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The classification of minimal matrices of size $2 \times q$ [☆]

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Abstract

Minimal matrices were introduced to give an algebraic characterization of sets of uniqueness, a notion of interest in Discrete Tomography. They have also been used to produce minimal summands in Kronecker products of complex irreducible characters of the symmetric group. In this paper, motivated by these two applications, we classify all minimal matrices of size $2 \times q$. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

Minimal matrices were introduced in [6], where they were used to give an algebraic characterization of sets of uniqueness. These sets are of interest in Discrete Tomography, see [2,3]. Later minimal matrices were used in [7] as a tool for producing minimal summands in Kronecker products of complex irreducible characters of the symmetric group. In these two applications lies our motivation for studying minimal matrices.

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Not much is known about minimal matrices but a characterization [7, Proposition 3.1], a sufficient condition for plane partitions to be minimal [8, Proposition 6.1] and the minimal matrices of size 2×2 [6, Theorem 4.1]. In this paper, we extend the last result and determine all minimal matrices of size $2 \times q$. It follows from this classification and Theorem $1'$ in [6] that the sets of uniqueness contained in a box of size $2 \times q \times r$ coincide with the pyramids, that is, with the diagrams of plane partitions of size $2 \times q$ and largest part at most r, see Section 6. We leave for a future paper the application of our classification to Kronecker products.

We now proceed to state our main result. For this we first introduce some notation and definitions. A vector $\lambda = (\lambda_1, \ldots, \lambda_p)$ of positive integers is called a *partition* of n, in symbols $\lambda \vdash n$, if $\lambda_1 \geqslant \cdots \geqslant \lambda_p > 0$ and $\sum_{i=1}^{p} \lambda_i = n$. The set of all partitions of n is a lattice under the *dominance order*, which is defined by

$$(\lambda_1, \ldots, \lambda_p) \trianglerighteq (\mu_1, \ldots, \mu_q)$$

if $\sum_{i=1}^{a} \lambda_i \geqslant \sum_{i=1}^{a} \mu_i$ for all $1 \leqslant a \leqslant \min\{p, q\}$. If $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$, we write $\lambda \triangleright \mu$, see [1] for more details on the dominance order.

Let $\lambda = (\lambda_1, \ldots, \lambda_p)$, $\mu = (\mu_1, \ldots, \mu_q)$ be partitions of an integer n. We denote by $\mathsf{M}(\lambda, \mu)$ the set of all matrices $A = (a_{ij})$ with non-negative integer coefficients of size $p \times q$, row sum vector λ and column sum vector μ, that is

$$\begin{aligned} \sum_{j=1}^{q} a_{ij} &= \lambda_i \quad \text{for } 1 \leqslant i \leqslant p, \\ \sum_{i=1}^{p} a_{ij} &= \mu_j \quad \text{for } 1 \leqslant j \leqslant q. \end{aligned} \tag{1}$$

If A is in $\mathsf{M}(\lambda, \mu)$, we denote by $\pi(A)$ the partition of n obtained from A by ordering its entries decreasingly. If A contains zeros, we omit them in $\pi(A)$. We say that A is *minimal* if A is in $\mathsf{M}(\lambda, \mu)$ and $\pi(A)$ is minimal in the set $\{\pi(B) \mid B \in \mathsf{M}(\lambda, \mu)\}$ with respect to the dominance order. Intuitively A is minimal if the diagram $D(A)$ associated to A, as explained below, is as flat as posible subject to the restrictions (1).

To any matrix $A = (a_{ij}) \in \mathsf{M}(\lambda, \mu)$ we associate a diagram $D(A)$: it is the set of triples (i, j, k) of natural numbers such that

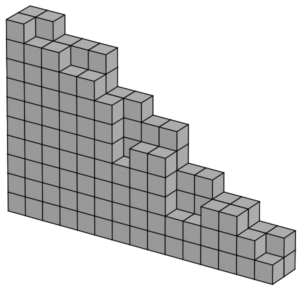
$$1 \leqslant i \leqslant p, \quad 1 \leqslant j \leqslant q, \quad 1 \leqslant k \leqslant a_{ij}.$$

We represent $D(A)$ graphically by stacking unit cubes as in the following example:

The diagram of the matrix

$$\begin{bmatrix} 10 & 10 & 9 & 9 & 9 & 7 & 7 & 6 & 6 & 4 & 4 & 3 & 3 & 2 & 2 \\ 10 & 9 & 9 & 8 & 8 & 7 & 4 & 5 & 5 & 2 & 2 & 3 & 3 & 2 & 1 \end{bmatrix} \tag{2}$$

is represented by



Note that the entries in the matrix count the number of unit cubes in each particular stack.

Recall that A is called a *plane partition* if its rows and columns are weakly decreasing. Finally, suppose that A is of size $2 \times q$. Then we say that A is of *type* I if A has weakly decreasing columns and any 2×2 submatrix

$$\begin{bmatrix} a_{1k} & a_{1l} \\ a_{2k} & a_{2l} \end{bmatrix}, \quad k < l,$$

is either a plane partition or has one of the forms

$$\begin{bmatrix} c+1 & c \\ d & d+1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} c & c+1 \\ d+1 & d \end{bmatrix};$$

and we say that A is of *type* II if A has at least one increasing column ($a_{1k} < a_{2k}$ for some $1 \leqslant k \leqslant q$) and $|a_{1j} - a_{2j}| \leqslant 1$ for all $1 \leqslant j \leqslant q$. For example, the matrix in (2) is of type I. An example of a matrix of type II is

$$\begin{bmatrix} 9 & 8 & 7 & 6 & 6 & 5 & 5 & 4 & 3 & 3 & 1 & 0 & 1 \\ 10 & 7 & 8 & 6 & 6 & 4 & 4 & 5 & 3 & 2 & 0 & 1 & 0 \end{bmatrix}.$$

The main theorem in this paper is:

Theorem 1.1. *Let $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1, \ldots, \mu_q)$ be partitions of n, and let A be in $\mathsf{M}(\lambda, \mu)$. Then A is minimal if and only if A is either of type* I *or of type* II.

Since all plane partitions are of type I, we obtain the following.

Corollary 1.2. *Every plane partition of size $2 \times q$ is minimal.*

This should be contrasted with the following example. Let $\lambda = (9, 4, 2)$, $\mu = (8, 5, 2)$ and let

$$A = \begin{bmatrix} 4 & 4 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Then A, B are in $\mathsf{M}(\lambda, \mu)$ and $\pi(A) \rhd \pi(B)$. Thus A is a plane partition that is not minimal.

The paper is organized as follows. Section 2 contains some new results on minimal matrices of arbitrary size. They give us information about minimal matrices that are not plane partitions. In Section 3, we consider matrices of type I. The main result in this section is Theorem 3.3, which says that the decreasing sequences of the entries of two matrices of type I, in standard form, are not comparable under the dominance order. Section 4 contains two results on matrices of type II. In Section 5, we prove Theorem 1.1. Finally, Section 6 contains an application of our classification to Discrete Tomography.

2. Minimal matrices

In this section, we prove some new results on minimal matrices of arbitrary size. They will be used in the proof of Theorem 1.1. One of them, Corollary 2.3, gives a necessary condition for a matrix, which is not a plane partition, to be minimal. We also construct a family of examples of minimal matrices which, in general, are not plane partitions. This family includes all matrices of type II.

The dominance order defined in the previous section can be extended to k-tuples of real numbers. In this context, it is known as *majorization order*. It has been used extensively in the Theory of Inequalities, see [4, p. 45] and [5, p. 7]. We will need the following result due to Rado, which we adapt to the notation and necessities of this paper. For any k-tuple α of non-negative integers we denote by $\pi(\alpha)$ the partition obtained from α by rearranging its coordinates in decreasing order. Then we have:

Proposition 2.1 ([4, p. 63] and [5, p. 121]). *Let* $\alpha = (\alpha_1, \dots, \alpha_a),\ \beta = (\beta_1, \dots, \beta_b)$ *be partitions of m and let* $\lambda = (\lambda_1, \dots, \lambda_p),\ \mu = (\mu_1, \dots, \mu_q)$ *be partitions of n. If* $\alpha \trianglerighteq \beta$ *and* $\lambda \trianglerighteq \mu$, *then*

$$\pi(\alpha_1, \dots, \alpha_a, \lambda_1, \dots, \lambda_p) \trianglerighteq \pi(\beta_1, \dots, \beta_b, \mu_1, \dots, \mu_q).$$

Moreover, if any of the first two inequalities is strict, then the third is also strict.

We sketch here, for the sake of completeness, an elementary proof of this Proposition: Let r be a positive integer and let $\gamma = \pi(\alpha_1, \dots, \alpha_a, r)$ and $\delta = (\beta_1, \dots, \beta_b, r)$. Then it is enough to show that $\gamma \trianglerighteq \delta$. This can be done by a case by case analysis, according to the positions of r in γ and δ. For the last statement it is enough to observe that if $\gamma = \delta$, then $\alpha = \beta$.

In the rest of this section, we assume that $\lambda = (\lambda_1, \dots, \lambda_p)$ and $\mu = (\mu_1, \dots, \mu_q)$ are two partitions of n.

Lemma 2.2. *Let* $A = (a_{ij})$ *be a minimal matrix in* $\mathsf{M}(\lambda, \mu)$, *and let* i, j, k *be such that* $a_{ij} - a_{kj} \geqslant 2$. *Then* $i < k$ *and* $a_{il} \geqslant a_{kl}$ *for all* $1 \leqslant l \leqslant q$.

Proof. Suppose there exists some $1 \leqslant l \leqslant q$ such that $a_{il} < a_{kl}$. Let B be obtained from A substituting the submatrix

$$S = \begin{bmatrix} a_{ij} & a_{il} \\ a_{kj} & a_{kl} \end{bmatrix} \quad \text{by} \quad T = \begin{bmatrix} a_{ij}-1 & a_{il}+1 \\ a_{kj}+1 & a_{kl}-1 \end{bmatrix}$$

(note that we are not assuming either $i < k$ or $j < l$). Since $\pi(a_{ij}, a_{kj}) \rhd \pi(a_{ij} - 1, a_{kj} + 1)$ and $\pi(a_{kl}, a_{il}) \trianglerighteq \pi(a_{kl} - 1, a_{il} + 1)$, then by Proposition 2.1 $\pi(S) \rhd \pi(T)$, and again by the same result $\pi(A) \rhd \pi(B)$. This is a contradiction to the minimality of A, hence $a_{il} \geqslant a_{kl}$ for all $1 \leqslant l \leqslant q$. So we have $\lambda_i = \sum_{l=1}^{q} a_{il} > \sum_{l=1}^{q} a_{kl} = \lambda_k$, but λ is weakly decreasing, therefore $i < k$. □

Corollary 2.3. *Let A be a minimal matrix in $\mathsf{M}(\lambda, \mu)$, and let i, j, k, l be such that $a_{ij} > a_{kj}$ and $a_{il} < a_{kl}$. Then $|a_{im} - a_{km}| \leqslant 1$ for all $1 \leqslant m \leqslant q$.*

These two results give some restrictions in the rows i and k of a minimal matrix. There are analogous results for columns.

We next construct a family of examples of minimal matrices which, in general, are not plane partitions. It includes all matrices of type II. Let $V(p, s)$ denote the set of all vectors $v = (v_1, \ldots, v_p)$ with non-negative integer coordinates such that $\sum_{i=1}^{p} v_i = s$. Let us write s in the form $s = \gamma p + r$ with $0 \leqslant r < p$ and let $\boldsymbol{m}(s) := (m_1, \ldots, m_p)$, where

$$m_i = \begin{cases} \gamma + 1 & \text{if } 1 \leqslant i \leqslant r; \\ \gamma & \text{if } r < i \leqslant p. \end{cases}$$

The following lemma is easy to prove.

Lemma 2.4. *The vector $\boldsymbol{m}(s)$ belongs to $V(p, s)$ and for every vector $v \in V(p, s)$ one has $\pi(v) \trianglerighteq \boldsymbol{m}(s)$. So, $\boldsymbol{m}(s)$ is, up to permutation of coordinates, the only minimal vector in $V(p, s)$.*

Let $\mathsf{c}_j(A)$ denote the jth column of A. Then we have:

Proposition 2.5. *Let $A \in \mathsf{M}(\lambda, \mu)$ be such that $\pi(\mathsf{c}_j(A)) = \boldsymbol{m}(\mu_j)$ for all $1 \leqslant j \leqslant q$, in other words each column $\mathsf{c}_j(A)$ of A is a minimal vector in $V(p, \mu_j)$. Then A is minimal. Moreover, if B is any other minimal matrix in $\mathsf{M}(\lambda, \mu)$, then also $\pi(\mathsf{c}_j(B)) = \pi(\mathsf{c}_j(A))$ for all $1 \leqslant j \leqslant q$. In particular $\pi(B) = \pi(A)$.*

Proof. It follows from Lemma 2.4 and Proposition 2.1 that for any matrix M in $\mathsf{M}(\lambda, \mu)$, $\pi(M) \trianglerighteq \pi(A)$. Let B be any minimal matrix in $\mathsf{M}(\lambda, \mu)$. By Lemma 2.4 $\pi(\mathsf{c}_j(B)) \trianglerighteq \boldsymbol{m}(\mu_j)$ for all $1 \leqslant j \leqslant q$. If for some $1 \leqslant k \leqslant q$ we have $\pi(\mathsf{c}_k(B)) \rhd \boldsymbol{m}(\mu_k)$, then, by the last part of Proposition 2.1, we would have $\pi(B) \rhd \pi(A)$, which contradicts the minimality of B. Therefore $\pi(\mathsf{c}_j(B)) = \boldsymbol{m}(\mu_j) = \pi(\mathsf{c}_j(A))$ for all $1 \leqslant j \leqslant k$. □

3. Matrices of type I

In this section, we state and prove the properties of matrices of type I needed in the proof of our theorem. We asume that $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \dots, \mu_q)$ are partitions of n. The main result in this section is Theorem 3.3, which says that the decreasing sequences of the entries of two matrices of type I, in standard form, are not comparable under the dominance order.

Recall that a matrix $A \in \mathsf{M}(\lambda, \mu)$ is of type I if it has weakly decreasing columns and each 2×2 submatrix

$$\begin{bmatrix} a_{1k} & a_{1l} \\ a_{2k} & a_{2l} \end{bmatrix}, \quad k < l,$$

is either a plane partition or has one of the forms

$$\begin{bmatrix} c+1 & c \\ d & d+1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} c & c+1 \\ d+1 & d \end{bmatrix}.$$

Lemma 3.1. *Let A be a matrix of type* I. *Suppose* $\left[\begin{smallmatrix} c+1 \\ d \end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix} d \\ d+1 \end{smallmatrix}\right]$ *are two columns of* A, *and let* $\left[\begin{smallmatrix} e \\ f \end{smallmatrix}\right]$ *be a third column of* A. *Then*:
(1) *If* $e + f = c + d + 1$, *then* $\left[\begin{smallmatrix} e \\ f \end{smallmatrix}\right]$ *is equal to either* $\left[\begin{smallmatrix} c+1 \\ d \end{smallmatrix}\right]$ *or* $\left[\begin{smallmatrix} c \\ d+1 \end{smallmatrix}\right]$.
(2) *If* $e + f > c + d + 1$, *then* $e \geqslant c + 1$ *and* $f \geqslant d + 1$.
(3) *If* $e + f < c + d + 1$, *then* $e \leqslant c$ *and* $f \leqslant d$.

Proof. First assume $e + f = c + d + 1$. If $e < c$, then it follows from the definition of type I that $\left[\begin{smallmatrix} e \\ f \end{smallmatrix}\right]$ is to the right of $\left[\begin{smallmatrix} c+1 \\ d \end{smallmatrix}\right]$ and that the submatrix

$$\begin{bmatrix} c+1 & e \\ d & f \end{bmatrix}$$

is a plane partition. This implies $c + d + 1 > e + f$, which contradicts our assumption. Thus $e \geqslant c$. If $e > c + 1$, we similarly obtain a contradiction. Therefore $e = c, c + 1$ and (1) follows. Secondly, assume $e + f > c + d + 1$. Then column $\left[\begin{smallmatrix} e \\ f \end{smallmatrix}\right]$ is to the left of columns

$$\begin{bmatrix} c+1 \\ d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c \\ d+1 \end{bmatrix},$$

and by definition of type I

$$\begin{bmatrix} e & c+1 \\ f & d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} e & c \\ f & d+1 \end{bmatrix}$$

are plane partitions. Thus (2) follows. Statement (3) is proved similarly. $\square$

Suppose that $A \in \mathsf{M}(\lambda, \mu)$ satisfies the hypothesis of the previous lemma. Then the block of A formed by all columns $\left[\begin{smallmatrix} e \\ f \end{smallmatrix}\right]$ with $e + f = c + d + 1$ will be called an *npp-block* (npp stands for non-plane partition). By reordering the columns of the block in such a way as to have the first row of the block weakly decreasing we obtain a matrix B in $\mathsf{M}(\lambda, \mu)$ with $\pi(B) = \pi(A)$ and such that its corresponding npp-block has the form

$$\begin{bmatrix} c+1 & \cdots & c+1 & c & \cdots & c \\ d & \cdots & d & d+1 & \cdots & d+1 \end{bmatrix}. \tag{3}$$

In general if A is a matrix of type I, we can reorder the columns within each of its npp-blocks as indicated above. The resulting matrix will be denoted by $\bar{A}$. It is a matrix in $\mathsf{M}(\lambda, \mu)$, of type I, with $\pi(\bar{A}) = \pi(A)$, such that its first row is weakly decreasing and each npp-block looks like (3). $\bar{A}$ will be called the *standard form* of A. If $A = \bar{A}$, we will say that A is in *standard form*. Note that if A is a plane partition, A has not npp-blocks and A is already in standard form.

Lemma 3.2. *Let A, $B \in \mathsf{M}(\lambda, \mu)$ be of type I, in standard form. Then $A = B$ if and only if $\pi(A) = \pi(B)$.*

Proof. Assume that $\pi(A) = \pi(B)$. Since A and B are in standard form, their first rows are weakly decreasing. Then the largest entries of A and B are precisely a_{11} and b_{11}. Then $\pi(A) = \pi(B)$ implies $a_{11} = b_{11}$, and therefore $a_{21} = b_{21}$. Denote by $\widetilde{A}$, respectively $\widetilde{B}$ the matrix obtained from A, respectively from B by deleting the first column. Then $\widetilde{A}$ and $\widetilde{B}$ have their first row and their columns weakly decreasing and $\pi(\widetilde{A}) = \pi(\widetilde{B})$. This implies that their first columns are equal. We proceed inductively and conclude that $A = B$. The converse is obvious. □

For any matrix $A \in \mathsf{M}(\lambda, \mu)$ of type I, in standard form, we define a total order $<_A$ on the set

$$P = \big\{(i, j) \,\big|\, i = 1, 2 \text{ and } 1 \leqslant j \leqslant q\big\}$$

of positions of A as follows: First, we consider an auxiliary matrix $\widehat{A} = (\widehat{a}_{ij})$ obtained from A by substituting each npp-block of the form (3) with g columns equal to $\left[\begin{smallmatrix} c+1 \\ d \end{smallmatrix}\right]$ and h columns equal to $\left[\begin{smallmatrix} c \\ d+1 \end{smallmatrix}\right]$ by the matrix

$$\begin{bmatrix} c + \frac{g}{g+h} & \cdots & c + \frac{g}{g+h} \\ d + \frac{h}{g+h} & \cdots & d + \frac{h}{g+h} \end{bmatrix}$$

with $g + h$ columns and rational entries. It follows from Lemma 3.1 that $\widehat{A}$ has weakly decreasing rows and columns; it also has row sum vector λ and column sum vector μ. We now define $<_A$ by

$$(i, j) <_A (k, l) \quad \text{if} \begin{cases} \widehat{a}_{ij} > \widehat{a}_{kl} & \text{or} \\ \widehat{a}_{ij} = \widehat{a}_{kl} & \text{and either} \quad (i < k \text{ or } i = k \text{ and } j < l). \end{cases} \tag{4}$$

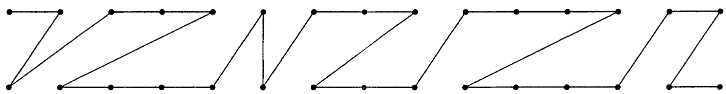
For example, if

$$A = \begin{bmatrix} 10 & 10 & 9 & 9 & 9 & 7 & 7 & 6 & 6 & 4 & 4 & 3 & 3 & 2 & 2 \\ 10 & 9 & 9 & 8 & 8 & 7 & 4 & 5 & 5 & 2 & 2 & 3 & 3 & 2 & 1 \end{bmatrix},$$

then

$$\widehat{A} = \begin{bmatrix} 10 & 10 & 9 & 9 & 9 & 7 & 6\frac{1}{3} & 6\frac{1}{3} & 6\frac{1}{3} & 3\frac{1}{2} & 3\frac{1}{2} & 3\frac{1}{2} & 3\frac{1}{2} & 2 & 2 \\ 10 & 9 & 9 & 8 & 8 & 7 & 4\frac{2}{3} & 4\frac{2}{3} & 4\frac{2}{3} & 2\frac{1}{2} & 2\frac{1}{2} & 2\frac{1}{2} & 2\frac{1}{2} & 2 & 1 \end{bmatrix}$$

and the total order on P is given by the diagram



We will denote by $I^A_{(k,l)}$ the interval $\{(i, j) \in P | (i, j) \leqslant_A (k, l)\}$. We use this order to prove the following.

Theorem 3.3. *Let $A, B \in \mathsf{M}(\lambda, \mu)$ be of type* I, *in standard form. If $A \neq B$, then $\pi(A)$ and $\pi(B)$ are not comparable in the dominance order.*

For the proof we need two technical lemmas.

Lemma 3.4. *$A \in \mathsf{M}(\lambda, \mu)$ be of type* I, *in standard form. Let $y = |I^A_{(1,s)}|$ denote the cardinality of the interval $I^A_{(1,s)}$ for $1 \leqslant s \leqslant q$. Then the y largest entries of A are precisely those a_{ij} with $(i, j) \in I^A_{(1,s)}$.*

Proof. If A is a plane partition, then $A = \widehat{A}$ and the claim follows from (4). If this is not the case, we just have to observe that if some $(2, j) \in I^A_{(1,s)}$ is a position corresponding to an entry lying in some npp-block of A, then by Lemma 3.1 and (4) all positions corresponding to entries in the second row of such npp-block belong to $I^A_{(1,s)}$. □

The following lemma is proved in a similar way.

Lemma 3.5. *Let A in $\mathsf{M}(\lambda, \mu)$ be of type* I, *in standard form. Suppose that either a_{2s} does not belong to an npp-block or a_{2s} lies in the last column of an npp-block for some $1 \leqslant s \leqslant q$, let $y = |I^A_{(2,s)}|$. Then the y largest entries of A are precisely those a_{ij} with $(i, j) \in I^A_{(2,s)}$.*

Proof of Theorem 3.3. Let $\pi(A) = (\alpha_1, \ldots, \alpha_{2q})$ and $\pi(B) = (\beta_1, \ldots, \beta_{2q})$. Let u be the position of the first column in which A and B differ. Since A and B play symmetric roles we may assume $a_{1u} > b_{1u}$. Let $X = I^B_{(1,u)}$ and $x = |X|$. Then, by Lemma 3.4

$$\sum_{h=1}^{x} \alpha_h \geqslant \sum_{(i,j)\in X} a_{ij} > \sum_{(i,j)\in X} b_{ij} = \sum_{h=1}^{x} \beta_h. \qquad (5)$$

To finish the proof we will find a y such that

$$\sum_{h=1}^{y} \alpha_h < \sum_{h=1}^{y} \beta_h. \tag{6}$$

Let $E = (e_{ij})$ be such that $A = B + E$. Note that $e_{1j} + e_{2j} = 0$ for all $1 \leqslant j \leqslant q$, that $\sum_{j=1}^{u} e_{1j} > 0$, and thus $\sum_{j=1}^{u} e_{2j} < 0$. Then

$$m = \min \left\{ \sum_{j=1}^{s} e_{2j} \;\middle|\; 1 \leqslant s \leqslant q \right\} < 0.$$

Let w be the position of the last column in which m is attained. Note that $w < q$. We will distinguish three cases. In each of them we define an interval Y of P, and using the following notation: $y = |Y|$, $k = \max\{j \mid (1, j) \in Y\}$ and $l = \max\{j \mid (2, j) \in Y\}$, we prove

$$\sum_{h=1}^{y} \alpha_h = \sum_{(i,j)\in Y} a_{ij}, \tag{7}$$

$$k > w \quad \text{and} \quad \sum_{j=1}^{l} e_{2j} = m. \tag{8}$$

Case 1: $(2, w) <_A (1, w + 1)$. Let $Y = I^A_{(1,w+1)}$. Then (7) follows from Lemma 3.4. Here $k = w + 1$ and $l = w$. Thus (8) holds.

Case 2: $(1, w + 1) <_A (2, w)$ *and either* a_{2w} *does not belong to an npp-block or it lies in the last column of an npp-block*. Let $Y = I^A_{(2,w)}$. Then (7) follows from Lemma 3.5. Here $k \geqslant w + 1$ and $l = w$. Thus (8) holds.

Case 3: $(1, w + 1) <_A (2, w)$ *and* a_{2w} *is in an npp-block-like* (3), *but not in the last column of the block*. Then (a_{2w}, a_{2w+1}) is either (d, d), $(d, d + 1)$ or $(d + 1, d + 1)$. By the definition of w we have $e_{2w} \leqslant 0$ and $e_{2w+1} > 0$. These inequalities, the identity $A = B + E$ and the fact that the first row of B is weakly decreasing imply that $(a_{2w}, a_{2w+1}) = (d, d + 1)$ and $(b_{2w}, b_{2w+1}) = (d, d)$. Then $e_{2w} = 0$. Let f denote the position of the first column of the npp-block, thus $f \leqslant w$. We claim that $b_{2s} = d$ for all $f \leqslant s \leqslant w$. This is shown by induction on s. It is true for $s = w$. Let $f \leqslant s < w$ and assume, by induction hypothesis, that $b_{2t} = d$ for all $s < t \leqslant w$. Then $e_{2t} = 0$ for all $s < t \leqslant w$, and $\sum_{j=1}^{s} e_{2j} = \sum_{j=1}^{w} e_{2j} = m$. Thus $e_{2s} \leqslant 0$. Since the first row of B is weakly decreasing, we must have $e_{2s} = 0$. Therefore $b_{2s} = a_{2s} = d$. Our claim follows. It implies $1 < f$.

If $a_{2f-1} \leqslant c$, we define $Y = I^A_{(2,f-1)}$. Then (7) follows from Lemma 3.5. Also $(1, s) \in Y$ for all $f \leqslant s \leqslant w + 1$. Hence $k > w$. Since $l = f - 1$ and $e_{2f} = \cdots = e_{2w} = 0$, (8) holds.

If $a_{2f-1} \geqslant c + 1$, we define $Y = I^A_{(1,w+1)}$. Then (7) follows from Lemma 3.4. Here $k = w + 1$ and $l = f - 1$, and as in the previous case (8) holds.

Then, in cases 1–3, we have, by (7) and (8), that

$$\begin{aligned}\sum_{h=1}^{y}\alpha_h &= \sum_{(i,j)\in Y} a_{ij}\\ &= \sum_{(i,j)\in Y} b_{ij} + \sum_{j=1}^{k} e_{1j} + \sum_{j=1}^{l} e_{2j}\\ &= \sum_{(i,j)\in Y} b_{ij} + m - \sum_{j=1}^{k} e_{2j}.\end{aligned}$$

Since $k > w$, we have $m - \sum_{j=1}^{k} e_{2j} < 0$. Therefore

$$\sum_{h=1}^{y}\alpha_h < \sum_{(i,j)\in Y} b_{ij} \leqslant \sum_{h=1}^{y}\beta_h,$$

and (6) holds. □

Remark 3.6. If $A \in \mathsf{M}(\lambda, \mu)$ is of type I, then

$$\lambda_1 - \lambda_2 \geqslant \left|\left\{1 \leqslant j \leqslant q \mid \mu_j \text{ is odd}\right\}\right|.$$

4. Matrices of type II

In this short section, we collect two results on matrices of type II needed in the proof of our main theorem. We assume that $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \ldots, \mu_q)$ are partitions of n. Recall that a matrix $A \in \mathsf{M}(\lambda, \mu)$ is of type II if it has at least one increasing column ($a_{1k} < a_{2k}$ for some $1 \leqslant k \leqslant q$) and $|a_{1j} - a_{2j}| \leqslant 1$ for all $1 \leqslant j \leqslant q$.

The first result is a particular case of Proposition 2.5.

Lemma 4.1. *If* $A \in \mathsf{M}(\lambda, \mu)$ *is of type* II, *then* A *is minimal. Moreover, if* B *is any other minimal matrix in* $\mathsf{M}(\lambda, \mu)$, *then* $\pi(\mathsf{c}_j(B)) = \pi(\mathsf{c}_j(A))$ *for all* $1 \leqslant j \leqslant q$, *and* B *is of type* II. *In particular* $\pi(A) = \pi(B)$.

Remark 4.2. If $A \in \mathsf{M}(\lambda, \mu)$ is of type II, then

$$\lambda_1 - \lambda_2 < \left|\left\{1 \leqslant j \leqslant q \mid \mu_j \text{ is odd}\right\}\right|.$$

Note that Remarks 3.6 and 4.2 imply that there cannot be simultaneously a matrix of type I and a matrix of type II in a set $\mathsf{M}(\lambda, \mu)$.

5. Proof of the main theorem

In this section, we conclude the proof of Theorem 1.1. We assume that $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \ldots, \mu_q)$ are partitions of n.

One direction of the proof of the theorem is contained in the following.

Proposition 5.1. *If A is minimal in $\mathsf{M}(\lambda, \mu)$, then A is either of type* I *or of type* II.

Proof. If A has at least one increasing column, then, since $\lambda_1 \geqslant \lambda_2$, it follows from Corollary 2.3 that $|a_{1j} - a_{2j}| \leqslant 1$ for all $1 \leqslant j \leqslant q$. Therefore A is of type II. Now suppose that all columns of A are weakly decreasing. Let $1 \leqslant j < k \leqslant q$. Then either

$$S = \begin{bmatrix} a_{1j} & a_{1k} \\ a_{2j} & a_{2k} \end{bmatrix}$$

is a plane partition or $a_{1j} < a_{1k}$ or $a_{2j} < a_{2k}$. If $a_{1j} < a_{1k}$, the condition $\mu_j \geqslant \mu_k$ implies $a_{2j} > a_{2k}$. Then by the corresponding result for columns of Corollary 2.3,

$$S = \begin{bmatrix} a_{1j} & a_{1j} + 1 \\ a_{2k} + 1 & a_{2k} \end{bmatrix}.$$

If $a_{2j} < a_{2k}$, then we show similarly that

$$S = \begin{bmatrix} a_{1k} + 1 & a_{1k} \\ a_{2j} & a_{2j} + 1 \end{bmatrix}.$$

Therefore A is of type I. □

For the other direction we know by Lemma 4.1 that matrices of type II are minimal. So it remains to show that matrices of type I are minimal. Let $A \in \mathsf{M}(\lambda, \mu)$ be of type I. Since $\mathsf{M}(\lambda, \mu)$ is a finite set, there exists a minimal matrix B in $\mathsf{M}(\lambda, \mu)$, such that $\pi(A) \trianglerighteq \pi(B)$. Proposition 5.1 implies that B is either of type I or of type II. Remarks 3.6 and 4.2 imply that B has to be of type I. Let $\bar{A}$, respectively, $\bar{B}$ be the standard form of A, respectively, of B; thus $\pi(\bar{A}) = \pi(A) \trianglerighteq \pi(B) = \pi(\bar{B})$. Then Theorem 3.3 implies $\bar{A} = \bar{B}$ and we conclude that A is minimal. This finishes the proof of Theorem 1.1.

6. Sets of uniqueness

Sets of uniqueness were introduced in [2] and minimal matrices were used in [6] to characterize them algebraically. In this section, we explain briefly how our classification of minimal matrices of size $2 \times q$ determine all sets of uniqueness contained in a box of size $2 \times q \times r$. We refer the interested reader to [6] for details. We need first:

Lemma 6.1. *If* $A \in \mathsf{M}(\lambda, \mu)$ *is a plane partition of size* $2 \times q$, *then there is no other* $B \in \mathsf{M}(\lambda, \mu)$ *with* $\pi(A) = \pi(B)$.

Proof. Let $B \in \mathsf{M}(\lambda, \mu)$ with $\pi(A) = \pi(B)$. Since A is a plane partition, A is of type I and therefore A is minimal by Theorem 1.1. Since $\pi(A) = \pi(B)$, B is also minimal. Then Theorem 1.1 and Remarks 3.6 and 4.2 imply that B is of type I. Let $\bar{B}$ be the standard form of B. Since $A = \bar{A}$, Lemma 3.2 implies $A = \bar{B}$. Thus $\bar{B}$ is a plane partition and $B = \bar{B} = A$. $\square$

For a natural number, let $[n] := \{1, \ldots, n\}$. Let $B(p, q, r) := [p] \times [q] \times [r]$ be a 3-dimensional box. A subset S of $B(p, q, r)$ is called a *pyramid* if for all $(a, b, c) \in S$ and all $(x, y, z) \in B(p, q, r)$ the conditions $x \leqslant a,\ y \leqslant b$ and $z \leqslant c$ imply $(x, y, z) \in S$.

Theorem 6.2. *Let S be a subset of the* 3*-dimensional box* $B(2, q, r)$ *and suppose S has weakly decreasing slice vectors. Then S is a set of uniqueness if and only if S is a pyramid.*

Proof. One implication is contained in Theorem $1'$ in [6] or in Corollary 3.2 in [8]. For the other, we assume that S is a pyramid and let $A \in \mathsf{M}(\lambda, \mu)$ be its associated plane partition, see [6, p. 446]. Since A is of type I, A is minimal. And it follows from Lemma 6.1 that there is no other $B \in \mathsf{M}(\lambda, \mu)$ with $\pi(A) = \pi(B)$. Then by Theorem $1'$ in [6], S is a set of uniqueness. $\square$

We note that Theorem 6.2 can also be proved using additive sets, see [8, Theorem 2].

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References

[1] T. Brylawski, The lattice of integer partitions, Discrete Math. 6 (1973) 201–219.
[2] P.C. Fishburn, J.C. Lagarias, J.A. Reeds, L.A. Shepp, Sets uniquely determined by projections on axes II. Discrete case, Discrete Math. 91 (1991) 149–159.
[3] P.C. Fishburn, L.A. Shepp, Sets of uniqueness and additivity in integer lattices, in: G.T. Herman, A. Kuba (Eds.), Discrete Tomography. Foundations, Algorithms and Applications, Birkhäuser, Boston, MA, 1999.

[4] G. Hardy, J.E. Littlewood, G. Pólya, Inequalities, 2nd ed., Cambridge University Press, London, 1952.
[5] A.W. Marshall, I. Olkin, Inequalities: Theory of Majorization and its Applications, Academic Press, New York, 1979.
[6] A. Torres-Cházaro, E. Vallejo, Sets of uniqueness and minimal matrices, J. Algebra 208 (1998) 444–451.
[7] E. Vallejo, Plane partitions and characters of the symmetric group, J. Algebraic Combin. 11 (2000) 79–88.
[8] E. Vallejo, A characterization of additive sets, 2000 (preprint).