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J. Math. Anal. Appl. 340 (2008) 487–497

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

# On the growth of solutions of a class of higher order linear differential equations with coefficients having the same order ☆

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Received 5 November 2006

Available online 29 August 2007

Submitted by A.V. Isaev

**Abstract**

In this paper, the authors investigate the growth of solutions of a class of higher order linear differential equations

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0 f = 0$$

when most coefficients in the above equations have the same order with each other, and obtain some results which improve previous results due to K.H. Kwon [K.H. Kwon, Nonexistence of finite order solutions of certain second order linear differential equations, Kodai Math. J. 19 (1996) 378–387] and Z.-X. Chen [Z.-X. Chen, The growth of solutions of the differential equation $f'' + e^{-z}f' + Q(z)f = 0$, Sci. China Ser. A 31 (2001) 775–784 (in Chinese); Z.-X. Chen, On the hyper order of solutions of higher order differential equations, Chinese Ann. Math. Ser. B 24 (2003) 501–508 (in Chinese); Z.-X. Chen, On the growth of solutions of a class of higher order differential equations, Acta Math. Sci. Ser. B 24 (2004) 52–60 (in Chinese); Z.-X. Chen, C.-C. Yang, Quantitative estimations on the zeros and growth of entire solutions of linear differential equations, Complex Var. 42 (2000) 119–133].


*Keywords:* Linear differential equations; Entire function; Type; Hyper order

## 1. Introduction and results

We shall assume that reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions (see e.g. [11,15]). In addition, we will use the notation $\sigma(f)$ to denote the order of growth of entire function $f(z)$, $\tau(f)$ to denote the type of $f(z)$ with $\sigma(f) = \sigma$, is defined to be

$$\tau(f) = \overline{\lim_{r\to\infty}} \frac{\log M(r, f)}{r^{\sigma}}.$$

☆ This work is supported by the Science Foundation of Education Department of Jiangxi Province in China.

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0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2007.08.041

We use $\sigma_2(f)$ to denote the hyper order of $f(z)$, is defined to be (see [18])

$$\sigma_2(f)=\overline{\lim_{r\to\infty}}\frac{\log\log T(r,f)}{\log r}.$$

We use $mE$ to denote the linear measure of a set $E\subset(0,+\infty)$ and use $m_lE$ to denote the logarithmic measure of a set $E\subset[1,+\infty)$. If $P(z)$ is a polynomial, we use the notation $\deg P$ to denote the degree of $P(z)$.

For second order linear differential equations

$$f''+B(z)f'+A(z)f=0, \tag{1.1}$$

many authors have investigated the growth of solutions of (1.1), where $A(z)\not\equiv 0$ and $B(z)$ are entire functions of finite order. It is well known that if either $\sigma(B)<\sigma(A)$ or $\sigma(A)<\sigma(B)\leqslant 1/2$, then every solution $f\not\equiv 0$ of (1.1) is of infinite order (see [9,13]). For the case $\sigma(A)<\sigma(B)$ and $\sigma(B)>1/2$, many authors have studied the problem. In 2000, I. Laine and P.C. Wu proved the following result.

**Theorem A.** *(See [16].) Suppose that $\sigma(A)<\sigma(B)<\infty$ and that $T(r,B)\sim\log M(r,B)$ as $r\to\infty$ outside a set of finite logarithmic measure. Then every non-constant solution $f$ of (1.1) is of infinite order.*

Thus a natural question is: what condition on $A(z)$, $B(z)$ when $\sigma(A)=\sigma(B)$ will guarantee that every solution $f\not\equiv 0$ of (1.1) has infinite order? For second order linear differential equations,

$$f''+h_1e^{P(z)}f'+h_0e^{Q(z)}f=0, \tag{1.2}$$

in 1996, K.H. Kwon investigated the growth of the solutions of (1.2) for the case $\deg P=\deg Q$ and obtained the following result.

**Theorem B.** *(See [14].) Let $P(z)=a_nz^n+\cdots$, $Q(z)=b_nz^n+\cdots$ $(a_nb_n\neq 0)$ be non-constant polynomials, $h_1(z)$ and $h_0(z)\not\equiv 0$ be entire functions with $\sigma(h_j)<n$ $(j=0,1)$, if $\arg a_n\neq\arg b_n$ or $a_n=cb_n$ $(0<c<1)$, then every solution $f\not\equiv 0$ of (1.2) has infinite order with $\sigma_2(f)\geqslant n$.*

In 2001, Z.-X. Chen investigated the problem and proved the following theorem.

**Theorem C.** *(See [2].) Let $A_j(z)\not\equiv 0$ $(j=0,1)$ be entire functions with $\sigma(A_j)<1$, $a,b$ be complex numbers such that $ab\neq 0$ and $a=cb$ $(c>1)$. Then every solution $f\not\equiv 0$ of the equation*

$$f''+A_1(z)e^{az}f'+A_0(z)e^{bz}f=0 \tag{1.3}$$

*has infinite order.*

Combining Theorems B and C, we obtain that if $ab\neq 0$ and $a\neq b$, then every solution $f\not\equiv 0$ of (1.3) has infinite order. Can we get the similar result in higher order linear differential equations which has the same form as (1.3)? The following Corollary 3 gives the affirmative answer.

For higher order linear differential equations

$$f^{(k)}+A_{k-1}f^{(k-1)}+\cdots+A_0f=0, \tag{1.4}$$

Z.-X. Chen obtained the following theorems.

**Theorem D.** *(See [6].) Let $A_j(z)$ $(j=0,\dots,k-1)$ be entire functions such that*

$$\max\{\sigma(A_j),\ j=1,\dots,k-1\}<\sigma(A_0)<+\infty.$$

*Then every solution $f\not\equiv 0$ of (1.4) satisfies $\sigma_2(f)=\sigma(A_0)$.*

**Theorem E.** *(See [3].) Suppose that $a_j$ $(j=0,\dots,k-1)$ are complex numbers. There exist $a_s$ and $a_l$ such that $s<l$, $a_s=d_se^{i\varphi}$, $a_l=-d_le^{i\varphi}$, $d_s>0$, $d_l>0$, and for $j\neq s,l$, $a_j=d_je^{i\varphi}$ $(d_j\geqslant 0)$ or $a_j=-d_je^{i\varphi}$, $\max\{d_j\mid j\neq s,l\}=d<\min\{d_s,d_l\}$. If $A_j=h_j(z)e^{a_jz}$, where $h_j$ are polynomials, $h_sh_l\not\equiv 0$, then every transcendental solution $f$ of (1.4) satisfies $\sigma(f)=\infty$ and $\sigma_2(f)=1$.*

**Theorem F.** *(See [4].) Let $h_j(z)$ $(j = 0, \dots, k-1)$ be entire functions with $\sigma(h_j) < 1$ and $A_j = h_j(z)e^{a_j z}$, where $a_j$ $(j = 0, \dots, k-1)$ are complex numbers. Suppose that there exists $a_s$ such that $h_s \not\equiv 0$, and for $j \neq s$, if $A_j \not\equiv 0$, $a_j = c_j a_s$, $0 < c_j < 1$; if $A_j \equiv 0$, we define $c_j = 0$. Then every transcendental solution $f$ of* (1.4) *satisfies $\sigma(f) = \infty$. Furthermore if* $\max\{c_1, \dots, c_{s-1}\} < c_0$, *then every solution $f \not\equiv 0$ of* (1.4) *has infinite order.*

The aim of our paper is to investigate the growth of the solutions of (1.4) when most coefficients in (1.4) have the same order with each other, and we obtain the following results.

**Theorem 1.** *Let $A_j(z)$ $(j = 0, \dots, k-1)$ be entire functions satisfying $\sigma(A_0) = \sigma$, $\tau(A_0) = \tau$, $0 < \sigma < \infty$, $0 < \tau < \infty$, and let $\sigma(A_j) \leqslant \sigma$, $\tau(A_j) < \tau$ if $\sigma(A_j) = \sigma$ $(j = 1, \dots, k-1)$, then every solution $f \not\equiv 0$ of* (1.4) *satisfies $\sigma_2(f) = \sigma(A_0)$.*

**Corollary 1.** *Let $A_j(z)$ $(j = 0, \dots, k-1)$ be entire functions satisfying $\sigma(A_j) = \sigma$ and* $\max\{\tau(A_j),\ j = 1, \dots, k-1\} < \tau(A_0)$, *where $0 < \sigma < \infty$, $0 < \tau(A_0) < \infty$. Then every solution $f \not\equiv 0$ of* (1.4) *satisfies $\sigma_2(f) = \sigma$.*

**Corollary 2.** *Let $h_j(z)$ $(j = 0, \dots, k-1)$ be entire functions with $\sigma(h_j) < n$, and let $A_j(z) = h_j(z)e^{P_j(z)}$, where $P_j(z) = a_{jn}z^n + \cdots + a_{j0}$ $(j = 0, \dots, k-1)$ are polynomials with degree $n \geqslant 1$, $a_{jn}$ $(j = 0, \dots, k-1)$ are complex numbers. If $|a_{0n}| >$* $\max\{|a_{jn}|,\ j = 1, 2, \dots, k-1\}$ *and $h_0(z) \not\equiv 0$, then every solution $f \not\equiv 0$ of* (1.4) *satisfies $\sigma_2(f) = n$.*

**Remark 1.** Theorem 1 is an extension of Theorem D. For example, when $k = 2$, $f_1(z) = e^{e^z}$ and $f_2(z) = e^z e^{e^z}$ are two linearly independent solutions of equation $f'' - (1 + 2e^z)f' + e^{2z} f = 0$, where $\sigma(1 + 2e^z) = \sigma(e^{2z}) = 1$, $\tau(e^{2z}) = 2 > \tau(1 + 2e^z) = 1$.

**Theorem 2.** *Let $h_j(z)$ $(j = 0, \dots, k-1)$ be not all vanishing entire functions with $\sigma(h_j) < n$, and let $A_j(z) = h_j(z)e^{P_j(z)}$, where $P_j(z) = a_{jn}z^n + \cdots + a_{j0}$ $(j = 0, \dots, k-1)$ are polynomials with degree $n \geqslant 1$. If $a_{jn}$ $(j = 0, \dots, k-1)$ are distinct complex numbers and $h_0(z) \not\equiv 0$, then every solution $f \not\equiv 0$ of* (1.4) *satisfies $\sigma(f) = \infty$.*

**Corollary 3.** *Let $h_j(z)$ $(j = 0, \dots, k-1)$ be not all vanishing entire functions with $\sigma(h_j) < 1$, and let $A_j(z) = h_j(z)e^{a_j z}$, if $a_j$ $(j = 0, \dots, k-1)$ are distinct complex numbers and $h_0(z) \not\equiv 0$, then every solution $f \not\equiv 0$ of* (1.4) *satisfies $\sigma(f) = \infty$.*

**Remark 2.** The hypothesis that $a_{jn}$ $(j = 0, \dots, k-1)$ are distinct complex numbers is necessary. For example, the equation $f''' + 2e^{3z} f'' + ze^{2z} f' - e^{2z} f = 0$ admits a polynomial solution $f(z) = z$, where $a_{11} = a_{01} = 2$.

**Theorem 3.** *Let $h_j(z)$ $(j = 0, \dots, k-1)$ be entire functions with $\sigma(h_j) < n$, and let $A_j(z) = h_j(z)e^{P_j(z)}$, where $P_j(z) = a_{jn}z^n + \cdots + a_{j0}$ $(j = 0, \dots, k-1)$ are polynomials with degree $n \geqslant 1$, $a_{jn}$ are complex numbers such that $a_{0n} = |a_{0n}|e^{i\theta_0}$, $a_{sn} = |a_{sn}|e^{i\theta_s}$, $a_{0n}a_{sn} \neq 0$ $(0 < s \leqslant k-1)$, $\theta_0, \theta_s \in [0, 2\pi)$, $\theta_0 \neq \theta_s$, $h_0 h_s \not\equiv 0$; for $j \neq 0, s$, $a_{jn}$ satisfies either $a_{jn} = d_j a_{0n}$ $(d_j < 1)$ or $\arg a_{jn} = \theta_s$, then every solution $f \not\equiv 0$ of* (1.4) *satisfies $\sigma_2(f) = n$.*

**Remark 3.** Theorem 3 is an extension of Theorem B, since Theorem B is just the case for $k = 2$, $\arg a_{1n} \neq \arg a_{0n}$ or $a_{1n} = d_1 a_{0n}$ $(0 < d_1 < 1)$.

From Theorem 3, we know that every solution $f \not\equiv 0$ of equation $f^{(4)} + e^{5iz} f^{(3)} + 2e^{-4z} f'' + ze^z f' + e^{3z} f = 0$ satisfies $\sigma_2(f) = 1$, in this example, we have $\sigma(A_j) = 1$ $(j = 0, 1, 2, 3)$, $\tau(A_0) = 3 < \max\{\tau(A_j),\ j = 1, 2, 3\} = 5$. So Theorem 3 is a complement to Theorem 1.

**Theorem 4.** *Let $h_j(z)$ $(j = 0, \dots, k-1)$ be entire functions with $\sigma(h_j) < n$, and let $A_j(z) = h_j(z)e^{P_j(z)}$, where $P_j(z) = a_{jn}z^n + \cdots + a_{j0}$ $(j = 0, \dots, k-1)$ are polynomials with degree $n \geqslant 1$. Suppose that there exist nonzero complex numbers $a_{sn}$ and $a_{ln}$ such that $0 < s < l \leqslant k-1$, $a_{sn} = |a_{sn}|e^{i\theta_s}$, $a_{ln} = |a_{ln}|e^{i\theta_l}$, $\theta_s, \theta_l \in [0, 2\pi)$, $\theta_s \neq \theta_l$, $h_s h_l \not\equiv 0$; for $j \neq s, l$, $a_{jn}$ satisfies either $a_{jn} = d_j a_{sn}$ $(0 < d_j < 1)$ or $a_{jn} = d_j a_{ln}$ $(0 < d_j < 1)$, then every transcendental solution of* (1.4) *satisfies $\sigma(f) = \infty$. Furthermore if $f(z)$ is a polynomial solution of* (1.4), *then* $\deg f \leqslant s - 1$, *if $s = 1$, then every non-constant solution $f(z)$ of* (1.4) *satisfies $\sigma(f) = \infty$.*

**Remark 4.** Theorem 4 is an extension of Theorem E, since Theorem E is just the case for $n=1$, $\theta_s=0$, $\theta_l=\pi$, $0<s<l\leqslant k-1$. Theorem 4 is also an extension of Theorem F, since Theorem F is just the case for $n=1$, $s=l$, $a_{s1}=a_{l1}$, $\theta_s=\theta_l=0$.

In Theorem 4, we can only obtain that every transcendental solution of (1.4) satisfies $\sigma(f)=\infty$, but the sharper result $\sigma_2(f)=n$ remains open. In Theorem 4, Eq. (1.4) may have polynomial solutions, for example, $f^{(5)}+2e^{3iz}f^{(4)}+e^{5iz}f^{(3)}+3e^{4z}f''+2ze^zf'-2e^zf=0$ admits a polynomial solution $f(z)=2z$, where $s=2$, $l=3$, $\theta_2=0$, $\theta_3=\frac{\pi}{2}$.

**Question 1.** Can we get the same result as Theorem 1 when all the coefficients in (1.4) are analytic in the unit disc $\{z\colon |z|<1\}$ (see Theorem 1.5 in [12]).

## 2. Lemmas for the proofs of theorems

**Lemma 1.** *(See [8].) Let $f(z)$ be a transcendental meromorphic function, and let $\alpha>1$ be a given constant, for any given $\varepsilon>0$,*

(i) *there exist a set $E\subset[0,\infty)$ that has finite logarithmic measure and a constant $B>0$ that depends only on $\alpha$ such that for all $z$ satisfying $|z|=r\notin E$, we have*

$$\left|\frac{f^{(k)}(z)}{f(z)}\right|\leqslant B\bigl[T(\alpha r,f)r^{\varepsilon}\log T(\alpha r,f)\bigr]^k \quad (k\in N); \tag{2.1}$$

(ii) *there exist a set $E\subset[0,2\pi)$ that has linear measure zero and a constant $B>0$ that depends only on $\alpha$, for any $\theta\in[0,2\pi)\backslash E$ there exists a constant $R_0=R_0(\theta)>1$ such that for all $z$ satisfying $\arg z=\theta$ and $|z|=r>R_0$, we have*

$$\left|\frac{f^{(k)}(z)}{f(z)}\right|\leqslant B\bigl[T(\alpha r,f)\log T(\alpha r,f)\bigr]^k \quad (k\in N). \tag{2.2}$$

**Lemma 2.** *(See [5].) Let $f(z)$ be an entire function of order $\sigma(f)=\alpha<+\infty$. Then for any given $\varepsilon>0$, there is a set $E\subset[1,\infty)$ that has finite linear measure and finite logarithmic measure such that for all $z$ satisfying $|z|=r\notin[0,1]\cup E$, we have*

$$\exp\bigl\{-r^{\alpha+\varepsilon}\bigr\}\leqslant\bigl|f(z)\bigr|\leqslant\exp\bigl\{r^{\alpha+\varepsilon}\bigr\}. \tag{2.3}$$

**Lemma 3.** *Let $A_j(z)$ $(j=0,\dots,k-1)$ be entire functions with $\sigma(A_j)\leqslant\sigma<\infty$, if $f(z)$ is a solution of* (1.6) *then $\sigma_2(f)\leqslant\sigma$.*

**Proof.** Using the Wiman–Valiron theory, we can easily prove Lemma 3.

**Lemma 4.** *(See [8,10].) Let $f(z)$ be a transcendental meromorphic function with $\sigma(f)=\sigma<\infty$, $\Gamma=\{(k_1,j_1),\dots,(k_m,j_m)\}$ be a finite set of distinct pairs of integers which satisfy $k_i>j_i\geqslant 0$ for $i=1,\dots,m$. And let $\varepsilon>0$ be a given constant, then there exists a set $E\subset[0,2\pi)$ which has linear measure zero, such that if $\varphi\in[0,2\pi)\backslash E$, there is a constant $R_1=R_1(\varphi)>1$ such that for all $z$ satisfying $\arg z=\varphi$ and $|z|\geqslant R_1$ and for all $(k,j)\in\Gamma$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\leqslant|z|^{(k-j)(\sigma-1+\varepsilon)}. \tag{2.4}$$

**Lemma 5.** *(See [1].) Let $P(z)$ be a polynomial of degree $n\geqslant 1$, where $P(z)=(\alpha+\beta i)z^n+\cdots$, $\delta(P,\theta)=\alpha\cos n\theta-\beta\sin n\theta$, $\alpha,\beta\in R$, and let $\varepsilon$ be a given constant, then we have*

(i) *If $\delta(P,\theta)>0$, then there exists an $r(\theta)>0$ such that for any $r\geqslant r(\theta)$,*

$$\bigl|e^{P(re^{i\theta})}\bigr|\geqslant\exp\bigl\{(1-\varepsilon)\delta(P,\theta)r^n\bigr\}; \tag{2.5}$$

(ii) *If* $\delta(P,\theta)<0$, *then there exists an* $r(\theta)>0$ *such that for any* $r\geqslant r(\theta)$,

$$\left|e^{P(re^{i\theta})}\right|\leqslant \exp\left\{(1-\varepsilon)\delta(P,\theta)r^n\right\}. \tag{2.6}$$

**Lemma 6.** *(See [10].) Let* $f(z)$ *be an entire function and suppose that* $|f^{(k)}(z)|$ *is unbounded on some ray* $\arg z=\theta$. *Then there exists an infinite sequence of points* $z_m=r_me^{i\theta}$ $(m=1,2,\ldots)$, *where* $r_m\to\infty$, *such that* $f^{(k)}(z_m)\to\infty$ *and*

$$\left|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\right|\leqslant |z_m|^{k-j}\big(1+o(1)\big)\quad (j=0,\ldots,k-1). \tag{2.7}$$

**Lemma 7.** *(See [7].) Let* $k\geqslant 1$, $P_1,P_2,\ldots,P_k$ *be non-constant polynomials with degree* $d_1,d_2,\ldots,d_k$, *respectively, such that* $\deg(P_i-P_j)=\max\{d_i,d_j\}$ *for* $i\neq j$. *Set* $A(z)=\sum_{j=1}^{k}B_j(z)e^{P_j(z)}$, *where* $B_j(z)\not\equiv 0$ *are entire functions with* $\sigma(B_j)<d_j$, *then* $\sigma(A)=\max_{1\leqslant j\leqslant n}\{d_j\}$.

**Lemma 8.** *Let* $f(z)$ *be an entire function with* $\sigma(f)=\sigma$, $\tau(f)=\tau$, $0<\sigma<\infty$, $0<\tau<\infty$, *then for any given* $\beta<\tau$, *there exists a set* $E\subset[1,+\infty)$ *that has infinite logarithmic measure such that for all* $r\in E$, *we have*

$$\log M(r,f)>\beta r^{\sigma}. \tag{2.8}$$

**Proof.** By definition, there exists an increasing sequence $\{r_m\}\to\infty$ satisfying $(1+\frac{1}{m})r_m<r_{m+1}$ and

$$\lim_{m\to\infty}\frac{\log M(r_m,f)}{r_m^{\sigma}}=\tau. \tag{2.9}$$

Then there exists a positive integer $m_0$ such that for all $m\geqslant m_0$ and for any given $\varepsilon$ $(0<\varepsilon<\tau-\beta)$, we have

$$\log M(r_m,f)>(\tau-\varepsilon)r_m^{\sigma}. \tag{2.10}$$

For any given $\beta<\tau$, there exists a positive integer $m_1$ such that for all $m\geqslant m_1$, we have

$$\left(\frac{m}{m+1}\right)^{\sigma}>\frac{\beta}{\tau-\varepsilon}. \tag{2.11}$$

By (2.10) and (2.11), for all $m\geqslant m_2=\max\{m_0,m_1\}$ and for any $r\in[r_m,(1+\frac{1}{m})r_m]$, we have

$$\log M(r,f)\geqslant \log M(r_m,f)>(\tau-\varepsilon)r_m^{\sigma}\geqslant(\tau-\varepsilon)\left(\frac{m}{m+1}r\right)^{\sigma}>\beta r^{\sigma}.$$

Set $E=\bigcup_{m=m_2}^{\infty}[r_m,(1+\frac{1}{m})r_m]$, then

$$m_lE=\sum_{m=m_2}^{\infty}\int_{r_m}^{(1+\frac{1}{m})r_m}\frac{dt}{t}=\sum_{m=m_2}^{\infty}\log\left(1+\frac{1}{m}\right)=\infty.$$

Therefore, we complete the proof of this lemma. □

## 3. Proof of Theorem 1

Assume that $f(z)$ is a non-trivial solution of (1.4). From (1.4) we have

$$\left|A_0(z)\right|\leqslant\left|\frac{f^{(k)}(z)}{f(z)}\right|+\left|A_{k-1}(z)\right|\left|\frac{f^{(k-1)}(z)}{f(z)}\right|+\cdots+\left|A_1(z)\right|\left|\frac{f'(z)}{f(z)}\right|. \tag{3.1}$$

By Lemma 1(i), we know there exist a set $E_1\subset[0,\infty)$ that has finite logarithmic measure and a constant $B>0$ such that

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\leqslant B\big(T(2r,f)\big)^{2k}\quad (j=1,\ldots,k) \tag{3.2}$$

holds for all $|z| = r \notin E_1$ and for sufficiently large $r$. If $\sigma(A_j) < \sigma$ $(j \neq 0)$, by Lemma 2, there exists a set $E_2 \subset [1, \infty)$ having finite logarithmic measure such that for all $z$ satisfying $|z| = r \notin E_2$, we have

$$\left|A_j(z)\right| \leqslant \exp\left\{r^{\alpha_1}\right\} \quad (j \neq 0), \tag{3.3}$$

where $0 < \alpha_1 < \sigma$. If $\sigma(A_j) = \sigma$, $\tau(A_j) < \tau$ $(j \neq 0)$, we choose $\alpha_2, \alpha_3$ satisfying $\max\{\tau(A_j),\ j \neq 0\} < \alpha_2 < \alpha_3 < \tau$ such that for sufficiently large $r$, we have

$$\left|A_j(z)\right| < \exp\left\{\alpha_2 r^{\sigma}\right\} \quad (j \neq 0). \tag{3.4}$$

By Lemma 8, there exists a set $E_0 \subset [1, +\infty)$ having infinite logarithmic measure such that for all $r \in E_0$, we have

$$M(r, A_0) > \exp\left\{\alpha_3 r^{\sigma}\right\}. \tag{3.5}$$

Hence from (3.1)–(3.5), for all $z$ satisfying $|A_0(z)| = M(r, A_0)$ and for sufficiently large $|z| = r \in E_0 \backslash (E_1 \cup E_2)$, we have

$$\exp\left\{\alpha_3 r^{\sigma}\right\} \leqslant kB \exp\left\{\alpha_2 r^{\sigma}\right\}\left(T(2r, f)\right)^{2k}. \tag{3.6}$$

By (3.6), we have $\sigma_2(f) \geqslant \sigma$. On the other hand, by Lemma 3, we have $\sigma_2(f) \leqslant \sigma$, hence, $\sigma_2(f) = \sigma = \sigma(A_0)$.

## 4. Proof of Theorem 2

Assume that $f(z)$ is a transcendental solution of (1.4), we show that $\sigma(f) = \infty$. Suppose to the contrary that $\sigma(f) = \sigma < \infty$. By Lemma 4, there exists a set $E_3 \subset [0, 2\pi)$ with linear measure zero, such that if $\theta \in [0, 2\pi)\backslash E_3$, there is a constant $R_1 = R_1(\theta) > 1$, such that for all $z$ satisfying $\arg z = \theta$ and $|z| = r > R_1$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leqslant |z|^{k\sigma} \quad (k \geqslant j > i \geqslant 0). \tag{4.1}$$

Set $a_{jn} = |a_{jn}|e^{i\varphi_j}$, and $E_4 = \{\theta \in [0, 2\pi)\colon\ \delta(P_j, \theta) = |a_{jn}|\cos(\varphi_j + n\theta) = 0,\ j = 0, 1, \ldots, k-1\} \cup \{\theta \in [0, 2\pi)\colon\ \delta(P_j - P_i, \theta) = 0, 0 \leqslant i < j \leqslant k-1\}$, then $mE_4 = 0$.

For each entire function $A_j(z) = h_j(z)e^{P_j(z)}$ $(j = 0, \ldots, k-1)$, by Lemma 5, there exists a set $H_j \subset [0, 2\pi)$ with linear measure zero, such that if $z = re^{i\theta}$, $\theta \in [0, 2\pi)\backslash H_j$, and for sufficiently large $r$, then $A_j$ satisfies (2.5) or (2.6). Set $E_5 = \bigcup_{j=0}^{k-1} H_j$, then $E_5$ is also a set having linear measure zero. For any given $\theta \in [0, 2\pi)\backslash(E_3 \cup E_4 \cup E_5)$, we have

$$\delta(P_j, \theta) \neq 0, \quad \delta(P_i, \theta) \neq \delta(P_j, \theta) \quad (0 \leqslant i < j \leqslant k-1).$$

Since $a_{jn}$ are distinct complex numbers, there exists only one $s \in \{0, \ldots, k-1\}$ such that $\delta(P_s, \theta) = \max\{\delta(P_j, \theta)\colon j = 0, \ldots, k-1\}$ for any given $\theta \in [0, 2\pi)\backslash(E_3 \cup E_4 \cup E_5)$. Set $\delta = \delta(P_s, \theta)$, $\delta_1 = \max\{\delta(P_j, \theta)\colon\ j \neq s\}$, then $\delta_1 < \delta$. We divide the proof into two cases:

(i) $\delta > 0$,
(ii) $\delta < 0$.

Case (i). $\delta > 0$. By Lemma 5, for any given constant $\varepsilon_1$ $(0 < 3\varepsilon_1 < \frac{\delta - \delta_1}{\delta_1})$, we obtain for sufficiently large $r$,

$$\left|A_s\left(re^{i\theta}\right)\right| \geqslant \exp\left\{(1 - \varepsilon_1)\delta r^n\right\}, \tag{4.2}$$

$$\left|A_j\left(re^{i\theta}\right)\right| \leqslant \exp\left\{(1 + \varepsilon_1)\delta_1 r^n\right\} \quad (j \neq s). \tag{4.3}$$

Now we prove that $|f^{(s)}(re^{i\theta})|$ is bounded on the ray $\arg z = \theta$. If $|f^{(s)}(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 6, there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, \ldots)$, such that $r_m \to \infty$, $f^{(s)}(z_m) \to \infty$ and

$$\left|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}\right| \leqslant |z_m|^{s-j}\left(1 + o(1)\right) \quad (j = 0, \ldots, s-1). \tag{4.4}$$

Substituting (4.1)–(4.4) into (1.4), we obtain

$$\begin{aligned}\exp\left\{(1-\varepsilon_1)\delta r_m^n\right\} &\leqslant \left|A_s(z_m)\right| \\ &\leqslant \left|\frac{f^{(k)}(z_m)}{f^{(s)}(z_m)}\right| + \cdots + \left|A_{s+1}(z_m)\right|\left|\frac{f^{(s+1)}(z_m)}{f^{(s)}(z_m)}\right| + \left|A_{s-1}(z_m)\right|\left|\frac{f^{(s-1)}(z_m)}{f^{(s)}(z_m)}\right| + \cdots \\ &\quad + \left|A_0(z_m)\right|\left|\frac{f(z_m)}{f^{(s)}(z_m)}\right| \\ &\leqslant k\exp\left\{(1+\varepsilon_1)\delta_1 r_m^n\right\}\cdot |z_m|^M, \end{aligned} \tag{4.5}$$

where $M > 0$ is a constant, not always the same at each occurrence. By (4.5), we obtain

$$\exp\left\{\frac{1}{3}(\delta-\delta_1)r_m^n\right\} \leqslant r_m^M. \tag{4.6}$$

This is a contradiction. Hence $|f^{(s)}(re^{i\theta})| \leqslant M$ on $\arg z = \theta$. We can easily obtain

$$\left|f\left(re^{i\theta}\right)\right| \leqslant Mr^k \tag{4.7}$$

on $\arg z = \theta$.

Case (ii). $\delta < 0$. By (1.4), we get

$$-1 = A_{k-1}(z)\frac{f^{(k-1)}(z)}{f^{(k)}(z)} + \cdots + A_j(z)\frac{f^{(j)}(z)}{f^{(k)}(z)} + \cdots + A_0(z)\frac{f(z)}{f^{(k)}(z)}. \tag{4.8}$$

Now we prove that $|f^{(k)}(re^{i\theta})|$ is bounded on the ray $\arg z = \theta$. If $|f^{(k)}(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 6, there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, \ldots)$, such that $r_m \to \infty$, $f^{(k)}(z_m) \to \infty$ and

$$\left|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\right| \leqslant |z_m|^{k-j}\left(1+o(1)\right) \quad (j = 0, \ldots, k-1). \tag{4.9}$$

By Lemma 5, for any given constant $\varepsilon_2$ $(0 < \varepsilon_2 < \frac{1}{2})$, we have

$$\left|A_j(z_m)\right| \leqslant \exp\left\{(1-\varepsilon_2)\delta r_m^n\right\} \quad (j = 0, \ldots, k-1). \tag{4.10}$$

Then by (4.9) and (4.10), we have for sufficiently large $r_m$

$$\left|A_j(z_m)\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\right| \leqslant \exp\left\{(1-\varepsilon_2)\delta r_m^n\right\}\cdot r_m^{k-j}\left(1+o(1)\right) \to 0 \quad (j = 0, \ldots, k-1). \tag{4.11}$$

Substituting (4.11) into (4.8), we get

$$1 \leqslant 0. \tag{4.12}$$

This is a contradiction. Hence $|f^{(k)}(re^{i\theta})| \leqslant M$ on $\arg z = \theta$. Therefore

$$\left|f\left(re^{i\theta}\right)\right| \leqslant Mr^k \tag{4.13}$$

holds on $\arg z = \theta$. Combining (4.7), (4.13) and the fact that $E_3 \cup E_4 \cup E_5$ has linear measure zero, by the standard Phragmén–Lindelöf theorem, we obtain that $f(z)$ is a polynomial, which contradicts our assumption. Therefore $\sigma(f) = \infty$.

In the following, we show that any non-constant polynomial cannot be a solution of (1.4). If $f(z)$ is a polynomial, then by Lemma 7, we have $\sigma(f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0 f) = n$. By a simple order consideration, we get a contradiction. If $f(z)$ is a constant solution of (1.4), then $f(z) \equiv 0$. Hence, every solution $f \not\equiv 0$ of (1.4) satisfies $\sigma(f) = \infty$.

## 5. Proof of Theorem 3

Assume that $f(z)$ is a non-trivial solution of (1.4). By (1.4), we have

$$-A_0=\frac{f^{(k)}}{f}+\cdots+A_j\frac{f^{(j)}}{f}+\cdots+A_s\frac{f^{(s)}}{f}+\cdots+A_1\frac{f'}{f}. \tag{5.1}$$

We suppose that $a_{j_1n},\dots,a_{j_mn}$ ($j_\alpha\in\{1,\dots,s-1,s+1,\dots,k-1\}$) satisfy $a_{j_\alpha n}=d_{j_\alpha}a_{0n}$ ($\alpha=1,\dots,m$) and $\arg a_{jn}=\theta_s$ for $j\in\{1,\dots,s-1,s+1,\dots,k-1\}\backslash\{j_1,\dots,j_m\}$. Choose a constant $c$ satisfying $\max\{d_{j_1},\dots,d_{j_m}\}<c<1$. We divide the proof into two cases:

(a) $c<0$;
(b) $0\leqslant c<1$.

Case (a). $c<0$. From [17, pp. 253–255], there exist constants $\theta_1$, $\theta_2$, $\alpha$, $R_2$ satisfying $\theta_1,\theta_2\in[0,2\pi)$, $\theta_1<\theta_2$, $\alpha>0$, $R_2>0$ such that for all $z$ satisfying $|z|=r>R_2$ and $\arg z=\theta\in(\theta_1,\theta_2)$, we have

$$\delta(P_0,\theta)=|a_{0n}|\cos(\theta_0+n\theta)>\alpha,\qquad \delta(P_s,\theta)=|a_{sn}|\cos(\theta_s+n\theta)<0,$$

and

$$\delta(P_j,\theta)<0\quad(j\neq 0).$$

By Lemma 5, for any given constant $\varepsilon_3$ ($0<\varepsilon_3<\frac{1}{2}$) and for sufficiently large $r$, we have

$$\bigl|A_0\bigl(re^{i\theta}\bigr)\bigr|\geqslant\exp\bigl\{(1-\varepsilon_3)\alpha r^n\bigr\}; \tag{5.2}$$

$$\bigl|A_j\bigl(re^{i\theta}\bigr)\bigr|\leqslant\exp\bigl\{(1-\varepsilon_3)\delta(P_j,\theta)r^n\bigr\}<M\quad(j\neq 0). \tag{5.3}$$

By Lemma 1(ii), we know there exist a set $E_6\subset[0,2\pi)$ with linear measure zero and constants $B>0$, $R_3>1$ such that for all $z$ satisfying $\arg z=\theta\in(\theta_1,\theta_2)\backslash E_6$ and $|z|=r>R_3$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\leqslant B\bigl(T(2r,f)\bigr)^{2k}\quad(j=1,\dots,k). \tag{5.4}$$

Substituting (5.2)–(5.4) into (5.1), we get for sufficiently large $r$,

$$\exp\bigl\{(1-\varepsilon_3)\alpha r^n\bigr\}\leqslant\bigl|A_0\bigl(re^{i\theta}\bigr)\bigr|\leqslant kM\cdot B\bigl(T(2r,f)\bigr)^{2k}. \tag{5.5}$$

By (5.5), we obtain $\sigma_2(f)\geqslant n$. On the other hand, by Lemma 3, we have $\sigma_2(f)\leqslant n$, therefore $\sigma_2(f)=n$.

Case (b). $0\leqslant c<1$. From [17, pp. 253–255] there exist constants $\theta_1$, $\theta_2$, $\alpha$, $R_2$ satisfying $\theta_1,\theta_2\in[0,2\pi)$, $\theta_1<\theta_2$, $\alpha>0$, $R_2>0$ such that for all $z$ satisfying $|z|=r>R_2$ and $\arg z=\theta\in(\theta_1,\theta_2)$, we have

$$\delta(P_0,\theta)=|a_{0n}|\cos(\theta_0+n\theta)>\alpha,\qquad \delta(P_s,\theta)=|a_{sn}|\cos(\theta_s+n\theta)<0,$$

hence

$$\delta\bigl(P_0-ca_{0n}z^n,\theta\bigr)>(1-c)\alpha,\qquad \delta\bigl(-ca_{0n}z^n,\theta\bigr)\leqslant 0,$$

and

$$\delta\bigl(P_j-ca_{0n}z^n,\theta\bigr)<0\quad(j\neq 0).$$

By Lemmas 5 and 2, for any given constant $\varepsilon_3$ ($0<\varepsilon_3<\frac{1}{2}$) and for sufficiently large $r$, we have

$$\bigl|A_0e^{-ca_{0n}z^n}\bigr|=\bigl|h_0e^{P_0-ca_{0n}z^n}\bigr|\geqslant\exp\bigl\{(1-\varepsilon_3)(1-c)\alpha r^n\bigr\} \tag{5.6}$$

and

$$\bigl|A_je^{-ca_{0n}z^n}\bigr|=\bigl|h_je^{P_j-ca_{0n}z^n}\bigr|\leqslant\exp\bigl\{o(1)r^n\bigr\}\quad(j\neq 0). \tag{5.7}$$

Substituting (5.4), (5.6), (5.7) into (5.1), for all $z$ satisfying $\arg z=\theta\in(\theta_1,\theta_2)\backslash E_6$ and $|z|=r>R_3$, we have

$$\exp\bigl\{(1-\varepsilon_3)(1-c)\alpha r^n\bigr\}\leqslant\bigl|A_0(z)e^{-ca_{0n}z^n}\bigr|\leqslant kB\exp\bigl\{o(1)r^n\bigr\}\cdot\bigl(T(2r,f)\bigr)^{2k}. \tag{5.8}$$

By (5.8), we get $\sigma_2(f)\geqslant n$. On the other hand, by Lemma 3, we have $\sigma_2(f)\leqslant n$, therefore $\sigma_2(f)=n$.

## 6. Proof of Theorem 4

Assume that $f(z)$ is a transcendental solution of (1.4), we show that $\sigma(f)=\infty$. Suppose to the contrary that $\sigma(f)=\sigma<\infty$. By Lemma 4, there exists a set $E_3\subset[0,2\pi)$ with linear measure zero, such that if $\theta\in[0,2\pi)\backslash E_3$, there is a constant $R_1=R_1(\theta)>1$, such that for all $z$ satisfying $\arg z=\theta$ and $|z|=r>R_1$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right|\leqslant |z|^{k\sigma}\quad (k\geqslant j>i\geqslant 0). \tag{6.1}$$

Set $E_7=\{\theta\in[0,2\pi)\colon |a_{sn}|\cos(\theta_s+n\theta)=\delta(P_s,\theta)=\delta(P_l,\theta)=|a_{ln}|\cos(\theta_l+n\theta)\}$, since $\theta_s\neq\theta_l$, then $mE_7=0$. For any given $\theta\in[0,2\pi)\backslash(E_3\cup E_7)$, we have

$$\delta(P_s,\theta)>\delta(P_l,\theta)\quad\text{or}\quad \delta(P_s,\theta)<\delta(P_l,\theta).$$

Set $c_1=\delta(P_s,\theta)$, $c_2=\delta(P_l,\theta)$, we divide the proof into two cases:

(i) $c_1>c_2$;
(ii) $c_1<c_2$.

Case (i). $c_1>c_2$. Here we also divide (i) into three cases:

(a) $c_1>c_2>0$;
(b) $c_1>0>c_2$;
(c) $0>c_1>c_2$.

Case (a). $c_1>c_2>0$. Set $c_3=\max\{\delta(P_j,\theta),\ j\neq s\}$, then $c_3<c_1$. By Lemma 5, for any given constant $\varepsilon_4$ $(0<3\varepsilon_4<\frac{c_1-c_3}{c_3})$ and for sufficiently large $r$, we have

$$\left|A_s\left(re^{i\theta}\right)\right|\geqslant\exp\left\{(1-\varepsilon_4)c_1r^n\right\}, \tag{6.2}$$

and

$$\left|A_j\left(re^{i\theta}\right)\right|\leqslant\exp\left\{(1+\varepsilon_4)c_3r^n\right\}\quad (j\neq s). \tag{6.3}$$

Now we prove that $|f^{(s)}(re^{i\theta})|$ is bounded on the ray $\arg z=\theta$. If $|f^{(s)}(re^{i\theta})|$ is unbounded on the ray $\arg z=\theta$, then by Lemma 6, there exists an infinite sequence of points $z_m=r_me^{i\theta}$ $(m=1,2,\ldots)$ such that $r_m\to\infty$, $f^{(s)}(z_m)\to\infty$ and

$$\left|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}\right|\leqslant |z_m|^{s-j}\left(1+o(1)\right)\quad (j=0,\ldots,s-1). \tag{6.4}$$

Substituting (6.2)–(6.4) into (1.4), we obtain

$$\begin{aligned}\exp\left\{(1-\varepsilon_4)c_1r_m^n\right\}&\leqslant\left|A_s(z_m)\right|\\ &\leqslant\left|\frac{f^{(k)}(z_m)}{f^{(s)}(z_m)}\right|+\cdots+\left|A_l(z_m)\right|\left|\frac{f^{(l)}(z_m)}{f^{(s)}(z_m)}\right|+\cdots+\left|A_{s+1}(z_m)\right|\left|\frac{f^{(s+1)}(z_m)}{f^{(s)}(z_m)}\right|\\ &\quad+\left|A_{s-1}(z_m)\right|\left|\frac{f^{(s-1)}(z_m)}{f^{(s)}(z_m)}\right|+\cdots+\left|A_0(z_m)\right|\left|\frac{f(z_m)}{f^{(s)}(z_m)}\right|\\ &\leqslant k\exp\left\{(1+\varepsilon_4)c_3r_m^n\right\}\cdot|z_m|^M.\end{aligned} \tag{6.5}$$

By (6.5), we obtain

$$\exp\left\{\frac{1}{3}(c_1-c_3)r_m^n\right\}\leqslant r_m^M.$$

This is a contradiction. Hence $|f^{(s)}(re^{i\theta})|\leqslant M$ on $\arg z=\theta$. We can easily obtain

$$\left|f\left(re^{i\theta}\right)\right| \leqslant Mr^k \tag{6.6}$$

on $\arg z = \theta$.

Case (b). $c_1 > 0 > c_2$. Using the same reasoning as in the case (a), we can also obtain

$$\left|f\left(re^{i\theta}\right)\right| \leqslant Mr^k \tag{6.7}$$

on $\arg z = \theta$.

Case (c). $0 > c_1 > c_2$. By (1.4), we get

$$-1 = A_{k-1}(z)\frac{f^{(k-1)}(z)}{f^{(k)}(z)} + \cdots + A_l(z)\frac{f^{(l)}(z)}{f^{(k)}(z)} + \cdots + A_s(z)\frac{f^{(s)}(z)}{f^{(k)}(z)} + \cdots + A_0(z)\frac{f(z)}{f^{(k)}(z)}. \tag{6.8}$$

By Lemma 5, for any given constant $\varepsilon_5$ $(0 < \varepsilon_5 < \frac{1}{2})$ and for sufficiently large $r$, we have

$$\left|A_j\left(re^{i\theta}\right)\right| \leqslant \exp\left\{(1-\varepsilon_5)c_1 r^n\right\} \quad (j = 0, \ldots, k-1). \tag{6.9}$$

Now we prove that $|f^{(k)}(re^{i\theta})|$ is bounded on the ray $\arg z = \theta$. If $|f^{(k)}(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 6, there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, \ldots)$ such that $r_m \to \infty$, $f^{(k)}(z_m) \to \infty$ and

$$\left|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\right| \leqslant |z_m|^{k-j}\left(1 + o(1)\right) \quad (j = 0, \ldots, k-1). \tag{6.10}$$

Substituting (6.9) and (6.10) into (6.8), we get

$$1 \leqslant 0.$$

This is a contradiction. Hence $|f^{(k)}(re^{i\theta})| \leqslant M$ on $\arg z = \theta$. Therefore

$$\left|f\left(re^{i\theta}\right)\right| \leqslant Mr^k \tag{6.11}$$

holds on $\arg z = \theta$. Combining (6.6), (6.7) and (6.11) and the fact that $E_3 \cup E_7$ has linear measure zero, by the standard Phragmén–Lindelöf theorem, we obtain that $f(z)$ is a polynomial, which contradicts our assumption. Therefore $\sigma(f) = \infty$.

Case (ii). $c_1 < c_2$. Using the same reasoning as in the case (i), we can also obtain that $f(z)$ is a polynomial, which contradicts our assumption. Therefore $\sigma(f) = \infty$.

In the following, we show that if $f(z)$ is a polynomial solution of (1.4), then $\deg f \leqslant s-1$. If $f(z)$ is a polynomial with $\deg f \geqslant s$. If $\theta_s \neq \theta_l + \pi$ or $\theta_l \neq \theta_s + \pi$, set $E_8 = \{\theta \in [0, 2\pi)\colon\ \delta(P_s, \theta) > \delta(P_l, \theta) > 0\}$, then $mE_8 > 0$. We can choose a curve $\Gamma = \{z\colon\ \arg z = \theta \in E_8\}$, by the same reasoning as in case (a) of Theorem 4, for all $z \in \Gamma$ and for sufficiently large $r$ we obtain

$$\begin{aligned}\exp\left\{(1-\varepsilon_4)c_1 r^n\right\} &\leqslant \left|A_s(z)f^{(s)}(z)\right| \\ &\leqslant \left|f^{(k)}(z)\right| + \cdots + \left|A_{s+1}(z)f^{(s+1)}(z)\right| + \left|A_{s-1}(z)f^{(s-1)}(z)\right| + \cdots + \left|A_0(z)f(z)\right| \\ &\leqslant k\exp\left\{(1+\varepsilon_4)c_3 r^n\right\}\cdot|z|^M.\end{aligned} \tag{6.12}$$

By (6.12), we obtain

$$\exp\left\{\frac{1}{3}(c_1 - c_3)r^n\right\} \leqslant r^M.$$

This is a contradiction. If $\theta_s = \theta_l + \pi$ or $\theta_l = \theta_s + \pi$, set $E_9 = \{\theta \in [0, 2\pi)\colon\ \delta(P_s, \theta) > 0 > \delta(P_l, \theta)\}$, then $mE_9 > 0$. By the same reasoning as in (6.12), we can also get a contradiction. Hence every polynomial solution of (1.4) satisfies $\deg f \leqslant s-1$.

## Acknowledgment

We thank the referee's valuable suggestions to improve our paper.

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