of an SP_L-cut. For a $\gamma < \beta$

$$\dfrac{\vdash_1^{\boldsymbol{\alpha}} \Gamma \Rightarrow \boldsymbol{A} \qquad \dfrac{\vdash_2^{\gamma} \Delta, \boldsymbol{A} \Rightarrow A_0 \quad \vdash_2^{\gamma} \Delta, \boldsymbol{A}, A_0 \Rightarrow C}{\vdash_2^{\beta} \Delta, \boldsymbol{A} \Rightarrow C}\,(cut)}{\Delta, \Gamma \Rightarrow C}$$

   with $rk(A_0) = 1$.

   MIH(=Main Induction Hypothesis) yields $\vdash_1^{(\sum \boldsymbol{\alpha})\times 3_2(\gamma)} \Delta, \Gamma \Rightarrow A_0$, and once again by MIH and

$$\left(\sum \boldsymbol{\alpha}\# \left(\sum \boldsymbol{\alpha}\right) \times 3_2(\gamma)\right) \times 3_2(\gamma) \leq \left(\sum \boldsymbol{\alpha}\right) \times 3_2(\beta)$$

   we conclude $\vdash_1^{(\sum \boldsymbol{\alpha})\times 3_2(\beta)} \Delta, \Gamma \Rightarrow C$.

   We will depict a 'derivation' to illustrate the arguments.

$$\dfrac{\vdash_1^{\boldsymbol{\alpha}} \Gamma \Rightarrow \boldsymbol{A} \qquad \dfrac{\vdash_1^{\boldsymbol{\alpha}} \Gamma \Rightarrow \boldsymbol{A} \quad \vdash_2^{\gamma} \Delta, \boldsymbol{A} \Rightarrow A_0}{\vdash_1^{(\sum \boldsymbol{\alpha})\times 3_2(\gamma)} \Delta, \Gamma \Rightarrow A_0}\,\text{MIH} \qquad \vdash_2^{\gamma} \Delta, \boldsymbol{A}, A_0 \Rightarrow C}{\vdash_1^{(\sum \boldsymbol{\alpha})\times 3_2(\beta)} \Delta, \Gamma \Rightarrow C}\,\text{MIH}.$$

   In what follows assume that $\Delta, \boldsymbol{A} \Rightarrow C$ is the lower sequent of an inference rule J other than an SP_L-cut.
3. If the principal formula of J if any is not in $\boldsymbol{A}$, then lift up the left upper part: for a $\gamma < \beta$

$$\dfrac{\vdash_1^{\boldsymbol{\alpha}} \Gamma \Rightarrow \boldsymbol{A} \qquad \dfrac{\cdots \quad \vdash_2^{\gamma} \Delta_i, \boldsymbol{A} \Rightarrow C_i \quad \cdots}{\vdash_2^{\beta} \Delta, \boldsymbol{A} \Rightarrow C}\,(J)}{\Delta, \Gamma \Rightarrow C}$$

$$\dfrac{\cdots \quad \dfrac{\vdash_1^{\boldsymbol{\alpha}} \Gamma \Rightarrow \boldsymbol{A} \quad \vdash_2^{\gamma} \Delta_i, \boldsymbol{A} \Rightarrow C_i}{\vdash_1^{(\sum \boldsymbol{\alpha})\times 3_2(\gamma)} \Delta_i, \Gamma \Rightarrow C_i}\,\text{MIH} \quad \cdots}{\vdash_1^{(\sum \boldsymbol{\alpha})\times 3_2(\beta)} \Delta, \Gamma \Rightarrow C}\,(J).$$

Note that $(\sum \boldsymbol{\alpha}) \times 3_2(\gamma) < (\sum \boldsymbol{\alpha}) \times 3_2(\beta)$, since by the definition $(\sum \boldsymbol{\alpha}) > 0$.

4. Finally suppose that the principal formula of J is a cut formula $A_i \in \boldsymbol{A}$ with $rk(A_i) \leq 1$. For simplicity suppose $i = 1$, and let J' denote the last rule in $\vdash_1^{\alpha_1} \Gamma \Rightarrow A_1$.
   (a) J' is a left rule or a cut.
      i. The case when J' is an $(L\exists)$ with an $\exists y D(y) \in \Gamma$.

$$\dfrac{\cdots \quad \vdash_1^{\alpha_0} \Gamma, D(\bar{n}) \Rightarrow A_1 \quad \cdots}{\vdash_1^{\alpha_1} \Gamma \Rightarrow A_1}\,(L\exists).$$

Then $\alpha_0 < \alpha_1$, and hence $\sum \boldsymbol{\alpha}_1 \# \alpha_0 < \sum \boldsymbol{\alpha}_1 \# \alpha_1 = \sum \boldsymbol{\alpha}$. Thus SIH yields

$$\vdash_1^{(\sum \boldsymbol{\alpha}_1 \# \alpha_0)\times 3_2(\beta)} \Delta, \Gamma, D(\bar{n}) \Rightarrow C$$

for each n.

$$\dfrac{\cdots \quad \dfrac{\vdash_1^{\boldsymbol{\alpha}_1} \Gamma \Rightarrow \boldsymbol{A}_1 \quad \vdash_1^{\alpha_0} \Gamma, D(\bar{n}) \Rightarrow A_1 \quad \vdash_2^{\beta} \Delta, \boldsymbol{A}_1, A_1 \Rightarrow C}{\vdash_1^{(\sum \boldsymbol{\alpha}_1 \# \alpha_0)\times 3_2(\beta)} \Delta, \Gamma, D(\bar{n}) \Rightarrow C}\,\text{SIH} \quad \cdots (n \in \mathbb{N})}{\vdash_1^{(\sum \boldsymbol{\alpha})\times 3_2(\beta)} \Delta, \Gamma \Rightarrow C}\,(L\exists).$$

      ii. The case when J' is an $(L \to)$ with a $D \to E \in \Gamma$.

$$\dfrac{\vdash_1^{\alpha_0} \Gamma \Rightarrow D \quad \vdash_1^{\alpha_0} \Gamma, E \Rightarrow A_1}{\vdash_1^{\alpha_1} \Gamma \Rightarrow A_1}\,(L \to).$$

Then

$$\dfrac{\vdash_1^{\alpha_0} \Gamma \Rightarrow D \quad \dfrac{\vdash_1^{\boldsymbol{\alpha}_1} \Gamma \Rightarrow \boldsymbol{A}_1 \quad \vdash_1^{\alpha_0} \Gamma, E \Rightarrow A_1 \quad \vdash_2^{\beta} \Delta, \boldsymbol{A}_1, A_1 \Rightarrow C}{\vdash_1^{(\sum \boldsymbol{\alpha}_1 \# \alpha_0)\times 3_2(\beta)} \Delta, \Gamma, E \Rightarrow C}\,\text{SIH}}{\vdash_1^{(\sum \boldsymbol{\alpha})\times 3_2(\beta)} \Delta, \Gamma \Rightarrow C}\,(L \to).$$

      iii. The case when J' is an L-cut with cut formula D.

$$\dfrac{\vdash_1^{\alpha_0} \Gamma \Rightarrow D \quad \vdash_1^{\alpha_0} \Gamma, D \Rightarrow A_1}{\vdash_1^{\alpha_1} \Gamma \Rightarrow A_1}\,(L\text{-cut}).$$

$$\dfrac{\vdash_1^{\alpha_0} \Gamma \Rightarrow D \quad \dfrac{\vdash_1^{\boldsymbol{\alpha}_1} \Gamma \Rightarrow \boldsymbol{A}_1 \quad \vdash_1^{\alpha_0} \Gamma, D \Rightarrow A_1 \quad \vdash_2^{\beta} \Delta, \boldsymbol{A}_1, A_1 \Rightarrow C}{\vdash_1^{(\sum \boldsymbol{\alpha}_1 \# \alpha_0)\times 3_2(\beta)} \Delta, \Gamma, D \Rightarrow C}\,\text{SIH.}}{\vdash_1^{(\sum \boldsymbol{\alpha})\times 3_2(\beta)} \Delta, \Gamma \Rightarrow C}\,(L\text{-cut})$$

      iv. Other cases are seen similarly.
   (b) J' is a right rule.
      i. In the case when $A_1 \equiv \exists x B(x)$ we have for an $\alpha_0 < \alpha_1$ and a $\gamma < \beta$

$$\dfrac{\vdash_1^{\alpha_0} \Gamma \Rightarrow B(\bar{n})}{\vdash_1^{\alpha_1} \Gamma \Rightarrow \exists x B(x)}\,(R\exists) \quad \text{and} \quad \dfrac{\cdots \quad \vdash_2^{\gamma} \Delta, \boldsymbol{A}, B(\bar{n}) \Rightarrow C \quad \cdots}{\vdash_2^{\beta} \Delta, \boldsymbol{A} \Rightarrow C}\,(L\exists).$$

Then

$$\dfrac{\vdash^{\boldsymbol{\alpha}} \Gamma \Rightarrow \boldsymbol{A} \quad \vdash_1^{\alpha_0} \Gamma \Rightarrow B(\bar{n}) \quad \vdash_2^{\gamma} \Delta, \boldsymbol{A}, B(\bar{n}) \Rightarrow C}{\vdash_1^{((\sum \boldsymbol{\alpha})\# \alpha_0)\times 3_2(\gamma)} \Delta, \Gamma \Rightarrow C}\,\text{MIH}$$

and

$$\left(\left(\sum \boldsymbol{\alpha}\right) \# \alpha_0\right) \times 3_2(\gamma) < \left(\sum \boldsymbol{\alpha}\right) \times 3_2(\beta). \tag{3}$$

ii. The case when $A_1 \equiv H \to A_0$ with an $H \in L$ and an $A_0 \in SP_L$. We have for an $\alpha_0 < \alpha_1$ and a $\gamma < \beta$

$$\frac{\vdash_1^{\alpha_0} \Gamma, H \Rightarrow A_0}{\vdash_1^{\alpha_1} \Gamma \Rightarrow A_1}\ (R\to) \quad \text{and} \quad \frac{\vdash_2^{\gamma} \Delta, \boldsymbol{A} \Rightarrow H \quad \vdash_2^{\gamma} \Delta, \boldsymbol{A}, A_0 \Rightarrow C}{\vdash_2^{\beta} \Delta, \boldsymbol{A} \Rightarrow C}\ (L\to).$$

Then by (3)

$$\frac{\dfrac{\vdash_1^{\boldsymbol{\alpha}} \Gamma \Rightarrow \boldsymbol{A} \quad \vdash_2^{\gamma} \Delta, \boldsymbol{A} \Rightarrow H}{\vdash_1^{(\sum \boldsymbol{\alpha})\times 3_2(\gamma)} \Delta, \Gamma \Rightarrow H}\ \text{MIH} \quad \dfrac{\vdash_1^{\boldsymbol{\alpha}} \Gamma \Rightarrow \boldsymbol{A} \quad \vdash_0^{\alpha_0} \Gamma, H \Rightarrow A_0 \quad \vdash_2^{\gamma} \Delta, \boldsymbol{A}, A_0 \Rightarrow C}{\vdash_1^{((\sum \boldsymbol{\alpha})\#\alpha_0)\times 3_2(\gamma)} \Delta, \Gamma, H \Rightarrow C}\ \text{MIH}.}{\vdash_1^{(\sum \boldsymbol{\alpha})\times 3_2(\beta)} \Delta, \Gamma \Rightarrow C}\ (L\text{-cut})$$

iii. In the case when $A_i \equiv \forall x B(x)$ we have for an $\alpha_0 < \alpha_1$ and a $\gamma < \beta$

$$\frac{\cdots \quad \vdash_1^{\alpha_0} \Gamma \Rightarrow B(\bar{n}) \quad \cdots (n \in \mathbb{N})}{\vdash_1^{\alpha_1} \Gamma \Rightarrow A_1}\ (R\forall) \quad \text{and} \quad \frac{\vdash_2^{\gamma} \Delta, \boldsymbol{A}, B(\bar{n}) \Rightarrow C}{\vdash_2^{\beta} \Delta, \boldsymbol{A} \Rightarrow C}\ (L\forall).$$

Then by (3)

$$\frac{\vdash_1^{\boldsymbol{\alpha}} \Gamma \Rightarrow \boldsymbol{A} \quad \vdash_1^{\alpha_0} \Gamma \Rightarrow B(\bar{n}) \quad \vdash_2^{\gamma} \Delta, \boldsymbol{A}, B(\bar{n}) \Rightarrow C}{\vdash_1^{(\sum \boldsymbol{\alpha})\times 3_2(\beta)} \Gamma, \Delta \Rightarrow C}\ \text{MIH}.$$

iv. In the case when $A_i \equiv I(t)$ we have for an $\alpha_0 < \alpha_1$ and a $\gamma < \beta$

$$\frac{\vdash_1^{\alpha_0} \Gamma \Rightarrow \Phi(I, t)}{\vdash_1^{\alpha_1} \Gamma \Rightarrow A_1}\ (RI) \quad \text{and} \quad \frac{\vdash_2^{\gamma} \Delta, \boldsymbol{A}, \Phi(I, t) \Rightarrow C}{\vdash_2^{\beta} \Delta, \boldsymbol{A} \Rightarrow C}\ (LI).$$

Then by (3)

$$\frac{\vdash_1^{\boldsymbol{\alpha}} \Gamma \Rightarrow \boldsymbol{A} \quad \vdash_1^{\alpha_0} \Gamma \Rightarrow \Phi(I, t) \quad \vdash_2^{\gamma} \Delta, \boldsymbol{A}, \Phi(I, t) \Rightarrow C}{\vdash_1^{(\sum \boldsymbol{\alpha})\times 3_2(\beta)} \Gamma, \Delta \Rightarrow C}\ \text{MIH}.$$

v. Other cases $A_1 \equiv B_0 \vee B_1, B_0 \wedge B_1$ are seen similarly.

This completes a proof of (2), and hence of Lemma 9.

5. Quick cut-elimination of monotone cuts with negative antecedents

We show that monotone cuts with negative antecedents can be eliminated more quickly. In this section we consider Heyting arithmetic HA and its infinitary counterpart HA^∞. First let us introduce a class $\mathcal{NM}$ of L_{HA}-formulas.

Definition 10. $\mathcal{N}$ denotes the class of negative formulas, i.e., formulas in which no disjunction and no existential quantifier occurs.

Define inductively a class of formulas $\mathcal{NM}$ in L_{HA} as follows.

1. Any atomic formula $s = t$ belongs to $\mathcal{NM}$.
2. If $R, S \in \mathcal{NM}$, then $R \vee S$, $R \wedge S$, $\exists xR$, $\forall xR \in \mathcal{NM}$.
3. If $L \in \mathcal{N}$ and $R \in \mathcal{NM}$, then $L \to R \in \mathcal{NM}$.

It is easy to see that $\mathcal{N} \subset \mathcal{NM}$.

Note that by the equivalence

$$[\exists x A(x) \to B] \leftrightarrow \forall x[A(x) \to B] \tag{4}$$

$\exists x A(x) \to B$ for $A \in \mathcal{N}$, $B \in \mathcal{NM}$ is equivalent to the $\mathcal{NM}$-formula $\forall x[A(x) \to B]$.

The rank $rk(A)$ of sentences A is redefined as follows.

Definition 11. The rank $rk(A)$ of a sentence A is defined by

$$rk(A) := \begin{cases} 0 & \text{if } A \in \mathcal{N} \\ 1 & \text{if } A \in (\mathcal{NM} \setminus \mathcal{N}) \\ 2 & \text{otherwise.} \end{cases}$$

Let HA^∞ denote the infinitary system in the language L_{HA}, whose initial sequents and inference rules are obtained from those of $\widehat{ID}^{i\infty}(SP_L)$ by deleting the initial sequents $\Gamma, P(t) \Rightarrow P(t)$ for $P \in (L(I) \setminus L_{HA})$ and inference rules (LI), (RI).

By restricting antecedents to negative (or Harrop) formulas we have a stronger inversion.

Lemma 12 (*Inversion Lemma with Negative Antecedents*). *Assume* $\vdash_1^\alpha \Gamma \Rightarrow A$.

1. *If* $A \equiv B_0 \wedge B_1$, *then* $\vdash_1^\alpha \Gamma \Rightarrow B_i$ *for any* $i \in \{0, 1\}$.
2. *If* $A \equiv \forall x B(x)$, *then* $\vdash_1^\alpha \Gamma \Rightarrow B(\bar{n})$ *for any* $n \in \omega$.
3. *If* $A \equiv B_0 \to B_1$, *then* $\vdash_1^\alpha \Gamma, B_0 \Rightarrow B_1$.
4. *If* $\Gamma \subseteq \mathcal{N}$ *and* $A \equiv B_0 \vee B_1$, *then* $\vdash_1^\alpha \Gamma \Rightarrow B_i$ *for an* $i \in \{0, 1\}$.
5. *If* $\Gamma \subseteq \mathcal{N}$ *and* $A \equiv \exists x B(x)$, *then* $\vdash_1^\alpha \Gamma \Rightarrow B(\bar{n})$ *for an* $n \in \omega$.

Theorem 13. *Let* C_0 *denote an* $\mathcal{NM}$*-sentence, and* Γ_0 *a finite set of* $\mathcal{N}$*-sentences. Suppose that* $\vdash_2^\beta \Gamma_0 \Rightarrow C_0$. *Then* $\vdash_1^{2^\beta} \Gamma_0 \Rightarrow C_0$.

Again Theorem 13 follows from the following Lemma 14 for quick cut-elimination in parallel.

$\boldsymbol{A}$ denotes a finite list $A_k, \ldots, A_2, A_1$ ($k \geq 0$) of $\mathcal{NM}$-formulas, and α an ordinal. Then $\vdash_1^\alpha \Gamma \Rightarrow \boldsymbol{A}$ designates that $\vdash_1^\alpha \Gamma \Rightarrow A_i$ for *any* $i \in \{1, \ldots, k\}$. Note here that the depth α of the derivations of $\Gamma \Rightarrow A_i$ is independent from i.

Lemma 14. *Suppose* $\Gamma \cup \Delta \subset \mathcal{N}$ *and* $\boldsymbol{A} \cup \{C\} \subset \mathcal{NM}$. *If*

$$\vdash_1^\alpha \Gamma \Rightarrow \boldsymbol{A} \quad \text{and} \quad \vdash_2^\beta \Delta, \boldsymbol{A} \Rightarrow C$$

then

$$\vdash_1^{\alpha+2^\beta} \Delta, \Gamma \Rightarrow C.$$

We can prove Lemma 14 by induction on β as Lemma 9. Case (1) is when $\Delta, \boldsymbol{A} \Rightarrow C$ is an initial sequent, i.e., we don't need to examine the left upper parts $\vdash_1^\alpha \Gamma \Rightarrow \boldsymbol{A}$. In Case (5) the Inversion lemma on the succedent is always available since the antecedent Γ consists solely of negative formulas. Note that in Case (ii) the remaining cut formula $H \in \mathcal{N}$ is in the class $\mathcal{NM}$.

This completes a proof of Lemma 14, and of Theorem 13.

Note that the procedure leaves cuts with negative cut formulas H in Case (ii). If we are restrict to eliminating monotone cuts, then cuts are eliminated quickly and completely.

Theorem 15. *Let* C_0 *denote an* $\mathcal{NM}$*-sentence, and* Γ_0 *a finite set of* $\mathcal{N}$*-sentences. Suppose that there exists a derivation of* $\Gamma_0 \Rightarrow C_0$ *in which any cut formula is a monotone formula, and whose depth is at most* β. *Then there exists a cut-free derivation of* $\Gamma_0 \Rightarrow C_0$ *with depth* 2^β.

Let us iterate this procedure for monotone cuts.

In what follows, Φ denotes a class of arithmetic formulas such that any atomic formula is in Φ, and Φ is closed under substitution of terms for variables and renaming of bound variables.

Given such a class Φ of formulas, introduce a hierarchy $\{\mathcal{M}_n(\Phi)\}_n$ of arithmetic formulas.

Definition 16. First set $\mathcal{M}_1(\Phi) = \Phi$.

Define inductively classes of formulas $\mathcal{M}_{n+1}(\Phi)$ ($n \geq 1$) in L_{HA} as follows.

1. $\mathcal{M}_n(\Phi) \subset \mathcal{M}_{n+1}(\Phi)$.
2. If $R, S \in \mathcal{M}_{n+1}(\Phi)$, then $R \vee S, R \wedge S, \exists x R, \forall x R \in \mathcal{M}_{n+1}(\Phi)$.
3. If $L \in \mathcal{M}_n(\Phi)$ and $R \in \mathcal{M}_{n+1}(\Phi)$, then $L \to R \in \mathcal{M}_{n+1}(\Phi)$.

We have $\bigcup_{n<\omega} \mathcal{M}_n(\Phi) = L_{HA}$.

For $\Phi = \Sigma_1$, $\mathcal{M}_n(\Sigma_1)$ coincides with the class Θ_n introduced by W. Burr [4]. Note that by (4) for any $n \geq 2$, each formula in $\mathcal{M}_n(\Sigma_1) = \Theta_n$ is equivalent to a formula in $\mathcal{M}_n(\Delta_0)$, where Δ_0 is the class of all atomic formulas. Also each formula in Θ_2 is equivalent to a monotone formula in $\mathcal{M}$.

The rank $rk(A; \Phi)$ of sentences A relative to the class Φ is defined.

Definition 17. The rank $rk(A; \Phi)$ of a sentence A is defined by

$$rk(A; \Phi) := \min\{n - 1 : A \in \mathcal{M}_n(\Phi)\}.$$

Let $\vdash_r^\alpha \Gamma \Rightarrow C$ designate that there exists an infinitary derivation of $\Gamma \Rightarrow C$ such that the depth of the derivation tree is bounded by α and any cut formula occurring in it has rank less than r. $\vdash_2^\alpha \Gamma \Rightarrow C$ means that in the witnessing derivation of depth α any cut formula is in the class $\mathcal{M}_2(\Phi)$.

Theorem 18. *Suppose that* $\vdash_{r+1}^\beta \Gamma_0 \Rightarrow C_0$ *and* $r \geq 1$. *Then* $\vdash_r^{3_2(\beta)} \Gamma_0 \Rightarrow C_0$.

Proof. This is seen as in the proof of Theorem 8, but leave the cut inference of cut formula H with $rk(H; \Phi) < r$ in Case (ii). □

6. Applications to fragments of Heyting arithmetic

Finally let us treat an application of quick cut-elimination to fragments of Heyting arithmetic.

Definition 19. Let Φ be a class of arithmetic formulas such that any atomic formula is in Φ, and Φ is closed under substitution of terms for variables and renaming of bound variables.

$i\Phi$ denotes the fragment of HA in which induction axioms are restricted to formulas in Φ.

$$A(0) \wedge \forall x[A(x) \to A(x+1)] \to \forall x\, A(x)\ (A \in \Phi).$$

For a class of formulas Ψ, $\mathrm{RFN}_\Psi(i\Phi)$ denotes the Ψ-(uniform) reflection principle for $i\Phi$:

$$\mathrm{RFN}_\Psi(i\Phi) = \{\mathrm{Pr}_{i\Phi}(\ulcorner\varphi(\dot{x})\urcorner) \to \varphi(x) : \varphi \in \Psi\}$$

where $\mathrm{Pr}_{i\Phi}$ denotes a standard provability predicate for $i\Phi$ and $\dot{x}$ is the x-th formalized numeral.

When $\Psi = L_{HA}$ the subscript Ψ in $\mathrm{RFN}_\Psi(i\Phi)$ is dropped.

By the result of Buchholz [3] we see that HA proves the consistency of the intuitionistic arithmetic $i\mathcal{M}$ for the class $\mathcal{M}$ of monotone formulas since $\widehat{ID}^i(\mathcal{M})$ can define the truth of monotone formulas, and the consistency statement $\mathrm{CON}(i\mathcal{M})$ is an almost negative formula. Observe that any prenex Π^0_k-formula is a monotone formula, and any monotone formula is equivalent to a prenex formula.

Moreover using a truth definition for Θ_n-formulas ($\Theta_n = \mathcal{M}_n(\Sigma_1)$) and a partial truth definition we see that for each $n \geq 2$ $\widehat{ID}^i_{n-1}(\mathcal{M})$ proves the soundness $\mathrm{RFN}(i\Theta_n)$ of $i\Theta_n$. Hence $\mathrm{HA} \vdash \mathrm{RFN}(i\Theta_n)$ by the full conservativity of $\widehat{ID}^i_n(\mathcal{M})$ over HA in [1].

However this does not show that $\{i\mathcal{M}_n(\Phi)\}_n$ forms a proper hierarchy. Burr [4], Corollary 2.25 shows that $I\Pi^0_n$ and $i\Theta_n$ prove the same Π^0_2-sentences for the fragments $I\Pi^0_n$ of Peano Arithmetic PA. Since $I\Pi^0_{n+1}$ proves the 2-consistency $\mathrm{RFN}_{\Pi^0_2}(I\Pi^0_n)$ of $I\Pi^0_n$ and hence of $i\Theta_n$, by the result of Burr we see that $i\Theta_{n+1}$ proves the 2-consistency of $i\Theta_n$. Thus $\{i\Theta_n\}_n$ forms a proper hierarchy.
Let us show that $i\Theta_3$ proves the soundness of $i\Theta_2$ with respect to Θ_2, $\mathrm{RFN}_{\Theta_2}(i\Theta_2)$. Recall that formulas in Θ_2, monotone formulas and formulas in prenex formulas are equivalent to each other.

Let $<$ denote a standard ε_0-well ordering. Let

$$Prg[A] :\Leftrightarrow \forall x[\forall y < x\, A(y) \to A(x)]$$

and for a class Φ of formulas, $TI(<\alpha, \Phi)$ denote the transfinite induction schema

$$Prg[A] \to \forall x < \beta\, A(x)$$

for each $\beta < \alpha$ and $A \in \Phi$.

Also let $\omega_1 := \omega$ and $\omega_{m+1} := \omega^{\omega_m}$.

Proposition 20. *If $m + k \leq n + 2$, then*

$$i\mathcal{M}_n(\Phi) \vdash TI(<\omega_m, \mathcal{M}_k(\Phi)).$$

Proof. Let

$$\mathrm{j}[A](\alpha) :\Leftrightarrow \forall\beta[\forall\gamma < \beta A(\gamma) \to \forall\gamma < \beta + \omega^\alpha A(\gamma)].$$

Then for $A \in \mathcal{M}_n(\Phi)$ we have $\mathrm{j}[A] \in \mathcal{M}_{n+1}(\Phi)$

$$i\mathcal{M}_n(\Phi) \vdash Prg[A] \to Prg[\mathrm{j}[A]]$$

and $i\mathcal{M}_n(\Phi) \vdash TI(<\omega_1, \mathcal{M}_{n+1}(\Phi))$. The proposition follows from these. □

Corollary 21. 1. *For $n \geq 2$*

$$i\Theta_{2n-1} \vdash \mathrm{RFN}_{\Theta_2}(i\Theta_n).$$

For example $i\Theta_3$ proves the soundness of prenex induction with prenex consequences.

2. *For any $m, k, n \geq 1$*

$$i\mathcal{M}_{2m+k}(\Pi^0_n) \vdash \mathrm{RFN}_{\mathcal{M}_k(\Pi^0_n)}(i\mathcal{M}_m(\Pi^0_n)).$$

Proof. 21. 1 follows from Theorems 18, 15 and Proposition 20. Namely transform a finitary derivation of a monotone sentence C in $i\mathcal{M}_n(\Delta_0)$ to an infinitary one. Apply first Theorem 18 $(n-2)$-times, to get a derivation of C such that any cut formula occurring in it is a monotone formula and its depth is bounded by $3_{2n-4}(\omega^2) = \omega_{2n-3}$. Then apply Theorem 15 to get a cut-free derivation of C in depth $2^{\omega_{2n-3}} = \omega_{2n-2}$. By Proposition 20 $TI(<\omega_{2n-1}, \Theta_2)$ is provable in $i\Theta_{2n-1}$. Since any formula occurring in the cut-free derivation is a subformula of the monotone $C \in \Theta_2$, by a Θ_2-truth definition of subformulas of C we know that C is true in $i\Theta_{2n-1}$.
21. 2 follows from Theorem 18, quick cut-elimination of monotone cuts with arbitrary antecedents and Proposition 20. Namely transform a finitary derivation of a sentence $C_0 \in \mathcal{M}_k(\Pi^0_n)$ in $i\mathcal{M}_m(\Pi^0_n)$ to an infinitary one. Eliminate cuts by applying Theorem 18 m-times, and get a derivation of C_0 of depth $3_{2m}(\omega^2) = \omega_{2m+1}$, in which any cut formula is in Π^0_n. Any formula occurring in the derivation is either a subformula of $C_0 \in \mathcal{M}_k(\Pi^0_n)$ or a Π^0_n-formula. Therefore using $\mathcal{M}_k(\Pi^0_n)$-truth definition of sequents occurring in the derivation and $TI(<\omega_{2m+2}, \mathcal{M}_k(\Pi^0_n))$ we conclude that C_0 is true in $i\mathcal{M}_{2m+k}(\Pi^0_n)$. □

Next consider conservations.

The following Corollary 22 shows, for example that $i\Theta_2$ is Π^0_k-conservative over $i\Pi^0_k$ for any k, and generalizes a theorem by Visser and Wehmeier (cf. Theorem 3 in [9] and Corollary 2.28 in [4]) stating that $i\Theta_2$ is Π^0_2-conservative over $i\Pi^0_2$.

Corollary 22. *For any $\Phi \subset \Theta_2$, $i\Theta_2$ is Φ-conservative over $i\Phi$.*

Proof. Transform a finitary derivation of a monotone sentence C in $i\mathcal{M}_2(\Delta_0)$ to an infinitary one. Apply Theorem 15 to get a cut-free derivation of C of depth less than ω_2. □

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