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Discrete Mathematics 130 (1994) 103–169

DISCRETE MATHEMATICS

General metrics and contracting operations☆

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Received 18 November 1990

Key words: Valued spaces and relational structures; Clones of contracting maps and their invariants; Metrics and ultrametrics; Graphs, automata and sequential machines

Introduction

The usual metric space (A, d) is a set A with a map d from A^2 into the set $\mathbb{R}_+$ (of non-negative reals) satisfying $d(x,y)=0 \Leftrightarrow x=y$, $d(x,y) \leqslant d(x,z)+d(z,y)$ and $d(x,y)=d(y,x)$ for all $x,y,z \in A$. Inspired by Quilliot's [21] combinatorial results, Jawhari et al. [13] extended in 1986 the concept of a metric space in several directions. Firstly, $(\mathbb{R}_+; \leqslant, +, 0,)$ is replaced by $\underline{V}=(V; \leqslant, +, 0, -)$ where (i) $(V; +, 0, \leqslant)$ is an ordered monoid (not necessarily abelian) whose neutral element 0 is the least element of $(V; \leqslant)$, (ii) the self map $v \to \bar{v}$ is an involutive order automorphism of $(V, \leqslant)$ such that $\overline{(v+w)}=\bar{w}+\bar{v}$ for all $v, w \in V$. Secondly, the map d from A^2 to V satisfies the same axioms as above, except that the third axiom is replaced by $d(y,x)=\overline{d(x,y)}$. A natural model for $\underline{V}$ is the set of all binary reflexive relations on a set E, where $\leqslant$ is $\subseteq$, $+$ is the composition $\circ$ of relations, 0 is the diagonal (that is the set $\{(v,v)\colon v \in V\}$) and $-$ is the inversion, usually denoted $^{-1}$, of binary relations (that is $\rho^{-1} := \{(y,x)\colon (x,y) \in \rho\}$). The latter paper developed certain aspects of the category of such spaces and contractions, and showed that graphs, directed graphs and ordered sets may be viewed in this context. It also extended absolute retracts, injective envelopes and fixed

☆ This research has been supported by the French PRC Math-Info. The partial financial support provided by the France-Québec projet intégré, the 'Centre Jacques-Cartier' from Lyon, the FCAR Québec Eq-0539 grant and the NSERC Canada A-5407 grant, is gratefully acknowledged.

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SSDI 0012-365X(92)00527-P

point properties to the more general spaces, linking these infinistic concepts to discrete structures provided $\underline{V}$ has special properties. In such cases a $\underline{V}$-space is called *hyperconvex*.

In this paper we go a step further. We do not require the neutral element 0 to be the least element of $(V, \leqslant)$ and we replace the first axiom by $d(x,y) \leqslant 0 \Leftrightarrow x = y$ (a model for such V is the set of all binary relations on E). This seemingly innocuous change has significant consequences leading to a more complex theory.

The paper is divided in four sections. The first one relates the category of such spaces and contractions to the category of relational systems and relational homomorphisms. This is done in a more general setting for (A,d), where d maps A^n to $(V, \leqslant)$ (for a positive integer n). In Section 2, for a join-semilattice we define m-ary contracting operations on A linking these n-ary spaces to universal algebras. In this context, we discuss the preservation of n-ary relations on A, the n-interpolation property and the extension property. In Section 3, we develop the proper theory of generalized $\underline{V}$-metric spaces mentionned earlier. In particular we study the elements of V appearing as the values of d for at least one $\underline{V}$-space (A,d). The central notion of hyperconvexity takes a more complex form but is still equivalent to the one-point extension property. Special $\underline{V}$'s may themselves be turned into metric spaces $(V, d_{\underline{V}})$ and in this case we can embed every $\underline{V}$-metric space into a power of $(V, d_{\underline{V}})$. This happens when $\underline{V}$ is an *Heyting algebra* i.e. satisfies an infinite distributive law (for $+$ and $\wedge$). In this case, the notions of injectivity, absolute retracts, retracts of powers of (V, d_V) and hyperconvexity coincide. Moreover, we can say something about the relations compatible with all contracting operations. Next, we look at the special case of a meet-semilattice with least element 0. A $\underline{V}$-metric space is called then an *ultrametric* and the contracting operations form exactly the sets of terms of a congruence affine algebra on A. In Section 4, we first extend an arbitrary binary space $\underline{V}$ into an Heyting algebra and we proceed to show that an arbitrary $\underline{V}$-space embeds isometrically into a power of $(V, d_{\underline{V}})$. We apply this to binary relational systems and to automata. A further application to the structural theory of automata studies the composition of an automaton from given building blocks which are themselves automata. Limiting ourselves to feedback-free compositions, but allowing an infinite number of states, we can describe the situation in terms of a hyperconvex ultrametric.

1. Valued spaces and relations

1.1. Spaces

Let $\underline{V} = (V, \leqslant)$ be an ordered set and n be a nonnegative integer. An *n-space over* $\underline{V}$ is a pair $\underline{A} = (A, \delta)$ where A is a set and δ is a map from A^n to V. If $\underline{A} = (A, \delta)$ and $\underline{A}' = (A', \delta')$ are such n-spaces over $\underline{V}$, a map f from A to A' is a *contraction from $\underline{A}$ to $\underline{A}'$* if the following inequality holds

$$\delta'(f(a_1), \ldots, f(a_n)) \leqslant \delta(a_1, \ldots, a_n) \tag{1.1}$$

for all $a_1, \ldots, a_n \in A$.

If $n=1$, a map f from A to A' is a contraction provided $\delta'(f(a))\leqslant\delta(a)$ for all $a\in A$. The contractions for $n>1$ may be perceived as unary ones if we consider the 1-spaces (A^n,δ) and (A'^n,δ') and the natural extension $f^{(n)}$ from A^n to A'^n defined by setting $f^{(n)}(a_1,\dots,a_n):=(f(a_1),\dots,f(a_n))$ for all $a_1,\dots,a_n\in A$. Trivially the composition of contractions is a contraction and the identity map on every n-space over $\underline{V}$ is a contraction. Thus the n-spaces over $\underline{V}$, equipped with the contractions form a category which will be denoted $\boldsymbol{S}_{n\underline{V}}$. In what follows we give another description of $\boldsymbol{S}_{n\underline{V}}$.

1.2. Relations

Let D be a set, let n be a positive integer and, for each $d\in D$, let ρ_d be an n-ary relation on A (i.e. a subset of A^n). We call $\underline{A}=(A,(\rho_d\colon d\in D))$ an *n-ary relational system on A of type D*. A map f from A to A' is a *relational homomorphism* from $\underline{A}=(A,(\rho_d\colon d\in D))$ to $\underline{A}'=(A',(\rho'_d\colon d\in D))$ if for all $d\in D$ we have $f(\rho_d)\subseteq\rho'_d$ (i.e. $(f(a_1),\dots,f(a_n))\in\rho'_d$ for all $(a_1,\dots,a_n)\in\rho_d$). The n-ary relational systems of type D and their relational homomorphisms form also a category which will be denoted $\boldsymbol{R}_{nD}$.

1.3. Correspondence between spaces and relations

Let $\underline{A}=(A,\delta)$ be an n-space over $\underline{V}$. For $v\in V$ put $(\delta)_v:=\{(a_1,\dots,a_n)\in A^n\colon \delta(a_1,\dots,a_n)\leqslant v\}$. For $D\subseteq V$ put $\underline{A}_{D\delta}=(A,((\delta)_v\colon v\in D))$. For D, $E\subseteq V$ we say that D is *meet-dense* in E if each $v\in E\backslash D$ is a meet of a (possibly infinite) set of elements of D (i.e. v is the greatest element of the set $\{x\in E\colon x\leqslant y$ for all $y\in D$, $y\geqslant v\}$). We have the following.

Lemma 1.3.1. *Let $\underline{A}=(A,\delta)$ and $\underline{A}'=(A',\delta')$ be n-spaces over $\underline{V}$; let $D\subseteq V$ and let f be a map from A to A'. If f is a contraction from $\underline{A}$ to $\underline{A}'$ then f is a relational homomorphism from $\underline{A}_{D\delta}$ to $\underline{A}'_{D\delta'}$. If, moreover, D is meet-dense in* Im δ *then every relational homomorphism from $\underline{A}_{D\delta}$ to $\underline{A}'_{D\delta'}$is a contraction.*

Proof. Let $(a_1,\dots,a_n)\in(\delta)_d$ (i.e. $\delta(a_1,\dots,a_n)\leqslant d$). Since f is a contraction, $\delta'(f(a_1),\dots,f(a_n))\leqslant\delta(a_1,\dots,a_n)\leqslant d$, proving $(f(a_1),\dots,f(a_n))\in(\delta')_d$, and thus f is a relational homomorphism. Conversely, let D be meet-dense in $\mathrm{Im}\,\delta$ and let f be a relational homomorphism from $\underline{A}_{D\delta}$ to $\underline{A}'_{D\delta'}$. Consider $(a_1,\dots,a_n)\in A^n$. For every $d\in D$ we have $\delta'(f(a_1),\dots,f(a_n))\leqslant d$ whenever $\delta(a_1,\dots,a_n)\leqslant d$ and, since D is meet-dense in $\mathrm{Im}\,\delta$, we obtain the required $\delta'(f(a_1),\dots,f(a_n))\leqslant\delta(a_1,\dots,a_n)$. □

Let T_D denote the functor assigning $\underline{A}_{D\delta}$ to $\underline{A}=(A,\delta)$ and mapping each contraction into itself.

We have the following proposition.

Proposition 1.3.2. *If D is meet-dense in V then T_D is a faithful functor from $\boldsymbol{S}_{n\underline{V}}$ into $\boldsymbol{R}_{nD}$. In other words, the category of n-spaces over $\underline{V}$ is isomorphic to a full subcategory of n-ary relational systems of type D.*

Moreover, if $\underline{V}$ *is a complete lattice then* T_D *has a left inverse, namely the functor* $U_{n\underline{V}}$ *from* $\boldsymbol{R}_{nD}$ *into* $\boldsymbol{S}_{n\underline{V}}$ *which to every* $\underline{A}=(A,(\rho_d\colon d\in D))$ *associates the n-space* $(A,\delta_{\underline{A}})$ *over* $\underline{V}$ *where* $\delta_{\underline{A}}$ *is the map from* A^n *to* V *defined by setting* $\delta_{\underline{A}}(a)=\operatorname{Inf}\{d\in D\colon a\in\rho_d\}$ *for every* $a\in A^n$.

Proof. The first statement is just the lemma above. For the second, first we show that $U_{n\underline{V}}$ is a functor. Let a map f from A to A' be a relational homomorphism. Then for every $a=(a_1,\ldots,a_n)\in A^n$, we have

$$\{d\in D\colon a\in\rho_d\}\subseteq\{d\in D\colon f^{(n)}(a)\in\rho'_d\},$$

where $f^{(n)}(a):=(f(a_1),\ldots,f(a_n))$. Hence

$$\delta_{\underline{A}'}(f^{(n)}(a)):=\operatorname{Inf}\{d\in D\colon f^{(n)}(a)\in\rho'_d\}\leqslant\operatorname{Inf}\{d\in D\colon a\in\rho_d\}=:\delta_{\underline{A}}(a),$$

proving that f is a contraction. We show $U_{n\underline{V}}\circ T_D=1_{S_{n\underline{V}}}$. Let $(A,\delta)\in\boldsymbol{S}_{n\underline{V}}$ and $a\in A^n$. Then

$$\delta_{\underline{A}_{D\delta}}(a)=\operatorname{Inf}\{d\in D\colon a\in(\delta)_d\}=\operatorname{Inf}\{d\in D\colon \delta(a)\leqslant d\}:=\delta(a)$$

because $a\in(\delta)_d$ means $\delta(a)\leqslant d$ and D is meet-dense. □

For a particular $\underline{V}$ we can prove more. Let $\wp(D)$ denote the family of all subsets of D.

Proposition 1.3.3. *Let D be a set,* $\underline{V}=(\wp(D),\supseteq)$ *and n an integer. Then the category* $\boldsymbol{R}_{nD}$ *is isomorphic to the category* $\boldsymbol{S}_{n\underline{V}}$.

Proof. Put $D':=\{\{d\}\colon d\in D\}$. The set D' is meet-dense in $\underline{V}$, which is a complete lattice, so by Proposition 1.3.2 we obtain that the functor $T_{D'}$ from $\boldsymbol{S}_{n\underline{V}}$ to $\boldsymbol{R}_{nD'}$ has a left inverse $U_{n\underline{V}}$. We show that $T_{D'}\circ U_{n\underline{V}}=1_{RnD'}$. Indeed, with

$$\underline{A}=(A,(\rho_{d'}\colon d'\in D'))\in\boldsymbol{R}_{nD'}$$

the functor $U_{n\underline{V}}$ associates $(A,\delta_{\underline{A}})$ where $\delta_{\underline{A}}(a)=\operatorname{Inf}\{d'\in D'\colon a\in\rho_{d'}\}$ for $a\in A^n$. Now to $(A,\delta_{\underline{A}})$ the functor $T_{D'}$ associates the relational system $\underline{B}=(A,((\delta_{\underline{A}})_{d'}\colon d'\in D'))$. Let $d'\in D'$ and $a\in A^n$; by definition $a\in(\delta_{\underline{A}})_{d'}$ means $\delta_{\underline{A}}(a)\leqslant d'$, that is $\delta_{\underline{A}}(a)\supseteq d'$. Since d' is a singleton and $\delta_{\underline{A}}(a)=\operatorname{Inf}\{t'\in D\colon a\in\rho_{\{t'\}}\}=\operatorname{Inf}\{t\in D\colon a\in\rho_{\{t\}}\}$, this inclusion means $a\in\rho_{d'}$, that is $(\delta_{\underline{A}})_{d'}=\rho_{d'}$. This gives $\underline{A}=\underline{B}$ and thus $\boldsymbol{R}_{nD'}=\boldsymbol{S}_{n\underline{V}}$. However, the difference between $\boldsymbol{R}_{nD'}$ and $\boldsymbol{R}_{nD}$ is purely notational and so the proposition is proven. □

Remark 1.3.4. For $\underline{V}=(\wp(D),\supseteq)$, the isomorphism between $\boldsymbol{R}_{nD}$ and $\boldsymbol{S}_{n\underline{V}}$ is explicitly given by the maps U from $\boldsymbol{R}_{nD}$ to $\boldsymbol{S}_{n\underline{V}}$ and T from $\boldsymbol{S}_{n\underline{V}}$ to $\boldsymbol{R}_{nD}$ defined as follows: to $\underline{A}=(A,(\rho_d\colon d\in D))$ associate $U(\underline{A}):=(A,\delta_{\underline{A}})$ where the map $\delta_{\underline{A}}$ from A^n to V is defined by setting $\delta_{\underline{A}}(a)=\{d\in D\colon a\in\rho_d\}$ for all $a\in A^n$. Conversely, to an n-space $\underline{A}=(A,\delta)$ over $\underline{V}$ associate the relational system $T(\underline{A}):=(A,((\delta)_{\{d\}}\colon d\in D))$.

Summing up, for a given $\underline{V}$, the category $\boldsymbol{S}_{n\underline{V}}$ of n-spaces over $\underline{V}$ is a full subcategory of the category $\boldsymbol{R}_{nD}$ provided that the poset $\underline{V}$ embeds into the poset $\underline{V}'=(\wp(D), \supseteq)$ (as this embedding induces an embedding of $\boldsymbol{S}_{n\underline{V}}$ into $\boldsymbol{S}_{n\underline{V}'}$ where the latter is isomorphic to $\boldsymbol{R}_{nD}$). Here the minimum size of such a set is what Kelly and Trotter [15] call the 2-*order-dimension* of $\underline{V}$. On the other hand, for a given set D, each ordered subset $\underline{V}$ of $(\wp(D), \supseteq)$ leads to a category of n-ary spaces, namely $\boldsymbol{S}_{n\underline{V}}$, and thus to a full subcategory of $\boldsymbol{R}_{nD}$. This provides a classification of relational systems by the category $\boldsymbol{S}_{n\underline{V}}$ to which they belong. We have as many structurally different categories as there are non-isomorphic subposets of $(\wp(D), \supseteq)$.

Proposition 1.3.5. *Let $\underline{V}$ and $\underline{V}'$ be two posets. The categories $\boldsymbol{S}_{n\underline{V}}$ and $\boldsymbol{S}_{n\underline{V}'}$ have exactly the same contractions if and only if $\underline{V}$ and $\underline{V}'$ are order-isomorphic.*

Proof. If ψ is an order-isomorphism from $\underline{V}$ onto $\underline{V}'$, then we have the functor F which to the object (A,δ) of $\boldsymbol{S}_{n\underline{V}}$ associates the object $(A,\psi\circ\delta)$ of $\boldsymbol{S}_{n\underline{V}'}$ and which maps each contraction onto itself. Conversely, suppose that F is an isomorphism from $\boldsymbol{S}_{n\underline{V}}$ onto $\boldsymbol{S}_{n\underline{V}'}$ mapping each contraction of $\boldsymbol{S}_{n\underline{V}}$ onto itself. The image of an object (A,δ) of $\boldsymbol{S}_{n\underline{V}}$ is of the form $(A,\psi(\delta))$ (because id_A is a contraction of (A,δ) onto itself). For a given A this defines a map Ψ from $\underline{V}^{A^n}$ onto $\underline{V}'^{A^n}$. These sets being ordered componentwise, Ψ is an order-isomorphism: indeed, the fact that id_A is a contraction from (A,δ_1) onto (A,δ_2) means $\delta_2\leqslant\delta_1$. Thus for a one element set A this map Ψ induces an order-isomorphism from $\underline{V}$ onto $\underline{V}'$. □

Remark 1.3.6. *For an isomorphism F from $\boldsymbol{S}_{n\underline{V}}$ onto $\boldsymbol{S}_{n\underline{V}'}$ mapping each contraction of $\boldsymbol{S}_{n\underline{V}}$ onto itself, there need not exist an order-isomorphism ψ from $\underline{V}$ onto $\underline{V}'$ such that $F(A,\delta)=(A,\psi\circ\delta)$ for every (A,δ) in $\boldsymbol{S}_{n\underline{V}}$.* Indeed, let $n=2$ and $\underline{V}=\underline{V}'$ be an antichain. Let π be a fixed permutation of V; given an object (A,δ) of $\boldsymbol{S}_{n\underline{V}}$ define $F(A,\delta)=(A,\delta')$ by setting $\delta'(a_1,a_2):=\delta(a_1,a_2)$ if $\delta(a_1,a_1)=\delta(a_2,a_2)$ and $\delta'(a_1,a_2):=\pi(\delta(a_1,a_2))$ otherwise. Since $\underline{V}$ is an antichain, in $\boldsymbol{S}_{n\underline{V}}$ a map f is a contraction from (A_1,δ_1) onto (A_2,δ_2) iff $\delta_2(f(a_1),f(a_2))=\delta_1(a_1,a_2)$ holds for all $a_1,a_2\in A_1$. With this fact one can show that F is an isomorphism preserving the contractions. For a nontrivial π there is no ψ such that $\delta'=\psi\circ\delta$ for all δ.

1.4. Products of relations and spaces

Recall that in $\boldsymbol{R}_{nD}$, for an index set I and relational systems $\underline{A}_i=(A_i,(\rho_{di}\colon d\in D))$, $(i\in I)$, their product $\underline{A}=(A,(\rho_d\colon d\in D))$ is defined by (1) $A:=\prod\{A_i\colon i\in I\}$ (the cartesian product), (2) for $f_1,\ldots,f_n\in A$, and $d\in D$ we put $(f_1,\ldots,f_n)\in\rho_d$ whenever $(f_1(i),\ldots,f_n(i))\in\rho_{di}$ for all $i\in I$. Concerning $\boldsymbol{S}_{n\underline{V}}$ we have the following.

Lemma 1.4.1. *The category $\boldsymbol{S}_{n\underline{V}}$ has finite (resp. arbitrary) products if and only if $\underline{V}$ is a join-semilattice (resp. a complete lattice). In this case, for a finite nonempty (resp.*

arbitrary) index set I, the product of n-spaces (A_i, δ_i) *over* $\underline{V}$, $(i \in I)$, *is* (A, δ), *where* $A := \prod\{A_i\colon i \in I\}$ *(the cartesian product) and for all* $f_1, \ldots, f_n \in A$:

$$\delta(f_1, \ldots, f_n) := \operatorname{Sup}\{\delta_i(f_1(i), \ldots, f_n(i))\colon i \in I\}.$$

Proof. If $\underline{V}$ is a join-semilattice (resp. a complete lattice) then a routine verification shows that (A, δ) defined in the lemma is a product of the (A_i, δ_i)'s. Conversely, if $S_{n\underline{V}}$ has finite (resp. arbitrary) products then each nonempty finite family (resp. an arbitrary family) of one-element spaces has a product in $S_{n\underline{V}}$. This insures that $\underline{V}$ is a join-semilattice (resp. a complete lattice) and these products are one-element spaces. Indeed, let W be a nonempty subset of V and $B := \{b\}$ a one element set; for $v \in V$ put $\underline{B}_v := (\{b\}, \delta_v)$, where $\delta_v(b, \ldots, b) = v$. Let $\underline{A} := (A, \delta)$ be a product of the family $\{\underline{B}_w\colon w \in W\}$; taking into account the contractions p_w from $\underline{A}$ to $\underline{B}_w$ we obtain $w = \delta_w(b, \ldots, b) \leqslant \delta(a_1, \ldots, a_n)$ *for all* $a_1, \ldots, a_n \in A$ and $w \in W$. In particular, the set U of upper bounds of W is non-empty. For $u \in U$ the map id_B is a contraction from $\underline{B}_u$ onto $\underline{B}_w$ for all $w \in W$ and so by definition of the product, there is a unique contraction Λ from $\underline{B}_u$ into $\underline{A}$ such that $p_w \circ \Lambda = \mathrm{id}_B$; that is for $u \in U$ there is a unique element x_u in A such that $\phi(u) := \delta(x_u, \ldots, x_u) \leqslant u$. Let $u, u' \in U$; the existence of a product of $\underline{B}_u$ and $\underline{B}_{u'}$ insures similarly that u and u' have an upper bound. Let v be such an element. Since $v \in U$, we get $x_u = x_v = x_{u'}$; thus $\phi(u)$ is the supremum of W, and A is a singleton. □

Example 1.4.2. If $\underline{V} = (\mathbb{R}_+, \leqslant)$, the non-negative reals with the natural order, and if the (A_i, δ_i) are ordinary metric spaces then their product is endowed with the so-called Sup-distance or ℓ^∞-distance.

2. The clone of contracting operations

2.1. *Operations and clones*

Let A be a set. For a positive integer n, an *n-ary operation* on A is a map f from A^n to A. We denote by $\underline{O}_A^{(n)}$ the set of all n-ary operations on A and put $\underline{O}_A := \bigcup\{\underline{O}_A^{(n)}\colon 1 \leqslant n < \omega\}$. We consider special subsets of $\underline{O}_A$, called *clones*, which are closed under the *composition* of operations, the *permutation* and *identification* of variables, and contain the *projections*. A clone is a direct multivariable analog of a monoid of transformations of A, or more specifically, a permutation group on A, whereby the projections play the same role as id_A. (Note that in universal algebra, 'nullary operations' are used to provide distinguished constants, e.g. 0 and 1 in a ring or a lattice; for our purpose they present notational difficulties and constants are introduced via constant unary operations (i.e. constant selfmaps of A)).

There are several formal definitions of clones in the literature. We briefly present one due to Mal'tsev [16] which is conceptually simple, algebraic, and easy to apply.

Composition (also called substitution or superposition) of operations parallels the composition of selfmaps. However, the presence of several variables makes the description more complex. To capture the replacement of a variable in an operation by the value of another operation on other variables, we postulate it only for the first variable. For this we define the following binary operation on $\underline{O}_A$.

Let $f \in \underline{O}_A^{(m)}$ and $g \in \underline{O}_A^{(n)}$. Put $p := m+n-1$ and define $h := f * g \in \underline{O}_A^{(p)}$ by setting $h(a_1, \ldots, a_p) := f(g(a_1, \ldots, a_n), a_{n+1}, \ldots, a_p)$ for all $a_1, \ldots, a_p \in A$. The operation $*$ is associative with neutral element id_A.

In universal algebra, logic and applications, often it is desirable to manipulate variables in one operation by introducing all operations obtained from it by permuting or fusing its variables. To describe this succintly we introduce self-maps ζ, τ and Δ on $\underline{O}_A$ which reduce to the identity on $\underline{O}_A^{(1)}$ whereas for $m>1$ and $f \in \underline{O}_A^{(m)}$ both $\zeta f, \tau f \in \underline{O}_A^{(m)}$ are defined by setting $(\zeta f)(a) := f(a_2, \ldots, a_m, a_1)$, $(\tau f)(a) := f(a_2, a_1, a_3, \ldots, a_m)$ for all $a = (a_1, \ldots, a_m) \in A^m$ while $\Delta f \in \underline{O}_A^{(m-1)}$ is defined by $(\Delta f)(a_1, \ldots, a_{m-1}) := f(a_1, a_1, a_2, \ldots, a_{m-1})$ for all $a_1, \ldots, a_{m-1} \in A$.

A subset P of $\underline{O}_A$ closed under $*, \zeta, \tau$ and Δ (i.e. $f * g, \zeta f, \tau f, \Delta f \in P$ for all $f, g \in P$) is a *preiterative set* (also called a closed class). For $1 \leqslant i \leqslant m$ the ith m-ary projection e_j^m is defined by $e_j^m\ (a_1, \ldots, a_m) := a_j$ for all $a_1, \ldots, a_m \in A$. A *clone* on A is a subset closed under $*, \zeta, \tau$ and Δ and containing all projections (or equivalently just e_1^2).

2.2. Contracting operations

Let (A, δ) be an n-space over an ordered set $\underline{V}$. If $\underline{V}$ is a join-semilattice, then by Lemma 1.4.1 an m-ary operation f on A is a contraction from $(A, \delta)^m$ into (A, δ) if for every $n \times m$ matrix X over A

$$\delta(f(X_{1*}), \ldots, f(X_{n*})) \leqslant \delta(X_{*1}) \vee \cdots \vee \delta(X_{*m}) \tag{2.1}$$

where X_{i*} and X_{*j} denote the ith row and jth column vector of X.

If $\underline{V}$ is not a join-semilattice then we can embed it into a join-semilattice $\underline{V}'$ as a meet-dense set (e.g. via the MacNeille completion [17]). Condition (2.1) then reduces to the following requirement:

$$\delta(X_{*j}) \leqslant v \ (j = 1, \ldots, m) \text{ implies } \delta(f(X_{1*}), \ldots, f(X_{n*})) \leqslant v \tag{2.2}$$

for every $n \times m$ matrix X over A and every $v \in V$. We say that f satisfying this condition is a *δ-contraction over $\underline{V}$*.

Lemma 2.2.1. *For an n-space (A, δ) the set $C_{\delta \underline{V}}$ of all δ-contracting operations is a clone.*

Proof. Direct verification. □

Remark 2.2.2. Even on a join-semilattice there are other possibilities to define contractions which are not based on the join operation. For example, for $n = 2$ and $\underline{V} := (\mathbb{R}_+, \leqslant)$, the nonnegative reals with the natural order, the Sup-distance is not the only one used; in fact the Euclidian l^2, and the l^1 or Hamming distances are much more common. This suggests that we may try to replace (2.1) by

$$\delta(f(X_{1*}),\ldots,f(X_{n*}))\leqslant g_m(\delta(X_{*1}),\ldots,\delta(X_{*m})). \tag{2.3}$$

where g_m are appropriate m-ary operations on $V\,(m=1,2,\ldots)$. More generally one can suppose each A^m to be endowed with a map δ_m from $(A^m)^n$ to V, and consider the contractions f from the n-spaces (A^m,δ_m) to (A,δ); they satisfy:

$$\delta(f(X_{1*}),\ldots,f(X_{n*}))\leqslant\delta_m(X_{1*},\ldots,X_{n*}). \tag{2.4}$$

One can observe that if the projections satisfy (2.4) then the δ-contractions (defined by (2.1) or (2.2)) do as well. Consequently, if the set of operations satisfying (2.4) (for $m=1,2,\ldots$) is a clone, then it contains the clone of all δ-contracting operations.

Problem. Do they coincide? Concerning this problem, note that if we consider the collection C of maps satisfying (2.3) (rather than (2.4)) then it contains the projections provided the g_m are *extensive* (that is to say $v_1,\ldots,v_m\leqslant g_m(v_1,\ldots,v_m)$); if the g_m come from an *associative* binary operation $+$ (that is $g_m(v_1,\ldots,v_m):=v_1+\cdots+v_m$) then C is closed under composition provided that $+$ is *order-preserving* (that is $v_1+v_2\leqslant v'_1+v'_2$ whenever $v_1\leqslant v'_1$ and $v_2\leqslant v'_2$) whereas it is closed under taking all operations obtained via identification of variables provided $+$ is *subidempotent* (that is $v+v\leqslant v$). Under these conditions $C=C_{\delta\underline{V}}$; indeed, if the ordered set $\underline{V}$ is endowed with a binary operation $+$ which is associative, order preserving, extensive and subidempotent, then it is, in fact, a join-semilattice and $+$ is the join operation. (Indeed, by extensivity, $v+w$ is an upper bound of v and w, while for $v\leqslant x$ and $w\leqslant x$ the fact that $+$ is order-preserving and subidempotent yields $v+w\leqslant x+x\leqslant x$, proving that $v+w$ is the join of v, w.)

2.3. Preservation

Let m and n be positive integers. A *partial m-ary operation f with domain D* is a map from a subset D of A^m into A. As defined in Section 1.2 an *n-ary relation on A* is a subset ρ of A^n. We say that *f preserves ρ* if $(f(X_{1*}),\ldots,f(X_{n*}))\in\rho$ whenever X is an $n\times m$ matrix whose row vectors X_{i*} belong all to D and whose column vectors X_{*j} belong all to ρ. In particular, if f is a full operation (i.e. $D=A^m$) f preserves ρ iff f is an homomorphism from $(A,\rho)^m$ to (A,ρ) or, equivalently, ρ is a subuniverse (i.e. the carrier or domain of a subalgebra) of the nth power of the algebra $\underline{A}=(A,f)$. This concept is more imporant than it seems at first glance. For example, if f is a full operation and ρ is unary, thus a subset of A, then f preserves ρ means $f(\rho^m)\subseteq\rho$, whereas if ρ is an equivalence relation then f preserves ρ iff ρ is a congruence of $\underline{A}=(A,f)$; similarly if ϕ is a map from A into itself and ρ is the graph of ϕ, that is $\{(a,\phi(a))\colon a\in A\}$, then f preserves ρ iff ϕ is an endomorphism of $\underline{A}$. For an ordering $\leqslant$ on A, the fact that the operation f preserves $\leqslant$ has the usual meaning of order-preserving map (also called isotone or monotone in the literature):

$f(x_1, \ldots, x_m) \leqslant f(x'_1, \ldots, (x'_m)$ whenever $x_1 \leqslant x'_1, \ldots, x_m \leqslant x'_m$. Several names are used for '*f preserves* ρ': f compatible with ρ or ρ-invariant, stable or homomorphic with respect to ρ, etc.

We will need the immediate extension of this concept to possibly infinite m and n. For an ordinal k let $\underline{k}$ denote the set of all ordinals less than k (e.g. $\underline{2} = \{0, 1\}$, $\underline{\omega} = \{0, 1, \ldots, n, \ldots\}$, in fact one could suppose $k = \underline{k}$ as well). Let m be an ordinal; as usual $A^{\underline{m}}$ denotes the set of all maps from $\underline{m}$ to A; we identify $A^{\underline{m}}$ and A^m for m finite. For $D \subseteq A^{\underline{m}}$ a map f from D to A is a *partial m-ary operation with domain D* (it is a full operation if $D = A^{\underline{m}}$). A subset ρ of $A^{\underline{n}}$ is an *n-ary relation on A*. For an application X from $m \times n$ into $A, i < n$ and $j < m$, put $X_{i*}(j) = X_{*j}(i) := X(i, j)$. For $f: D \subseteq A^{\underline{m}} \to A$, and X such that all $X_{i*} \in D$, let $f_X \in A^{\underline{n}}$ be defined by setting $f_X(i) := f(X_{i*})$ for all $i < n$. We say that *f preserves* ρ if $f_X \in \rho$ whenever all $X_{i*} \in D$ and all $X_{*j} \in \rho$.

For an ordinal m let $\underline{O}_A^{(m)}$, resp. $\underline{O}_A^{<m}$, denote the set of operations on A with arity m, resp. with arity less than m (and nonzero). Let $\boldsymbol{R}_A^{(n)}$ denote the set of n-ary relations on A and $\boldsymbol{R}_A^{<n}$ the set of relations on A with arity less than n (and nonzero). Given a relation ρ on A, let $\mathrm{Pol}^{(m)}\rho$, resp. $\mathrm{Pol}^{<m}\rho$, denote the set of $f \in \underline{O}_A^{(m)}$, resp. $f \in \underline{O}_A^{<m}$, preserving ρ. Given a set R of relations put $\mathrm{Pol}^{(m)} R := \bigcap \{\mathrm{Pol}^{(m)}\rho : \rho \in R\}$ and define $\mathrm{Pol}^{<m} R$ in a similar way. If in these definitions we replace operations by partial operations, we obtain successively the sets $\underline{\boldsymbol{P}}_A^{(m)}$ and $\underline{\boldsymbol{P}}_A^{<m}$, $\mathrm{Polp}^{(m)}\rho$ and $\mathrm{Polp}^{<m}\rho$, $\mathrm{Polp}^{(m)} R$ and $\mathrm{Polp}^{<m} R$. For a set F of partial operations on A, put $\mathrm{Inv}^{(n)} F := \{\rho \in \boldsymbol{R}_A^{(n)}$: every $f \in F$ preserves $\rho\}$ and define similarly $\mathrm{Inv}^{<m} F$. In all these notations, we omit the exponent '$<m$' if $m = \omega$.

The relation of preservation induces a Galois connection between various sets of operations and sets of relations. For instance, the sets of the form $\mathrm{Pol}^{(m)} R$, $\mathrm{Inv}^{(n)} F$ are the Galois closed subsets of $\underline{O}_A^{(m)}$, $\boldsymbol{R}_A^{(n)}$, induced by *f preserves* ρ. Some of their general properties as well as intrinsic characterizations are in [19] for A finite, and in [25] for A infinite. We mention just the fact that all Pol R are clones and conversely each clone on A is of the form Pol R where R is a countable set of finitary relations if A is finite and $R = \{\rho\}$ where ρ is an m-ary relation on A, with $m = |A|$, if A is infinite.

Finally a clone C on A has the *n-interpolation property* if C contains each $f \in \underline{O}_A^{(m)}$ $(m = 1, \ldots)$ such that for every subset B of A^m, with size at most n, there is some $g_B \in C$ which coincides with f on B. In other words, to test whether f belongs to C it suffices to verify that for every $B := \{a_1, \ldots, a_n\} \subseteq A^m$ we have $f(a_i) = g_B(a_i)$, $(i = 1, \ldots, n)$ for some $g_B \in C$. We mention in passing that clones with the n-interpolation property for all finite n, called *local clones*, play an important role in universal algebra.

2.4. The clone of contracting operations

Let L be a lattice; let us recall [12] that an element $v \in L$ is *compact* if $v \leqslant \mathrm{Sup}\{v_i: i \in I\}$ implies $v \leqslant \mathrm{Sup}\{v_i: i \in J\}$ for some finite subset J of I; denote by $c(L)$ the set of compact elements of L. The lattice L is *algebraic* if it is complete and every element is a supremum of a set (possibly infinite) of compact elements; it is *trivial* if it has one element.

Let $\underline{V}$ be an ordered set and let (A,δ) be an n-space over $\underline{V}$. As in Section 1.3, for $v\in V$ we put $(\delta)_v := \{a\in A^n\colon \delta(a)\leqslant v\}$. Clearly the map $\phi: v\to(\delta)_v$ is an order-preserving map from $\underline{V}$ into $(\boldsymbol{R}_A^{(n)}, \subseteq)$. We say that (A,δ) is *algebraic* if: (a1) $\underline{V}$ is a nontrivial algebraic lattice, (a2) $c(\underline{V})$ contains $\delta(A^n)$ and is finitely sup-generated by $\delta(A^n)$, and (a3) ϕ is injective.

Proposition 2.4.1. *The following are equivalent for $C\subseteq \underline{\boldsymbol{O}}_A$ and a positive integer n:*

(i) *C is the set of δ-contracting operations of an n-space (A,δ) over an ordered set $\underline{V}$;*

(ii) *C is the set of δ-contracting operations of an algebraic n-space (A,δ);*

(iii) *$C=\operatorname{Pol} R$ for a set R of n-ary relations on A;*

(iv) *C is a clone with the n-interpolation property.*

Proof. We prove (ii) ⇒(i)⇒(iii)⇒(iv)⇒(ii). The equivalence of (iii) and (iv) is known (cf. [27]), but for reader's convenience we prove it here.

(ii)⇒(i) Evident.

(i)⇒(iii) Evident: take $R := \{(\delta)_v\colon v\in V\}$.

(iii)⇒(iv) Let $f\in \underline{O}_A^{(m)}$ be such that on every subset B of A^m with size at most n, the operation agrees with some $g_B\in C$. We prove that $f\in C$. Let $\rho\in R$ and let X be an $n\times m$ matrix with $X_{*1},\ldots,X_{*m}\in\rho$ and let B be the set consisting of $X_{1*},\ldots,X_{n*}$. By assumption there is $g_B\in C\subseteq\operatorname{Pol}\rho$ such that $(f(X_{1*}),\ldots,f(X_{n*}))=(g_B(X_{1*}),\ldots, g_B(X_{n*}))\in\rho$ proving that $f\in\operatorname{Pol}\rho$. Thus $f\in\bigcap\{\operatorname{Pol}\rho\colon \rho\in R\}=C$.

(iv)⇒(ii) Put $V := \operatorname{Inv}^{(n)} C$ and $\underline{V} := (V,\subseteq)$. It is well known and easy to see (cf. Section 2.3) that V is the set of subuniverses of the nth power of the algebra (A, C) and hence a non-trivial algebraic lattice (since the empty set and A^n belong to V) (cf. [12] Section 0.6). For $a\in A^n$ put $\delta(a) := \bigcap\{\rho\in V\colon a\in\rho\}$. Clearly $\delta(a)$, called sometimes the *orbit* of a, is the least n-ary relation from V (i.e preserved by all operations from C) containing a. For $v\in V$, by the minimality of $\delta(a)$ we have $(\delta)_v := \{a\in A^n\colon \delta(a)\leqslant v\}=v$ whence $\phi: v\to(\delta)_v$ is obviously injective. As it is well known, the members of $c(\underline{V})$ are exactly the finitely generated subalgebras of the nth power of (A, C). Consequently (A,δ) is algebraic.

We next show that C is the set $C_{\delta\underline{V}}$ of δ-contracting operations.

(a) We prove $C\subseteq C_{\delta\underline{V}}$ (even without (iv)). Let $f\in C$ be m-ary and let X be an $n\times m$ matrix over A. Put $\rho := \delta(X_{*1})\vee\cdots\vee\delta(X_{*m})$ and note that $X_{*i}\in\delta(X_{*i})\subseteq\rho$ for $i=1,\ldots,m$. Since $f\in C$ preserves $\rho\in V=\operatorname{Inv}^{(n)} C$, we have

$$(f(X_{1*}),\ldots,f(X_{n*}))\in\rho$$

proving the required $\delta(f(X_{1*}),\ldots,f(X_{n*}))\subseteq\rho$.

(b) To prove $C_{\delta\underline{V}}\subseteq C$ let $f\in C_{\delta\underline{V}}$ be m-ary and let X be an $n\times m$ matrix over A. Put

$$\sigma := \{(g(X_{1*}),\ldots,g(X_{n*}))\colon g\in C\cap\underline{O}_A^{(m)}\}. \tag{2.5}$$

It is known and it may be shown easily that $\sigma\in\operatorname{Inv}^{(n)} C$. Now the m-ary projection e_j^m belongs to C and so $X_{*i}\in\sigma$ for $i=1,\ldots,m$; whence $\delta(X_{*1})\vee\ldots\vee\delta(X_{*m})\subseteq\sigma$. Finally from $f \in C_{\delta \underline{V}}$ we get

$$\delta(f(X_{1*}),\dots,f(X_{n*})) \leqslant \delta(X_{*1}) \vee \cdots \vee \delta(X_{*m}) \subseteq \sigma$$

and hence from $(f(X_{1*}),\dots,f(X_{n*})) \in \delta(f(X_{1*}),\dots,f(X_{n*}))$ and (2.5) we get $(f(X_{1*}),\dots,f(X_{n*}))=(g(X_{1*}),\dots,g(X_{n*}))$ for some $g \in C$. Applying the n-interpolation property we get the required $f \in C$ and so $C=C_{\delta \underline{V}}$. □

Example 2.4.2. Consider the case $n=1$. The unary relation $(\delta)_v$ is a subset of A. According to Proposition 2.4.1, the clone C consists of all operations admitting all $(\delta)_v$ as subuniverses (such an algebra (A, C) with $1<|A|<\omega$ is called semi-primal in [9]). It follows from [3] (cf. [12, 0.9 Theorem 2]]) that $\underline{V}$ from Proposition 2.4.1(iv) may be any non-trivial algebraic lattice. Examples or $n=2$ will be discussed in Section 3.

2.5. Invariants of the clone of contracting operations

We have seen in Proposition 2.4.1 that the clone $C_{\delta \underline{V}}$ is the clone Pol R determined by a set R of n-ary relations on A. What can be said about the structure of Inv $C_{\delta \underline{V}}$? In general there is not much we can say. Nonetheless, we note a sufficient condition for a k-ary relation ρ to be preserved by all δ-contracting maps, wherein k need not be finite.

Let (A, δ) be an n-space over a join-semilattice $\underline{V}$. For $r\colon \underline{k} \to A$ define the following n-space $(\underline{k}, \delta_r)$ over $\underline{V}$ by setting

$$\delta_r(k_1,\dots,k_n) := \delta(r(k_1),\dots,r(k_n)) \tag{2.6}$$

for all $k_1,\dots,k_n<k$. Note that $W := V^{\underline{k}^n}$ inherits the semilattice structure from $\underline{V}$: for μ, $\mu' \in V^{\underline{k}^n}$ the supremum $\mu \vee \mu'$ is defined by $(\mu \vee \mu')(x) := \mu(x) \vee \mu'(x)$ for all $x \in V^{\underline{k}^n}$. We say that a k-ary relation ρ on A is *δ-closed* if for each positive integer m, all $r_1,\dots,r_m \in \rho$ and all $r \in A^{\underline{k}}$

$$\delta_r \leqslant \delta_{r_1} \vee \cdots \vee \delta_{r_m} \text{ implies } r \in \rho. \tag{2.7}$$

For example, if $m=1$ the condition (2.7) states that $r \in \rho$ whenever $\delta_r \leqslant \delta_{r_1}$ for some $r_1 \in \rho$. According to (2.6) this means that

$$\delta(r(k_1),\dots,r(k_n)) \leqslant \delta(r_1(k_1),\dots,r_1(k_n)) \tag{2.8}$$

holds for all $k_1,\dots,k_n<k$. We write $r \preccurlyeq r_1$ if (2.8) holds. Note that $\preccurlyeq$ is a quasi-order (i.e. a reflexive and transitive relation). In fact, if D denotes the map from $A^{\underline{k}}$ to $V^{\underline{k}^n}$ defined by $D(r)=\delta_r$ for all $r \in A^{\underline{k}}$, then this quasi-order is the inverse image of the order on $V^{\underline{k}^n}$ discussed above. Let $\underline{W}$ denote the semilattice consisting of $V^{\underline{k}^n}$ equipped with this order. Recall that a subset I of a join-semilattice L is an *ideal* if $x \leqslant y$ and $y \in I$ implies $x \in I$ and $x, y \in I$ implies $x \vee y \in I$. First we slightly reformulate the definition of δ-closure. To a set I of maps from $\underline{k}^n$ to V associate $D^{-1}(I) := \{r \in A^{\underline{k}}: \delta_r \in I\}$.

Lemma 2.5.1. *A k-ary relation ρ on A is δ-closed if and only if $\rho = D^{-1}(I)$ for some ideal I of $\underline{V}^{\underline{k}^n}$.*

Proof. Necessity. Put $J := \{\delta_r \colon r \in \rho\}$ and let I be the ideal of $\underline{W}$ generated by J. We show that $\rho = D^{-1}(I)$. Indeed, let $r \in D^{-1}(I)$. Then there exist a positive integer m and $r_1, \ldots, r_m \in \rho$ such that $\delta_r \leqslant \delta_{r_1} \vee \cdots \vee \delta_{r_m}$ (in $\underline{W}$), whence $r \in \rho$ by (2.7). Conversely, if $r \in \rho$ then $\delta_r \in J \subseteq I$ and $r \in D^{-1}(I)$.

Sufficiency. Let I be an ideal of $\underline{W}$, $m > 0$, and $r_1, \ldots, r_m \in D^{-1}(I)$. Let r be such that $\delta_r \leqslant \delta_{r_1} \vee \cdots \vee \delta_{r_m}$. Then $\delta_{r_1}, \ldots, \delta_{r_m} \in I$, hence $\delta_r \in I$ and $r \in D^{-1}(I)$, proving that $D^{-1}(I)$ is δ-closed. □

The δ-closed relations are described in the following very special case. For $r \in A^{\underline{k}}$ put $(r] := \{r' \in A^{\underline{k}} \colon r' \preccurlyeq r\}$ (where $\preccurlyeq$ is the quasi-order defined above), that is $(r] = D^{-1}((\delta_r])$, where $(\delta_r] := \{\mu \in V^{\underline{k}^n} \colon \mu \preccurlyeq \delta_r\}$.

Corollary 2.5.2. *Let $\underline{V}$ be a complete join-semilattice in which every element is compact (e.g. the chain of negative integers). Then a k-ary relation ρ is δ-closed if and only if $\rho = (r]$ for some $r \colon \underline{k} \to A$.*

We show that $\rho \in \mathrm{Inv}^{(k)} \mathrm{C}_{\delta \underline{V}}$ for a δ-closed ρ.

Proposition 2.5.3. *If ρ is δ-closed then every δ-contracting operation on A preserves ρ.*

Proof. Let f be an m-ary δ-contracting operation on A, let $r_1, \ldots, r_m \in \rho$ and $k_1, \ldots, k_n < k$. Define the map $r := f(r_1, \ldots, r_m) \in A^{\underline{k}}$ by setting $r(k') := f(r_1(k'), \ldots, r_m(k'))$ for all $k' < k$. Let X denote the $n \times m$ matrix $(r_j(k_i))_{ij}$. Since f is δ-contracting, we have:

$$\begin{aligned}\delta(r(k_1), \ldots, r(k_n)) &= \delta(f(X_{1*}), \ldots, f(X_{n*})) \leqslant \delta(X_{*1}) \vee \cdots \vee \delta(X_{*m}) \\ &= \delta(r_1(k_1), \ldots, r_1(k_n)) \vee \cdots \vee \delta(r_m(k_1), \ldots, r_m(k_n))\end{aligned}$$

proving by (2.6) that $\delta_r \leqslant \delta_{r_1} \vee \cdots \vee \delta_{r_m}$.

Now (2.7) yields the required $r \in \rho$. □

In a special situation $\mathrm{Inv}^{(k)} \mathrm{C}_{\delta \underline{V}}$ may be the set of δ-closed relations. We describe this in terms of the following extension property. A partial m-ary operation f on A with domain B is *δ-contracting* if it is a contraction from the n-ary space induced on B by the m-power of (A, δ), that is to say: for every $n \times m$ matrix X whose rows are all in B we have $\delta(f(X_{1*}), \ldots, f(X_{n*})) \leqslant \delta(X_{*1}) \vee \ldots \vee \delta(X_{*m})$. Let κ be a cardinal. We say that (A, δ) has the *κ-extension property* if for every $m > 0$ each δ-contracting partial m-ary operation $f \colon B \to A$ with $|B| \leqslant \kappa$ extends to (or equivalently, is a restriction of) a full δ-contracting operation. We have the following.

Proposition 2.5.4. *Let (A, δ) be an n-space over a semilattice $\underline{V}$; let $\kappa > 0$ be a cardinal and let k be an ordinal such that $|k| = \kappa$.*

(i) *if all relations in* $\mathrm{Inv}^{(k)} C_{\delta \underline{V}}$ *are* δ*-closed, then* (A, δ) *has the* κ*-extension property; and*

(ii) *if* $n > 1$, (A, δ) *has the* κ*-extension property and there is an element* $0 \in V$ *such that*

$$\delta(a_1, \ldots, a_n) \leqslant 0 \text{ iff } a_1 = \cdots = a_n \quad \text{for all } a_1, \ldots, a_n \in A, \tag{2.9}$$

then all relations in $\mathrm{Inv}^{(k)} C_{\delta \underline{V}}$ *are* δ*-closed.*

Proof. Write C for $C_{\delta \underline{V}}$.

(i) Let f be a partial m-ary δ-contracting operation on A with domain B, where $|B| \leqslant \kappa$. If B is empty, then any projection, e.g. e_1^m, extends f. Thus, let B be nonempty, let $\phi: \underline{k} \to B$ be surjective. Put $\sigma := \{g \circ \phi : g \in C \cap O_A^{(m)}\}$. As mentioned in the proof of Proposition 2.4.1 (cf. [2.5]) it is known, and may be checked easily, that the k-ary relation σ belongs to $\mathrm{Inv}\, C$ and so σ is δ-closed. Put $r := f \circ \phi$ and $r_j := e_j^m \circ \phi$ where e_j^m is the jth m-ary projection $(j = 1, \ldots, m)$. Taking into account that $e_j^m \in C$ we have $r_1, \ldots, r_m \in \sigma$. We verify that $\delta_r \leqslant \delta_{r_1} \vee \cdots \vee \delta_{r_m}$. Indeed, let $k' = (k_1, \ldots, k_n) \in \underline{k}^n$, and let X be the $n \times m$ matrix with rows $\phi(k_1), \ldots, \phi(k_n)$. According to (2.6) $\delta_r(k') = \delta(r(k_1), \ldots, r(k_n)) = \delta(f(X_{1*}), \ldots, f(X_{n*}))$. Similarly for $j = 1, \ldots, m$ we have $\delta_{r_j}(k') = \delta(r_j(k_1), \ldots, r_j(k_n)) = \delta(X_{*j})$. Since f is δ-contracting, we have $\delta_r(k') \leqslant \delta_{r_1}(k') \vee \cdots \vee \delta_{r_m}(k')$ and so $\delta_r \leqslant \delta_{r_1} \vee \cdots \vee \delta_{r_m}$. As σ is δ-closed and $r_1, \ldots, r_m \in \sigma$, from (2.7) we get $r \in \sigma$, hence $f \circ \phi = \underline{f} \circ \phi$ for some $\underline{f} \in C$ where $\underline{f}$ is the required extension of f.

(ii) Let $\rho \in \mathrm{Inv}^{(k)} C_{\delta \underline{V}}$, let $r_1, \ldots, r_m \in \rho$ and $\delta_r \leqslant \delta_{r_1} \vee \cdots \vee \delta_{r_m}$. Define $\psi: \underline{k} \to A^m$ by setting $\psi(k') := (r_1(k'), \ldots, r_m(k'))$ for all $k' \in \underline{k}$. Further put $B := \mathrm{im}\, \psi$. We show that $\mathrm{Ker}\, \psi \subseteq \mathrm{Ker}\, r$. Indeed, let $\psi(k') = \psi(k'')$. Setting $a_j := r_j(k') = r_j(k'')$ for $j = 1, \ldots, m$ and using the assumption we get $\delta(r_j(k'), r_j(k''), \ldots, r_j(k'')) = \delta(a_j, \ldots, a_j) \leqslant 0$ for all $j = 1, \ldots, m$. From $\delta_r \leqslant \delta_{r_1} \vee \cdots \vee \delta_{r_m}$ we get $\delta(r(k'), r(k''), \ldots, r(k'')) \leqslant 0 \vee \cdots \vee 0 = 0$. Again from the assumption we get the required $r(k') = r(k'')$.

Now we define a partial m-ary operation $f: B \to A$ by setting $f \circ \psi = r$. We show that f is δ-contracting. Indeed, let X be an $n \times m$ matrix whose rows are all in B. Then $X_{i*} = \psi(k_i)$ for some $k_i \in \underline{k}$ $(i = 1, \ldots, m)$, and from $f \circ \psi = r$ and $\delta_r \leqslant \delta_{r_1} \vee \cdots \vee \delta_{r_m}$ we get the required

$$\begin{aligned} \delta(f(X_{1*}), \ldots, f(X_{n*})) &= \delta(f(\psi(k_1)), \ldots, f(\psi(k_n))) = \delta(r(k_1), \ldots, r(k_n)) \\ &\leqslant \delta(r_1(k_1), \ldots, r_1(k_n)) \vee \cdots \vee \delta(r_m(k_1), \ldots, r_m(k_n)) \\ &= \delta(X_{*1}) \vee \cdots \vee \delta(X_{*m}). \end{aligned}$$

In view of $|B| \leqslant \kappa$ and the assumption, the partial operation f has a full δ-contracting extension $\underline{f}$. Since $\rho \in \mathrm{Inv}^{(k)} C$, we have $\underline{f}(r_1, \ldots, r_m) \in \rho$ proving the required $r = f \circ \psi = \underline{f}(r_1, \ldots, r_m) \in \rho$. $\square$

2.6. The extension property

The following property is stronger than the κ-extension property. We say that (A, δ) has the *extension property* if every partial δ-contracting operation extends to a full

δ-contracting operation (i.e. (A,δ) has the κ-extension property for all κ). We give equivalent conditions for the extension property. Let f: $B \to A$ be a partial m-ary δ-contracting operation. If for some $z \in A^m \setminus B$ there is a δ-contracting extension f': $B \bigcup \{z\} \to A$ of f we call f' a *one-point extension* of f. We say that (A,δ) satisfies the condition $(*)$ provided:

if $n>1$, *then* δ is totally symmetric (invariant under all exchanges of variables) and there is $0 \in V$ such that $\delta(a_1,\ldots,a_n)=0$ whenever $a_1=a_2$.

We have the following.

Proposition 2.6.1. *The following are equivalent for an n-space* (A,δ):

(i) (A,δ) *has the extension property;*

(ii) *each partial* δ*-contracting operation with a proper domain has a one point extension;*

(iii) *for every partial m-ary* δ*-contracting operation* $f: B \to A$ *and each* $z \in A^m \setminus B$ *there is a* δ*-contracting extension* $f': B \bigcup \{z\} \to A$.

If, moreover, (A,δ) *satisfies the condition* $(*)$ *then the above conditions are also equivalent to:*

(iv) *for every partial m-ary* δ*-contracting operation* $f: B \to A$, *for each* $\phi: B^{n-1} \to V$ *and every* $z \in A^m \setminus B$ *such that*

$$\delta(X_{*1}) \vee \cdots \vee \delta(X_{*m}) \leqslant \phi(X_{2*},\ldots,X_{n*}) \tag{2.10}$$

holds for every $n \times m$ *matrix* X *with first row* z *and the other rows in* B, *there exists* $t \in A$ *such that*

$$\delta(t, f(x_2),\ldots,f(x_n)) \leqslant \phi(x_2,\ldots,x_n) \tag{2.11}$$

holds for all $x_2,\ldots,x_n \in B$.

Proof. (i)$\Rightarrow$(iii)$\Rightarrow$(ii) is obvious. (ii)$\Rightarrow$(i). Let $f: B \to A$ a partial m-ary δ-contracting operation. We can well order the set $A^m \setminus B$ and, by transfinite induction, we can easily extend f to A^m.

Let (A,δ) be such that the condition $(*)$ holds.

(i)$\Rightarrow$(iv). Let f, ϕ and z be as in (iv) and let $\hat{f}$ be a δ-contracting extension of f. Put $t:=\hat{f}(z)$, let $x_2,\ldots,x_n \in B$ and let X be the $n \times m$ matrix with rows $z, x_2,\ldots,x_n$. Since $\hat{f}$ is δ-contracting and (2.10) holds, we have

$$\delta(t, f(x_2),\ldots,f(x_n)) = \delta(\hat{f}(X_{1*}),\ldots,\hat{f}(X_{n*})) \leqslant \delta(X_{*1}) \vee \cdots \vee \delta(X_{*m})$$
$$\leqslant \phi(X_{2*},\ldots,X_{n*}) = \phi(x_2,\ldots,x_n)$$

proving (2.11).

(iv)$\Rightarrow$(iii). Let f and z be as in (iii). Given an $n \times m$ matrix X with the first row z and the other rows in B put $\phi(X_{2*},\ldots,X_{n*}) := \delta(X_{*1}) \vee \cdots \vee \delta(X_{*m})$. Then (2.10) holds and so by assumption there is t satisfying (2.11) for all $x_2,\ldots,x_n \in B$. Extend f to $f': B \bigcup \{z\} \to A$ by setting $f'(z) := t$. We verify that f' is contracting; due to $(*)$ it suffices to consider an $n \times m$ matrix with first row z and the other rows in B. The required inequality is then a consequence of (2.10) and our choice of ϕ. □

We have the following sufficient condition for one-point extensions.

Lemma 2.6.2. *Let (A, δ) satisfy the condition $(*)$. Let B be a subset of A^m, $u \in B$ and $z \in A^m \setminus B$ be such that*

$$\delta(X_{*1}) \vee \cdots \vee \delta(X_{*m}) \leqslant \delta(Y_{*1}) \vee \cdots \vee \delta(Y_{*m}) \tag{2.12}$$

for all $m \times n$ matrices X and Y with $X_{1*} := u$, $Y_{1*} := z$ and $X_{j*} = Y_{j*} \in B$ $(j = 2, \ldots, n)$. Then every δ-contracting f with domain B extends to a δ-contracting f' with domain $B \bigcup \{z\}$.

Proof. Extend f to f' by setting $f'(z) := f(u)$. To prove that f' is δ-contracting consider two $m \times n$ matrices X and Y with $X_{1*} := u$, $Y_{1*} := z$ and $X_{j*} = Y_{j*} \in B$ $(j = 2, \ldots, n)$. As f is δ-contracting and (2.12) holds, we have

$$\begin{aligned}\delta(f'(X_{1*}), \ldots, f'(X_{n*})) = \delta(f(Y_{1*}), \ldots, f(Y_{n*})) &\leqslant \delta(X_{*1}) \vee \cdots \vee \delta(X_{*m}) \\ &\leqslant \delta(Y_{*1}) \vee \cdots \vee \delta(Y_{*m}). \quad \square\end{aligned}$$

2.7. The extension property and the invariants of partial operations

For a set K of ordinals and a set R of relations put $\mathrm{Pol}^K R := \bigcup_{k \in K} \mathrm{Pol}^{(k)} R$ and $\mathrm{Polp}^K R := \bigcup_{k \in K} \mathrm{Polp}^{(k)} R$. For a cardinal κ, let $\mathrm{Polp}^{K, \leqslant\kappa} R$ denote the set of all $f \in \mathrm{Polp}^K R$ with domain of size $\leqslant \kappa$.

Note that $\mathrm{Inv}^{(k)} \mathrm{Pol}^K R$ consist of the k-ary relations on R preserved by every operation on A of arity $l \in K$ which in turn, preserves all $\rho \in R$. Since $\mathrm{Pol}\, R \subseteq \mathrm{Polp}\, R$ we have always

$$\mathrm{Inv}^{(k)} \mathrm{Pol}^K R \supseteq \mathrm{Inv}^{(k)} \mathrm{Polp}^K R. \tag{2.13}$$

We relate the extendability of all partial operations on A to the equality in (2.13).

Proposition 2.7.1. *The following conditions are equivalent for a set R of relations on A, a set K of ordinals and a cardinal κ:*

(i) *every operation from* $\mathrm{Polp}^{K, \leqslant\kappa} R$, *is a restriction of a (full) operation from* $\mathrm{Pol}\, R$,
(ii) $\mathrm{Inv}^{(\kappa)} \mathrm{Pol}^K R = \mathrm{Inv}^{(\kappa)} \mathrm{Polp}^{K, \leqslant\kappa} R$,
(iii) $\mathrm{Inv}^{(\kappa)} \mathrm{Pol}^K R = \mathrm{Inv}^{(\kappa)} \mathrm{Polp}^K R$.

Proof. Put $C := \mathrm{Pol}^K R$ and $E := \mathrm{Polp}^K R$.

(iii)⇒(i) Let $f \in \mathrm{Polp}^{K, \leqslant\kappa} R$ have domain $D \subseteq A^m$. We may assume D nonempty; as $|D| \leqslant \kappa$, there is surjection s from κ onto D. Put $\rho := \{g \circ s \colon g \in C \cap O_A^{(m)}\}$. Clearly ρ is a κ-ary relation on A. We show that $\rho \in \mathrm{Inv}^{(\kappa)} C$. Let $g \in C$ be p-ary and let $r_1, \ldots, r_p \in \rho$. Then $r_i = g_i \circ s$ for m-ary $g_i \in C$ $(i = 1, \ldots, p)$. Define $h \in O_A^{(m)}$ by setting $h(a) := g(g_1(a), \ldots, g_p(a))$ for all $a \in A^m$. As C is a clone, we have $h \in C$ and $g(r_1, \ldots, r_p) = h \circ s \in \rho$ proving our claim. By (iii) we have $\rho \in \mathrm{Inv}^{(\kappa)} C \subseteq \mathrm{Inv}^{(\kappa)} E$. Since the projections e_i^m belong to C, we have $s_i := e_i^m \circ s \in \rho$ $(i=1,\ldots,m)$. Now the partial operation $f \in E$ preserves ρ and therefore $f(s_1,\ldots,s_m) \in \rho$. By the definition of ρ, we have $f(s_1,\ldots,s_m) = \hat{f} \circ s$ for some $\hat{f} \in C$. It is easy to verify that $\hat{f}$ agrees with f on D and so, $\hat{f}$ is the required extension.

(i)⇒(ii) As remarked in (2.13) we have always $\supseteq$ in (ii). To prove $\subseteq$, let $\rho \in \mathrm{Inv}^{(\kappa)} C$. Suppose $f \in E$ has domain $D \in A^m$ such that $|D| \leqslant \kappa$. Applying (i), extend f to $\hat{f} \in C$. Suppose $r_1,\ldots,r_m \in \rho$ are such that $(r_1(k),\ldots,r_m(k)) \in D$ for all $k \in \underline{\kappa}$. In view of $h := f(r_1,\ldots,r_p) = \hat{f}(r_1,\ldots,r_m) \in \rho$ we get the required $\rho \in \mathrm{Inv}^{(\kappa)} E$.

(ii)⇒(iii) Again only $\subseteq$ is needed. Let $\rho \in \mathrm{Inv}^{(\kappa)} C$. Let $f \in E$ be m-ary with domain D and let $r_1,\ldots,r_m \in \rho$ be such that $D' := \{(r_1(k),\ldots,r_m(k)) : k \in \underline{\kappa}\} \subseteq D$. Denoting by f' the restriction of f to D', clearly $f' \in \mathrm{Polp}_{\leqslant\kappa} R$. By (ii) we have $f(r_1,\ldots,r_m) = f'(r_1,\ldots,r_m) \in \rho$, proving $\rho \in \mathrm{Inv}^{(k)} E$. □

3. Metric and ultrametric spaces

3.1. Metric over an ordered monoid

3.1.1. Let $\underline{V} = (V; \leqslant, +, 0, -)$ be such that

(i) $(V; +, 0, \leqslant)$ is an ordered monoid (i.e. the binary operation $+$ is associative and 0 its neutral element and $p \leqslant p'$ and $q \leqslant q'$ implies $p+q \leqslant p'+q'$; note that $+$ need not be commutative).

(ii) $v \to \bar{v}$ is an automorphism of $\leqslant$ which is involutive and reverses $+$ in the sense that $\bar{\bar{v}} = v$, $\overline{(v+w)} = \bar{w} + \bar{v}$ holds for all $v, w \in V$.

Note that $\bar{0} = 0$ follows easily from (i) and (ii).

A *$\underline{V}$-predistance* on A is a map $d : A^2 \to V$ satisfying:

(d1) $d(x,x) \leqslant 0$;

(d2) $d(x,y) \leqslant d(x,z) + d(z,y)$ (the *triangle inequality* or *△-inequality*);

(d3) $d(y,x) = \overline{d(x,y)}$

for all $x, y, z \in A$.

The pair (A,d) is called a *$\underline{V}$-premetric* space. A pair (A,d) satisfying (d1′) $d(x,y) \leqslant 0 \Leftrightarrow x = y$ and (d2) and (d3) is called a *$\underline{V}$-metric space* or shortly a *$\underline{V}$-metric* and d is referred to as *$\underline{V}$-distance*. As usual, the same letter d may represent different $\underline{V}$-distances provided there is no danger of confusion.

3.1.2. Let $\underline{A} = (A,d)$ and $\underline{A}' = (A',d')$ be binary spaces. Recall that $f : A \to A'$ is a *contraction from $\underline{A}$ to $\underline{A}'$* provided $d'(f(x), f(y)) \leqslant d(x,y)$ holds for all $x, y \in A$. The map f is an *isometry* if it is injective and

$$d'(f(x), f(y)) = d(x,y) \quad \text{for all } x, y \in A. \tag{3.1}$$

Observe that for a $\underline{V}$-metric space $\underline{A}$ and a $\underline{V}$-premetric space $\underline{A}'$ a map f satisfying (3.1) is injective (indeed, if $f(x) = f(y)$, then $0 \geqslant d'(f(x), f(y)) = d(x,y)$ implies $x = y$). In real metric spaces contractions are usually termed *non-expansive maps*.

For a $\underline{V}$-metric space $\underline{A}=(A,d)$ and $B\subset A$ let $d\restriction_B$ denote the restriction of d to B (i.e. $d\restriction_B$ maps B^2 into V and $(d\restriction_B)(x,y):=d(x,y)$ for all $x,y\in B$). Call $(B,d\restriction_B)$ the *subspace induced* by B (or an *induced subspace*). Usually we write (B,d) instead of $(B,d\restriction_B)$.

The spaces considered in [13] are distinguished among $\underline{V}$-metric spaces by the fact that the neutral element 0 (of the operation +) is also a least element of $(V,\leqslant)$. In this case the Property (d1) reduces to the standard property $d(x,y)=0 \Leftrightarrow x=y$. Such spaces are discussed in detail in [13] together with several combinatorial and other applications.

The $\underline{V}$-premetric and the $\underline{V}$-metric spaces are closely related: Given a $\underline{V}$-premetric space (B,δ) define a binary relation $\approx$ on B by setting $b\approx b'$ whenever $d(b,b')\leqslant 0$. It is completely straightforward to verify that, modulo $\approx$, the $\underline{V}$-premetric space (B,δ) becomes a $\underline{V}$-metric space.

Lemma 3.1.3. *The relation $\approx$ is an equivalence relation on B. The map δ is constant on each product $C_1\times C_2$ of blocks (i.e. equivalence classes) C_1 and C_2 of $\approx$. If $D\subseteq B$ and δ' and $\approx'$ are the restrictions of δ and $\approx$ to D and f a contraction from (D,δ') into a $\underline{V}$-metric space (A,d), then $f(x)=f(x')$ whenever $x\approx x'$.*

In view of Lemma 3.1.3 we concentrate on $\underline{V}$-metric spaces. The $\underline{V}$-premetric spaces are needed only in the proof of Theorem 3.3.4.

3.2. The possible values of a distance

3.2.1. For the main result of this section we need the following rather technical concepts. Let $v\in V$. Call v *idempotent* if $v+v=v$, *selfdual* if $\bar{v}=v$ and *small* whenever v is simultaneously idempotent, selfdual and $v\leqslant 0$; we denote by V_σ the set of small elements of V. Call v a *distance value* if $v=d(x,y)$ for some $\underline{V}$-metric space (A,d) and $x,y\in A$. We have:

Lemma 3.2.2. *An element $v\in V$ is small if and only if $v=d(a,a)$ for some $\underline{V}$-metric space (A,d) and $a\in A$.*

Proof. ($\Rightarrow$) Consider $(\{v\},d)$ where $d(v,v)=v$. The axioms of a $\underline{V}$-metric space are satisfied due to $v\leqslant 0$, $v=v+v$ and $v=\bar{v}$.

($\Leftarrow$) The element $v=d(a,a)$ satisfies $v\leqslant 0$ by (d1). Next

$$v:=d(a,a)\leqslant d(a,a)+d(a,a)=v+v\leqslant v+0=v$$

by (d2), isotony and $v\leqslant 0$. Moreover, $\bar{v}=\overline{d(a,a)}=d(a,a)=v$. Thus v is small. □

Lemma 3.2.3. *An element $v\in V$ is a distance value if and only if*

$$v+\bar{v}\geqslant a,\ \bar{v}+v\geqslant b,\ a+v=v=v+b \tag{3.2}$$

for some small $a,b\in V$.

Proof. ($\Rightarrow$) Let (A,d) be a $\underline{V}$-metric space and $v=d(x,y)$ for some $x,y\in A$. Put $a=d(x,x)$ and $b=d(y,y)$. By Lemma 3.2.2 both a and b are small. We have

$$a=d(x,x)\leqslant d(x,y)+d(y,x)=v+\bar{v}$$

and similarly $b\leqslant\bar{v}+v$. Using $a\leqslant 0$ we obtain

$$v=d(x,y)\leqslant d(x,x)+d(x,y)=a+v\leqslant v$$

proving $a+v=v$. The equality $v=v+b$ is derived in a similar way.

($\Leftarrow$) Let v,a,b satisfy (3.2) and let a,b be small. We have two cases:

(1) Let $v\nleqslant 0$. Put $A=\{0,1\}$ and define d: $\{0,1\}\to V$ by setting,

$$d(0,0):=a,\ d(0,1):=v,\ d(1,0):=\bar{v},\ d(1,1):=b.$$

Using (3.2) and a,b small, it may be verified directly that d is a predistance. For example, $d(0,0)=a\leqslant 0$, $d(1,1)=b\leqslant 0$. Now, if $d(x,y)\leqslant 0$, then $d(x,y)\in\{a,b\}$ due to $v\nleqslant 0$, and $\bar{v}\nleqslant\bar{0}=0$ and so (d1′) holds.

(2) Let $v\leqslant 0$. Then also $\bar{v}\leqslant 0$ and from (3.2) we have:

$$v=a+v\leqslant a+0=a\leqslant v+\bar{v}\leqslant v$$

proving $v=a$. Thus v is small and so a distance value by Lemma 3.2.2. □

Put $I:=\{u\in V\colon u+u=u\leqslant 0\}$. Note that $u\in I$ implies $\bar{u}\in I$, hence $\bar{I}\subseteq I=\bar{\bar{I}}\subseteq\bar{I}$ shows $I=\bar{I}$. We have the following.

Lemma 3.2.4. *Let T be a non empty subset of I. If $s:=\operatorname{Sup}T$ exists then $s\in I$. If, moreover, $\bar{T}=T$ then $\bar{s}$ is selfdual.*

Proof. Clearly $s\leqslant 0$. For each $t\in T$ we have $t=t+t\leqslant s+s$ and hence $s\leqslant s+s$. As $s\leqslant 0$, we also have $s+s\leqslant s+0=s$. Now, $^-$ being an order automorphism, we have $\bar{s}=\operatorname{Sup}\{\bar{t}\colon t\in T\}=\operatorname{Sup}\bar{T}=\operatorname{Sup}T=s$. □

3.2.5. For $v\in V$, put $I(v):=\{u\in I\colon u\leqslant v\}$ and let $\lceil v\rceil:=\operatorname{Sup}I(v)$ provided it exists. Suppose $\lceil v\rceil$ exists. According to Lemma 3.2.4 the element $\lceil v\rceil$ is the greatest element of $I(v)$. We show that $\lceil\bar{v}\rceil$ exists and $\lceil\bar{v}\rceil=\overline{\lceil v\rceil}$. Indeed, since $^-$ is an order automorphism, we have:

$$\overline{\lceil v\rceil}=\operatorname{Sup}\{\bar{u}\colon u\in I,\ u\leqslant v\}=\operatorname{Sup}\{w\in I\colon\bar{w}\leqslant v\}=\operatorname{Sup}\{w\in I\colon w\leqslant\bar{v}\}=\lceil\bar{v}\rceil.$$

In particular, for v selfdual we have $\overline{\lceil v\rceil}=\lceil\bar{v}\rceil=\lceil v\rceil$ and, as $\lceil v\rceil\in I$, the element $\lceil v\rceil$ is small. Note that $b\leqslant\lceil v\rceil$ whenever $b\leqslant v$ and b is small.

We return to the distance values characterized in Lemma 3.2.3. Note that from (3.2) we get the necessary condition $v\leqslant a+v\leqslant v+\bar{v}+v$. The question is when the condition $v\leqslant v+\bar{v}+v$ is also sufficient (for v to be a distance value). We start with a technical lemma. For brevity put $v^0:=v+\bar{v}$.

Lemma 3.2.6. *Consider the following statements*:

(i) *both* $\lceil v^0 \rceil$ *and* $\lceil \bar{v}^0 \rceil$ *exist and* $\lceil v^0 \rceil + v = v = v + \lceil \bar{v}^0 \rceil$;

(ii) *both* $\lceil v^0 \rceil$ *and* $\lceil \bar{v}^0 \rceil$ *exist and* $\lceil v^0 \rceil + v + \lceil \bar{v}^0 \rceil = v$;

(iii) v *is a distance value.*

We have (i)⇒(ii)⇒(iii) *for each* $v \in V$. *Moreover, if* $\underline{V}$ *is such that* $\lceil v^0 \rceil$ *exists whenever* $v \leqslant v + \bar{v} + v$, *then* (i), (ii) *and* (iii) *are equivalent.*

Proof. (i)⇒(ii) $v = \lceil v^0 \rceil + v = \lceil v^0 \rceil + v + \lceil \bar{v}^0 \rceil$.

(ii)⇒(iii) Put $a := \lceil v^0 \rceil$ and $b := \lceil \bar{v}^0 \rceil$. From $v = a + v + b$ and a idempotent $a + v = a + a + v + b = a + v + b = v$ and similarly $v + b = v$. This proves (3.2) and (iii).

Suppose $\lceil v^0 \rceil$ exists whenever $v \leqslant v + \bar{v} + v$. We prove (iii)⇒(i).

Let v be a distance value. Then we have $v \leqslant v + \bar{v} + v$. Applying $^-$ to this we get $\bar{v} \leqslant \bar{v} + \bar{\bar{v}} + \bar{v}$. Consequently both $\lceil v^0 \rceil$ and $\lceil \bar{v}^0 \rceil$ exist. By Lemma 3.2.3 the relation (3.2) holds for some small a and b. Then we have $a \leqslant \lceil v^0 \rceil \leqslant 0$ and therefore $v = a + v \leqslant \lceil v^0 \rceil + v \leqslant v$.

The equality $v + \lceil \bar{v}^0 \rceil = v$ follows in a similar fashion and so (i) holds. □

In the sequel we adapt the standard notational convention: $u + v \wedge w$ stands for $u + (v \wedge w)$ and $v^\alpha := v \wedge 0$. In the next lemma we consider $\underline{V}$ such that $(V, \leqslant)$ is a meet-semilattice satisfying:

$$(D_l) \quad u + v^{0\alpha} = u \wedge (u + v^0) \quad \text{for all } u, v \in V. \tag{3.3}$$

Note that this law is equivalent to:

$$(D_r) \quad v^{0\alpha} + u = (v^0 + u) \wedge u \quad \text{for all } u, v \in V. \tag{3.4}$$

(Indeed, it suffices to replace u by $\bar{u}$ in (3.3) and apply the order automorphism $^-$ to both sides). We have the following lemma.

Lemma 3.2.7. *Let* $(V, \leqslant)$ *be a meet-semilattice and let* $\underline{V}$ *satisfy* (3.3). *The following are equivalent for* $v \in V$:

(i) v *is a distance value,*

(ii) $v \leqslant v + \bar{v} + v$,

(iii) $\lceil v^0 \rceil$ *exists and* $\lceil v^0 \rceil + v = v$.

If one of (i)–(iii) *holds then*

(iv) $v^{0\alpha}$ *is idempotent.*

Moreover, (iv) *implies*

(v) $\lceil v^0 \rceil$ *exists and* $\lceil v^0 \rceil = v^{0\alpha}$.

Proof. (i)⇒(ii) As noted in 3.2.5, (ii) follows from (3).

(ii)⇒(iv) Put $r := v^{0-}$,

$$v^0 + v^0 = v + \bar{v} + v + \bar{v} \geqslant v + \bar{v} = v^0, \qquad v^0 \wedge r = v^0 \wedge v^0 \wedge 0 = v^0 \wedge 0 = r. \tag{3.5}$$

Applying (3.3) (3.5) we get:

$$r+r=r+v^{0\alpha}=(r+v^0)\wedge r=(v^0+v^0)\wedge v^0\wedge r=v^0\wedge r=r,$$

(iv)⇒(v) Follows from (iv) and $r\leqslant 0$, $r\leqslant v^0$.

(ii)⇒(iii) From (v), (3.5) and (ii) we get

$$\lceil v^0 \rceil+v=v^{0\alpha}+v=(v^0+v)\wedge v=(v+\bar{v}+v)\wedge v=v.$$

(iii)⇒(ii) $v=\lceil v^0 \rceil+v\leqslant v^0+v=v+\bar{v}+v$.

(iii)⇒(i) Applying $^-$ to $v\leqslant v+\bar{v}+v$, we get $\bar{v}\leqslant\bar{v}+\bar{\bar{v}}+\bar{v}$ and so $\bar{v}$ also satisfies (ii). It follows that (iii) holds for both v and $\bar{v}$. Now the hypothesis as well as (ii) in Lemma 3.2.6 hold and so (i) holds. □

Remarks 3.2.8. (1) The implication (iii)⇒(ii) in Lemma 3.2.7 holds in every $\underline{V}$. We do not know whether (ii)⇒(iii) holds under other assumptions than those of Lemma 3.2.7.

(2) Suppose

$$\lceil v^0 \rceil \text{ exists and } v=a+v,\ a\leqslant v^0 \tag{3.6}$$

holds for some small $a\in V$. Then a small and $a\leqslant v^0$ implies $a\leqslant\lceil v^0 \rceil\leqslant 0$, hence $v=a+v\leqslant\lceil v^0 \rceil+v\leqslant v$ proving $\lceil v^0 \rceil+v=v$. Let the assumptions of Lemma 3.2.7 hold. Since $v=\lceil v^0 \rceil+v\leqslant v+\bar{v}+v$, from (ii)⇒(iv)⇒(v) we get that $v^{0\alpha}$ is idempotent and so $\lceil v^0 \rceil=v^{0\alpha}$. To see that an element a with the above properties may exist, consider $\underline{V}$ such that $(V,\leqslant)$ has a least element O' and let $v=O'+c$ for some $c\in V$. It is easy to see that O' is small and $O'+v=v$, $O'\leqslant v^0$. For every $O'\leqslant a\leqslant\lceil v^0 \rceil$, we have

$$v=O'+v\leqslant a+v\leqslant\lceil v^0 \rceil+v=v,\ a\leqslant\lceil v^0 \rceil\leqslant v^0$$

and so all small a in the interval $(O',\lceil v^0 \rceil)$ satisfy (3.6). For an example of such non trivial interval, let V be the set of languages over a finite alphabet A and v be the set of all non-empty words over A (cf. 4.3). Then $\lceil v^0 \rceil$ is the set of words of length different from 1. Such an example shows that we may have $d(a,b)=d(a',b')$, without $d(a,a)=d(a',a')$ and $d(b,b)=d(b',b')$.

Motivated by (iii) in Lemma 3.2.7 for $u\in V$ put

$$A_u:=\{v\in V\colon v\leqslant u,\ \lceil v^0 \rceil \text{ exists and } \lceil v^0 \rceil+v=v\}.$$

We have the following.

Lemma 3.2.9. *Let* $\lceil v^0 \rceil$ *exist. Then* $\lceil v^0 \rceil+v$ *is the greatest element of* A_v. *If, moreover,* $\underline{V}$ *satisfies the assumptions of Lemma* 3.2.7, *then* $\lceil v^0 \rceil+v$ *is the largest distance value* $\leqslant v$. *Finally, if* $\lceil \bar{v}^0 \rceil$ *also exists then*

$$\lceil v^0 \rceil+v=v+\lceil \bar{v}^0 \rceil=\lceil v^0 \rceil+v+\lceil \bar{v}^0 \rceil \tag{3.7}$$

Proof. Put $r:=\lceil v^0 \rceil$ and $w:=r+v$. By 3.2.5, the element r is selfdual and so r is small. Next $\bar{w}=\bar{v}+r$ and

$$r=r+r+r\leqslant r+v+\bar{v}+r=w+\bar{w}\leqslant v+\bar{v} \tag{3.8}$$

The set $I(u)$ has been introduced in 3.2.5. From (3.8) we get $r \in I(w+\bar{w}) \subseteq I(v+\bar{v})$. Moreover for every $x \in I(w+\bar{w})$ we have $x \leqslant \lceil v+\bar{v} \rceil = r$ and $\lceil w+\bar{w} \rceil$ exists and equals r. We have:

$$\lceil w+\bar{w} \rceil + w = r+r+v = r+v = w$$

and $w = r+v \leqslant v$ due to $r \leqslant 0$. Thus $w \in A_v$. Let $x \in A_v$ be arbitrary. Then $\lceil x+\bar{x} \rceil$ exists; from $x \leqslant v$ we have $\lceil x+\bar{x} \rceil \leqslant \lceil v+\bar{v} \rceil = r$ and $x = \lceil x+\bar{x} \rceil + x \leqslant r+v = w$ proving that w is the largest element of A_v. Let the assumptions of Lemma 3.2.7 hold. From (i)⇔(iii) in Lemma 3.2.7 we see that w is the largest distance value $\leqslant v$. Finally suppose that $\lceil \bar{v}^0 \rceil$ also exists. From what has just been shown, $\lceil \bar{v}^0 \rceil + \bar{v}$ is the largest distance value $\leqslant \bar{v}$. Noting that $\lceil \bar{v}^0 \rceil$ is selfdual and applying $^-$ we obtain that $z := v + \lceil \bar{v}^0 \rceil$ is the greatest distance value below v. Thus $w = z$ proving the first equality in (3.7). By Lemma 3.2.7 the element $\lceil v^0 \rceil$ exists whenever $v \leqslant v+\bar{v}+v$. Applying Lemma 3.2.6 (ii)⇔(iii) we obtain the second equality in (3.7). □

Remarks 3.2.10. (1) Even under the assumptions of Lemma 3.2.7 we do not know whether the existence of $\lceil v^0 \rceil$ implies the existence of $\lceil \bar{v}^0 \rceil$. Suppose that $\lceil v^0 \rceil$ exists. By Lemma 3.2.9 the element $w := \lceil v^0 \rceil + v$ is a distance value, whence $\bar{w}$ is a distance value and $\lceil \bar{w}+w \rceil$ exists by Lemma 3.2.7. However, we do not know whether $\lceil \bar{w}+w \rceil = \bar{v}+v$.

(2) We can prove the following: *Let* $\lceil v^0 \rceil$ *and* $\lceil \bar{v}^0 \rceil$ *be defined and let*

$$\lceil v^0 \rceil \leqslant v + \lceil \bar{v}^0 \rceil + \bar{v}, \qquad \lceil \bar{v}^0 \rceil \leqslant \bar{v} + \lceil \bar{v}^0 \rceil + v. \tag{3.9}$$

Then $u := \lceil v^0 \rceil + v + \lceil \bar{v}^0 \rceil$ *is the largest distance value below* v.

Indeed, put $r := \lceil v^0 \rceil$ and $s := \lceil \bar{v}^0 \rceil$. Noting that both r and s are small and using (3.9) we get

$$r = r+r+r \leqslant r+v+s+\bar{v}+r = r+v+s+s+\bar{v}+r = u+\bar{u}. \tag{3.10}$$

We have $u = \lceil v^0 \rceil + v + \lceil \bar{v}^0 \rceil \leqslant v$ and so $\lceil v^0 \rceil = r \leqslant u+\bar{u} \leqslant v+\bar{v} = v^0$. Thus $\lceil u+\bar{u} \rceil$ exists and equals r. Now from $u = r+v+s$ and r idempotent we get $\lceil u+\bar{u} \rceil + u = r+u = u$. Proceeding in a similar fashion we get $\lceil u+\bar{u} \rceil = s$, and $u + \lceil \bar{u}+u \rceil = u$. From (i)⇒(iii) in Lemma 3.2.6 we obtain that u is a distance value. As shown above $u \leqslant v$. Let x be a distance value and let $x \leqslant v$. Let a and b be the corresponding small elements from (3.2) then $a \leqslant x+\bar{x} \leqslant \lceil v^0 \rceil$ and $b \leqslant \lceil \bar{v}^0 \rceil$ and $x = a+x = a+x+b \leqslant \lceil v^0 \rceil + v + \lceil \bar{v}^0 \rceil = v$. Thus u is the largest distance value $\leqslant v$.

3.3. Extension property, convexity and hyperconvexity

3.3.1. Let **M** be a class of $\underline{V}$-metric spaces and κ a cardinal. A space $\underline{A} \in \mathbf{M}$ has the *one-point* κ*-extension property for* **M** if for every $\underline{B} = (B, d) \in \mathbf{M}$, each contraction from a subspace (D, d) of $\underline{B}$ into A such that $|D| \leqslant \kappa$, extends to a contraction from $(D \cup \{u\}, d)$ into $\underline{A}$ for every $u \in B \setminus D$. If $\underline{A}$ has the one-point κ-extension property for every κ we say that $\underline{A}$ has the *one-point extension property* for **M**. For instance, if $\underline{V}$ is a join-semilattice and **M** is the class of finite powers of a fixed $\underline{V}$-metric space $\underline{A}$ we get the extension property defined in Section 2.

Let (A, d) be a $\underline{V}$-metric space, $x \in A$, $r \in V$ and $t \in V$ small. Put

$$U_t := \{a \in A\colon d(a,a) \leqslant t\}, \; B(x,r) := \{a \in A\colon d(x,a) \leqslant r\}$$

and call $B_t(x,r) := B(x,r) \cap U_t$ a *t-ball.* We have:

Lemma 3.3.2. *Let* $\underline{A} = (A,d)$ *and* $\underline{B} = (B,\delta)$ *be* $\underline{V}$*-metric spaces,* $D = \{e_i\colon i < \kappa\} \subseteq B$, $u \in B \setminus D$ *and f a contraction from the induced subspace* (D,δ) *into* $\underline{A}$. *Put* $t := \delta(u,u)$ *and* $r_i := \delta(e_i, u)$, $x_i := f(e_i)$ $(i < \kappa)$. *Then*

(1) *The contraction f extends to a contraction from* $(D \cup \{u\}, \delta)$ *into* $\underline{A}$ *if and only if* $\bigcap \{B_t(x_i, r_i)\colon i < \kappa\}$ *is nonempty.*

(2) *Let* $\lceil v + \bar{v} \rceil$ *exist for all* $v \in V$. *Then* $\lceil r_i + t + \bar{r}_i \rceil$ *exists and*

$$t \leqslant \bar{r}_i + \lceil r_i + t + \bar{r}_i \rceil + r_i, \tag{3.11$_i$}$$

$$d(x_i, x_j) \leqslant r_i + t + \bar{r}_j \tag{3.12$_{ij}$}$$

holds for all i,j $< \kappa$.

Proof. (1) ($\Rightarrow$) Let g be the extension of f *to* $D \cup \{u\}$. For all $i < \kappa$ we have $d(g(e_i), g(u)) \leqslant \delta(e_i, u) = r_i$ proving that $z := g(u) \in B(x_i, r_i)$. Moreover, $d(z,z) \leqslant \delta(u,u) = t$ and so $z \in B_t(x_i, r_i)$ for all $i < \kappa$.

($\Leftarrow$) Let $z \in B_t(x_i, r_i)$ for all $i < \kappa$. Put $g(u) := z$ and $g(e_i) = f(e_i)$ for all $i < \kappa$. It suffices to show that g is a contraction from $(D \cup \{u\}, \delta)$ into (A, d). For this we only need to verify

$$d(g(e_i),\, g(u)) = d(x_i, z) \leqslant r_i \delta(e_i, u), \qquad d(g(u),\, g(u)) = d(z,z) \leqslant t = \delta(u,u).$$

(2) For $i < \kappa$, we have

$$\delta(e_i, e_i) \leqslant \delta(e_i, u) + \delta(u,u) + \delta(u, e_i) = r_i + t + \bar{r}_i.$$

By Lemma 3.2.2, both the elements t and $\delta(e_i, e_i)$ are small. Setting $v := r_i + t$ we have

$$v^0 := v + \bar{v} = r_i + t + \bar{t} + \bar{r}_i = r_i + t + \bar{r}_i.$$

By assumption $\lceil r_i + t + \bar{r}_i \rceil$ exists and by 3.2.2 also $\delta(e_i, e_i) \leqslant \lceil r_i + t + \bar{r}_i \rceil$.

Now

$$t := \delta(u,u) \leqslant \delta(u, e_i) + \delta(e_i, e_i) + \delta(e_i, u) \leqslant \bar{r}_i + \lceil \bar{r}_i + t + r_i \rceil + r_i$$

thus proving (3.11$_i$). Since f is a contraction,

$$d(x_i, x_j) = d(f(e_i), f(e_j)) \leqslant \delta(e_i, e_j) \leqslant \delta(e_i, u) + \delta(u,u) + \delta(u, e_j) = r_i + t + \bar{r}_j,$$

proving (3.1.2$_{ij}$). □

3.3.3. In the remainder of this section we assume that $\lceil v+\bar{v} \rceil$ exists for all $v \in V$. Motivated by Lemma 3.3.2, we say that a $\underline{V}$-metric space (A,d) is *κ-convex* if for all $x_i \in A$, $r_i \in V$ $(i<\kappa)$ and all $t \in V_\sigma$ satisfying

$$t \leqslant \bar{r}_i + \lceil r_i + t + \bar{r}_i \rceil + r_i, \tag{3.11$_i$}$$

$$d(x_i, x_j) \leqslant r_i + t + \bar{r}_j \tag{3.12$_{ij}$}$$

there is $z \in \bigcap \{B_t(x_i, r_i)\colon i<\kappa\}$ i.e. satisfying

$$d(z,z) \leqslant t, \tag{3.13}$$

$$d(x_i, z) \leqslant r_i \tag{3.14$_i$}$$

for all $i<\kappa$. If $\underline{A}$ is *κ-convex* for all κ, we call $\underline{A}$ *hyperconvex*.

The main result of this section is the following theorem.

Theorem 3.3.4. *A $\underline{V}$-metric is κ-convex if and only if it has the one-point κ-extension property in the class of $\underline{V}$-metric spaces.*

Proof. (⇒) Lemma 3.3.2.

(⇐) Let (A,d) be a $\underline{V}$-metric space with the one-point κ-extension property and let $x_i \in A$, $r_i \in V$ $(i<\kappa)$ and $t \in V_\sigma$ satisfy (3.11$_i$) and (3.12$_{ij}$) for all $i,j<\kappa$. For each $i<\kappa$ put $s_i := \lceil r_i + t + \bar{r}_i \rceil$ and $u_i := s_i + r_i$. Put $\underline{\kappa} := \{i\colon i<\kappa\}$ and $B := \underline{\kappa} \cup \{u\}$ where u is an element outside $\underline{\kappa}$. Finally define $\delta\colon B^2 \to V$ by setting

$$\delta(u,u) := t, \quad \delta(i,i) := s_i, \quad \delta(i,u) := u_i + t, \quad \delta(u,i) = t + \bar{u}_i,$$
$$\delta(i,j) = u_i + t + \bar{u}_j \text{ for all } i,j<\kappa,\ i \neq j.$$

We need the following.

Fact 1. *The space (B,δ) is a $\underline{V}$-premetric space.*

Proof of Fact 1. Recall that each $s_i \in V_\sigma$ by 3.2.5. From this and $t \in V_\sigma$ we see that $\delta(b,b) \leqslant 0$ and $\delta(b,b) = \overline{\delta(b,b)}$ holds for all $b \in B$. Next for $i,j<\kappa$, $i \neq j$ we have

$$\delta(j,i) = u_j + t + \bar{u}_i = u_j + \bar{t} + \bar{u}_i = \overline{(u_i + t + \bar{u}_j)} = \overline{\delta(i,j)},$$

$$\delta(u,i) = t + \bar{u}_i = \bar{t} + \bar{u}_i = \overline{u_i + t} = \overline{\delta(i,u)}.$$

It remains to check the validity of the △-inequality $\delta(i,j) \leqslant \delta(i,k) + \delta(k,j)$ for all i, j, $k \in B$. First note that for $i,j \in \underline{\kappa}$ we have $\delta(i,j) \leqslant u_i + t + \bar{u}_j$. Indeed, if $i \neq j$ we have even equality while for $i=j$ taking into account $s_i = \lceil r_i + t + \bar{r}_i \rceil \in V_\sigma$ we have

$$\delta(i,i) = s_i = s_i + s_i + \bar{s}_i \leqslant s_i + r_i + t + \bar{r}_i + \bar{s}_i = u_i + t + \bar{u}_i.$$

Note that due to (3.11_k) and $t, s_k \in V_\sigma$, we have

$$t=t+t+t\leqslant t+\overline{r_k}+s_k+r_k+t=t+\bar{r}_k+\overline{s_k}+s_k+r_k+t=t+\overline{u_k}+u_k+t,$$

hence

$$t\leqslant t+\overline{u_k}+u_k+t. \tag{3.15$_k$}$$

We distinguish the following cases.

(1) Let $i, j, k \in \underline{\kappa}$.

(a) Let $i\neq k\neq j$. Then from (3.15_k)

$$\delta(i,j)=u_i+t+\overline{u_j}\leqslant u_i+t+\overline{u_k}+u_k+t+\overline{u_j}=\delta(i,k)+\delta(k,j).$$

(b) Let $i=k\neq j$. We have

$$u_i+t=s_i+r_i+t=s_i+s_i+r_i+t=s_i+u_i+t \tag{3.16$_i$}$$

and so

$$\delta(i,j)=u_i+t+\overline{u_j}=s_i+u_i+t+\overline{u_j}=\delta(i,i)+\delta(i,j)$$

(c) Let $i\neq j=k$. Applying $^-$ to (3.16_j) we get

$$t+\overline{u_j}=t+\overline{u_j}+s_j$$

leading to

$$\delta(i,j)=u_i+t+\overline{u_j}=u_i+t+\overline{u_j}+s_j=\delta(i,j)+\delta(j,j).$$

(d) Let $i=j=k$. Then

$$\delta(i,i)=s_i=s_i+s_i=\delta(i,i)+\delta(i,i).$$

(2) Let exactly one of i, j and k equal u

(a) Let $k=u$. Then, since $i, j<\kappa$, we have

$$\delta(i,j)=u_i+t+\overline{u_j}=u_i+t+t+\overline{u_j}=\delta(i,u)+\delta(u,j).$$

(b) Let $j=u$. Then by (3.15_k)

$$\delta(i,u)=u_i+t\leqslant u_i+t+\overline{u_k}+u_k+t=\delta(i,k)+\delta(k,u). \tag{3.17$_i$}$$

(c) Let $i=u$. Apply $^-$ to (3.17_j).

(3) Let exactly two of the elements i, j and k equal u.

(a) Let $i<\kappa$. Then

$$\delta(i,u)=u_i+t=u_i+t+t=\delta(i,u)+\delta(u,u). \tag{3.18$_i$}$$

(b) Let $j<\kappa$. Apply $^-$ to (3.18_j).

(c) Let $k<\kappa$. Then by (3.15_k).

$$\delta(u,u)=t\leqslant t+\overline{u_k}+u_k+t=\delta(u,k)+\delta(k,u).$$

(4) Finally let $i=j=k=u$. Then

$$\delta(u,u)=t=t+t=\delta(u,u)+\delta(u,u). \quad \square$$

Fact 2. *The map $f(i):=x_i(i<\kappa)$ is a contraction of $(\underline{\kappa},\delta)$ into (A,d).*

Proof of Fact 2. Let $i,j<\kappa$. Note that by (3.12_{ii}) we have $d(x_i,x_i)\leqslant r_i+t+\overline{r_i}$. Since $d(x_i,x_i)$ is small, by Lemma 3.2.4 we have $d(x_i,x_i)\leqslant s_i=\lceil r_i+t+\overline{r_i}\rceil$. Now by (3.12_{ij}) we have

$$d(f(i),f(j))=d(x_i,x_j)\leqslant r_i+t+\overline{r_j}.$$

Thus for $i\neq j$ we have

$$d(x_i,x_j)=d(x_i,x_i)+d(x_i,x_j)+d(x_j,x_j)\leqslant s_i+r_i+t+\overline{r_j}+s_j=u_i+t+\overline{u_j}=\delta(i,j)$$

whereas

$$d(f(i),f(i))=d(x_i,x_i)\leqslant s_i=\delta(i,i). \quad \square$$

Let $\approx$ be the equivalence on B defined in Section 3.1. By Lemma 3.1.3 it is an equivalence. Let B° denote the set of blocks of $\approx$. Also by Lemma 3.1.3, we may define a map $\delta^\circ: B^{\circ 2}\to V$ assigning to C_1, $C_2\in B^\circ$ the element $\delta(c_1,c_2)$ where $c_i\in C_i$ are arbitrary $(i=1,2)$. For $D\subseteq B$ put $D^\circ:=\{C\in B^\circ : C\cap D\neq 0\}$ (the hull of D in $\approx$). Let f be the map from Fact 2. Define $f^\circ: D^\circ\to A$ by setting $f^\circ(C):=f(c)$ for all $C\in B^\circ$ and any $c\in C$. We have the following.

Fact 3. *(B°,δ°) is a $\underline{V}$-metric space and f° is a contraction from (D°,δ°) into (A,d).*

Proof of Fact 3. In view of Lemma 3.1.3 it is clear that (B°,δ°) is a $\underline{V}$-premetric. Let $\delta^\circ(C_1,C_2)\leqslant 0$. Then $\delta(c_1,c_2)=\delta^\circ(C_1,C_2)\leqslant 0$ for some $c_i\in C_i$ $(i=1,2)$ whence $c_1\approx c_2$ and $C_1=C_2$.Thus (B°,δ°) is a $\underline{V}$-metric space. The other statements follow from Fact 2. $\square$

Proof of Theorem 3.3.4 (*conclusion*). Put $D:=\underline{\kappa}$ and let f denote the map from Fact 2. Consider the contraction f° from the metric subspace $(\underline{\kappa}^\circ,\delta^\circ)$ of (B°,δ°) into the metric space (A,d). By the one-point κ-extension property f° extends to a contraction g from (B°,δ°) into (A,d). Let Z denote the block of $\approx$ containing u and put $z:=g(Z)$. Similarly for $i<\kappa$ let $i\in X_i\in B^\circ$. Taking into account Lemma 3.1.3. we obtain

$$d(z,z)=d(g(Z),\, g(Z))\leqslant\delta^\circ(Z,Z)=\delta(z,z)=t,$$

$$d(x_i,z)=d(g(X_i),g(Z))\leqslant\delta^\circ(X_i,Z)=\delta(x_i,u)=r_i+t\leqslant r_i$$

for all $i<\kappa$. This proves that (A,δ) is κ-convex. $\square$

3.3.5. The following special case provides an illustration of the meaning of 2-convexity. Let (A, d) be a $\underline{V}$-metric space. Recall that

$$(d)_v := \{(x, y) \in A^2 \colon d(x, y) \leqslant v\}$$

for all $v \in V$ (cf. 1.3). Let ρ and σ be binary relations on A. The *composition* (relational or de Morgan product) and the *inverse* (converse) relations are

$$\rho \circ \sigma := \{(x, y) \in A^2 \colon (x, u) \in \rho,\ (u, y) \in \sigma \text{ for some } u \in A\}$$

and $\rho^{-1} := \{(y, x) \colon (x, y) \in \rho\}$. The relations ρ and σ *permute* (commute) if $\rho \circ \sigma = \sigma \circ \rho$. Finally ρ is *symmetric* if $\rho = \rho^{-1}$ and *reflexive* if $\rho \supseteq \{(a, a) \colon a \in A\}$. Recall that $(\rho \circ \sigma)^{-1} = \sigma^{-1} \circ \rho^{-1}$.

The following is immediate.

Fact. $(d)_u^{-1} = (d)_{\bar{u}}$, $(d)_u \circ (d)_v \subseteq (d)_{u+v}$ *holds for all* $u, v \in V$.

We have the following.

Lemma 3.3.6. *Assume that* 0 *is the least element of* $(V, \leqslant)$; *let* (A, d) *be a* 2*-convex* $\underline{V}$*-metrix space and* $u, v \in V$. *Then* $(d)_u$ *is reflexive and* $(d)_u \circ (d)_v = (d)_{u+v}$. *Consequently,* $(d)_u$ *and* $(d)_v$ *permute whenever* $u + v = v + u$.

Proof. The relation $(d)_u$ is reflexive on account of $d(x, x) \leqslant 0 \leqslant u$ for all $x \in A$. Let $(x_1, x_2) \in (d)_{u+v}$ hence $d(x_1, x_2) \leqslant u + \overline{(\bar{v})}$. By 2-convexity $d(x_1, z) \leqslant u$ and $d(x_2, z) \leqslant \bar{v}$ for some $z \in A$, proving $(x_1, x_2) \in (d)_u \circ (d)_v$. Combining this with the Fact above, we have $(d)_u \circ (d)_v \subseteq (d)_{u+v} \subseteq (d)_u \circ (d)_v$. $\square$

Lemma 3.3.7. *Let* $\underline{V}$ *be such that* $\lceil \bar{v} + v \rceil$ *exists for all* $v \in \underline{V}$.

(1) *The following are equivalent:*

(i) $\lceil \bar{u} + u \rceil = \lceil \bar{u} + \lceil u + \bar{u} \rceil + u \rceil$ *for every* u.

(ii) $t \leqslant \bar{u} + u$ *if and only if* $t \leqslant \bar{u} + \lceil u + t + \bar{u} \rceil + u$ *for every* u, *and every small* t.

(2) *If moreover,* $w \wedge 0$ *exists for every self-dual element* w *and for all* $v, b \in \underline{V}$

(R2) $(\bar{v} + (\bar{b} + b) \wedge 0 + v) \wedge 0 = (\bar{v} + \bar{b} + b + v) \wedge (\bar{v} + v) \wedge 0$,

(R3) $(v + \bar{v}) \wedge 0 + (v + \bar{v}) \wedge 0 = (v + \bar{v} + v + \bar{v}) \wedge (v + \bar{v}) \wedge 0$,

then the above conditions (i) *and* (ii) *are satisfied.*

Proof. (1) (ii)⇒(i) Since $\lceil u + \bar{u} \rceil \leqslant 0$, we have $\bar{u} + \lceil u + \bar{u} \rceil + u \leqslant \bar{u} + u$, and thus $I(\bar{u} + \lceil u + \bar{u} \rceil + u) \subseteq I(\bar{u} + u)$. For the converse, let $t = \lceil \bar{u} + u \rceil$. Then t is small and $t \leqslant \bar{u} + u$, thus from (ii) we have $t \leqslant \bar{u} + \lceil u + t + \bar{u} \rceil + u \leqslant \bar{u} + \lceil u + \bar{u} \rceil + u$, giving $I(\bar{u} + u) \subseteq I(\bar{u} + \lceil u + \bar{u} \rceil + u)$. From $I(\bar{u} + u) = I(\bar{u} + \lceil u + \bar{u} \rceil + u)$ and the existence of $\lceil u + \bar{u} \rceil$ the equality in (i) follows.

(i) ⇒(ii) We only need to prove that if $t \leqslant \bar{u} + u$ then $t \leqslant \bar{u} + \lceil u + t + \bar{u} \rceil + u$ (the converse follows from the fact that $\bar{u} + \lceil u + t + \bar{u} \rceil + u \leqslant \bar{u} + u$ for all small t). Applying

(i) to $u'=u+t$, we get: $\lceil t+\bar{u}+u+t\rceil=\lceil \bar{t}+\bar{u}+\lceil u+t+\bar{t}+\bar{u}\rceil+u+t\rceil$. Using the fact that t is small (i.e. $t\leqslant 0$, $t+t=t$, $\bar{t}=t$), from $t\leqslant\bar{u}+u$ we get $t\leqslant\bar{t}+\bar{u}+u+t$ and thus

$$t\leqslant\lceil t+\bar{u}+u+t\rceil=\lceil t+\bar{u}+\lceil u+t+\bar{u}\rceil+u+t\rceil\leqslant\lceil \bar{u}+\lceil u+t+\bar{u}\rceil+u\rceil$$

(2) For a small t such that $t\leqslant\bar{u}+u$, let $\alpha=(u+t+\bar{u})\wedge 0$ We claim that $\lceil u+t+\bar{u}\rceil=\alpha$. It suffices to show that α is idempotent. Applying (R3) to $v=u+t$ and using $t=t+t=t+\bar{t}$ we get

$$\alpha+\alpha=(u+t+\bar{u}+u+t+\bar{u})\wedge(u+t+\bar{u})\wedge 0.$$

Since $t\leqslant\bar{u}+u$, it follows

$$u+t+\bar{u}=u+t+t+t+\bar{u}\leqslant u+t+\bar{u}+u+t+\bar{u},$$

yielding $\alpha+\alpha=\alpha$.

We claim that for every small t, $t\leqslant\bar{u}+u$ implies $t\leqslant\bar{u}+(u+t+\bar{u})\wedge 0+u$. Indeed, from (R2) we get

$$(\bar{u}+(u+t+\bar{u})\wedge 0+u)\wedge 0=(\bar{u}+u+t+\bar{u}+u)\wedge(\bar{u}+u)\wedge 0 \text{ and } t\leqslant\bar{u}+u \text{ implies}$$

$$t=t+t+t+t\leqslant\bar{u}+u+t+\bar{u}+u.$$

From the fact that $(u+t+\bar{u})\wedge 0=\lceil u+t+\bar{u}\rceil$ we get condition (ii). □

Lemma 3.3.8. *Assume that every subset X of V_σ has a supremum in $\underline{V}$. Then*

(1) *For every family $r=\{r_i\in V\colon i<\kappa\}$ in V and each $a\in V$,* the set

$$T_a=\{t\in V_\sigma\colon t\leqslant a,\ t\leqslant\bar{r}_i+\lceil r_i+t+\bar{r}_i\rceil+r_i \text{ for all } i<\kappa\}$$

has a largest element $f_r^{\rightarrow}(a)$.

(2) *A $\underline{V}$ metric space (A,d) is κ-convex if for all families $r=\{r_i\in V\colon i<\kappa\}$, $\{x_i\in A\colon i<\kappa\}$, and $a\in V_\sigma$, the intersection $\cap\{B_t(x_i,r_i)\colon i<\kappa\}$ is non empty if and only if $d(x_i,x_j)\leqslant r_i+f_r^{\rightarrow}(a)+\bar{r}_j$ holds for all $i,j<\kappa$.*

(3) *If, moreover, the set $((0]\leqslant)$ of elements below 0 is a complete lattice satisfying the distributivity conditions* (R2), (R3) from Lemma 3.3.7 *then*

$$f_r^{\rightarrow}(a)=\lceil \wedge\{(\bar{r}_i+r_i)\wedge a\colon i<\kappa\}\rceil.$$

Proof. (1) From our assumption V has a least element O'. This element is small and belongs to T_a. From Lemma 3.2.4 and $T_a\subseteq V_\sigma$ we get that $f_r^{\rightarrow}(a):=\operatorname{Sup} T_a$ exists and belongs to V_σ. Now for $t\in T_a$ and $i<\kappa$, applying the fact that $v\mapsto\lceil v\rceil$ is order preserving, we obtain

$$t\leqslant\bar{r}_i+\lceil r_i+t+\bar{r}_i\rceil+r_i\leqslant\bar{r}_i+\lceil r_i+f_r^{\rightarrow}(a)+\bar{r}_i\rceil+r_i$$

and so $f_r^{\rightarrow}(a)\leqslant\bar{r}_i+\lceil r_i+f_r^{\rightarrow}(a)+\bar{r}_i\rceil+r_i$ for all $i<\kappa$ proving $f_r^{\rightarrow}(a)\in T_a$.

(2) Since $f_r^{\rightarrow}(a)\leqslant\bar{r}_i+\lceil r_i+f_r^{\rightarrow}(a)+\bar{r}_i\rceil+r_i$, a κ-convex metric space must satisfy the condition of the lemma.

Conversely, given $x_i \in A$, $r_i \in V$ $(i<\kappa)$ suppose a small t satisfies

$$t \leqslant \bar{r}_i + \lceil r_i + t + \bar{r}_i \rceil + r_i \text{ and } d(x_i, y_j) \leqslant r_i + t + \bar{r}_j \text{ for all } i, j < \kappa.$$

Then $t \leqslant f_r^{\rightarrow}(t)$ and thus $d(x_i, x_j) \leqslant r_i + f_r^{\rightarrow}(t) + \bar{r}_j$ for all $i, j < \kappa$.

The condition of the lemma implies that $\bigcap\{B_t(x_i, r_i)\colon i<\kappa\} \neq \phi$ and thus $\underline{A}$ is κ-convex.

(3) Immediate from Lemma 3.3.7(ii). □

3.3.9. A family $\{\xi_i \in V\colon i<\kappa\}$ is *a subzero family* if $\xi_i \leqslant 0$ for some $i<\kappa$. Call $\underline{V}$ a *weak Heyting algebra*, or briefly *weakly Heyting* if:

(1) The set $((0], \leqslant)$ of elements below 0 is a complete lattice.

(2) For every subzero family $\{\xi_i \in V\colon i<\kappa\}$ and all $u, v \in V$, the following equality holds:

$$u + \wedge\{\xi_i\colon i<\kappa\} + v = \wedge\{u + \xi_i + v\colon i<\kappa\}. \tag{3.19}$$

For example, if 0 is the least element of $\underline{V}$ then trivially $\underline{V}$ is weakly-Heyting.

For $v \in V$ and $n<\omega$ define inductively $v \cdot n$ by setting $v \cdot 0 = 0$ and $v \cdot (n+1) = v \cdot n + v$. We have the following.

Lemma 3.3.10. *If $\underline{V}$ is weakly Heyting then for all $u, v \in V$ and all subzero families $\{\xi_i \in V\colon i<\kappa\}$ and $\{\eta_j \in V\colon j<\lambda\}$ we have:*

(i) $$\lceil v \rceil = \wedge\{v \cdot n\colon n<\omega\} \tag{3.20}$$

(ii) $$\wedge\{\xi_i\colon i<\kappa\} + \wedge\{\eta_j\colon j<\lambda\} = \wedge\{\xi_i + \eta_j\colon i<\kappa, j<\lambda\} \tag{3.21}$$

(iii) $$u + \lceil \wedge\{\xi_i\colon i<\kappa\} \rceil + v = \wedge\left\{u + \sum_{j<n} \xi_{\sigma(j)} + v\colon \sigma \in \kappa^n, n<\omega\right\} \tag{3.22}$$

Proof. Put $\xi = \wedge\{\xi_i\colon i<\kappa\}$, $\eta = \wedge\{\eta_j\colon j<\lambda\}$.

(ii) Applying (3.19) twice we get

$$\begin{aligned}\xi + \eta &= \xi + \wedge\{\eta_j\colon j<\lambda\} = \wedge\{\xi + \eta_j\colon j<\lambda\} \\ &= \wedge\{\wedge\{\xi_i\colon i<\kappa\} + \eta_j\colon j<\lambda\} = \wedge\{\wedge\{\xi_i + \eta_j\colon i<\kappa\}\colon j<\lambda\} \\ &= \wedge\{\xi_i + \eta_j\colon i<\kappa, j<\lambda\}.\end{aligned}$$

(i) Put $v' = \wedge\{v \cdot n\colon n<\omega\}$. From (3.19) above, we get $v' + v' = \wedge\{v \cdot n + v \cdot m\colon n<\omega, m<\omega\}$. Since $v \cdot n + v \cdot m = v \cdot (n+m)$ for all $n, m<\omega$, this gives $v' + v' = v'$, thus $v' \in I(v)$. For $u \in I(v)$ we have $u = u \cdot n \leqslant v \cdot n$ and so $u \leqslant \wedge\{v \cdot n\colon n<\omega\} = v'$. Consequently v' is the largest element of $I(v)$, that is $v' = \lceil v \rceil$.

(iii) We have $\lceil \xi \rceil = \wedge\{\xi \cdot n\colon n<\omega\}$ thus

$$u + \lceil \xi \rceil + v = \wedge\{u + \xi \cdot n + v\colon n<\omega\}.$$

It suffices to show that

$$u+\xi\cdot n+v=\wedge\left\{u+\sum_{j<n}\xi_{\sigma(j)}+v\colon\ \sigma\in\kappa^{\underline{n}}\right\} \tag{3.23}$$

The proof goes by induction on n. For $n=0$, we have $\sum_{j<0}\xi_{\sigma(j)}=0$ thus both sides of (3.23) reduce to $u+v$ and the result follows. Suppose (3.23) holds for $n-1$ and all u, $v\in V$. Applying (3.19) and (3.23) we have

$$\begin{aligned}u+\xi\cdot n+v=u+\xi\cdot(n-1)+\xi+v&=\wedge\left\{u+\sum_{j<n-1}\xi_{\sigma(j)}+\xi+v\colon\ \sigma\in\kappa^{\underline{n-1}}\right\}\\&=\wedge\left\{u+\sum_{j<n-1}\xi_{\sigma(j)}+\wedge\{\xi_i\colon\ i<\kappa\}+v\colon\ \sigma\in\kappa^{\underline{n-1}}\right\}\\&=\wedge\left\{u+\sum_{j<n}\xi_{\tau(j)}+v\colon\ \tau\in\kappa^{\underline{n}}\right\}\end{aligned}$$

so (3.23) holds for n concluding the induction and the proof. □

Lemma 3.3.11. *If $\underset{\sim}{V}$ is weakly Heyting then it satisfies* (R2) *and* (R3) *of Lemma* 3.3.7.

Proof. Applying (3.19) to $\xi_0:=\bar{b}+b$ and $\xi_1:=0$ we get

$$(v+(\bar{b}+b)\wedge 0+v)\wedge 0=(\bar{v}+\bar{b}+b+v)\wedge(\bar{v}+v)\wedge 0$$

proving (R2).
Put $\alpha=(v+\bar{v})\wedge 0$. Applying twice (3.19) we get

$$\begin{aligned}\alpha+\alpha&=\alpha+((v+\bar{v})\wedge 0)=(\alpha+v+\bar{v})\wedge\alpha\\&=((v+\bar{v}+v+\bar{v})\wedge(v+\bar{v}))\wedge\alpha=(v+\bar{v}+v+\bar{v})\wedge\alpha,\end{aligned}$$

thus proving (R3). □

Theorem 3.3.12. *If $\underset{\sim}{V}$ is weakly Heyting then a $\underset{\sim}{V}$-metric space $\underset{\sim}{A}$ is κ-convex if and only if it satisfies the following condition*:

(HC_κ) *For all $x_i\in A$, $r_i\in V$ $(i<\kappa)$ and all $t\in V_\sigma$, the intersection $\bigcap\{B_t(x_i,r_i)\colon\ i<\kappa\}$ is nonempty if and only if $d(x_i,x_j)\leqslant r_i+t+\bar{r}_j$ for all $i,j<\kappa$.*

Proof. Let $x_i\in A$, $r_i\in V$ for $i<\kappa$ and let $t\in V_\sigma$. Put $r=\{r_i,\ i<\kappa\}$. According to Lemma 3.3.8 it suffices to prove the following.

Claim. *If $d(x_i,x_j)\leqslant r_i+t+\bar{r}_j$ for all $i,j<\kappa$ then $d(x_i,x_j)\leqslant r_i+f_{\vec{r}}(t)+\bar{r}_j$*

Proof of the Claim. Put $\kappa'+1=\kappa\bigcup\{\kappa\}$, $\xi_i=\bar{r}_i+r_i$ for $i<\kappa$ and $\xi_i=t$ for $i=\kappa$. Since V is weakly Heyting, from Lemma 3.3.11 and Lemma 3.3.8, we get

$$f_{\vec{r}}(t)=\ulcorner\ \wedge\{(\bar{r}_i+r_i)\wedge t\colon i<\kappa\}\ \urcorner.$$

Since $t \leqslant 0$, the family $\{\xi_i: i < \kappa+1\}$ is subzero, thus from Lemma 3.3.10(iii) we get

$$r_i + f_{\vec{r}}(t) + \bar{r}_j = \wedge \left\{ r_i + \sum_{l<n} \xi_{\sigma(l)} + \bar{r}_j : v \in \kappa'^{\underline{n}}, n < \omega \right\}$$

and so we only need to prove that for every $n < \omega$

$$d(x_i, x_j) \leqslant r_i + \sum_{l<n} \xi_{\sigma(l)} + \bar{r}_j \tag{3.24}$$

holds for all $\sigma \in \kappa'^{\underline{n}}$, $i, j < \kappa$.

We proceed by induction on n. From $d(x_i, y_j) \leqslant r_i + t + r_j$ and $t \leqslant 0$ we deduce $d(x_i, y_j) \leqslant r_i + \bar{r}_j$ and so (3.24) holds for $n = 0$.

Suppose (3.24) holds for $n-1$. Let $\sigma \in \kappa'^{\underline{n}}$. We have two cases:

(1) $\xi_{\sigma(l)} = t$ for every $l < n$; in this case

$$r_i + \sum_{l<n} \xi_{\sigma(l)} + \bar{r}_j = r_i + tn + \bar{r}_j = r_i + t + \bar{r}_j \geqslant d(x_i, x_j).$$

(2) $\xi_{\sigma(l_0)} = \bar{r}_{\sigma(l_0)} + r_{\sigma(l_0)}$ for some l_0.

We write

$$r_i + \sum_{l<n} \xi_{\sigma(l)} + \bar{r}_j = r_i + \sum_{l<j_0} \xi_{\sigma(l)} + \bar{r}_{\sigma(l_0)} + r_{\sigma(l_0)} + \sum_{l_0<l<n} \xi_{\sigma(l)} + \bar{r}_j.$$

The induction hypothesis ensures that

$$r_i + \sum_{l<j_0} \xi_{\sigma(l)} + \bar{r}_{\sigma(l_0)} \geqslant d(x_i, x_{\sigma(l_0)}) \quad \text{and} \quad r_{\sigma(l_0)} + \sum_{l_0<l<n} \xi_{\sigma(l)} + \bar{r}_j \geqslant d(x_{\sigma(l_0)}, x_j).$$

From the $\triangle$-inequality we get

$$r_i + \sum_{l<n} \xi_{\sigma(l)} + \bar{r}_j \geqslant d(x_i, x_{\sigma(l_0)}) + d(x_{\sigma(l_0)}, x_j) \geqslant d(x_i, x_j).$$

This concludes the inductive proof of the claim, and of the theorem itself. □

3.3.13. A collection $\mathbf{B}$ of sets has the *2-Helly-property* if $\bigcap \mathbf{Y}$ is nonempty whenever $\mathbf{Y} \subseteq \mathbf{B}$ consists of pairwise intersecting sets; equivalently $\bigcap \{X_i: X_i \in \mathbf{B}, i \in I\} \neq \emptyset$ whenever $X_i \bigcap X_j \neq \emptyset$ for every $i, j \in I$.

Proposition 3.3.14. *Let $\underline{V}$ be a weakly Heyting algebra, $\kappa > 1$ be a cardinal and $\underline{A}$ be a $\underline{V}$-metric space. Then $\underline{A}$ is κ-convex if and only if $\underline{A}$ is 2-convex and for each small $t \in \underline{V}$, every κ-collection of t-balls has the 2-Helly property.*

Proof. ($\Rightarrow$) Let $t \in V$ be small and $x_i \in A$, $r_i \in V$ $(i < \kappa)$ be such that for all $i, j < \kappa$ there exists $z_{ij} \in B_t(x_i, r_i) \bigcap B_t(x_j, r_j)$. Then

$$d(x_i, x_j) \leqslant d(x_i, z_{ij}) + d(z_{ij}, z_{ij}) + d(z_{ij}, x_j) \leqslant r_i + t + \bar{r}_j.$$

The κ-convexity insures the existence of an $z \in A$ such that $d(z, z) \leqslant t$ and $d(x_i, z) \leqslant r_i$ for all $i < \kappa$. Now $z \in \bigcap \{B_t(x_i, r_i): i < \kappa\}$ as required.

($\Leftarrow$) Let $t \in V$ be small and $x_i \in A$, $r_i \in V$ $(i < \kappa)$ be such that $d(x_i, x_j) \leqslant r_i + t + \bar{r}_j$ holds for all $i, j < \kappa$. By 2-convexity there is $z_{ij} \in B_t(x_i, r_i) \bigcap B_t(x_j, r_j)$. From the 2-Helly property there is $z \in \bigcap \{B_t(x_i, r_i): i < \kappa\}$. Clearly z has the required property. □

3.4. Retraction, injectivity and hyperconvexity

3.4.1. Let $\underline{E}$ *and* $\underline{F}$ be $\underline{V}$-metric spaces. A contraction $f: \underline{E} \to \underline{F}$ is a *coretraction* (*retraction*) provided $g \circ f = \mathrm{id}_E$ $(f \circ g = \mathrm{id}_F)$ for some contraction $g: \underline{F} \to \underline{E}$. Call $\underline{E}$ a *retract* of $\underline{F}$ and write $\underline{E} \lhd \underline{F}$ if there is a coretraction from $\underline{E}$ to $\underline{F}$ (or, equivalently, a retraction from $\underline{F}$ to $\underline{E}$).

Coretractions from $\underline{E}$ to $\underline{F}$ are isometric embeddings from $\underline{E}$ into $\underline{F}$. For an arbitrary $\underline{E}$ the converse is in general false. Spaces $\underline{E}$ for which this holds true play an important role. Formally they are defined as follows.

Let $\mathbf{M}$ be a class of $\underline{V}$-metric spaces. Call $\underline{E} \in \mathbf{M}$ an *absolute retract* with respect to $\mathbf{M}$ (and with respect to the isometries between the members of $\mathbf{M}$) if every isometry from $\underline{E}$ into $\underline{F} \in \mathbf{M}$ is a coretraction. (In this paper the absolute retracts, abbreviated AR, are all with respect to the isometries but in other contexts they may refer to other morphisms (cf. [13]). Next $\underline{E} \in \mathbf{M}$ is *injective* with respect to $\mathbf{M}$ (and with respect to the isometries between members of $\mathbf{M}$) if for all $\underline{E}, \underline{F}, \underline{G} \in \mathbf{M}$, every contraction $f: \underline{E} \to \underline{F}$ and each isometry $h: \underline{E} \to \underline{G}$ we have $f = g \circ h$ for some contraction $g: \underline{G} \to \underline{F}$.

Absolute retracts and injectives are linked by the following fact.

Theorem 3.4.2. *With respect to* $\mathbf{M}$, *every injective is an absolute retract and every retract of an injective is an injective; moreover, every product of injectives is an injective.*

This fact is purely categorical and has not much to do with metric spaces. Nevertheless, in our content, we can derive it from the following lemma.

Lemma 3.4.3. *If* $\mathbf{M}$ *is closed under isometric subspaces* (i.e. if $(A, d) \in \mathbf{M}$, and $X \subseteq A$, then $(X, d \restriction_X) \in M$), *then* (A, d) *is injective with respect to* $\mathbf{M}$ *if and only if* (A, d) *has the one-point extension property for* $\mathbf{M}$.

Proof. Transfinite induction. □

Lemma 3.4.4. *Let* $\mathbf{M}$ *be a class of metric spaces, and* κ *be a cardinal. The class of* $(A, d) \in \mathbf{M}$ *which have the one-point* κ*-extension property is closed under retracts, and under products* (*where such products exist*).

Proof. (a) Let $\underline{A} \in \mathbf{M}$, $\underline{A}'$ be a retract of $\underline{A}$, $\underline{B} \in \mathbf{M}$, $\underline{D}$ be an isometric subspace of $\underline{B}$, f: $\underline{D} \to \underline{A}'$ be a contraction and let $u \in B \setminus D$. In order to show that f extends to u, select

h: $\underline{A}'\to\underline{A}$, k: $\underline{A}\to\underline{A}'$ so that $k\circ h=1_{A'}$. Since $h\circ f$: $\underline{D}\to\underline{A}$ and $\underline{A}$ has the one-point κ-extension property, there is g extending $h\circ f$ on $D\cup\{u\}$, but then $k\circ g$ is a contraction extending f to $D\cup\{u\}$.

(b) Let ($\underline{A}_i$: $i\in I$) be a family of members of **M** having the one-point κ-extension property. Assume that the (categorical) product $\underline{P}=\Pi\{\underline{A}_i: i\in I\}$ is defined. For each $i\in I$, let p_i: $\underline{P}\to\underline{A}_i$ be the ith projection. Let $\underline{B}\in\mathbf{M}$, $\underline{D}$ be an isometric subspace of $\underline{B}$, f: $\underline{D}\to\underline{P}$, and let $u\in B\setminus D$. For each $i\in I$, the space $\underline{A}_i$ has the one-point κ-extension property, thus the map $p_i\circ f$: $\underline{D}\to\underline{A}_i$ has an extension f_i to $D':=D\cup\{u\}$. Since $\underline{P}$ is the product of the $\underline{A}_i$'s, there is a map g:$\underline{D}'\to\underline{P}$ such that $p_i\circ g=f_i$ for all $i\in I$. This map extends f. □

A similar result holds for $\underline{V}$-metric spaces (A,d) satisfying the following convexity condition (HC_κ) (cf. Theorem 3.3.12):

> For all $x_i\in A$, $v_i\in V$, $(i<\kappa)$ and all $t\in V_\sigma$: the intersection $\bigcap\{B_t(x_i,r_i): i<\kappa\}$ is non-empty if and only if $d(x_i,x_j)\leqslant r_i+t+\overline{r_j}$ for all $i,j<\kappa$.

Lemma 3.4.5. *Let κ be a cardinal. The class $C_{\underline{V}_\kappa}$ of $\underline{V}$-metric spaces satisfying* (HC_κ) *is closed under retracts and under products (where such products exist).*

Proof. Let $\underline{A}=(A,d)\in C_{\underline{V}_\kappa}$ and let $\underline{B}:=(B,\delta)$ be retract of $\underline{A}$. Without loss of generality we may assume that $B\subseteq A$ and δ is the restriction of d to B. Let t, $x_i\in B$ and $r_i(i\in\kappa)$ be such that $d(x_i,x_j)\leqslant r_i+t+\overline{r_j}$, for all $i,j<\kappa$. Since $\underline{A}$ satisfies (HC_κ) there is $z\in A$ such that $d(z,z)\leqslant t$ and $d(x_i,z)\leqslant r_i$ for all $i<\kappa$. Let f: $\underline{A}\to\underline{B}$ denote the corresponding retraction. Put $z':=f(z)$. We have $d(z',z')\leqslant d(z,z)\leqslant t$ and similarly $d(x_i,z')\leqslant r_i$ for all $i<\kappa$ proving that $\underline{B}\in C_{\underline{V}_\kappa}$.

Let $\underline{A}_j=(A_j,d_j)\in C_{\underline{V}_\kappa}$ for all $j\in J$. Let $\underline{A}:=(A,d)$ denote the direct product of the $\underline{A}_j$ $(j\in J)$. Let t be small, let $x_i\in A$ and $r_i\in V$ $(i<\kappa)$ be such that $d(x_i,x_j)\leqslant r_i+t+\overline{r_j}$ for all i, $j<\kappa$. For a fixed $\ell\in J$, from the definition of d we get $d_\ell(x_i(\ell)$, $(x_j(\ell))\leqslant d(x_i,x_j)\leqslant r_i+t+\overline{r_j}$ for all $i,j<\kappa$. Now from the HC_k property in $\underline{A}_\ell$ we obtain the that there is $z(\ell)\in A_\ell$ such that $d(z(\ell), z(\ell))\leqslant t$ and $d_\ell(x_i(\ell), z(\ell))\leqslant r_i$ for all $i<\kappa$. The element $z\in A$ thus defined satisfies $d(z,z)=\mathrm{Sup}\{d_\ell(z(\ell), z(\ell))$: $\ell\in J\}\leqslant t$ and $d(x_i,z)\leqslant r_i$ for all $i<\kappa$. □

3.5. Metric spaces over a Heyting algebra

3.5.1. Let $\underline{V}=\langle V;\leqslant,+,0,{}^{-}\rangle$ be as in 3.1.1. For p, $q\in V$ put

$$D_{pq}:=\{x\in V: p+x\geqslant q,\ q+\bar{x}\geqslant p\}. \tag{3.25}$$

If the set D_{pq} has a least element we denote it by $d_{\underline{V}}(p,q)$ and say that *$d_{\underline{V}}(p,q)$ exists.* We say that *$d_{\underline{V}}$ exists* if $d_{\underline{V}}(p,q)$ exists for all p, $q\in V$. We have the following.

Lemma 3.5.2. *If $d_{\underline{V}}$ exists then $(V, d_{\underline{V}})$ is a $\underline{V}$-metric space.*

Proof. For every $p \in V$ we have $0 \in D_{pp}$ and so $d_{\underline{V}}(p,p) \leqslant 0$. Conversely if $d_{\underline{V}}(p,q) \leqslant 0$, then $0 \in D_{pq}$ and so $p \geqslant q \geqslant p$ proving (d1′). To prove (d2) it suffices to note that $^{-}$ being an order automorphism, we have

$$D_{qp} = \{x \in V\colon\ q+x \geqslant p,\ p+\bar{x} \geqslant q\} = \{\bar{x} \in V\colon\ q+\bar{x} \geqslant p,\ p+x \geqslant q\} = \bar{D}_{pq},$$

thus proving (d2). Finally let p, q, $r \in V$ and $x := d_{\underline{V}}(p,r)$, $y := d_{\underline{V}}(p,q)$, $z := d_{\underline{V}}(q,r)$. By (3.25) we have $p+y \geqslant q$, $q+\bar{y} \geqslant p$, $q+z \geqslant r$, $r+\bar{z} \geqslant q$, and so $p+y+z \geqslant q+z \geqslant r$, $r+\bar{z}+\bar{y} \geqslant q+\bar{y} \geqslant p$, proving that $y+z \in D_{pr}$ and the required $x = d_{\underline{V}}(p,r) \leqslant y+z$. □

The $\underline{V}$-metric space $(V, d_{\underline{V}})$ is related to an arbitrary $\underline{V}$-premetric space by the following formula.

Lemma 3.5.3. *Let $d_{\underline{V}}$ exist and let (A, d) be an arbitrary $\underline{V}$-premetric space. Then*:

$$d(x,y) = d_{\underline{V}}(d(x,x), d(x,y)) = \operatorname{Sup}\{d_{\underline{V}}(d(z,x), d(z,y))\colon z \in A\} \tag{3.26}$$

for all x, $y \in A$.

Proof. Let x, y, $z \in A$ and

$$a := d(x,y) \qquad b := d(z,x), \qquad c := d(z,y), \qquad e := d(x,x).$$

From (d2) and (d3) we have $c \leqslant b+a$, $b \leqslant c+\bar{a}$ proving $a \in D_{bc}$ and hence $d_{\underline{V}}(b,c) \leqslant a$. Thus a is an upper bound of the set $S := \{d_{\underline{V}}(d(z,x), d(z,y))\colon z \in A\}$. On the other hand, from (3.25) and $e := d(x,x) \leqslant 0$ we get $a \leqslant e + d_{\underline{V}}(e,a) \leqslant d_{\underline{V}}(e,a)$ whence $d_{\underline{V}}(e,a) \in S$. Thus $\operatorname{Sup} S$ exists and equals both a and $d_{\underline{V}}(e,a)$. □

Remark 3.5.4. Suppose $d_{\underline{V}}$ exists. Its *companion* $\bar{d}_{\underline{V}}$) is defined by setting $\bar{d}_{\underline{V}}(p,q) = d_{\underline{V}}(\bar{q}, \bar{p})$ for all p, $q \in V$. Under the assumptions of Lemma 3.5.3 we have

$$d(x,y) = \bar{d}_{\underline{V}}(d(x,y), d(y,y)) = \operatorname{Sup}\{\bar{d}_{\underline{V}}(d(x,z), d(y,z))\colon z \in A\}. \tag{3.27}$$

Example 3.5.5. Let $\underline{V} = \langle V; \leqslant, \vee, 0, \mathrm{id}_V \rangle$ where $\langle V; \vee, \wedge, ', 0, 1 \rangle$ is a boolean algebra. Then $D_{pq} := \{x \in V\colon\ p \vee x \geqslant q,\ q \vee \bar{x} \geqslant p\} = \{x \in V\colon\ x \geqslant p+q\}$ where $p+q = (p \wedge q') \vee (q \wedge p')$ and so $d_{\underline{V}}(p,q) = p+q$. If V is a boolean algebra of sets then $d_{\underline{V}}(p,q)$ is the symmetric difference $p \Delta q$. (If we put $\delta(p,q) := |p \Delta q|$ we obtain an $\mathbb{R}_+$-metric space (V, δ) related to the Hamming distance and widely used in combinatorial applications). If V consists of sentences then $d_{\underline{V}}(p,q)$ measures how far the sentence p is from being logically equivalent to the sentence q, e.g. $d_{\underline{V}}(p,q) = 0$ iff they are logically equivalent. Distances over boolean algebras were considered in [5].

3.5.6. Call $\underline{V}$ *solid* if $(V, \leqslant)$ is a complete join-semilattice and $d_{\underline{V}}$ exists. Recall that for $(V, \leqslant)$ a complete join-semilattice, $\underline{A} := (A, d)$ a $\underline{V}$-metric space and I a set, the power $\underline{A}^I$ is (A^I, d) where $d(f,g) := \operatorname{Sup}\{d(f(i), g(i))\colon i \in I\}$ for all f, $g \in A^I$.

Proposition 3.5.7. *Let $\underline{V}$ be solid. Then every $\underline{V}$-metric space embeds isometrically into a power of $(V, d_{\underline{V}})$.*

Proof. Let $\underline{A}$ and $\underline{B}$ be $\underline{V}$-metric spaces and let $\underline{H}=\mathrm{Hom}(\underline{A}, \underline{B})$ denote the set of contractions from $\underline{A}$ to $\underline{B}$ equipped with the $\underline{V}$-distance of $\underline{B}^A$ (i.e. $d(f, g):= \mathrm{Sup}\{d(f(a), g(a)): a\in A\}$. Denote by $\underline{G}$ the set $\mathrm{Hom}(\underline{H}, \underline{B})$ equipped with the $\underline{V}$-distance of $\underline{B}^H$. For a fixed $a\in A$ define $\varphi_a: H\to B$ by setting $\varphi_a(f):=f(a)$ for every $f\in H$. Now $\varphi_a\in G$ (because $d(\varphi_a(f), \varphi_a(g))=d(f(a), g(a))\leqslant d(f, g)$. Finally define $\varphi: A\to G$ by setting $\varphi(a):=\varphi_a$ for all $a\in A$. Again $\varphi\in\mathrm{Hom}(\underline{A}, \underline{G})$ as for all x, $y\in A$ we have

$$d(\varphi(x), \varphi(y))=d(\varphi_x, \varphi_y)=\mathrm{Sup}\{d(\varphi_x(f), \varphi_y(f): f\in H\}$$
$$=\mathrm{Sup}\{d(f(x), f(y)): f\in H\}\leqslant d(x, y). \tag{3.28}$$

The above is a true for any $\underline{B}$. Choosing $\underline{B}:=(V, d_{\underline{V}})$ we show that φ is the required isometrical embedding. Indeed, (for given x, $y\in A$) define $f: A\to B$ by setting $f(a):=d(x, a)$ for all $a\in A$. By Lemma 3.5.3 the map f is a contraction and by (3.26) we have $d(x, y)= d_{\underline{V}}(d(x, x), d(x, y))= d_{\underline{V}}(f(x), f(y))$ and so (3.28) is an equality. Since all our spaces are $\underline{V}$-metric, this proves that φ is an isometry. □

Remarks 3.5.8. Let $\underline{V}$ be solid and $\underline{A}=(A, d)$ a $\underline{V}$-metric space. For $y\in A$ define δ_y: $A\to V$ by setting $\delta_y(x):=d(x, y)$ for all $x\in A$. Further let δ: $A\to V^A$ be defined by $\delta(y):=\delta_y$ for all $y\in A$. Now δ is an isometry from $\underline{A}$ into $(V, d_{\underline{V}})^A$. Indeed, applying the definition of sup-distance and Lemma 3.5.3, for all y, $z\in A$ we obtain

$$d(\delta(y), \delta(z))=d(\delta_y, \delta_z)=\mathrm{Sup}\{d_{\underline{V}}(\delta_y(x), \delta_z(x)): x\in A\}$$
$$=\mathrm{Sup}(d_{\underline{V}}(d(x, y), d(x, z)): x\in A\}=d(y, z).$$

Here the maps δ_y need not be contractions. However, proceeding in a similar fashion we get an isometrical embedding of $\underline{A}$ into $(V, \bar{d}_{\underline{V}})^A$ (cf. Remark 3.5.4) whose images are all contractions. We list a few facts relating the image of $d_{\underline{V}}$ and $\lceil\quad\rceil$ (introduced in 3.2.5).

Lemma 3.5.9. *Let $d_{\underline{V}}$ exist, let $p\in V$ and $q:=d_{\underline{V}}(p, p)$, $r:=\lceil p+\bar{p}\rceil$. Then:*

(1) *q is the least selfdual element of V such that $p+q=p$. In particular, $q=p$ if and only if p is small.*

(2) *$d_{\underline{V}}(r, r+p)=r+p$ and*

(3) *$r+p$ is the largest element of* $\mathrm{Im}\, d_{\underline{V}}$ *below p.*

Proof. (1) From (3.25) and $q\leqslant 0$ we get $p\leqslant p+q\leqslant p$ proving $p+q=p$. Let w be selfdual and satisfy $p+w=p$. Since $p+\bar{w}=p+w=p$, from (3.25) we get $w\in D_{pp}$ and $w\geqslant q$. We know that q is small by Lemma 3.2.2. If $p=q$ then p is small. Conversely, let p be small. Since $p+p=p$, by the first part of the proof we have $q\leqslant p$. However using $p\leqslant 0$ we get $p=p+q\leqslant q$.

(2) By its definition (cf. 3.2.5) the element r is idempotent and $r \leqslant (p+\bar{p}) \wedge 0$. Since $p+\bar{p}$ is selfdual, by Lemma 3.2.4, the element r is also selfdual. Put $e := d_{\underline{V}}(r, r+p)$. First we prove $e \leqslant r+p$. Indeed, $r+p = r+r+p$ and $r = r+r+r \leqslant r+p+\bar{p}+r = r+p+\overline{(r+p)}$ whence $r+p \in D_{r,r+p}$ and so $e \leqslant r+p$. By definition $e \in D_{r,r+p}$ and so $r+p \leqslant r+e \leqslant e$ due to $r \leqslant 0$, proving $e = r+p$.

(3) By $\lceil 2 \rceil$ we have $r+p \in \operatorname{Im} d_{\underline{V}}$ and $r+p \leqslant p$ inview of $r \leqslant 0$. To prove that $r+p$ is the largest element with this property let $b := d_{\underline{V}}(x, y) \leqslant p$ and $c := d_{\underline{V}}(x, x)$. We have $c \leqslant b+\bar{b} \leqslant p+\bar{p}$ and in particular $c \leqslant \lceil p+\bar{p} \rceil = r$ so $b = c+b \leqslant r+p$. □

In the preceding sections the existence of $d_{\underline{V}}$ has been postulated. A sufficient condition for its existence is based on the following well-known concept [2, Ch. 14, Section 5].

3.5.10. Let $\langle V; \leqslant, + \rangle$ be an ordered groupoid and $p, q \in V$. If the set $R_{pq} := \{r \in V: p \leqslant q+r\}$ has a least element, it is called *the right residual* of p by q and is denoted $p \mathbin{\cdot\!\!:} q$. The groupoid is *right-residuated* if all right residuals exist. The *left-residual* $p \mathbin{:\!\!\cdot} q$, *left-residuation* and *residuation*, (meaning the both-sided one) are defined in a similar way. For $\underline{V} = \langle V; \leqslant, +, 0, ^{-} \rangle$ one can show easily that $p \mathbin{\cdot\!\!:} q$ exists if and only if $\bar{p} : \bar{q}$ exists and $p \mathbin{\cdot\!\!:} q = \overline{(\bar{p} : \bar{q})}$.
We have:

Lemma 3.5.11. *If both $q:p$ and $p:q$ exist and $s := (q:p) \vee \overline{(p:q)}$ exists then $d_{\underline{V}}(p, q)$ exists and equals s. In particular, if $\underline{V}$ is residuated and $(q:p) \vee \overline{(p:q)}$ exists for all $p, q \in V$ then $d_{\underline{V}}$ exists and $d_{\underline{V}}(p, q) = (q:p) \vee \overline{(p:q)}$ for all $p, q \in V$.*

Proof. Put $u := q:p$, $v := p:q$ and $w := u \vee \bar{v}$. We have $p+w \geqslant p+u \geqslant q$, $q+\bar{w} \geqslant q+v \geqslant p$ and so $w \in D_{pq}$. On the other hand, for $x \in D_{pq}$ from (3.25) we have $x \geqslant q:p = u$ and $\bar{x} \geqslant p:q = v$ and so $x \geqslant w$ proving $d_{\underline{V}}(p, q) = w$. □

3.5.12. The following property will play an important role in the sequel. As usual, we say that $\langle V; \leqslant, + \rangle$ *is left κ-distributive* if whenever $p \in V$ and $Q \subseteq V$ are such that $|Q| \leqslant \kappa$ and either $\wedge Q$ or $\wedge \{p+q: q \in Q\}$ exists, then both exist and

$$p + \wedge Q = \wedge \{p+q: q \in Q\}. \tag{3.29}$$

The *right κ-distributivity* and *κ-distributivity* are defined in a similar way. Note that in (3.29) we could replace $=$ by $\geqslant$ (as $\leqslant$ is automatically true).

Remark 3.5.13. A routine verification shows that $\langle V; \leqslant, + \rangle$ *is left 2-distributive if and only if it is left n-distributive for all $n < \omega$.*

For our $\underline{V}$ these properties coincide.

Lemma 3.5.14. *The following are equivalent:*

(i) *$\underline{V}$ is left κ-distributive,*

(ii) *$\underline{V}$ is right κ-distributive,*

(iii) *$\underline{V}$ is κ-distributive,*

(iv) *$\underline{V}$ has the following property: if $P, Q \subseteq V$ are such that* (a) $|P| \leqslant \kappa$, $|Q| \leqslant \kappa$ *and* (b) *all three infima in* (3.30) *exist whenever at least two exist, then*

$$\bigwedge P + \bigwedge Q = \bigwedge \{p+q\colon p \in V,\ q \in Q\}. \tag{3.30}$$

The straightforward proof is omitted (to prove (i)⇒(ii) use the properties of the involution $^-$; in particular, the fact that it is an order automorphism).

Call *$\underline{V}$ fully distributive* if it is κ-distributive for all the $\kappa \geqslant 1$. We relate right residuation and full distributivity.

Lemma 3.5.15. (a) *If $\underline{V}$ is residuated then it is fully distributive.* (b) *If $(V, \leqslant)$ is a complete meet-semilattice and $\underline{V}$ is fully distributive then $\underline{V}$ is residuated.*

The somewhat technical proof is omitted. We give a sufficient condition for the κ-convexity (cf. 3.3.3) of $(V, d_{\underline{V}})$.

Proposition 3.5.16. *If* (i) *$(V, \leqslant)$ has all infima of subsets of cardinality $\leqslant \kappa$,* (ii) *$\underline{V}$ is κ-distributive and* (iii) *$d_{\underline{V}}$ exists, then $(V, d_{\underline{V}})$ is κ-convex.*

Proof. Let t, x_i, r_i $(i<\kappa)$ be as in 3.3.3. Put $z := \bigwedge\{x_j + r_j + t\colon j<\kappa\}$. We show that $t \in D_{zz}$. Note that for t selfdual the conditions in (3.25) reduce to $z+t \geqslant z$. Applying (3.29) and noting that t is idempotent we have

$$z+t = \bigwedge\{x_j+r_j+t\colon j<\kappa\}+t = \bigwedge\{x_j+r_j+t+t\colon j<\kappa\} = z;$$

hence $t \in D_{zz}$ and $d_{\underline{V}}(z, z) \leqslant t$. Similarly we prove that $r_i \in D_{x_i z}$ for all $i<\kappa$. Indeed, from $d_{\underline{V}}(x_i, x_j) \leqslant r_i + t + \overline{r_j}$ and the second condition of (3.25) we get $x_j + r_j + t + \overline{r_i} \geqslant x_i$ for all $j<\kappa$. Applying the κ-distributive law

$$z+\overline{r_i} = \bigwedge\{x_j+r_j+t\colon j<\kappa\}+\overline{r_i} = \bigwedge\{x_j+r_j+t+\overline{r_i}\colon j<\kappa\} \geqslant x_i. \tag{3.31}$$

On the other hand, in view of $t \leqslant 0$ we have

$$z = \bigwedge\{x_j+r_j+t\colon j<\kappa\} \leqslant x_i+r_i+t \leqslant x_i+r_i. \tag{3.32}$$

Now (3.31) and (3.32) show that $r_i \in D_{x_i z}$ proving the required $d_{\underline{V}}(x_i, z) \leqslant r_i$. □

3.5.17. We say that $\underline{V} = \langle V; \leqslant, +, 0, - \rangle$ is a *Heyting algebra* (or shortly $\underline{V}$ is Heyting) if $(V, \leqslant)$ is a complete lattice and $\underline{V}$ is fully distributive.

Now for Heyting algebras we relate hyperconvexity, absolute retracts and injectivity (with respect to the class of $\underline{V}$-metric spaces and their isometries) and $d_{\underline{V}}$.

Theorem 3.5.18. *The following are equivalent for a Heyting algebra $\underline{V}$ and a $\underline{V}$-metric space $\underline{A}$.*

(i) *$\underline{A}$ is injective,*

(ii) *$\underline{A}$ is an absolute retract,*
(iii) *$\underline{A}$ is a retract of a power of $(V, d_{\underline{V}})$*
(iv) *$\underline{A}$ is hyperconvex.*

Proof. (i) $\Rightarrow$ (ii) A general categorical fact proved here for the reader's convenience. Let $f: \underline{A} \to \underline{B}$ be an isometry. By injectivity $\mathrm{id}_A = g \circ f$ for a contraction $g: \underline{B} \to \underline{A}$, i.e. f is a coretraction.

(ii) $\Rightarrow$ (iii) By Proposition 3.5.7 the space $\underline{A}$ embeds isometrically into a power of $(V, d_{\underline{V}})$. Since $\underline{A}$ is an absolute retract, $\underline{A}$ is a retract of this power.

(iii) $\Rightarrow$ (iv) From Lemma 3.5.15(b) we know that $\underline{V}$ is residuated. Now $d_{\underline{V}}$ exists by Lemma 3.5.11. From Proposition 3.5.16 we see that $(V, d_{\underline{V}})$ is hyperconvex. Since, in our case, hyperconvexity amounts to the satisfaction of the condition (HC_κ) for all κ, it follows from Lemma 3.4.5 that a power as well as a retract of a hyperconvex space is hyperconvex.

(iv) $\Rightarrow$ (i) It has been shown in Theorem 3.3.4 that one-point κ-extension property and κ-convexity are equivalent and in Lemma 3.4.3 that one-point extension property is equivalent to injectivity. □

Remarks 3.5.19. For Heyting algebras $\underline{V}$ the above proof of 'injectivity implies hyperconvexity' (i.e. (i) $\Rightarrow$ (iv)) seems to be simpler than the proof given in 3.3.

For a Heyting algebra $\underline{V}$ we can slightly reformulate the condition for hyperconvexity.

Lemma 3.5.20. *Let $\underline{V}$ be a Heyting algebra. Then a $\underline{V}$-metric space (A, d) is hyperconvex if and only if for every small $t \in V$ and each map $f: A \to V$ such that*

$$d(x, y) \leqslant f(x) + t + \overline{f(y)}$$

holds for all $x, y \in A$ there is $z \in A$ satisfying $d(z, z) \leqslant t$ and $d(x, z) = f(x)$ for all $x \in A$.

Proof. ($\Rightarrow$). Write $A = \{x_i;\ i < \kappa\}$ (where $\kappa = |A|$) and put $r_i := f(x_i)$ for all $i < \kappa$

($\Leftarrow$). Let $t \in V$ be small and let $x_i \in A$ and $r_i \in V (i < \kappa)$ be such that

$$d(x_i, x_j) \leqslant r_i + t + \overline{r_j}$$

holds for all $i, j < \kappa$. For $x \in A$ put $D_x = \{r_i: x_i = x\}$ and $f(x) := \wedge D_x$. Note that $f(x)$ is the greatest element 1 of $(V, \leqslant)$ whenever $D_x = \emptyset$. Applying the distributive law we have for all $x, y \in A$.

$$d(x, y) \leqslant \wedge \{r_i + t + \overline{r_j}:\ r_i \in D_x \text{ and } r_j \in D_y\} = \wedge D_x + t + \overline{\wedge D_y} = f(x) + t + \bar{f}(y).$$

Now by the assumptions and the definition of f we have $d(x_i, z) \leqslant f(x_i) = r_i$ for all $i < \kappa$. □

For a Heyting algebra $\underline{V}$ and the metric $d_{\underline{V}}$ the sets $U_t := \{x \in V: d_{\underline{V}}(x, x) \leqslant t\}$, $B(x, r) = \{y \in V: d_{\underline{V}}(x, y) \leqslant r\}$ $(x, t, r \in V)$ have a simple form.

Lemma 3.5.21. *Let t, x, $r \in V$.*

(a) *If $(V, \leqslant)$ is a (complete) join-semilattice then both U_t and $B(x, r)$ are (complete) join-subsemilattices of $(V, \leqslant)$. Moreover, $B(x, r)$ is convex.*

(b) *If $\underline{V}$ is Heyting then U_t is a complete nonempty sublattice of $(V, \leqslant)$ and $B(x, r)$ is either empty or an interval of $(V, \leqslant)$.*

Proof. (a) By (3.25)

$$U_t := \{x \in V: x + t \geqslant x,\ x + \bar{t} \geqslant x\}, \tag{3.33}$$

$$B(x, r) = \{y \in V: x + r \geqslant y,\ y + \bar{r} \geqslant x\}. \tag{3.34}$$

Let $X \subseteq U_t$, $M := \vee X$ and $a \in \{t, \bar{t}\}$. Then $M + a \geqslant x + a \geqslant x$ for each $x \in X$ and so $M + a \geqslant M$ proving $M \in U_t$. The proof for $B(x, r)$ is quite similar. Let u, $w \in B(x, r)$ and $u \leqslant v \leqslant w$. From (3.34) we have $x + r \geqslant w \geqslant v$ and $v + \bar{r} \geqslant u + \bar{r} \geqslant x$ and so $v \in B(x, r)$.

(b) Let $X \subseteq U_t, m := \wedge X$ and $a \in \{t, \bar{t}\}$. Then

$$m + a = \wedge \{x + a: x \in X\} \geqslant \wedge X = m$$

and so $m \in U_t$. The proof for $B(x, r)$ is similar. Clearly the least element O' of $(V, \leqslant)$ belongs to U_t. $\square$

3.6. Ultrametrics

3.6.1. In the section we consider special $\underline{V}$-metric spaces. Put $V^* := V \setminus \{0\}$. Let (A, d) be a $\underline{V}$-metric space. For $x \in A$ and $v \in V$ a *closed ball* is the set

$$B(x, v) := \{a \in A: d(x, a) \leqslant v\}.$$

For a subset X of A the *v-hull* and the *hull* of X are the sets

$$[X]_v := \bigcup_{x \in X} B(x, v), \qquad [X] := \bigcap \{[X]_v: v \in V^*\}.$$

As usual, a down-directed ordered set (each pair of elements in the set has a lower bound) is a *filter*. A self map $X \to X^\nabla$: of the set $\mathscr{P}(A)$ (of subsets of A) is a *closure* if $X \subseteq X^\nabla$, $X^\nabla \subseteq Y^\nabla$, $X^\nabla = X^\nabla$ holds for all $X \subseteq Y \subseteq A$. A closure is *topological* if $(X \cup Y)^\nabla = X^\nabla \cup Y^\nabla$ holds for all X, $Y \subseteq A$. We have the following.

Lemma 3.6.2. *Let $\langle V; + \rangle$ be idempotent and 0 the least element of $(V, \leqslant)$. If (A, d) is a $\underline{V}$-metric space then $X \to [X]$ is a closure on A. This closure is topological provided $(V^*, \leqslant)$ is a filter.*

Proof. Let $X \subseteq Y \subseteq A$ and $Z := [X]$. Note that $d(x, x) = 0 \leqslant v$ and so $x \in B(x, v)$ for all $x \in A$ and $v \in V$. Clearly we have $X \subseteq [X]_v$ for all $v \in V$ and so $X \subseteq Z$. Let $z \in Z$ and

$v \in V^*$. From $Z \subseteq [X]_v$ we see that $z \in B(x, v)$ for some $x \in X \subseteq Y$ whence $z \in [Y]_v$ and $z \in [Y]$ proving $[X] = Z \subseteq [Y]$. We prove $[Z] \subseteq Z$. Let $t \in [Z]$ and let $v \in V^*$. From $[Z] \subseteq [Z]_v$ we have $t \in B(z, v)$ for some $z \in Z$. By the same token we have $z \in B(x, v)$ for some $x \in X$. From $d(x, z) \leqslant v$, $d(x, t) \leqslant v$ and the idempotency we obtain:

$$d(x, t) \leqslant d(x, z) + d(z, t) \leqslant v + v = v$$

i.e. $t \in B(x, v)$ and $t \in [X]_v$. Since $v \in V^*$ was arbitrary, we get $t \in [X] = Z$. Thus $[Z] \subseteq Z$ and $[Z] = Z$.

Finally, let $(V^*, \leqslant)$ be a filter and $X_1, X_2 \subseteq A$. Let $z \in A \setminus ([X_1] \cup [X_2])$. Then there are $v_i \in V^*$ such that X_i is disjoint from $B(z, v_i)$ $(i = 1, 2)$. Choose $v \in V^*$ so that $v < v_1$ and $v < v_2$. Now $B(z, v) \cap X_i \subseteq B(z, v_i) \cap X_i = \emptyset$ for $i = 1, 2$. Setting $X := X_1 \cup X_2$ we have $B(z, v) \cap X = \emptyset$ whence $z \notin [X]_v$ and $z \notin [X]$. Altogether $[X] \subseteq [X_1] \cup [X_2]$. The converse inclusion being evident, this proves that the closure is topological. $\square$

We turn to very special V-metric spaces.

3.6.3. Let $\langle V; \vee \rangle$ be a join semilattice and 0 the least element of the associated order $\leqslant$. Let id_V denote the identity selfmap on V. It is easy to verify that $\underline{V} := \langle V; \leqslant, \vee, 0, \mathrm{id}_V \rangle$ satisfies the assumptions of Section 1.1. Call such a $\underline{V}$-metric space (A, d) an *ultrametric*. The axioms of an ultrametric are:

(d1′) $d(x, y) = 0 \Leftrightarrow x = y$

(d2) $d(y, x) = d(x, y)$,

and

(d3) $d(x, y) \leqslant d(x, z) \vee d(z, y)$.

In the particular case of $V = \mathbb{R}_+$ (with the usual order and $x \vee y = \max(x, y)$) an ultrametric is also called a non-archimedian metric. It is well known that this ultrametric is quite different from the euclidean metric. This is true in general. As an example consider the following property. Let $x, y, z \in A$. Setting $a := d(x, y)$, $b := d(x, z)$ and $c := d(y, z)$ from (d3) we obtain $a \leqslant \vee c$, $b \leqslant a \vee c$ and $c \leqslant a \vee b$. It follows that

$$a \vee b \leqslant a \vee a \vee c = a \vee c \leqslant a \vee a \vee b = a \vee b,$$

and by symmetry,

$$a \vee b = a \vee c = b \vee c = a \vee b \vee c. \tag{3.35}$$

For example, if $a \leqslant b$ then $b = b \vee c$ shows $c \leqslant b$ and $b = a \vee c$. In particular, if $(V, \leqslant)$ is a chain (i.e. a totally or linearly ordered set), from $a \leqslant b$ we get $b = c$ and so among the elements a, b and c always at least two are equal. On account of this we call (3.35) the *isosceles property*.

We describe ultrametrics in terms of the relations

$$(d)_v := \{(x, y) \in A^2 : d(x, y) \leqslant v\} \quad \text{for } v \in V.$$

Put $\Delta_A := \{(a, a) : a \in A\}$.

Lemma 3.6.4. *Let* $(\underline{V}, \leqslant)$ *be a join semilattice and* 0 *its least element. Then* (A, d) *is an ultrametric space if and only if all relations* $(d)_v$ $(v \in V)$ *are equivalence relations on* A *and* $(d)_0 = \Delta_A$.

Proof. $(\Rightarrow)$ $(d)_0 = \Delta_A$ follows from (d1′). Let $v \in V$. Then $(d)_v \supseteq (d)_0 = \Delta_A$ show that $(d)_v$ is reflexive. Clearly $(d)_v$ is symmetric due to (d2) and transitive by (d3).

$(\Leftarrow)$ We have $\Delta_A = (d)_0$ proving (d1′). Let $x, y, z \in A$ and $r := d(x, y)$. Now $(x, y) \in (d)_r$ implies $(y, x) \in (d)_r$, i.e. $d(y, x) \leqslant d(x, y)$ and (d2) follows by symmetry. Finally, put $s := d(x, z)$, $t := d(z, y)$ and $u := s \vee t$. From $s \leqslant u$, $t \leqslant u$ we get $(x, z) \in (d)_u$, $(z, y) \in (d)_u$ proving (d3). □

Note that, according to Lemma 3.6.4 for a fixed $v \in V$, the closed balls $B(x, v)$ $(x \in A)$ partition A.

3.6.5. A clone C on A is *congruence affine* if $C = \operatorname{Pol} R$ where R is a set of equivalence relations on A (cf. 2.1 and 2.3 for the definitions). Thus Lemma 3.6.4 states that for a $\underline{V}$-ultrametric space (A, d) the clone $C_{d\underline{V}}$ of d-contracting operations is congruence affine. Let $\operatorname{Eq} A$ denote the set of equivalence relations on A. For an algebra $\underline{A} = \langle A; F \rangle$ the set $\operatorname{Con} \underline{A} := \operatorname{Eq} A \cap \operatorname{Inv}_2 F$ (ordered by $\subseteq$) is the *congruence lattice* of A. Note that $D := \{(d)_v : v \in V\} \subseteq \operatorname{Con} \langle A; C_{d\underline{V}} \rangle$. The set $\operatorname{Con} \langle A; C_{d\underline{V}} \rangle$ is closed under arbitrary intersections and directed unions and so it contains all the intersections and directed unions of members of D. In general, it may contain other equivalences constructed in different ways from equivalences in D [31]. Note that the map $v \to (d)_v$ is an order preserving map from $\underline{V}$ into the set $(\operatorname{Eq} A, \subseteq)$ such that $0 \mapsto \Delta_A$. There is not much we can say in the general case and so we turn to special cases.

3.6.6. We consider κ-convex $\underline{V}$-ultrametrics (cf. Section 3.3). For our $\underline{V}$ we have $V_\sigma = \{0\}$, hence $t = 0$ and all (3.11_i) $(i < \kappa)$ as well as (3.13) are trivial assertions. Hence a $\underline{V}$-ultrametric is κ-*convex* whenever to all $x_i \in A$ and $r_i \in V (i < \kappa)$ satisfying

$$d(x_i, x_j) \leqslant r_i \vee r_j \tag{3.36$_{ij}$}$$

for all $i, j < \kappa$, $i \neq j$ there is $z \in A$ such that

$$d(x_i, z) \leqslant r_i \tag{3.37$_i$}$$

holds for all $i < \kappa$. For example, (A, d) is 2-convex if for all $x_0, x_1 \in A$ and $r_0, r_1 \in V$ such that $d(x_0, x_1) \leqslant r_0 \vee r_1$ we have $d(x_0, z) \leqslant r_0$, $d(x_1, z) \leqslant r_1$ for some $z \in A$ or, equivalently, $B(x_0, r_0)$ meets $B(x_1, r_1)$.

Two equivalences ε and η are *permutable* (or *commute*) if $\varepsilon \circ \eta = \eta \circ \varepsilon$. We say that $D \subseteq \operatorname{Eq} A$ is *permutable* if the equivalences in D are pairwise permutable. It is well known (cf. [6, 2–6.6]) and easy to check $\varepsilon \vee \eta = \varepsilon \circ \eta$ if and only if ε and η are permutable (here $\vee$ is in $(\operatorname{Eq} A, \subseteq)$). We describe the 2-convex ultrametrics in terms of permutability.

Proposition 3.6.7. *A $\underline{V}$-ultrametric space (A, d) is 2-convex if and only if all $(d)_v$ ($v \in V$) are pairwise permutable and $(d)_v \circ (d)_{v'} = (d)_v \vee (d)_{v'} = (d)_{v \vee v'}$ holds for all $v, v' \in V$.*

Proof. By Lemma 3.6.4 in every ultrametric

$$(d)_v \circ (d)_{v'} \subseteq (d)_v \vee (d)_{v'} \subseteq (d)_{v \vee v'}$$

holds for all v, $v' \in V$. Now 2-convexity is equivalent to $(d)_{v \vee v'} \subseteq (d)_v \circ (d)_{v'}$ for all $v, v' \in V$. □

Note that for a 2-convex $\underline{V}$-ultrametric space the map $v \to (d)_v$ is a join-semilattice homomorphism (from $\langle V; \vee \rangle$ into $\langle \mathrm{Eq}\, A; \vee \rangle$).

3.6.8. We can translate κ-convexity into the following property. A family F of sets has the *κ-Helly property* if the intersection $\bigcap Y$ is non-empty for every $Y \subseteq F$ of cardinality $\leqslant \kappa$ consisting of pairwise intersecting sets.

Consider the closed balls $B(x, v) = \{a \in A: d(x, a) \leqslant v\}$ ($x \in A$, $v \in V$) introduced in 3.6.1. For a 2-convex $\underline{V}$-ultrametric space the condition (3.36_{ij}) is equivalent to $B(x_i, r_i)$ meets $B(x_j, r_j)$ and so we obtain:

> *For $\kappa > 2$ a V-ultrametric space (A, d) is κ-convex if and only if it is 2-convex and the family $\{B(x, v): x \in A,\ v \in V\}$ has the κ-Helly property.*

Remark 3.6.9. Let $\langle V; \vee, \wedge \rangle$ be a lattice, (A, d) a $\underline{V}$-ultrametric space, $D := \{(d)_v: v \in V\}$ and let $v \mapsto (d)_v$ be meet-preserving. Directly from the definitions and Proposition 3.6.7 it follows that the $\aleph_0$-convexity is just the condition of the Chinese remainder theorem (cf. [1]). It is known (cf. [12, p. 211 ex. 68]) that this condition is equivalent to $(D, \subseteq)$ *arithmetical* (i.e. permutable and distributive).

3.6.10. We need the following lattice-theoretical concept. Let $\kappa > 2$ be a cardinal, $\underline{L} := \langle L; \vee, \wedge \rangle$ be a lattice and let $D \subseteq L$ be a sublattice of L closed under infima (in L) of subsets of cardinality $\leqslant \kappa$. The set D is *κ-meet-distributive* if

$$\wedge \{v \vee y: y \in Y\} = v \vee \wedge Y \tag{3.38}$$

holds for each $v \in D$ and $Y \subseteq D$ with $|Y| \leqslant \kappa$.

Note that 2-meet-distributivity means that the familiar distributive law

$$(v \vee y) \wedge (v \vee y') = v \vee (y \wedge y') \tag{3.39}$$

holds in D. It is immediate that D is then n_0-meet-distributive for all $n < \omega$: (cf. 3.5.14, cf. also [2g]) The condition (3.38) may be strengthened:

Lemma 3.6.11. *Let κ, $\underline{L}$ and D be as in* 3.6.10. *Then D is κ-meet-distributive if and only if*

$$\wedge \{x \vee y: x \in X, y \in Y\} = \wedge X \vee \wedge Y \tag{3.40}$$

holds for all X, $Y \subseteq D$ with $|X|$, $|Y| \leqslant \kappa$.

We have the following.

Lemma 3.6.12. *Let $\kappa > 2$ be a cardinal and (A, d) a κ-convex $\underline{V}$-ultrametric space. Then $D := \{(d)_v : v \in V\}$ is a κ-meet-distributive subset of* (Eq A, $\subseteq$).

Proof. Let $\lambda \leqslant \kappa$, $r_0 \in V$ and $r_i \in V$ $(0 < i < \lambda)$. Since (A, d) is 2-convex, by Proposition 3.6.7 and the last remark in 3.6.9 we must prove

$$\sigma := \bigcap_{0<i<\lambda} ((d)_{r_0} \circ (d)_{r_i}) \subseteq (d)_{r_0} \circ \bigcap_{0<i<\lambda} (d)_{r_i} = \tau. \tag{3.41}$$

Let $(x_0, y) \in \sigma$. Then there are $x_i \in A$ such that $d(x_0, x_i) \leqslant r_0$ and $d(x_i, y) \leqslant r_i$ for all $0 < i < \lambda$. For all $0 < i, j \leqslant \lambda$ we have

$$d(x_i, x_j) \leqslant d(x_i, y) \vee d(y, x_j) \leqslant r_i \vee r_j, \qquad d(x_0, x_i) \leqslant r_0 \leqslant r_0 \vee r_i.$$

By κ-convexity (cf. 3.6.6) we have $d(x_i, z) \leqslant r_i$ for some $z \in A$ and all $i < \lambda$.
Thus $(x_0, z) \in (d)_{r_0}$ and taking into account that $(d)_{r_i}$ are equivalences also $(z, y) \in (d)_{r_i}$ for all $0 < i < \lambda$. It follows that $(x_0, y) \in \tau$. □

3.7. Clones of contracting operations for convex metric spaces

3.7.1. Let $(V, \leqslant)$ be a join-semilattice and (A, d) a $\underline{V}$-metric space. Recall that in 2.2.1 an n-ary operation f on A (i.e. a map from A^n into A) has been called a *d-contraction* (on A) provided

$$d(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \leqslant d(x_1, y_1) \vee \cdots \vee d(x_n, y_n) \tag{3.42}$$

holds for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$. The set $C_{d\underline{V}}$ of all d-contractions on A is a clone (i.e. it is composition closed and contains all projections, cf. 2.2.1). Let $\kappa > 0$ be an ordinal number; put $\underline{\kappa} := \{i : i < \kappa\}$. A subset ρ of $A^{\underline{\kappa}}$ (i.e a set of maps $\underline{\kappa} \to A$) is a κ-ary *relation* on A. For $r_1, \ldots, r_n \in A^{\underline{\kappa}}$ let $f[r_1, \ldots, r_n]$ denote the map $h \in A^{\underline{\kappa}}$ defined by setting $h(i) := f(r_1(i), \ldots, r_n(i))$ for all $i < \kappa$. Recall (cf. 2.3) that f *preserves* ρ if $f[r_1, \ldots, r_n] \in \rho$ whenever all $r_1, \ldots, r_n \in \rho$. For a set F of (finitary) operations on A let $\mathrm{Inv}_\kappa F$ denote the set of κ-ary relations ρ on A preserved by all $f \in F$. In this context, it suffices to consider only $\kappa \leqslant |A|$ for A infinite and $\kappa < \omega$ for A finite (cf. [19, 25].

Let $W \subseteq V$, let B be a set such that $\underline{\kappa} \subseteq B$ and let $\{\alpha_w : w \in W\}$ be a family of binary relations on B. Put

$$\tau := \{g \in A^B : (x, y) \in \alpha_w \Rightarrow d(g(x), g(y)) \leqslant w\}. \tag{3.43}$$

The set $\tau|_{\underline{\kappa}}$ (of restrictions of $g \in \tau$ to $\underline{\kappa}$) is called a *derived relation*. It is straightforward to verify that a derived relation belongs to $\mathrm{Inv}_\kappa C_{d\underline{V}}$. From the general theory, (cf. [19, 25]), it follows that $\mathrm{Inv}_\kappa C_{d\underline{V}}$ consist of the directed unions of derived relations. In certain cases we may improve (3.43). Let $\delta : B^2 \to V$ be such that $\delta(b, b) \leqslant 0$ for all $b \in B$.

Let $\underline{B}=(B,\delta)$ be the corresponding binary space and denote by σ the set $\mathrm{Hom}(\underline{B},\underline{A})$ (of contractions from $\underline{B}$ into $\underline{A}$). Call $\sigma|_{\kappa}$ a *quasi-metric* κ-ary relation. We have:

Lemma 3.7.2. (1) *A quasi-metric κ-ary relation is a derived κ-ary relation.*

(2) *If $(V, \leqslant)$ is a complete meet-semilattice, then a κ-ary relation is derived if and only if it is quasi-metric.*

Proof. (1) Let ρ be a quasi-metric κ-ary relation on A, $\underline{B}=(B,\delta)$ the corresponding binary space and $\sigma:=\mathrm{Hom}(\underline{B},\underline{A})$. Put $W:=\mathrm{Im}\,\delta$ and $\alpha_w:=\delta^{-1}(w)$ for each $w\in W$. We claim that τ given by (3.43) equals ρ. Let $g\in\sigma$. Then g is a contraction from $\underline{B}$ to $\underline{A}$. Let $(x,y)\in\alpha_w$ for some $w\in W$. Then $\delta(x,y)=w$ and $d(g(x),g(y))\leqslant\delta(x,y)=w$ proving $g\in\tau$ and $\sigma\subseteq\tau$. The proof of $\tau\subseteq\sigma$ is quite similar.

(2) Let the assumptions of (2) hold and let a derived κ-ary relation be given by (3.43). For $x,y\in B$ with $x\neq y$ put $U_{xy}:=\{w\in W\colon (x,y)\in\alpha_w\}$ and $\delta(x,y):=\wedge\, U_{xy}$. Further put $\delta(x,x):=0$ for all $x\in B$. Put $\underline{B}=(B,\delta)$ and $\sigma:=\mathrm{Hom}(\underline{B},\underline{A})$. Let $g\in\tau$ and $x,y\in B$, $x\neq y$. From (3.43) we have $v:=d(g(x),g(y))\leqslant w$ for every $w\in U_{xy}$ and hence $v\leqslant\wedge\, U_{xy}=\delta(x,y)$. Clearly $d(g(x),g(x))=0=\delta(x,x)$ for all $x\in B$ proving $g\in\sigma$ and $\tau\subseteq\sigma$. For the converse let $h\in\sigma$ and $(x,y)\in\alpha_w$ for some $w\in W$. Then $w\in U_{xy}$, hence $d(h(x),h(y))\leqslant\delta(x,y)\leqslant w$ proving $h\in\tau$ and $\sigma\subseteq\tau$. □

Lemma 3.7.3. *Let $(V,\leqslant)$ be a meet-semilattice, $\underline{B}=(B,\delta)$ a binary space and (A,d) a $\underline{V}$-metric space.* For *$u,v\in B$ put $\delta'(u,v):=\delta(u,v)\wedge\overline{\delta(v,u)}$ and $\underline{B}':=(B,\delta')$. Then $\underline{B}'$ satisfies* (d3) *and* $\mathrm{Hom}(\underline{B},\underline{A})=\mathrm{Hom}(\underline{B}',\underline{A})$.

Proof. *Let $f\in\mathrm{Hom}\,(\underline{B},\underline{A})$, $u,v\in B$* and $a:=d(f(u),f(v))$. Then

$$a\leqslant\delta(u,v),\ \bar{a}=d(f(v),f(u))\leqslant\delta(v,u)$$

and so $a\leqslant\overline{\delta(v,u)}$, and therefore

$$a\leqslant\delta(u,v)\wedge\overline{\delta(v,u)}=\delta'(u,v).$$

Conversely, let $g\in\mathrm{Hom}(\underline{B}',\underline{A})$ and $u,v\in B$. Then

$$d(g(u),g(v))\leqslant\delta'(u,v)\leqslant\delta(u,v).$$

It is easy to verify that δ satisfies (d3). □

3.7.4. In the sequel $(V,\leqslant)$ is a complete lattice. Let $\underline{A}=(A,d)$ be a $\underline{V}$-metric space. Combining Lemmas 3.7.2. and 3.7.3 we obtain that each derived κ-relation is the restriction to $\underline{\kappa}$ of $\mathrm{Hom}(\underline{B},\underline{A})$ where $\underline{B}=(B,\delta)$ is a suitable binary $\underline{V}$-space satisfying $\delta(b,b)\leqslant 0$ for all $b\in B$, the axiom (d3) and $B\supseteq\underline{\kappa}$. In general, $\underline{B}$ need not be a premetric (cf. Section 1.1), i.e. satisfy the $\triangle$-inequality. We construct a binary $\underline{V}$-space $\hat{B}=(B,\hat{\delta})$ which under special assumptions is a $\underline{V}$-premetric. Under additional strong assumptions we may replace $\underline{B}$ by $\hat{\underline{B}}$ (and even assume $\hat{\underline{B}}$ to be a $\underline{V}$-metric space on $\underline{\kappa}$).

For $u, v \in B$ denote by $P(u, v)$ the set of finite sequences $p = \langle b_1, \ldots, b_m \rangle$ over B with $b_1 := u$, $b_m = v$ and $b_i \in B$. In 3.7.5 we use the obvious bijection from $P(u, v)$ onto $P(v, u)$ assigning the reverse sequence $p' := \langle b_m, \ldots, b_1 \rangle$ to $p = \langle b_1, \ldots, b_m \rangle$. For $p = \langle b_1, \ldots, b_m \rangle$ put:

$$p^{\#} := \delta(b_1, b_2) + \cdots + \delta(b_{m-1}, b_m), \qquad \hat{\delta}(u, v) := \bigwedge\nolimits_{p \in P(u,v)} p^{\#}. \tag{3.44}$$

Since $\langle u, v \rangle \in P(u, v)$, we have:

$$\hat{\delta}(u, v) \leqslant \delta(u, v) \tag{3.45}$$

for all $u, v \in B$. Put $\hat{B} := (B, \hat{\delta})$. We start with the following lemma.

Lemma 3.7.5. *Let $\underset{\sim}{V}$ be a complete lattice, $\underset{\sim}{B} = (B, \delta)$ a binary $\underset{\sim}{V}$-space and $\hat{B} = (B, \hat{\delta})$ as in* 3.7.4. *Then*:

(i) $\mathrm{Hom}(\underset{\sim}{B}, \underset{\sim}{A}) = \mathrm{Hom}(\hat{\underset{\sim}{B}}, \underset{\sim}{A})$,

(ii) $\hat{\delta}(b, b) \leqslant 0$ *provided* $\delta(b, b) \leqslant 0$ $(b \in B)$,

(iii) $\hat{B}$ *satisfies* (d3) *provided* $\underset{\sim}{B}$ *does.*

(iv) *If* $\underset{\sim}{V}$ *is Heyting* (*cf.* 3.5.18) *then* $\hat{B}$ *satisfies the* $\triangle$-inequality (d2).

Proof. Put $\sigma := \mathrm{Hom}(\underset{\sim}{B}, \underset{\sim}{A})$ and $\hat{\sigma} := \mathrm{Hom}(\hat{\underset{\sim}{B}}, \underset{\sim}{A})$.

(i) By (3.45) we have $\hat{\sigma} \subseteq \sigma$. For the converse, consider $f \in \sigma$, $u, v \in B$ and $p = \langle b_1, \ldots, b_m \rangle \in P(u, v)$. Put $c_i := f(b_i)$ $(i = 1, \ldots, m)$. Taking into account that f is a contraction, we have $d(c_i, c_{i+1}) \leqslant \delta(b_i, b_{i+1})$ for $i = 1, \ldots, m-1$. In the $\underset{\sim}{V}$-metric space $\underset{\sim}{A}$ we have

$$d(f(u), f(v)) = d(c_1, c_m) \leqslant \sum_{0<i<m} d(c_i, c_{i+1}) \leqslant \sum_{0<i<m} \delta(b_i, b_{i+1}) = p^{\#},$$

$$d(f(u), f(v)) \leqslant \bigwedge_{p \in P(u,v)} s(p) = \hat{\delta}(u, v)$$

proving $f \in \hat{\sigma}$ and $\sigma \subseteq \hat{\sigma}$

(ii) Apply (3.45).

(iii) Let $\hat{B}$ satisfy (d3). Let $u, v \in B$ and $p = \langle b_1, \ldots, b_m \rangle \in P(u, v)$. Then

$$\begin{aligned} p^{\#} &= \delta(b_1, b_2) + \cdots + \delta(b_{m-1}, b_m) = \overline{\delta(b_2, b_1)} + \cdots + \overline{\delta(b_m, b_{m-1})} \\ &= \overline{\delta(b_m, b_{m-1}) + \cdots + \delta(b_2, b_1)} = \overline{p'^{\#}} \end{aligned}$$

where $p' := \langle b_m, \ldots, b_1 \rangle \in P(v, u)$. Taking into account that $v \to \bar{v}$ is an order automorphism of $(V, \leqslant)$ and $p \to p'$ a bijection of $P(u, v)$ onto $P(v, u)$ we get the required

$$\hat{\delta}(u, v) = \bigwedge_{p \in P(u,v)} p^{\#} = \bigwedge_{p \in P(u,v)} \overline{p'^{\#}} = \overline{\bigwedge_{p \in P(u,v)} p'^{\#}} = \overline{\bigwedge_{p \in P(v,u} p^{\#}} = \overline{\hat{\delta}(v, u)}\cdot$$

(iv) Consider $u, v, w \in B$ and put

$$C := P(u, v), \qquad D := P(v, w), \qquad E := P(u, w).$$

For $p=\langle b_1,\ldots,b_m\rangle\in C$ and $q=\langle b_m,\ldots,b_{m+n}\rangle\in D$ denote the concatenation $\langle b_1,\ldots,b_{m+n}\rangle$ (of p and q) by $p<>q$. Obviously $p<>q\in E$ and $(p<>q)^{\#}=p^{\#}+q^{\#}$. Applying full distributivity we get the required

$$\hat\delta(u,v)+\hat\delta(v,w)=\bigwedge_{p\in C}p^{\#}+\bigwedge_{q\in D}g^{\#}=\bigwedge_{p\in C,\ q\in D}(p^{\#}+q^{\#})$$

$$=\bigwedge_{p\in C,\ q\in D}(p<>q)^{\#}\leqslant\bigwedge_{r\in E}r^{\#}=\hat\delta(u,w). \qquad \square$$

As usual, for $f:\underline{\kappa}\to A$ put $\operatorname{Ker}f:=\{(x,y)\in\underline{\kappa}^2:\ f(x)=f(y)\}$. For an equivalence relation ε on $\underline{\kappa}$ put $\Delta_\varepsilon:=\{f\in A^{\kappa}:\ \operatorname{Ker}f\supseteq\varepsilon\}$ (i.e. Δ_ε consists of all maps of $\underline{\kappa}$ into A constant on each block of ε). We have the following.

Proposition 3.7.6. *Let $\underline{V}$ be Heyting (cf. 3.5.18) and $\underline{A}=(A,d)$ a $\underline{V}$-metric space. Then:*

(i) *a κ-ary relation σ on A is derived if and only if*

$$\sigma=\{f\in\Delta_\varepsilon:\ f|_{\underline{\kappa}'}=g|_{\underline{\kappa}'}\ \text{for some}\ g\in\operatorname{Hom}(\underline{B},\underline{A})\} \tag{3.46}$$

where $\kappa'\leq\kappa$, ε is an equivalence on $\underline{\kappa}$ and $\underline{B}=(B,\delta)$ is a $\underline{V}$-metric space with $B\supseteq\underline{\kappa}'$, and

(ii) *For a κ-ary relation ρ on $\underline{A}$* the clone $\operatorname{Pol}\rho$ *contains all the d-contracting operations if and only if ρ is the union of a directed family of derived relations.*

Proof. It suffices to prove (i). From Lemmas 3.7.2, 3.7.3 and 3.7.5 we know that $\sigma=\{f|_{\underline{\kappa}}:\ f\in\operatorname{Hom}(\underline{B},\underline{A})\}$ where $\underline{B}=(B,\delta)$ is a $\underline{V}$-premetric space (cf. 3.1.1) such that $B\supseteq\underline{\kappa}$. Recall that in 3.12 and 3.13 we have defined an equivalence $\approx$ on B by setting $u\approx v$ whenever $\delta(u,v)\leqslant 0$. As in Fact 3 of a 3.3.4 let $B^0:=B/\approx$. According to Fact 3 the space B^0 is a $\underline{V}$-metric space. It is easy to see that an operation f on A preserves $\tau:=\operatorname{Hom}(\underline{B},\underline{A})$ if and only if it preserves $\tau^0:=\operatorname{Hom}(\underline{B}^0,\underline{A})$ (use the fact that each $t\in\tau$ is constant on each block of $\approx$). Denote by K the set of blocks of $\approx$ meeting the subset $\underline{\kappa}$ of B. Clearly $\kappa':=|K|\leqslant\kappa$ and it suffices to index K by $\underline{\kappa}'$ to obtain the statement of (i) (where ε is the equivalence $\approx$). $\square$

In 3.7.6 (i) we determined the general form of a derived κ-ary relation. If we are interested in $\operatorname{Pol}\sigma$ only we can simplify (3.46); in particular, the use of Δ_ε is superfluous. This is based on the following general lemma whose easy proof is omitted. For a κ-ary relation ρ on A and $\varphi:\underline{\kappa}'\to\underline{\kappa}$ put $\rho_{(\varphi)}:=\{r\circ\varphi:\ r\in\rho\}$.

Lemma 3.7.7. *Let ρ be a κ-ary relation on A. $\varphi:\underline{\kappa}'\to\underline{\kappa}$ and ε an equivalence relation on $\underline{\kappa}$ such that* im φ *meets each block of ε exactly in a singleton. Then,*

(i) $\operatorname{Pol}\rho\subseteq\operatorname{Pol}\rho_{(\varphi)}$;

(ii) $\operatorname{Pol}\rho=\operatorname{Pol}\rho_{(\varphi)}$ *if $\rho\subseteq\Delta_\varepsilon$, or $\kappa=\kappa'$ and φ is a permutation of $\underline{\kappa}$.*

Corollary 3.7.8. *Let $\underline{A}$ and $\underline{V}$ be as in* 3.7.6 *and σ a derived κ-ary relation. Then* $\operatorname{Pol}\sigma=\operatorname{Pol}\rho$ *with $\rho=\operatorname{Hom}(\underline{B},\underline{A})|_{\underline{\kappa}'}$ where $\kappa'\leqslant\kappa$ and $\underline{B}=(B,\delta)$ a $\underline{V}$-metric space such that $B\supseteq\underline{\kappa}'$.*

3.7.9. Call a κ-ary relation σ on A *non-expansive* if it is of the form (3.46) for $B=\underline{\kappa}'$ (instead of $B\supseteq\underline{\kappa}'$) i.e. if there exist $\kappa'\leqslant\kappa$, an equivalence ε on $\underline{\kappa}$ and a $\underline{V}$-metric space $\underline{B}=(\underline{\kappa}',\delta)$ such that,

$$\sigma=\{f\in\Delta_\varepsilon\colon f|_{\underline{\kappa}'}\in\mathrm{Hom}(\underline{B},\underline{A}). \tag{3.47}$$

In other words, each $f\in\sigma$ is exactly an extension to $\underline{\kappa}$ of a contraction from $\underline{B}$ to $\underline{A}$ such that f is constant on each block of ε. The following interpretation of non-expansive relation is perhaps worth mentioning. For simplicity let $\kappa=\kappa'$ and ε the least equivalence. Then $\sigma:=\mathrm{Hom}(\underline{B},\underline{A})$ may be interpreted as the set of the solutions $\langle x_i\colon i<\kappa\rangle$ of the inequality system:

$$d(x_i,x_j)\leqslant\delta(i,j)\quad(\forall\, i,j<\kappa). \tag{3.48}$$

In the more general case of $B\supset\underline{\kappa}$, we may also interpret $\mathrm{Hom}(\underline{B},\underline{A})$ in terms of the solutions $\langle x_b\colon b\in B\rangle$ of the system

$$d(x_b,x_{b'})\leqslant\delta(b,b')\quad(\forall\, b,b'\in B); \tag{3.49}$$

however in σ we only monitor the part $\langle x_i\colon i<\kappa\rangle$ of a solution of (3.49) and so σ consists of $\langle x_i\colon i<\kappa\rangle$ for which there exist x_b $(b\in B\backslash\kappa)$ so that the inequalities (3.49) hold. This hidden part makes things more complex; eg. the congruence problems in universal algebra are due to this fact. In an extreme case it suffices to use non-expansive relations only. This is the main result of this section.

Theorem 3.7.10. *Let $\underline{V}$ be Heyting, $\underline{A}=(A,d)$ a hyperconvex $\underline{V}$-metric space and ρ a κ-ary relation. Then all d-contracting operations on A preserve ρ if and only if ρ is the directed union of non-expansive relations.*

Proof. From Proposition 3.7.6 we know that each derived operation is of the form (3.46). Denote by $K'=(\underline{\kappa}',\delta)$ the restriction of $\underline{B}$ to $\underline{\kappa}'$. By Theorem 3.3.4 the metric space $\underline{A}$ has the extension property. Consequently, each $h\in\mathrm{Hom}(K',\underline{A})$ extends to some $h'\in\mathrm{Hom}(\underline{B},\underline{A})$ and so σ may be replaced by the non-expansive relation

$$\{f\in\Delta_\varepsilon\colon f|_{\underline{\kappa}'}\in\mathrm{Hom}(K',\underline{A})\}.\qquad\square$$

In 4.4.14, we shall need the non-expansive relations (cf. 3.7.9) in a very special case.

3.7.11. Suppose $(V,\leqslant)$ is a chain with a least element 0. Denote by $\vee$ the corresponding join (i.e. $v\vee v':=\max(v,v')$) and put $\underline{V}:=\langle V,\leqslant,\vee,0\rangle$. Let $\underline{A}=(A,d)$ be a hyperconvex $\underline{V}$-ultrametric on A. According to 3.7.10 and 3.7.7 we have $\mathrm{Pol}\,\sigma=\mathrm{Pol}\,\rho$ for $\rho=\mathrm{Hom}(\underline{B},\underline{A})$ where $\underline{B}=(\underline{\kappa}',\delta)$ is a $\underline{V}$-ultrametric space. We give a better description of ρ. By the definition of a contraction (cf. 3.1.2) we have:

$$\rho=\{f\in B^{\underline{\kappa}'}\colon d(f(i),f(j))\leqslant\delta(i,j)\text{ for all } i<j<\kappa'\}. \tag{3.50}$$

Recall (cf. 3.6.4) that for $v \in V$ the binary relation $(d)_v := \{(x, y) \in B^2 : d(x, y) \leqslant v\}$ is an equivalence relation on B. We show that the clone Pol ρ (of all operations preserving ρ) equals $\bigcap_{v \in I} \operatorname{Pol}(d)_v$ where $I = \operatorname{im} \delta$. For $i < j$ call

$$pr_{ij}\,\rho := \{(f(i), f(j)) : f \in \rho\}$$

the *(i, j)th projection of ρ*. We need the following.

Lemma 3.7.12. *If ρ is given by* (3.50), *$i < j < \kappa'$ and $v := \delta(i, j)$ then $pr_{ij}\,\rho = (d)_v$.*

Proof. For $f \in \rho$ we have $d(f(i), f(j)) \leqslant v$ proving $(f(i), f(j)) \in (d)_v$ and $pr_{ij}\,\rho \subseteq (d)_v$.

For $\supseteq$ let $(x, y) \in (d)_v$ or, equivalently, $d(x, y) \leqslant v$. Define $f: \underline{\kappa}' \to A$ by setting $f(m) = x$ whenever $\delta(i, m) < v$ and $f(m) = y$ otherwise. We claim that $f \in \rho = \operatorname{Hom}(\underline{B}, \underline{A})$. Suppose $f \notin \rho$. Then there are $p, q < \kappa'$ such that

$$d(f(p), f(q)) \nleqslant \delta(p, q). \tag{3.51}$$

Since $\operatorname{Im} f = \{x, y\}$, we may choose the notation so that $f(p) = x, f(q) = y$. Setting $z := \delta(p, q)$ and using the assumptions on $(V, \leqslant)$, from (3.51) we get $v > z$. Put $r := \delta(i, p)$ and $s := \delta(i, q)$. In view of $f(q) = y$ we have $s \geqslant v$. From the isosceles property (for i, p and q) and $r < v \leqslant s$ we get $z = \delta(p, q) = s \geqslant v$ in contradiction to $v > z$. Thus $f \in \rho$ and $(x, y) = (f(i), f(j)) \in pr_{ij}\,\rho$ proving $\supseteq$. □

Proposition 3.7.13. *Let $(V, \leqslant)$ be a chain with the least element 0, $\underline{V} = \langle V, \leqslant, \vee, 0 \rangle$ and $\underline{A} = (A, d)$ a $\underline{V}$-ultrametric. If σ is a non-expansive relation for $\underline{A}$ then*:

$$\operatorname{Pol}\sigma = \bigcap_{w \in W} \operatorname{Pol}(d)_w \tag{3.52}$$

where $W := \operatorname{Im} \delta$ for a $\underline{V}$-ultrametric δ.

Proof. Let ρ be as in (3.50). Put $W := \operatorname{Im} \delta$ and denote by C the clone on the right-hand side of (3.52). It is known and easy to prove from (3.50) that $C \subseteq \operatorname{Pol}\sigma$. On the other hand, it is known that $\operatorname{Pol}\rho \subseteq \operatorname{Pol} pr_{ij}\,\rho$ for all $i < j < \kappa'$. From Lemma 3.7.12 it follows that $\operatorname{Pol}\rho \subseteq \operatorname{Pol}(d)_w$ for all $w \in W$ and so $\operatorname{Pol}\rho \subseteq C$. □

4. Relations, graphs, automata and sequential machines as metric spaces over a Heyting algebra

4.1. Binary spaces revisited

A binary $\underline{V}$-space (A, δ), as defined in Section 1.1, may in general seem to be far from a metric space; especially, if there is no Heyting structure on $\underline{V}$ for which δ can be viewed as a distance. However, as we shall see in this section, that impression is false. We need a few technical lemmas.

4.1.1. Let $V=(V, \leqslant)$ be a poset. Put $\underline{2}=(\{0,1\}, \leqslant)$, $U_1:=\{(v,v,0): v\in V\}$, $U_2:=\{(u,v,1): u,v\in V\}$ and $V_H:=U_1\cup U_2$. Clearly $V_H\subset V^2\times\underline{2}$ and so V_H is equipped with the (component wise) order $\leqslant$ inherited from $V^2\times\underline{2}$.

We shall turn V_H into a Heyting algebra. We start with the following (most likely known) extension of a semigroup.

Lemma 4.1.2. *Let $\langle U_1; \cdot\rangle$ be a semigroup, let U_2 be a set disjoint from U_1, $e\in U_2$ and $T:=U_1\cup U_2$. Define a binary operation $+$ on T by setting $a+b:=a\cdot b$ if $a,b\in U_1$, $a+b:=e$ if $a,b\in U_2$, $a+b:=b+a=b$ if $a\in U_1$, $b\in V_2$. Then $\langle T;+\rangle$ is a semigroup. If, moreover, $\langle U_1;\cdot\rangle$ is commutative then $\langle T;+\rangle$ is commutative.*

Proof. Let $a,b,c\in T$. If $a,b,c\in U_1$ then $(a+b)+c=a\cdot b\cdot c=a+(b+c)$. If two of a,b,c belong to U_1 while the third, say α, is in U_2 then $(a+b)+c=\alpha=a+(b+c)$. Finally, if a least two of a,b,c belong to U_2 then $(a+b)+c=e=a+(b+c)$. □

Let $(V,\leqslant)$ be the order of a meet-semilattice $\langle V;\wedge\rangle$ with a greatest element. We identify this greatest element with 1 and the least element, if any, with 0. Denote by $+$ the semigroup operation defined in Lemma 4.1.2 for $\langle U_1;\wedge\rangle$ and $e:=(1,1,1)$. Put $w:=(1,1,0)$. We have:

Lemma 4.1.3. *$\underline{T}:=\langle V_H;\leqslant,+,w\rangle$ is an ordered commutative monoid.*

Proof. Let $a,b,c\in V_H$, $a\leqslant b$. We have three cases. (1) Let $a=(u,u,0)\in U_1$ and $b=(v,v,0)\in U_1$. If $c\in U_2$ we have $a+c=e=b+c$ and if $c=(t,t,0)$ then $a+c=(u\wedge t,u\wedge t,0)\leqslant(v\wedge t,v\wedge t,0)=b+c$. (2) Let $a,b\in U_2$. If $c\in U_2$ then $a+c=e=b+c$. Thus let $c\in U_1$. Then $a+c=a\leqslant b=b+c$. (3) Finally let $a=(u,u,0)\in U_1$ and $b=(v,t,1)\in U_2$. If $c=(r,r,0)\in U_1$ then $a+c=(r\wedge u,r\wedge u,0)\leqslant b=b+c$. If $c=(r,s,1)\in U_2$ we have $a+c=c\leqslant e=b+c$. The element w is a neutral element of $+$. Indeed, $w\in U_1$ and so $a+w=a$ for all $a\in U_2$ while for $a=(v,v,0)\in U_1$ we have $a+w=(v\wedge 1,v\wedge 1,0)=a$. □

In our notation $\wedge$ takes precedence over $+$; e.g. $a+\alpha_1\wedge\alpha_2$ stands for $a+(\alpha_1\wedge\alpha_2)$.

Lemma 4.1.4. *If $(V,\leqslant)$ is a complete lattice then $\underline{T}$ (from Lemma 4.1.3) is Heyting.*

Proof. Let $a\in V_H$, $X_i\subseteq U_i$ $(i=1,2)$ and $X:=X_1\cup X_2$. Put $\alpha_i:=\wedge X_i$ and $\beta_i:=\wedge\{a+x;x\in X_i\}$ $(i=1,2)$. We must prove $a+\alpha_1\wedge\alpha_2=\beta_1\wedge\beta_2$.

We have two cases: (1) Let $a\in U_1$. Then $\beta_1=\wedge\{a\wedge x: x\in X_1\}$ and so $\beta_1=a\wedge\alpha_1$ if $X_1\neq\emptyset$ and $\beta_1=e$ otherwise. Similarly $\beta_2=\alpha_2$ if $\chi_2\neq\emptyset$ and $\beta_2=e$ otherwise. If $X_1\neq\emptyset$ we have $\alpha_1\wedge\alpha_2\in U_1$ and so $a+\alpha_1\wedge\alpha_2=a\wedge\alpha_1\wedge\alpha_2=\beta_1\wedge\beta_2$. If $X_1=\emptyset$ then in view of $\alpha_2\in U_2$ we have $a+\alpha_1\wedge\alpha_2=a+e\wedge\alpha_2=a+\alpha_2=\alpha_2=e\wedge\alpha_2=\beta_1\wedge\beta_2$.

(2) Let $a \in U_2$. Note that $\beta_1 = a$ if $X_1 \neq \emptyset$ and $\beta_1 = e$ otherwise. Next $\beta_2 = e$ whence $\beta_1 \wedge \beta_2 = a$ if $X_1 \neq \emptyset$ and $\beta_1 \wedge \beta_2 = e$ otherwise. If $X_1 \neq \emptyset$ we have $\alpha_1 \wedge \alpha_2 \in U_1$ and so $a + \alpha_1 \wedge \alpha_2 = a = \beta_1 \wedge \beta_2$ whereas for $X_1 = \emptyset$ we have $\alpha_1 \wedge \alpha_2 \in U_2$ and $a + \alpha_1 \wedge \alpha_2 = e = \beta_1 \wedge \beta_2$. □

Let $\underline{T}$ be as in Lemma 4.1.3. For every $x = (x_1, x_2, x_3) \in V_H$ put $\bar{x} = (x_2, x_1, x_3)$. It is immediate that $x \mapsto \bar{x}$ is an involutive order automorphism of $(V_H, \leqslant)$ fixing each element of U_1. According to 3.5.1 we can define the following metric: $d_T(a, b) := \wedge \{r \in V_H : b \leqslant a + r, a \leqslant b + \bar{r}\}$ for all $a, b \in V_H$. According to our convention, 0 denote the least element of $(V, \leqslant)$. Put $\lambda := (0, 0, 1)$. The map d_T may be easily described.

Lemma 4.1.5. *The values of d_T are given by the following tables.*

	$b \in U_1$	$b \in U_2$
$a \in U_1$	$a \vee b \vee \lambda$	b
$a \in U_2$	$\bar{a}$	λ

$d_T(a, b)$ *for* $a \neq b$

a	$a \in U_1$	$a \in U_2$
$d_T(a, a)$	a	$(0, 0, 0)$

$d_T(a, a)$

Proof. Let, $a, b \in V_H$, $a \neq b$. (1) Let $a, b \in U_1$. If $r \in U_1$ then $\bar{r} = r$ and the inequalities

$$b \leqslant a + r, \qquad a \leqslant b + \bar{r} \tag{4.1}$$

mean $b \leqslant a \wedge r$, $a \leqslant b \wedge r$ and so $b \leqslant a \leqslant b$. Let $r \in U_2$ satisfy (4.1). Then $b \leqslant r$ and $a \leqslant \bar{r}$ and $a \vee b \vee \lambda$ is the least such r. (2) Let $a \in U_1$ and $b \in U_2$. If $r \in U_1$ satisfies (4.1) then $b \leqslant r \wedge a \in U_1$, a contradiction. Now for $r \in U_2$ the system (4.1) becomes $b \leqslant r$, $a \leqslant e$ and so b is the least solution. (3) The case $a \in U_2$, $b \in U_1$ is similar. (4) Let $a, b \in U_2$. Were $r \in U_1$ a solution of (4.1) it would yield $b \leqslant a \leqslant b$, whereas every $r \in U_2$ satisfies (4.1) and so $\lambda = \wedge\ U_2$ is the least solution of (4.1).

Verification of the values given in right-hand Table is quite similar. □

Lemma 4.1.6. *The map d_T satisfies the triangle inequality* (d2).

Proof. Let $a, b, c \in V_H$. Put $\alpha := d_T(a, b)$, $\beta := d_T(a, c)$ and $\gamma := d_T(c, b)$. From the tables in Lemma 4.1.5 we see that $d_T(x, y) \in U_2$ if $x \neq y$ and $d_T(x, x) \in U_1$. If $a \neq c \neq b$ then we have the required $\alpha \leqslant e = \beta + \gamma$. *If* $a = c \neq b$ then $\beta \in U_1$ while $\alpha = \gamma \in U_2$ and so $\alpha \leqslant \gamma = \beta + \gamma$. The same argument applies if $a \neq c = b$. Finally, if $a = b = c$ we have $\alpha = \beta = \gamma \in U_1$ and $\alpha \leqslant \alpha \wedge \alpha = \alpha + \alpha$. □

Let $\underline{A} = (A, \delta)$ be a binary $\underline{V}$-space. Define $\delta^* : A^2 \to V_H$ by setting:

$$\delta^*(x, x) := (\delta(x, x), \delta(x, x), 0), \tag{4.2}$$

$$\delta^*(x, y) := (\delta(x, y), \delta(y, x), 1) \tag{4.3}$$

for all $x, y \in A$, $x \neq y$. Finally denote the binary $\underline{T}$-space (A, δ^*) by $\underline{A}^*$. We have the following.

Lemma 4.1.7. *Let $\underline{V}$ be as in* 4.1.1 *and let $\underline{A}=(A,\delta)$ and $\underline{B}=(\underline{B},\delta')$ be binary $\underline{V}$-spaces. Then,*

(i) *$\underline{A}^*$ satisfies* (d 1): *$\delta^*(x,y)\leqslant 0:=w$ if and only if $x=y$ and* (d 3): *$\delta^*(y,x)=\overline{\delta^*(x,y)}$ for all $x,y\in A$, and*

(ii) $\mathrm{Hom}(\underline{A},\underline{B})=\mathrm{Hom}(\underline{A}^*,\underline{B}^*)$.

Proof. (i) Direct verification. (ii) Let $f\in\mathrm{Hom}(\underline{A},\underline{B})$, $x,y\in A$ and $u:=f(x)$, $v:=f(y)$. Then $\delta'(u,v)\leqslant\delta(x,y)$ and $\delta'(v,u)\leqslant\delta(y,x)$. If $u\neq v$ then by (4.3),

$$\delta^*(u,v)=(\delta'(u,v),\delta'(v,u),1)\leqslant(\delta(x,y),\delta(y,x),1)=\delta^*(x,y).$$

Thus let $u=v$. Put $a:=0$ if $x=y$ and $a:=1$ otherwise. Then

$$\delta^*(u,u)=(\delta'(u,u),(\delta'(u,u),0)\leqslant(\delta(x,y),\delta(y,x),a)=\delta^*(x,y)$$

proving that $f\in\mathrm{Hom}(\underline{A}^*,\underline{B}^*)$ and the inclusion $\subseteq$ in (ii). For $\supseteq$ let $f\subseteq\mathrm{Hom}(\underline{A}^*,\underline{B}^*)$, $x,y\in A$ and $u:=f(x)$, $v:=f(y)$. The first coordinates in $\delta^*(u,v)\leqslant\delta^*(x,y)$ show the required $\delta'(u,v)\leqslant\delta'(x,y)$. □

The equations (4.2) and (4.3) define a map $\varphi:(A,\delta)|\mapsto(A,\delta^*)$ from the $\underline{V}$-spaces into the metric $\underline{T}$-spaces. We show that φ has an inverse. As usual, $pr_1:V_H\to V$ assigns u to $(u,v,i)\in V_H$. We have the following lemma.

Lemma 4.1.8. *Let (A,d) be a $\underline{T}$-space satisfying* (d 1), (d 2) *and let $\delta:=pr_1\circ d$. Then $\delta^*=d$.*

Proof. Let $a,b\in A$ where $a\neq b$. Since by (d1) we have $d(a,a)\leqslant w:=(1,1,0)$, the element $d(a,a)$ is of the form $(v,v,0)$ for some $v\in V$. Thus $\delta(a,a)=v$ and by (4.2) also $\delta^*(a,a)=(v,v,0)=d(a,a)$. Similarly $d(a,b)\nleqslant w$ whence $d(a,b)=(u,v,1)$ for some $u,v\in V$, and so $\delta(a,b)=u$. From $d(b,a)=\overline{d(a,b)}=(v,u,1)$ we get $\delta(b,a)=v$ and by (4.3) finally $\delta_*(a,b)=(u,v,1)=d(a,b)$. □

Combining Lemmas 4.1.4 and 4.1.8 we obtain the following.

Theorem 4.1.9. *Let $\underline{V}$ be a complete lattice. Then $\underline{V}$ extends to a Heyting algebra $\underline{T}$ and there is a contraction preserving bijection $\varphi:(A,\delta)\mapsto(A,\delta^*)$ from the binary $\underline{V}$-spaces onto the metric $\underline{T}$-spaces.*

As an example we calculate $\varphi^{-1}(d_T)=pr_1\circ d_T$.

Example 4.1.10. *The values of $\Delta_T:=pr_1\circ d_T$ are given in the following tables.*

a \ b	$(u',u',0)$	$(u',v',1)$
$(u,u,0)$	$u\vee u'$	u'
$(u,v,1)$	v	0

$\Delta_T(a,b)$ for $a\neq b$

a	$(u,u,0)$	$(u,v,1)$
	u	0

$\Delta_T(a,a)$

Now we have the following result.

Theorem 4.1.11. *Let V be a complete lattice. Every binary space over $\underline{V}$ embeds isometrically into a power of* $\mathcal{V}:=(V_H, \Delta_T)$.

Proof. Let (A,δ) be a binary space over V and as before let (A,δ^*) denote the metric $\underline{T}$-space defined by (4.2) and (4.3). The Heyting algebra $\underline{T}$ is solid in the sense of 3.5.6 and so by Proposition 3.5.7 there exists an isometry ψ from (A,δ^*) into a power $(V_H, d_T)^I$ (defined in 3.5.6). For $f: I \to V_H$ define $h:=Pr\,f$ by setting $h(i):=pr_1(f(i))$ for all $i \in I$. Writing $(V_H, d_T)^I$ as (V_H^I, m) according to 3.5.7, for $f, g: I \to V_H$ we have,

$$m(f,g)=\bigvee_{i\in I} d_T(f(i), g(i))$$

(where the sup is in $(V_H, \leqslant)$). From the definitions we get,

$$\mathrm{pr}_1(m((f,g)))=\bigvee_{i\in I} \Delta_T(\mathrm{Pr}(f(i)), \mathrm{Pr}(g(i)))$$

(with sup in $(V, \leqslant)$) showing that the map $a \mapsto \mathrm{Pr}(\psi)(a))$ is the desired isometry. $\square$

Remark 4.1.12. Let $(V, \leqslant)$ be a complete lattice with a least element 0. We have seen in Theorem 4.1.9 the existence of a contraction preserving bijection $\varphi:(A,\delta)\mapsto(A,\delta^*)$ (from binary $\underline{V}$-spaces onto metric $\underline{T}$-spaces). The notions of (i) injectivity and (ii) absolute retracts (cf. 3.4.1) are both defined solely in terms of contractions and so may be identified via φ (i.e. (A,δ) has exactly the properties (A,δ^*) does). We have also seen in Lemma 1.4.1 that powers are defined by morphisms and thus (iii) the retracts of powers may be identified as well. In this way the equivalence of (i)–(iii) in 3.5.19 may be transferred from the category of metric $\underline{T}$-spaces to $S_{2\underline{V}}$.

In general, the concrete characterization of spaces satisfying any of (i)–(iii) becomes more exacting the smaller the category is, and so it is not surprising that for the largest category $S_{2\underline{V}}$ these spaces are quite simple. There is a direct and elementary proof of this. Indeed, for a binary space (A,δ) over $\underline{V}$ call an element $x \in A$ *central* if $d(x,y)=d(y,x)=0$ for all $y \in A$ (where 0 is the least element of V) and call (A, δ) *central* if it contains a central element. For example, in (V_H, Δ_T) from 4.1.10 the element $\lambda:=(0,0,1)$ is central, and so (V_H, Δ_T) is central.

Trivially the *absolute retracts* (with respect to the injective isometries) *are central*; for an absolute retract (A,δ) extend A by a new element x, and extend δ by setting $\delta(x,y)=\delta(y,x):=0$ for all $y \in A \bigcup \{x\}$ obtaining in this way an isometric extension of (A,δ). Next the central *spaces are injective* (extend every partial map with values in a central space (A, δ) by sending any extra element onto a central element of (A,δ)). The fact that centrality is preserved under products and retracts (a consequence of the identity between injectivity and centrality) is in this case also trivial. In this way we obtain the following.

Theorem 4.1.13. *In the category $S_{2\underline{V}}$ the absolute retracts, the injectives (both with respect to the injective isometries), the central spaces and the retracts of powers of $\mathscr{V}$ coincide.*

4.2. Relational structures and graphs

4.2.1. Let D be a set and $\underline{V}=(\wp(D), \supseteq)$. As we saw in 1.3.3 the category $S_{2\underline{V}}$ is isomorphic to the category $R_{2\underline{D}}$ of binary relational systems of type D (cf. Section 1.2). In 4.1.10 we introduced the $\underline{V}$-space $\mathscr{V}=(V_H, d_T)$. We shall need an explicit description of the relational system $U_D:=(V_H; \langle \rho_i: i\in D\rangle)$ associated to $\mathscr{V}$ (cf. 1.3.4). Let $i\in D$; according to 1.3.4, the introduction to 1.3 and the fact that $\supseteq$ is the order on $P:=\wp(D)$ we have:

$$\rho_i=(\Delta_T)_{\{i\}}=\{(\alpha,\beta)\in V_H: i\in\Delta_T(\alpha,\beta)\}. \tag{4.4}$$

Since D and $\emptyset$ are the least and greatest element of P, we also have:

$$V_H=\{(u,u,D): u\in P\}\bigcup\{(u,v,\emptyset): u, v\in P\}.$$

To simplify the notation put $g(u,u,D):=u$ and $g(u,v,\emptyset):=(u,v)$ for all $u,v\in P$ and replace V_H by its image $U:=g(V_H)=P^2\cup P$ (where $P^2:=P\times P$). Put $Q:=\{\alpha\in P: i\in\alpha\}$ (=all subsets of D containing i). Using the Tables of 4.1.10 and recalling that $u\vee u'=u\cap u'$ and $0=D$ we obtain:

$$g(\rho_i)=(P^2\times P^2)\cup(Q\times Q)\cup((P\times Q)\times P)\cup(P\times(Q\times P)). \tag{4.5}$$

Thus $g(\rho_i)$ consists of (i) a clique (=complete graph including all loops) on P^2, (ii) a clique on Q (iii) all arcs (=oriented edges) going from the vertices of $P\times Q$ (members of P^2) to the vertices of P, and (iv) all arcs from the vertices of P to the vertices from $Q\times P$ (members of P^2).

Example 4.2.2. Let $D=\{1\}$ (i.e. our a relational system is just a binary relation ρ_1). Then $P=\{\emptyset,\{1\}\}$, $Q=\{\{1\}\}$ and ρ_1 is displayed in Fig. 1.

The translation of Theorem 4.1.13 to relations (in terms of relational homomorphisms, embeddings, products, etc.) yields the following theorem.

Theorem 4.2.3. *Every binary relational system of type D embeds into a power of $\underline{U}_D$. Moreover, the absolute retracts and the injectives (with respects to embeddings) as well as the central relational systems and the retracts of powers of $\underline{U}_D$ coincide.*

Remark 4.2.4. Theorem 4.2.3 extends to n-ary relational systems, but so far the corresponding $\underline{U}_D$ has been obtained by ad hoc and technical constructions (see [18, 20]) and we do not know whether metric ideas and Heyting algebra can yield a more transparent construction.

Certain Heyting substructures of $\underline{V}_H$ lead to subclasses of $S_{2\underline{V}}$ satisfying the analogs of Theorems 4.1.11 and 4.1.13. For $\underline{V}=(\wp(D), \supseteq)$ this produces interesting classes of relational systems. Some of them have already been studied, or are well known.

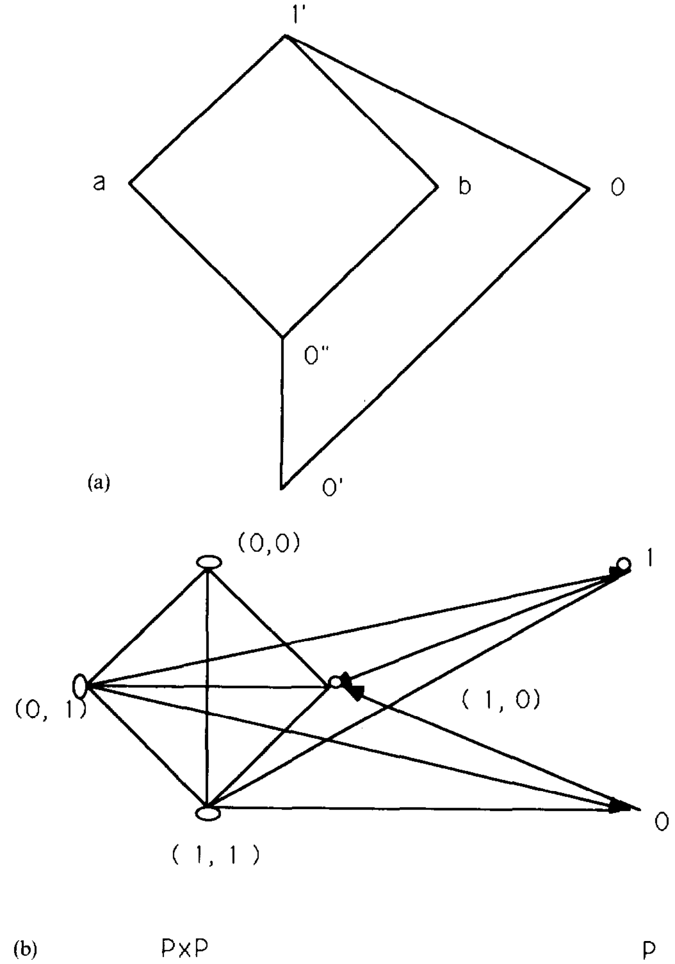

Fig. 1.

For example, $V^*_H := (V \times V \times \{1\}) \bigcup \{(0,0,0)\}$ is obviously a subsemigroup of $(V_H, +)$ (i.e. closed under $+$) and $0' := (0,0,0)$ its neutral element. It is also closed under $^-$ and obviously a Heyting algebra in the induced order. Consider a binary $\underline{V}$ – space (A, δ) such that (A, δ^*), defined by (4.2) and (4.3) in 4.1.6–4.1.7 happens to be a $\underline{V}^*_H$-space (i.e. all values of δ^* are in $\underline{V}^*_H$). Obviously for all $x \in A$ we have $\delta(x,x) = 0$ (where 0 is the least element of V). In the particular case of $\underline{V} := (\wp(D) \supseteq)$ the binary spaces correspond to relational D-systems of reflexive binary relations. Proceedings as in 4.2.1 for every $i \in D$ we construct the binary relation

$$\rho_i = (P^2 \times P^2) \cup \{(0', 0')\} \bigcup (\{0'\}) \times (Q \times P)) \bigcup (P \times Q) \times \{0'\}$$

or $P^2 \bigcup \{0'\}$ where again $P := \wp(D)$ and $Q := \{\alpha \in P : i \in \alpha\}$. For $D = \{1\}$ (i.e. a single binary reflexive relation) the lattice V^*_H and ρ_1 are indicated on the upper right corner

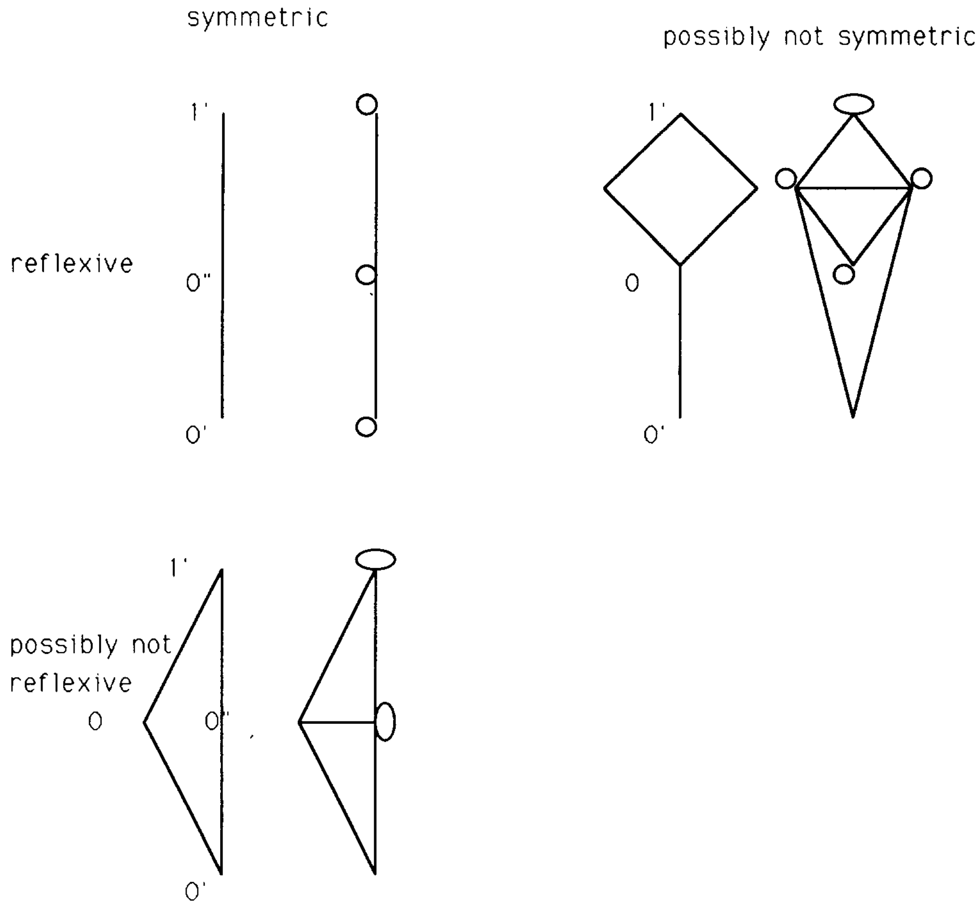

Fig. 2.

of Fig. 2. For the special case of relational systems consisting of binary reflexive relations (or, equivalently, for (A, δ) satisfying $\delta(x, y) = 0$ if and only if $x = y$) a similar approach is explored in [13].

Symmetry is another important property of binary relations. In Fig. 2 the four possible cases of the lattice and ρ_1 for a single binary relation (with respect to symmetry and reflexivity) are displayed. For example, every reflexive and symmetric binary relation (a graph with all loops) is isomorphic to an induced subgraph of a suitable power of the graph with 2 consecutive edges and 3 loops (in the left upper corner of Fig. 2).

4.3. Automata

4.3.1. Let A be a fixed set called an *alphabet*. The elements $a, b, \ldots$ of A are *letters* and finite sequences of letters are *words*. The empty word is denoted by $\square$. The word

$a=a_0a_1\cdots a_{n-1}$ has *length* $\ell(a):=n$ (and $\ell(\square):=0$). Denote by A^n the set of all words of length n, next put $A^*=\bigcup_{n=0}^{\infty} A^n$ and identify A^1 with A. For two words $r=r_0r_1\cdots r_{n-1}$ and $s=s_0s_1\cdots s_{m-1}$ denote by $r+s$ the concatenation $r_0r_1\cdots r_{n-1}s_0s_1\cdots s_{m-1}$ of r and s (we prefer this notation instead of the more natural $r\cdot s$). Clearly $\langle A^*;\ +\rangle$ is a monoid; in fact it is the so called *free monoid generated by A*. A *language* is a subset of A^*. Clearly, the operation $+$ may be extended to the set $L:=\wp(A^*)$ of languages by setting $X+Y:=\{x+y\colon x\in X, y\in Y\}$ for all X, $Y\subseteq A^*$ (Note that $X+\emptyset=\emptyset+X=\emptyset$ while $X+\{\square\}=\{\square\}+X=X$ for all $X\subseteq A^*$). Let $x\mapsto\bar{x}$ be a (fixed) involutive bijection of A (i.e. $\bar{\bar{x}}=x$ for all $x\in A$). Put $\bar{\square}:=\square$ and for $n>0$ and for $a=a_0a_1\cdots a_{n-1}\in A^*$ put $\bar{a}:=\bar{a}_{n-1}\bar{a}_{n-2}\cdots\bar{a}_0$. Clearly $\overline{r+s}=\bar{s}+\bar{r}$ for all $r,s\in A^*$. For $X\subseteq A^*$ set $\bar{X}:=\{\bar{x}\colon x\in X\}$. Clearly $^-$ is an order automorphism of $(L,\supseteq)$ and $+$ is an ordered monoid with respect to the complete lattice $L_A=(L,\supseteq)$ (where A^* and $\emptyset$ are the least and greatest elements). Moreover, we have full distributivity:

$$X+\bigwedge_{i\in I} X_i=X+\bigcup_{i\in I} X_i=\bigcup_{i\in I}(X+X_i)=\bigwedge_{i\in I}(X+X_i)$$

and so the following holds.

Fact 1. $\underline{L}_A:=\langle L, \supseteq, +, \{\square\}, -\rangle$ *is an involutive Heyting algebra.*

4.3.2. Classically a *non-deterministic automaton over A* is a quadruple $\mathsf{A}=\langle Q,T,I,F\rangle$ where Q is a set called the set of *states*, $I\subseteq Q$ and $F\subseteq Q$ are the sets of *initial and final states* and $T\subseteq Q\times A\times Q$ is the set of *transitions*. The language $L(\mathsf{A})$ *accepted* by Q is the set of $a_0\cdots a_{n-1}\in A^n$ for which there exist $p_0\in I$, $p_1,\ldots,p_{n-1}\in Q$, $p_n\in F$ such that $(p_i,a_i,p_{i+1})\in T$ for $i=0,\ldots,n-1$. For example, the empty word $\square$ belongs to $L(\mathsf{A})$ exactly if $I\cap F\neq\emptyset$. For $q,q'\in Q$ put $\mathsf{A}_{q,q'}:=\langle Q,T,\{q\},\{q'\}\rangle$ (i.e. $I=\{q\}$ and $F:=\{q'\}$) and consider $L(\mathsf{A}_{q,q'})$ as a 'distance' from q to q' (given T). Call the pair $\mathsf{Q}=\langle Q,T\rangle$ a *transition system* over A. To $a\in A$ we may associate the binary relation

$$\rho_a:=\{(q,q')\colon (q,a,q')\in T\}$$

on Q. (There is also the relation $\rho_\square=\{(q,q),\ q\in Q\}$). To Q associate the binary relational system $\mathsf{Q}_R:=(Q,\ (\rho_a\colon\ a\in A))$ of type A. We can define the *category of transition systems over A* whose objects are the transition systems over A and whose morphisms are defined by $\mathrm{Hom}(\mathsf{Q},\mathsf{Q}')=\mathrm{Hom}(\mathsf{Q}_R,\mathsf{Q}'_R)$ (where the latter are in the category of binary relational systems of type A).

We have seen in 1.3.3 how to turn a binary relational system of type A over Q into a binary space (Q,δ) over $\underline{V}:=(\wp(A),\supseteq)$. In our situation for distinct $q,q'\in Q$ we have concretely $\delta(q,q'):=\{a\in A\colon\ (x,a,y)\in T\}$ (i.e. $\delta(q,q')$ is the set of letters permitting a direct transition from the state q to the state q'). As noted above, for every $q\in Q$ we have

$$\delta(q,q):=\{\square\}\bigcup\{a\in A\colon\ (q,a,q)\in T\}. \tag{4.6}$$

Motivated by (4.6) we put $A^{\leqslant 1} := \{\square\} \bigcup A$ (considering $\square$ as an element). We introduce a particular binary operation $\dot{+}$ on $\wp(A^{\leqslant 1})$ as follows. Let $X_0, X_1 \subseteq A^{\leqslant 1}$.

(1) If $\square \in X_0 \cap X_1$ put $X_0 \dot{+} X_1 := X_0 \cup X_1$. (2) If $\square \in X_i \setminus X_{1-i}$ for some $i \in \{0,1\}$ put $X_0 \dot{+} X_1 = X_1 \dot{+} X_0 := X_{1-i}$ (3) If $\square \notin X_0 \cup X_1$ put $X_0 \dot{+} X_1 := \emptyset$. We show that the structure thus obtained has the required properties.

Lemma 4.3.3. *$\underline{U}_A := \langle \wp(A^{\leqslant 1}), \supseteq, \dot{+}, \{\square\} \rangle$ is an ordered commutative monoid.*

Proof. (1) Clearly $\dot{+}$ is commutative. (2) To prove associativity it suffices to show:

Claim. *Let $X, Y, Z \subseteq A^{\leqslant 1}$. Then $W := (X \dot{+} Y) \dot{+} Z \neq \emptyset$ if and only if $\square$ belongs to at least two of X, Y, Z. If this is the case, then $W = X \cup Y \cup Z$ whenever $\square \in X \cap Y \cap Z$ while otherwise W equals to the one of X, Y, Z not containing $\square$.*

Proof of the claim. Note that for $U, V \subseteq A^{\leqslant 1}$ we have $\square \in U \dot{+} V \Leftrightarrow \square \in U \cap V$.

($\Rightarrow$) Let $W := (X \dot{+} Y) \dot{+} Z \neq \emptyset$. If $\square \in W$ then $\square \in (X \dot{+} Y) \cap Z$ and so $\square \in X \cap Y \cap Z$. Thus let $\square \notin W$. If $W = Z$ then $\square \in (X \dot{+} Y) \setminus Z$ and $\square \in (X \cap Y) \setminus Z$. Finally let $W = X \dot{+} Y$. Then $\square \in Z \setminus (X \dot{+} Y)$. As $X \dot{+} Y = W \neq \emptyset$, we have either $\square \in X \setminus Y$ and $W = Y$ or $\square \in Y \setminus X$ and $W = X$. Note that we also verified the second part of the claim.

($\Leftarrow$) If $\square \in X \cap Y \cap Z$ clearly $W = X \cup Y \cup Z \neq \emptyset$. In the other 3 cases the sole set not containing $\square$ dominates the sum in W. $\square$

From the commutativity and the claim above, clearly $(X \dot{+} Y) \dot{+} Z$ and $X \dot{+} (Y \dot{+} Z)$ are equal.

(3) Note $\{\square\} \dot{+} X = X$ regardless whether $\square \in X$ or $\square \notin X$.

(4) To verify the isotony, let $X, Y, Z \subseteq A^{\leqslant 1}$ and $X \supseteq Y$. To verify $X \dot{+} Z \supseteq Y \dot{+} Z$ we have 4 cases. (a) There is nothing to prove if $Y \dot{+} Z = \emptyset$ (b) If $\square \in Y \cap Z$ then $\square \in X \cap Z$ and $X \dot{+} Z = X \cup Z \supseteq Y \cup Z = Y \dot{+} Z$. (c) If $\square \in Y \setminus Z$ then $\square \in X \setminus Z$ and $X \dot{+} Z = Z = Y \dot{+} Z$. (d) Finally if $\square \in Z \setminus Y$, then either $\square \in Z \setminus X$ and $X \dot{+} Z = X \supseteq Y = Y \dot{+} Z$ or $\square \in X \cap Z$ and $X \dot{+} Z = X \cup Z \supseteq Y = Y \dot{+} Z$. This proves Lemma 4.3.3. $\square$

Consider a binary space (E, d) over $\underline{U}_A$. The axiom (d1): $d(x,y) \leqslant 0 \Leftrightarrow x = y$ translates into $\square \in d(x,y) \Leftrightarrow x = y$. We show that (d1) implies the $\triangle$-inequality (d2). Suppose (d2) does not hold for some $x, y, z \in E$. Then $\varepsilon := d(x,z) \dot{+} d(z,y) \neq \emptyset$ (since $\emptyset$ is the greatest element of $\underline{U}_A$) and so $\square \in d(x,z)$ or $\square \in d(z,y)$ proving $x = z$ or $z = y$. If $x = z \neq y$ then $\square \notin d(z,y)$ and $\varepsilon = d(z,y) = d(x,y)$, giving a contradiction since we supposed that (d2) does not hold. The same argument applies if $x \neq z = y$. Finally, if $x = y = z$ then $\square \in d(x,z) \cap d(z,y)$ and $\varepsilon = d(x,z) \cup d(z,y) = d(x,y)$, giving again a contradiction.

To (E,d) satisfying (d1) associate the transition system (E, ε_d) where

$$\varepsilon_d := \{(x,a,y) : x, y \in E,\ a \in d(x,y) \cap A\};$$

conversely, to every transition system (E, ε) assign the binary space (E, δ_ε) over $\underline{U}_A$ where for all $x, y \in E$, $x \neq y$ we put:

$$\delta_\varepsilon(x, y) := \{a \in A\colon (x, a, y) \in \varepsilon\},\ \delta_\varepsilon(x, x) := \{\square\} \bigcup \{a \in A\colon (x, a, x) \in \varepsilon\}. \tag{4.7}$$

Let $^-$ be an involution on A. We can extend it to $A^{\leqslant 1}$ by setting $\bar{\square} = \square$ and also to $\wp(A^{\leqslant 1})$ (by setting $\bar{X} := \{\bar{x}\colon x \in X\}$ for all $X \subseteq A^{\leqslant 1}$). With this involution the above $\underline{U}_A$ is an involutive Heyting algebra (to check $X \dot{+} \bigcup_{i \in I} X_i = \bigcup_{i \in I} (X \dot{+} X_i)$ we have to distinguish four cases according to $\square \in X$ and $\square \in X_i$). A transition system (E, ε) is called *selfdual* if $(x, a, y) \in \varepsilon \Leftrightarrow (y, \bar{a}, x)$ holds for all $x, y \in E$ and $a \in A$. For example, if $|A| = 1$ then a selfdual automaton can be viewed as a binary symmetric relation (i.e. a graph). For $|A| = 2$ a self dual automaton is essentially a binary relation. For a self dual automaton (E, ε) the binary space δ_ε introduced above is a metric space over U_A [as it satisfies both (d1) and (d3)].

We can make a transition system (E, ε) over A into a selfdual one. For this it suffices to duplicate the alphabet A. To each letter $a \in A$ associate a new letter a' so that $a \mapsto a'$ is a bijection from A onto $A' := \{a'\colon a \in A\}$ where A and A' are disjoint. Now it suffices to put $B := A \cup A'$, set $\bar{a} := a'$, $\bar{a}' := a$ for all $a \in A$ and finally put $\bar{\varepsilon} := \{(y, \bar{a}, x)\colon (x, a, y) \in \varepsilon\}$ and $\varepsilon' := \varepsilon \cup \bar{\varepsilon}$. The automaton (E, ε') is clearly self-dual and so $(E, \delta_{\varepsilon'})$ is a metric space over $\underline{U}_B$.

The approach outlined in this section is solely based on the transition graph of an automaton, while the theory of automata commonly concentrates on a study of the language accepted by an automaton. We deal with the latter aspect in the next paragraph.

4.3.4. Let A be an alphabet, let $^-$ be an involution on A and let (E, ε) be a transition system over A. For $x, y \in E$, denote by $d_\varepsilon(x, y)$ the language accepted by the automaton $(E, \varepsilon, \{x\}, \{y\})$ (having a single initial state x and a single final state y). Denote by δ_ε the map associated with ε (defined in (4.7)).

Fact 1. *(E, d_ε) is a binary space on $\underline{L}_A$ and d_ε is the largest map $d\colon E \times E \to L_A$ satisfying:*

(D1) $d(x, y) \leqslant 0$ *if and only if* $x = y$;

(D2) $d(x, y) \leqslant d(x, z) + d(z, y)$;

(L) $d(x, y) \leqslant \delta_\varepsilon(x, y)$;

for all $x, y, z \in E$. If, moreover, (E, ε) is self dual then d_ε satisfies:

(D3) $d_\varepsilon(x, y) = \overline{d_\varepsilon(y, x)}$,

for all $x, y \in E$, and thus (E, d_ε) is an $\underline{L}_A$-metric space.

(Recall that $\underline{L}_A$, the family of subsets of A^*, is ordered by the reversed inclusion and that $+$ denotes the concatenation, cf. 4.3.1.)

Proof. Observe that:

$$d_\varepsilon(x, y) = \bigcup \{\delta_\varepsilon(p_0, p_1) + \cdots + \delta_\varepsilon(p_{n-1}, p_n)\colon n < \omega, p_1, \ldots, p_{n-1} \in E,\ x = p_0,\ y = p_n\}. \tag{4.8}$$

We check that d_ε satisfies (L), (D1), (D2) and (D3). The inequality (L) follows immediately from (4.8) (for $n=1$). For (D1) observe that $d_\varepsilon(x,y)\leqslant 0$ implies that there exist $n<\omega$, $p_0,\dots,p_n\in E$, $p_0=x$, $p_n=y$ such that $\delta_\varepsilon(p_i,p_{i+1})=\{\square\}$ for all $i<n$. By the definition of δ_ε this implies $p_i=p_{i+1}$ for all $i<n$ and thus $x=y$. Conversely, for every $x\in E$ we have $\square\in\delta_\varepsilon(x,x)\subseteq d_\varepsilon(x,x)$ and so $d_\varepsilon(x,x)\leqslant 0$. To prove (D2) let $x,y,z\in E$. We have

$$d_\varepsilon(x,y)\leqslant \bigwedge\left\{\sum_{i<n}\delta_\varepsilon(p_i,p_{i+1})+\sum_{j<m}\delta_\varepsilon(q_j,q_{j+1})\colon n<\omega,\ m<\omega,\right.$$
$$\left. p_0,\dots,p_n,q_0,\dots,q_m\in E,\ p_0=x,\ p_n=q_0=z,\ q_m=y\right\}.$$

Now $\underline{L}_A$ being Heyting, the distributivity guarantees that the right-hand side equals $d_\varepsilon(x,z)+d_\varepsilon(z,y)$.

Finally, if (E,ε) is selfdual then $\delta_\varepsilon(x,y)=\overline{\delta_\varepsilon(x,y)}$ and so $d_\varepsilon(x,y)=\overline{d_\varepsilon(x,y)}$.

Let $d\colon E\times E\to\underline{L}_A$ satisfy (D1), (D2) and (L). From (D2) we deduce that $d(p_0,p_n)\leqslant\sum_{i<n}d(p_i,p_{i+1})$ for $p_0,\dots,p_n\in E$. From this and (L) it follows that for $x,y\in E$ we have

$$d(x,y)\leqslant\sum_{i<n}d(p_i,p_{i+1})\leqslant\sum_{i<n}\delta_\varepsilon(p_i,p_{i+1})\quad\text{for all } p_0,\dots,p_n\in E$$

such that $p_0=x$ and $p_n=y$. It follows that $d(x,y)\leqslant d_\varepsilon(x,y)$ for all $(x,y)\in E$ and so $d\leqslant d_\varepsilon$. □

Fact 2. *A binary space (E,d) over $\underline{L}_A$ is of the form (E,d_ε) for some transition system (E,ε) if and only if d satisfies the following convexity condition:*

(D4) *If $u,v\in A^*,x,y\in E$, and $d(x,y)\leqslant\{u\}+\{v\}$ then $d(x,z)\leqslant\{u\}$ and $d(z,y)\leqslant\{v\}$ for some $z\in E$.*

Proof. First d_ε satisfies (D4). Indeed, let $x,y\in E$, $u=u_0\cdots u_{n-1}$ and $v=v_0\cdots v_{m-1}$ be such that $d_\varepsilon(x,y)\leqslant\{u\}+\{v\}$. Since $\{u\}+\{v\}=\{u+v\}$, we have $u+v\in d_\varepsilon(x,y)$. Thus there are $p_0,\dots,p_{n+m}\in E$ such that $x=p_0$, $y=p_{n+m}$ and $(p_i,u_i,p_{i+1})\in\varepsilon$ for all $i<n$ and $(p_{n+j},v_j,p_{n+j+1})\in\varepsilon$ for $j<m$. Setting $z:=p_n$ we have $u\in d_\varepsilon(x,z)$ and $v\in d_\varepsilon(z,y)$ proving $d_\varepsilon(x,z)\leqslant\{u\}$ and $d_\varepsilon(z,y)\leqslant\{v\}$. Conversely, let d satisfy the condition (D4); let ε be such that $\delta_\varepsilon(x,y)=d(x,y)\cap A^{\leqslant 1}$. We prove that $d_\varepsilon=d$. Since $d(x,y)\leqslant\delta_\varepsilon(x,y)$ the previous fact shows that $d\leqslant d_\varepsilon$. For the converse $d_\varepsilon\leqslant d$, we prove by induction on the length of a word a that $d(x,y)\leqslant\{a\}$ implies $d_\varepsilon(x,y)\leqslant\{a\}$. This is clear for words of length 0 and 1. Let $a=a_0\cdots a_n\in A^*$. If $d(x,y)\leqslant\{a\}$ then, since $a=a_0\cdots a_{n-1}+a_n$, the condition (D4) guarantees that $d(a,z)\leqslant\{a_0\cdots a_{n-1}\}$ and $d(z,y)\leqslant\{a_n\}$ for some $z\in E$. From the induction hypothesis, $d_\varepsilon(a,z)\leqslant\{a_0\cdots a_{n-1}\}$ and by the definition of d also $\delta_\varepsilon(z,y)\leqslant\{a_n\}$. From the $\triangle$-inequality we get

$$d_\varepsilon(x,y)\leqslant d_\varepsilon(x,z)+d_\varepsilon(z,y)\leqslant d_\varepsilon(x,z)+\delta_\varepsilon(z,y)\leqslant\{a_0\cdots a_{n-1}\}+\{a_n\}=\{a_0\cdots a_n\}$$

proving the induction step and hence our claim. □

Fact 3. *Let (E,ε) and (E',ε') be two transition systems over A. The following are equivalent for $f: E\to E'$:*

(i) *f is a transition system morphism from (E,ε) into (E',ε'),*

(ii) *f is a contraction from (E,δ_ε) into $(E',\delta_{\varepsilon'})$,*

(iii) *f is a contraction from (E,d_ε) into $(E',d_{\varepsilon'})$.*

Proof. (i)$\Leftrightarrow$(ii) This was already indicated in 4.3.2. As we have seen above, $\delta_\varepsilon(x,y)=A^{\leqslant 1}\vee d_\varepsilon(x,y)$ holds for all $x,y\in E$.

(ii) $\Rightarrow$(iii) If f satisfies (ii) then,

$$d_{\varepsilon'}(f(x),f(y))\leqslant\left\{\sum_{i<n}\delta_{\varepsilon'}(f(p_i),f(p_{i+1})):\ n<\omega, p_0,\ldots,p_n\in E,\ p_0=x,\ p_n=y\right\}$$

$$\leqslant\left\{\sum_{i<n}\delta_\varepsilon(p_i,p_{i+1}):\ n<\omega,\ p_0,\ldots,p_n\in E,\ p_0=x,\ p_n=y\right\}=d_\varepsilon(x,y).$$

(iii) $\Rightarrow$(ii) Let f satisfy (iii). Then,

$$\delta_{\varepsilon'}(f(x),f(y))=A^{\leqslant 1}\cap d_{\varepsilon'}(f(x),f(y))\supseteq A^{\leqslant 1}\cap d_\varepsilon(x,y)=\delta_\varepsilon(x,y). \qquad \square$$

Corollary 4.3.5. (i) *The categories AT_A and $\mathrm{AT}_{\bar{A}}$ consisting of automata and of selfdual automata on A and equipped with the automaton morphisms are full subcategories of the category of binary spaces over $\underline{L}_A$.*

(ii) *The category AT_A is a full subcategory of the category of metric spaces over $\underline{L}_B$ where $B=A\cup A'$ and A' is a copy of A. In particular, the product of automata corresponds to the product of spaces with the sup-distance.*

Proof. The first statement is clear. For the second, duplicate A as indicated in 4.3.4 and on $B=A\cup A'$ define $\bar{\ }$ by setting $\bar{a}=a'$, $\bar{a}'=a$. Clearly AT_A and AT_B are, as categories, identical. Now, apply the first statement. $\square$

Fact 4. *Let A be an alphabet with an involution $\bar{\ }$. Let $\underline{L}(A^*):=(\mathscr{P}(A^*),\mathscr{L}_A)$ be the transition system over A defined by $\mathscr{L}_A:=\{(X,\alpha,Y):\ X\supseteq Y+\{\bar{\alpha}\},\ Y\supseteq X+\{\alpha\},\ X,Y\in L_A,\alpha\in A\}$. Denote by $d_{\mathscr{L}_A}$ the corresponding distance and by $d_{\underline{L}_A}$ the distance associated to the Heyting algebra $\underline{L}_A=(\wp(A^*),\supseteq,+,-)$. Then,*

$$\delta_{\mathscr{L}_A}(X,Y)=d_{\underline{L}_A}(X,Y)\cap A^{\leqslant 1} \textit{ for all } X, Y\in L_A; \textit{ in particular } d_{\underline{L}_A}\leqslant d_{\mathscr{L}_A}.$$

Proof. Let $\alpha\in\delta_{\mathscr{L}_A}(X,Y)$. Then, according to the definition of $\mathscr{L}_A$, either $\alpha=\square$, (and $X=Y$) or $\alpha\in A$ and $X\supseteq Y+\{\bar{\alpha}\}$ and $Y\supseteq X+\{\alpha\}$. In the first case, since $X=Y$ we have $\square\in d_{\underline{L}_A}(X,Y)\cap A^{\leqslant 1}$; in the second case the two inequalities mean that $d_{\underline{L}_A}(X,Y)\leqslant\{\alpha\}$; that is $\alpha\in d_{\underline{L}_A}(X,Y)$. The converse is similar.

Since $d_{\mathscr{L}_A}$ is the largest distance satisfying $d_{\mathscr{L}_A}\leqslant\delta_{\mathscr{L}_A}$ it follows that $d_{\underline{L}_A}\leqslant d_{\mathscr{L}_A}$. $\square$

F. Saïdane and the first author have identified a subset D_A of $\mathscr{P}(A^*)$ with the following properties: the distance induced by $d_{\underset{\sim}{L}_A}$ on D_A coincide with the distance associated with the transition system $\underset{\sim}{D}_A$ induced by $\underset{\sim}{L}_A$ on D_A; moreover, every involutive transition system embeds isometrically into some power of $\underset{\sim}{D}_A$ (see F. Saïdane, Graphes et Langages: une approche métrique, Thèse, n° 206-91, Lyon, November 1991).

4.4. Sequential machines

4.4.1. In the sequel the *alphabet* A is a fixed non-singleton set and ξ a fixed indecomposable ordinal number (that is one satisfying $\tau+\xi=\xi$ for all $\tau<\xi$; thus for example ω is such ordinal). Denote by B the set A^ξ (of all maps from $\underset{\sim}{\xi}:=\{\tau\colon \tau<\xi\}$ into A). For $n<\omega$ an n-ary operation f on B is a *retrospective* if for all $\tilde{b}_1,\dots,\tilde{b}_n\in B$ the value $\tilde{c}=f(\tilde{b}_1,\dots,\tilde{b}_n)$ is such that for each $\tau<\xi$ the restriction $\tilde{c}|_{\underset{\sim}{\tau}}$ depends at most on $\tilde{b}_1|_{\underset{\sim}{\tau}},\dots,\tilde{b}_n|_{\underset{\sim}{\tau}}$. The first case, and perhaps the most interesting, is the one with $\xi=\omega$. We may associate to f a black box consuming n infinite tapes and producing a single infinite tape. On the ith input tape is the infinite sequence $\tilde{b}^i=\langle b_0^i,b_1^i,\dots\rangle (i=1,\dots,n)$ while the output tape at the (discrete) time τ carries the finite sequence $\langle c_0,c_1,\dots,c_{\tau-1}\rangle$. At the time τ the box reads (or swallows) the symbols $b_\tau^1,\dots,b_\tau^n$. The machine is myopic in the sense that at the time τ it may react only to the part $b_0^1,\dots,b_0^n,\dots,b_\tau^1,\dots,b_\tau^n$ already read (or swallowed). In other words, the machine is no oracle and so it is completely unaware what it will read (or swallow) in the future. Thus, if the input sequences $\tilde{b}^i$ and $\tilde{d}^i$ have identical initial segments of length $\tau (i=1,\dots,n)$ then the two output sequences $\tilde{c}:=f(\tilde{b}_1,\dots,\tilde{b}_n)$ and $\tilde{e}:=f(\tilde{d}_1,\dots,\tilde{d}_n)$ also agree on the first τ terms. For A finite and $\xi=\omega$ the unary retrospectives were introduced by Raney [22] (cf. [11, Section 4.7]).

4.4.2. For $\tau<\xi$, $\tilde{p}\in A^{\underset{\sim}{\tau}}$ and $\tilde{x}\in B:=A^{\underset{\sim}{\xi}}$ denote by $\tilde{p}\cdot\tilde{x}$ the map $\tilde{h}$ defined by setting $\tilde{h}(\lambda):=\tilde{p}(\lambda)$ for $\lambda<\tau$ and $h(\tau+\lambda):=\tilde{x}(\lambda)$ for all $\lambda<\xi$ (here we need the fact that $\tau+\xi=\xi$ for all $\tau<\xi$). To an n-ary retrospective f on B, $\tau<\xi$, $\tilde{p}^1,\dots,\tilde{p}^n\in A^{\underset{\sim}{\tau}}$, $\pi:=\langle\tilde{p}^1,\dots,\tilde{p}^n\rangle$ assign $\tilde{p}=f^\pi\in A^{\underset{\sim}{\tau}}$ and an n-ary retrospective f_π on B so that

$$f_\pi(\tilde{p}^1\cdot\tilde{x}^1,\dots,\tilde{p}^n\cdot\tilde{x}^n)=\tilde{p}\cdot f_\pi(\tilde{x}^1,\dots,\tilde{x}^n) \tag{4.9}$$

holds for all $\tilde{x}^1,\dots,\tilde{x}^n\in B$. Note that both $\tilde{p}=f^\pi$ and f_π are unique (by virtue of the definition of a retrospective). Further let Q_f denote the set of all f_π obtained in this way. Note that the cardinality of Q_f is at most,

$$\alpha=\alpha_{|A|,\xi}:=\sum_{\tau<\xi}|A^{\underset{\sim}{\tau}}|$$

(where $\sum$ is the cardinal sum). For $\xi=\omega$ clearly $\alpha_{|A|,\omega}=\max(\aleph_0,|A|)$ and so $\alpha_{|A|,\omega}=\aleph_0$ for $|A|\leqslant\aleph_0$. In 4.4.3–4.4.7 we consider only the case $\xi=\omega$. As usual, $\tilde{x}\in B:=A^{\underset{\sim}{\omega}}$ is interpreted as an ω-sequence $\langle x_0,x_1,\dots\rangle$ over A, a notation used throughout 4.4.3–4.4.7.

4.4.3. Following the ideas from [8] we show that for $\xi=\omega$ a retrospective may be interpreted as a (possibility infinite) initial deterministic Mealy automaton (IDMA for short). As usual, an IDMA is given by a sixtuple,

$$\underset{\sim}{M}=\langle I, O, Q, \delta, \lambda, q_0\rangle \tag{4.10}$$

where the *input* and *output alphabets* I and O and the *set of states* Q are non-empty sets, the *initial state* q_0 is an element of Q and the *transition function* δ and *output function* λ map $I\times Q$ into Q and O, respectively. In the sequel we assume that $I=A^n$ for some positive integer n and $O=A$ (i.e. $\underset{\sim}{M}$ has n inputs and a single output that all work over the same alphabet A) and refer to $\underset{\sim}{M}$ as an n-ary IDMA *over A*. Such an IDMA *realizes* an n-ary operation f on $B:=A^{\omega}$ provided for all $\tilde{x}_1,\dots,\tilde{x}_n\in B$ the sequence $\tilde{y}:=f(\tilde{x}^1,\dots,\tilde{x}^n)$ satisfies

$$y_\tau=\lambda((x_\tau^1,\dots,x_\tau^n),\, q_\tau) \tag{4.11}$$

for all $\tau<\omega$ where q_τ is defined inductively by setting

$$q_{m+1}:=\delta((x_m^1,\dots,x_m^n),\, q_m) \tag{4.12}$$

for all $m<\omega$. (In the sequel we shorten the right-hand side of (4.11) to $\lambda(x_\tau^1,\dots,x_\tau^n,q_\tau)$ and similarly for (4.12)). We have the following.

Lemma 4.4.4. *An n-ary operation f on $B:=A^{\omega}$ is a retrospective if and only if it is realized by an n-ary IDMA over A with at most $\alpha:=\max(\aleph_0,|A|)$ states.*

Proof. ($\Rightarrow$) Let f be a retrospective. Put $\underset{\sim}{M}:=\langle A^n, A, Q_f, \delta, \lambda, f\rangle$ where Q_f has been described in 4.4.2 and the transition function δ and output function λ are defined as follows. Let $\underset{\sim}{a}=\langle a_1,\dots,a_n\rangle\in A^n$ and $k\in Q_f$. By the definition $k=f_\pi$ for some $\pi=\langle\tilde{p}^1,\dots,\tilde{p}^n\rangle$ where $\tilde{p}^1,\dots,\tilde{p}^n\in A^\tau$ for some $0<\tau<\omega$. For $i=1,\dots,n$ denote by r^i the element $\langle p^i,a_i\rangle\in A^{\tau+1}$ and put $\pi':=\langle r^1,\dots,r^n\rangle$. Let f^π and $f^{\pi'}$ be the elements of A^τ and $A^{\tau+1}$ from (4.9). From the definition of retrospective we have $f^{\pi'}=f^\pi\cdot a$ for a unique $a\in A$. Now put

$$\delta(\underset{\sim}{a},k):=f_{\pi'},\qquad \lambda(\underset{\sim}{a},k):=a \tag{4.13}$$

(where again $f_{\pi'}\in Q_f$ is determined by (4.9)).

A straight-forward check shows that (4.13) does not depend on our choice of π (in $k=f_\pi$) and so it defines an n-ary IDMA over A. Observe that this IDMA realizes f.

($\Leftarrow$) It is easy to verify that every n-ary IDMA over A realizes an n-ary retrospective over $B:=A^{\omega}$. □

4.4.5. The structural theory of automata studies the construction of new automata from a given set $\underset{\sim}{M}=\{\underset{\sim}{M}_i\colon i\in I\}$ of IDMA's over A. The $\underset{\sim}{M}_i$'s may be seen as small IDMAs available on the market (each in potentially unlimited quantity of copies). The copies of these IDMAs serve as building blocks for a net of IDMAs which constitutes a new IDMA.

In this paper we limit ourselves to feedback-free (i.e. tree-like) nets. Although feedbacks largely enhance the construction capability, already their definition is based on various assumptions concerning real time functioning. This and the technical difficulties in feedback handling largely exceed the framework of this paper. Moreover, the feedback-free constructions directly translate into universal algebra permitting the application of clone techniques.

4.4.6. Put $\alpha := \max(\aleph_0, |A|)$ and denote by I_A the set of all IDMA's of type (4.10) over A with $|Q| \leqslant \alpha$ (i.e. with at most α states). For a positive integer n denote by $I_A^{(n)}$ the set of all $\underline{M} \in I_A$ with exactly n inputs. To formalize the building of tree-like nets, we introduce Mal'tsev type operations (cf. [16]) on I_A. Let

$$\underline{M} = \langle A^m, A, Q, \delta, \lambda, q_0 \rangle, \qquad N = \langle A^n, A, Q', \delta', \lambda', q_0' \rangle$$

(where δ and λ map $A^m \times Q$ into Q and A and δ' and λ' map $A^n \times Q'$ into Q' and A'). Put $r := m + n - 1$ and denote by

$$\underline{M} * \underline{N} = \langle A^r, A, Q'', \delta'', \lambda'', q_0'' \rangle$$

the IDMA obtained by joining the output of $\underline{N}$ to the first input of $\underline{M}$. It is easy to see that we may take $Q'' := Q \times Q'$, $q_0'' = (q_0, q_0')$ and define δ'' and λ'' by setting

$$\delta''(a_1, \ldots, a_r, q, q') := \delta(\delta'(a_1, \ldots, a_n, q'),\ a_{n+1}, \ldots, a_r,\ q),$$

$$\lambda''(a_1, \ldots, a_r, q, q') := \lambda(\delta'(a_1, \ldots, a_n, q'),\ a_{n+1}, \ldots, a_r,\ q)$$

for all $a_1, \ldots, a_r \in A$, $q' \in Q'$ and $q \in Q$. The IDMAs $\zeta \underline{M}$, $\tau \underline{M}$ and $\Delta \underline{M}$ are defined in an analogous fashion. Finally $e_1^2 \in I_A^{(2)}$ connects the first input directly to the output. In this way we obtain the algebra

$$\underline{I}_A := \langle I_A;\ *, \zeta, \tau, \Delta, e_1^2 \rangle$$

(of type (2, 1, 1, 1, 0)).

For an indecomposable ordinal ξ denote by S_A^ξ the set of all retrospectives on $B := A^\xi$. The composition of retrospectives from S_A^ω coincides with the above composition of the corresponding IDMAs. This is stated in the following lemma whose proof is omitted.

Lemma 4.4.7. *The map $\varphi : \underline{I}_A \to S_A^\omega$ which to each $\underline{M} \in \underline{I}_A$ assigns the retrospective realized by $\underline{M}$ is a homomorphism from $\underline{I}_A$ into $\langle S_A^\omega;\ *, \zeta,\ \tau, \Delta, e_1^2 \rangle$.*

The following subclones of S_A^ξ are determined by the cardinalities of Q_f (the sets of states of the corresponding IDMA if $\xi = \omega$). For an infinite cardinal κ put $T_{A\xi\kappa} := \{ f \in S_A^\xi \colon |Q_f| < \kappa \}$.

We have the following proposition.

Proposition 4.4.8. *For every infinite cardinal κ the set $T_{A\xi\kappa}$ is a subclone of S_A^ξ.*

Proof. Put $T:=T_{A\xi\kappa}$. Let $f\in T$ be n-ary and $g\in T$ m-ary. We show that $Q_{\zeta f}\subseteq\zeta(Q_f)$. Indeed, let $k\in Q_{\zeta f}$. According to 4.2 we have $k=(\zeta f)_\pi$ for some $\pi=\langle\tilde{p}^1,\dots,\tilde{p}^n\rangle$ where $\tilde{p}^1,\dots,\tilde{p}^n\in A^{\underline{\tau}}$ for some $0<\tau<\xi$. Put $\pi':=\langle\tilde{p}^2,\dots,\tilde{p}^n,\tilde{p}^1\rangle$. Then for all $\tilde{x}^1,\dots,\tilde{x}^n\in B$ from (4.9) we obtain:

$$(\zeta f)^\pi\cdot(\zeta f)_\pi(\tilde{x}^1,\dots,\tilde{x}^n)=(\zeta f)(p^1\cdot\tilde{x}^1,\dots,p^n\cdot\tilde{x}^n)=f(\tilde{p}^2\cdot\tilde{x}^2,\dots,\tilde{p}^n,\tilde{x}^n,\tilde{p}^1\cdot\tilde{x}^1)$$
$$=f^{\pi'}\cdot f_{\pi'}(\tilde{x}^2,\dots,\tilde{x}^n,\tilde{x}^1)=\zeta(f_{\pi'})(\tilde{x}^2,\dots,\tilde{x}^n,\tilde{x}^1).$$

Comparing the first and last part we get $k=(\zeta f)_\pi=\zeta(f_{\pi'})\in\zeta(Q_f)$ proving the above assertion. Now,

$$|Q_{\zeta f}|\leqslant|\zeta(Q_f)|\leqslant|Q_f|<\kappa$$

(actually, the first two inequalities are equalities) proving $\zeta f\in T$. The proof that $Q_{\tau f}\subseteq\tau(Q_f)$ is quite similar. In an analogous way for $n>1$ and $\pi=\langle\tilde{p}^1,\dots,\tilde{p}^{n-1}\rangle$ with $\tilde{p}^1,\dots,\tilde{p}^{n-1}\in A^{\underline{\tau}}$ and $\pi':=\langle\tilde{p}^1,\tilde{p}^1,\tilde{p}^2,\dots,\tilde{p}^{n-1}\rangle$ one can prove that $(\Delta f)_\pi=\Delta f_{\pi'}$ and so again $|Q_{\Delta f}|\leqslant|\Delta Q_f|\leqslant|Q_f|<\kappa$ proving that $\Delta f\in T$.

It remains to consider $h:=f*g$. Put $r:=m+n-1$. Let $0<\tau<\xi$, $\tilde{p}^1,\dots,\tilde{p}^r\in A^{\underline{\tau}}$ and $\pi:=\langle\tilde{p}^1,\dots,\tilde{p}^r\rangle$. Put,

$$\pi'':=\langle\tilde{p}^1,\dots,\tilde{p}^m\rangle,\qquad \pi':=\langle g^{\pi''},\tilde{p}^{m+1},\dots,\tilde{p}^r\rangle.$$

We have the following.

Fact. $(f*g)_\pi=f_{\pi'}*g_{\pi''}$.

Proof of the Fact. Let $\pi:=\tilde{x}^1,\dots,\tilde{x}^r\in B$. Put $\tilde{x}^0:=g_{\pi''}(\tilde{x}^1,\dots,\tilde{x}^m)$. Then

$$\begin{aligned}
h^\pi\cdot h_\pi(\tilde{x}^1,\dots,\tilde{x}^r)&=h(\tilde{p}^1\cdot\tilde{x}^1,\dots,\tilde{p}^r\cdot x^r)\\
&=f(g(\tilde{p}^1\cdot\tilde{x}^1,\dots,\tilde{p}^m\cdot\tilde{x}^m),\ \tilde{p}^{m+1}\cdot\tilde{x}^{m+1},\dots,\tilde{p}^r\cdot\tilde{x}^r)\\
&=f(g^{\pi''}\cdot g_{\pi''}(\tilde{x}^1,\dots,\tilde{x}^m),\ \tilde{p}^{m+1}\cdot\tilde{x}^{m+1},\dots,\tilde{p}^r\cdot\tilde{x}^r)\\
&=f(g^{\pi''}\cdot\tilde{x}^0,\tilde{p}^{m+1}\cdot\tilde{x}^{m+1},\dots,\tilde{p}^r\cdot\tilde{x}^r)\\
&=f^{\pi'}\cdot f_{\pi'}\tilde{x}^0,\tilde{x}^{m+1},\dots,\tilde{x}^r)\\
&=f^{\pi'}\cdot(f_{\pi'}(g_{\pi''}(\tilde{x}^1,\dots,\tilde{x}^m),\ \tilde{x}^{m+1},\dots,\tilde{x}^r)\\
&=f^{\pi'}\cdot(f_{\pi'}*g_{\pi''})(\tilde{x}^1,\dots,\tilde{x}^r),
\end{aligned}$$

proving the fact. □

From the fact we get $Q_{f*g}\subseteq Q_f*Q_g$ and finally $|Q_{f*g}|\leqslant|Q_f||Q_g|<\kappa^2=\kappa$ (as κ is an infinite cardinal).

Call the clone $T_{A\omega\aleph_0}$ the *clone of finite retrospectives*. Clearly it corresponds to the set of all finite state IDMA's on A. For $1<|A|<\aleph_0$ the single clone of the form $T_{A\omega\kappa}$ different from $T_{A\omega\aleph_0}$ is $S_A^\omega=T_{A\omega\aleph_1}$.

4.4.9. We describe $S:=S_A^\xi$ in terms of the following ultrametric d on $B:=A^\xi$. Put $V=\underline{\xi+1}=\{\kappa\colon \kappa\leqslant\xi\}$ and $\underline{V}\colon\langle V;\geqslant,\wedge,\xi\rangle$ where $v\wedge v':=\min(v,v')$ is the join for the order $\geqslant$ and ξ its least element. For distinct $\tilde{x},\tilde{y}\in B$ put

$$d(\tilde{x},\tilde{y}):=\bigwedge\{\lambda\colon \lambda<\omega\colon \tilde{x}(\lambda)\neq\tilde{y}(\lambda)\};\tag{4.14}$$

here again $\wedge$ is the usual inf in the well-ordered set $(V,\leqslant)$ and $d(\tilde{x},\tilde{x})=\xi$ for all $\tilde{x}\in B$. Thus for distinct $\tilde{x}$ and $\tilde{y}$ the value of $d(\tilde{x},\tilde{y})$ is the least ordinal on which they differ (as maps from ξ into A). We have:

Lemma 4.4.10. *The above map d is is a $\underline{V}$-ultrametric on B.*

Proof. Clearly it suffices to verify the $\triangle$-inequality. Let $\tilde{x},\tilde{y},\tilde{z}\in B$ and $v:=d(\tilde{x},\tilde{z})\leqslant d(\tilde{z},\tilde{y})$. Then $\tilde{x}(\lambda)=\tilde{z}(\lambda)=\tilde{y}(\lambda)$ for all $\lambda<v$ and so $d(\tilde{x},\tilde{y})\leqslant v$. □

For $\xi=\omega$ and A finite, Csakany and Gecseg [7] (cf. [11, Section 4.7]) used the related relational metric δ defined by setting $\delta(\tilde{x},\tilde{y}):=1/(d(\tilde{x},\tilde{y})+1)$ for $\tilde{x}\neq\tilde{y}$ and $\delta(\tilde{x},\tilde{x})=0$.

We relate retrospectives and $\underline{V}$-contracting operations.

Theorem 4.4.11. *An operation f on $B=A^\xi$ is retrospective if and only if f is contracting with respect to the above $\underline{V}$-ultrametric on B.*

Proof. ($\Rightarrow$) Let $\tilde{x}^1,\ldots,\tilde{x}^n,\tilde{y}^1,\ldots,\tilde{y}^n\in B$. Put

$$\tilde{u}:=f(\tilde{x}^1,\ldots,\tilde{x}^n),\tilde{v}:=f(\tilde{y}^1,\ldots,\tilde{y}^n),\ \delta:=\bigwedge_{i=1}^{n} d(\tilde{x}^i,\tilde{y}^i).\tag{4.15}$$

First consider the case $\delta=\xi$. Then $d(\tilde{x}^i,\tilde{y}^i)=\xi$ and consequently $\tilde{x}^i=\tilde{y}^i$ for all $i=1,\ldots,n$. Then $\tilde{u}=\tilde{v}$ and $d(\tilde{u},\tilde{v})=\xi=\delta$ as required. Thus let $\delta<\xi$. Then $\tilde{x}^i(\lambda)=\tilde{y}^i(\lambda)$ for all $\lambda<\delta$. Now f being retrospective, we get $\tilde{u}(\lambda)=\tilde{v}(\lambda)$ for all $\lambda<\delta$ and consequently $d(\tilde{u},\tilde{v})\geqslant\delta$.

($\Leftarrow$) Suppose an n-ary operation f on B is d-contracting but not retrospective. Then there are $\tilde{x}^1,\ldots,\tilde{x}^n,\tilde{y}^1,\ldots,\tilde{y}^n\in B$ and $\tau<\xi$ such that $\tilde{x}^i(\lambda)=\tilde{y}^i(\lambda)$ for all $\lambda\leqslant\tau$ while $\tilde{u}$ and $\tilde{v}$ defined by (4.15) satisfy $\tilde{u}(\tau)\neq\tilde{v}(\tau)$. Now $d(\tilde{x}^i,\tilde{y}^i)>\tau$ for $i=1,\ldots,n$ show that δ defined by (4.15) satisfies $\delta>\tau$. On the other hand $\tau\geqslant d(\tilde{u},\tilde{v})$ and, since f is $\underline{V}$-contracting, also $d(\tilde{u},\tilde{v})\geqslant\delta$ proving $\tau\geqslant\delta$ and leading to the contradiction $\delta>\tau\geqslant\delta$. □

The paper [7] (cf. [11, Section 4.7]) deals with the special case $|A|=n$ and f permutation of B (whereby a contracting f is an isometry). We show that (B,d) is hyperconvex (cf. 3.3.3).

Proposition 4.4.12. *The ultrametric (B,d) is hyperconvex.*

Proof. Let κ be a cardinal and $\tilde{x}^i\in B$ and $r_i\in V$ for all $i<\kappa$ such that for all $i,j<\kappa$,

$$d(\tilde{x}^i,\tilde{x}^j)\geqslant r_i\wedge r_j.\tag{4.16}$$

Define $\varphi:\underline{\kappa}\to V$ by setting $\varphi(i):=r_i$ for all $i<\kappa$. Since both $\underline{\kappa}$ and $(V,\leqslant)$ are well-ordered, we may arrange the notation so that φ is isotone (i.e. $\varphi(i)\leqslant\varphi(j)$ whenever $i<j<\kappa$). Put $\theta:=\operatorname{Ker}\varphi$ and let $\{Y_i: i<\lambda\}$ denote the equivalence classes of θ (where $\lambda\leqslant\xi+1$) and the intervals Y_i are arranged so that Y_i precedes Y_j whenever $i<j<\lambda$). For $\gamma<\lambda$ denote $y(\gamma)$ the least element of Y_γ. We have two cases: (1) Let λ be an isolated ordinal. Put $\tilde{z}:=\tilde{x}^j$ where $j\in Y_{\lambda-1}$ is arbitrary. Now for all $i<\kappa$ from (4.16_{ij}) we get

$$d(\tilde{x}^i,\tilde{z})=d(\tilde{x}^i,\tilde{x}^j)\geqslant r_i\wedge r_j=r_i$$

as r_j is the greatest element of $\{r_i; i<\kappa\}$. Note that if $r_j=\xi$ and $k\in Y_{\lambda-1}$ from (4.16_{kj}) we get $d(\tilde{x}^k,\tilde{x}^j)\geqslant\xi\wedge\xi=\xi$, whence $d(\tilde{x}^k,\tilde{x}^j)=\xi$ and $\tilde{x}^k=\tilde{x}^j$ proving that $Y_{\lambda-1}$ is a singleton and our choice of $\tilde{z}$ unique. In particular, this happens if $\lambda=\xi+1$ (because then $r_{y(\xi)}=\xi$).

(2) Let λ be a limit ordinal. For each $\gamma<\lambda$ put $\mu_\gamma:=r_{y(\gamma)}$ and $\tilde{z}_\gamma:=\tilde{x}^{y(\gamma)}|_{\mu_\gamma}$ (recall that $y(\gamma)$ is the first element of the block Y_γ of θ). We have:

Fact. *If $\gamma<\delta<\lambda$ then $\tilde{z}_\delta|_{\underline{\gamma}}=\tilde{z}_\gamma$.*

Proof. Put $i:=\mu_\gamma$. By (4.16_{ii}) and the isotony of φ, we have

$$d(\tilde{x}^{y(\gamma)},\tilde{x}^{y(\delta)})\geqslant\mu_\gamma\wedge\mu_\delta=\mu_\gamma \tag{4.17}$$

and therefore $\tilde{x}^{y(\gamma)}$ and $\tilde{x}^{y(\delta)}$ agree on $\underline{\mu}_\gamma$ proving the fact.

Let $\tilde{z}\in B$ be such that $\tilde{z}|_{\underline{\mu}_\gamma}=\tilde{z}_\gamma$ for all $\gamma<\lambda$. In view of the fact above, such $\tilde{z}$ exists. Moreover, $\tilde{z}$ is unique whenever $\lambda=\xi$. It remains to prove that $d(\tilde{x}^i,\tilde{z})\geqslant r_i$ for all $i<\kappa$. Let $i<\kappa$. Then $i\in Y_\gamma$ for some $\gamma<\kappa$. Put $j:=y(\gamma)$. From $i,j\in Y_\gamma$ we get $r_i=r_j=\mu_\gamma$. Now from (4.16_{ij}) we get

$$d(\tilde{x}^i,\tilde{x}^j)\geqslant r_i\wedge r_j=r_i. \tag{4.18}$$

By the definition of $\tilde{z}$ we have $d(\tilde{x}^j,\tilde{z})\geqslant\mu_\gamma=r_i$. Combining this with (4.18) and the $\triangle$-inequality we get the required $d(\tilde{x}^i,\tilde{z})\geqslant d(\tilde{x}^i,\tilde{x}^j)\wedge d(\tilde{x}^j,\tilde{z})\geqslant r_i\wedge r_i=r_i$. □

Remark 4.4.13. We have also proved: *If the set $R:=\{r_i\colon i<\kappa\}$ is cofinal with ξ or contains ξ then $\tilde{z}$ is unique* (in particular, this happens for $\xi=\omega$ whenever R is infinite).

Non-expansive relations were defined in 3.7.9. In our special case we have the following.

Proposition 4.4.14. *Let σ be a non-expansive relation for the above $\underline{V}$-ultrametric $\underline{A}=(A,d)$. Then the clone* $\operatorname{Pol}\sigma$ *either equals* $\operatorname{Pol}(d)_v$ *for some $v\in V$ or it equals* $\bigcap_{w<\lambda}\operatorname{Pol}(d)_w$ *for some limit ordinal $\lambda\leqslant\xi$.*

Proof. By Proposition 3.7.13 the clone $\operatorname{Pol}\sigma$ is of the form $\bigcap_{w\in W}\operatorname{Pol}(d)_w$ for $W=\operatorname{im}\delta$ where δ is a $\underline{V}$-ultrametric. First we characterize such W.

Fact. *Let $W \subseteq V$. Then $W = \mathrm{im}\,\delta$ for a $\underline{V}$-ultrametric if and only if $\xi \in W$ and $|W| > 1$.*

Proof of the fact. ($\Rightarrow$) Evident. ($\Leftarrow$) For $w, w' \in W$ put $\delta(w, w') := \xi$ if $w = w'$ and $\delta(w, w') := w \wedge w'$ otherwise. It suffices to prove the $\triangle$-inequality. Let $u, v, w \subseteq W$. Taking into account that ξ is the greatest element of V we may assume that u, v and w are pairwise distinct. Then,

$$\delta(u,v) = u \wedge v \geqslant u \wedge w \wedge w \wedge v = \delta(u,w) \wedge \delta(w,v).$$

Note that $(d)_\xi$ is the least quivalence on A and so $\mathrm{Pol}(d)_\xi$ is the clone of all operations on A. We have two cases: (1) $U := W \backslash \{\xi\}$ has a greatest element w. Then $\mathrm{Pol}\,\sigma = \mathrm{Pol}(d)_w$ and we are done. (2) U is cofinal with $\{w\colon w < \lambda\}$ for a limit ordinal $\lambda \leqslant \xi$. It is easy to see that then $\mathrm{Pol}\,\sigma$ equals $\bigcap_{w<\lambda} \mathrm{Pol}\,(d)_w$. □

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