Existence Theorems of Solutions for Certain Classes of Functional Equations Arising in Dynamic Programming

Zeqing Liu

Department of Mathematics, Liaoning Normal University, Dalian, Liaoning, 116029, People’s Republic of China

Submitted by William F. Ames

Received December 29, 1998

In this paper we study the existence, uniqueness, and iterative approximation of solutions for certain classes of functional equations arising in dynamic programming of multistage decision processes. The results presented here generalize, improve, and unify the corresponding results of other authors.

Key Words: dynamic programming; systems of functional equations; solution; coincidence solutions; fixed point.

1. INTRODUCTION

Bellman [2, 3] was the first to study the existence and uniqueness problems of solutions for certain classes of functional equations arising in dynamic programming. Bellman and Roosta [5] obtained an approximation solution for a class of the infinite-stage equation arising in dynamic programming.

As stated in Bellman and Lee [4], the basic form of the functional equations of dynamic programming is

\[ f(x) = \text{opt}_{y} H(x, y, f(T(x, y))), \quad (1.1) \]

where \( x \) and \( y \) represent the state and decision vectors, respectively, \( T \) represents the transformation of the process, and \( f(x) \) represents the optimal return function with initial state \( x \) (here opt denotes sup or inf).
As suggested in Wang [11–13] and Chang and Ma [9], the form (1.1) may be generalized to

\[ f(x) = \operatorname{opt}_y H(x, y, g(T(x, y))) \quad (1.2) \]

and

\[
\begin{align*}
f(x) &= \operatorname{opt}_y \left[ u(x, y) + G(x, y, g(T(x, y))) \right], \\
g(x) &= \operatorname{opt}_y \left[ u(x, y) + F(x, y, f(T(x, y))) \right]. \quad (1.3)
\end{align*}
\]

Baskaran and Subrahmanyan [1], Bhakta and Choudhury [6], and Bhakta and Mitra [7] considered the existence and uniqueness problems of solutions of functional equation (1.1) under certain assumptions. Chang [8] obtained the existence theorems of coincidence solutions of system of functional equations (1.3).

In this paper, we study the existence, uniqueness, and iterative approximation of solutions of functional equation (1.1) by using fixed point theorems and obtain the existence theorems of coincidence solutions for several new classes of systems of functional equations arising in dynamic programming of multistage decision processes. Our results generalize, improve, and unify the corresponding results of Bellman [2], Bellman and Roosta [5], Bhakta and Choudhury [6], Bhakta and Mitra [7], and Chang [8].

Throughout this paper, let \( N \) and \( \omega \) be the sets of positive and nonnegative integers, respectively, \( \mathbb{R} = (-\infty, \infty) \) and \( \mathbb{R}_+ = [0, \infty) \). Define

\[
\Phi_1 = \{ \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is nondecreasing, right continuous,} \}
\]

and \( \varphi(t) < t \) for \( t > 0 \}, \]

\[
\Phi_2 = \{ \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is nondecreasing and } \lim_{n \to \infty} \varphi^n(t) = 0 \text{ for } t > 0 \}, \]

\[
\Phi_3 = \{ \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is nondecreasing and } \sum_{n=0}^{\infty} \varphi^n(t) < \infty \text{ for } t > 0 \}, \]

\[
\Phi_4 = \{ \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is nondecreasing, right continuous,} \}
\]

and satisfies condition \((\ast)\)

\[
\text{where condition } (\ast) \text{ is defined as follows:}
\]

\[
(\ast) \quad \text{For any } a \in \mathbb{R}_+, \text{ there exists } c = c(a) \in \mathbb{R}_+ \text{ such that for all } t \in \mathbb{R}_+,
\]

\[
t \leq a + \varphi(t) \Rightarrow t \leq c \text{ and } \varphi^n(c) \to 0 \text{ as } n \to \infty.
\]
Clearly, \( \Phi_2 \subseteq \Phi_1 \cup \Phi_3 \) and \( \Phi_1 \supset \Phi_2 \). If \( r \in (0, 1) \) and if \( \varphi(t) = rt \) for all \( t \in \mathbb{R} \), then \( \varphi \in \Phi_3 \cap \Phi_3 \). The following result is evident.

**Lemma 1.1.** Let \( a, b, \) and \( c \) be in \( R \). Then
\[
|\text{opt}(a, c) - \text{opt}(b, c)| \leq |a - b|.
\]

## 2. Fixed Point Theorems

Let \( X \) be a nonempty set and let \( \{d_n\}_{n \in \mathbb{N}} \) be a countable family of pseudometrics on \( X \) such that for any distinct \( x, y \in X \), \( d_k(x, y) \neq 0 \) for some \( k \in \mathbb{N} \). Define
\[
d(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(x, y)}{1 + d_k(x, y)} \quad \text{for } x, y \in X.
\]

It is easy to see that \( d \) is a metric on \( X \). A sequence \( x \in \mathbb{N} \) converges to a point \( x \in X \) iff \( d(x, x) \to 0 \) as \( n \to \infty \) and \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence iff \( d_k(x_m, x_n) \to 0 \) as \( n, m \to \infty \) for each \( k \in \mathbb{N} \). Let \( f: (X, d) \to (X, d) \) be a mapping. For \( A \subseteq X \), \( x, y \in X \), and \( k, n \in \mathbb{N} \), let
\[
O_f(x, n) = \{f^i : 0 \leq i \leq n\}, \quad O_f(x) = \{f^i : i \in \omega\},
\]
\[
O_f(x, y) = O_f(x) \cup O_f(y), \quad \delta_k(A) = \sup\{d_k(a, b) : a, b \in A\}.
\]

\( X \) is said to be \( f \)-orbitally complete iff every Cauchy sequence that is contained in \( O_f(x) \) for \( x \in X \) converges in \( X \). In the rest of this section, we assume that the metric space \((X, d)\) is \( f \)-orbitally complete.

**Theorem 2.1.** Suppose that the following conditions hold.

(a1) For any \( k \in \mathbb{N} \) and \( x \in X \), there exists \( p(k, x) > 0 \) such that \( \delta_k(O_f(x)) \leq p(k, x) \).

(a2) There exists \( \varphi \in \Phi_1 \) such that
\[
d_k(f^k x, f^k y) \leq \varphi(\delta_k(O_f(x, y)))
\]

for any \( k \in \mathbb{N} \) and \( x, y \in X \).

Then \( f \) has a unique fixed point \( w \) in \( X \) and the iterative sequence \( \{f^n x\}_{n \in \mathbb{N}} \) converges to \( w \) for each \( x \in X \).

**Proof.** Let \( k \) be in \( \mathbb{N} \) and let \( x_0 \) be an arbitrary point of \( X \). Define \( x_0 = f^n x_0 \) and \( d_{k_1} = d_{k_1}(x_0) = \delta_k(O_f(x)) \) for \( n \in \mathbb{N} \) and \( i \in \omega \). Since \( \{d_{k_i}\}_{i \in \omega} \) is nonincreasing and bounded below by \( 0 \), it converges to a number \( r \geq 0 \). Suppose that \( r > 0 \). Given \( \varepsilon > 0 \), there exists \( j \in \mathbb{N} \) with...
$r \leq d_{ki} < r + \varepsilon$ for all $i > j$. It follows from (a1) that
\[
d_k(f^{n+1}x_0, f^{m+1}x_0) \leq \varphi\left(\delta_k\left(O_j(f^n x_0, f^m x_0)\right)\right) \\
\leq \varphi\left(\delta_k\left(O_j(f^{i+j} x_0)\right)\right) = \varphi(d_{ki+j})
\]
for all $n, m \geq i + j$. This implies that
\[
d_{ki+j+1} \leq \varphi(d_{ki+j}).
\]
Letting $j \to \infty$, by the right continuity of $\varphi$ we have
\[
0 < r \leq \varphi(r) < r,
\]
which is a contradiction. Therefore,
\[
\lim_{i \to \infty} d_{ki}(x_0) = r = 0 \tag{2.1}
\]
and $(f^n x_0)_{n \in N}$ is a Cauchy sequence. Thus $(f^n x_0)_{n \in N}$ converges to some point $w \in X$, because $(X, d)$ is $f$-orbitally complete.

Set $c_{kn} = \sup\{d_k(f^m x_0, f^m w) : m \geq n\}$ for all $n \in N$. Then $(c_{kn})_{n \in N}$ is nonincreasing and $c_{kn} \to s$ for some $s \geq 0$. We claim that $s = 0$; otherwise, $s > 0$. Note that $d_{kn}(x_0) \downarrow 0$, $d_{kn}(w) \downarrow 0$, and $c_{kn} \downarrow s$ as $n \to \infty$. Thus for any $\varepsilon > 0$, there exists $m \in N$ such that $s \leq c_{kn} < s + \varepsilon$, $d_{kn}(x_0) < \varepsilon$ and $d_{kn}(w) < \varepsilon$ for all $n \geq m$. Let $i, j \in \omega$. It follows from (a1) that
\[
d_k(f^{m+i+1}x_0, f^{m+j+1}w) \\
\leq \varphi\left(\delta_k\left(O_j(f^{m+i}x_0, f^{m+j}w)\right)\right) \\
\leq \varphi\left(\delta_k\left(O_j(f^m x_0, f^m w)\right)\right) \\
\leq \varphi(d_{km}(x_0) + d_{km}(w) + c_{km}),
\]
which implies that
\[
s \leq c_{km+1} \leq \varphi(3\varepsilon + s).
\]
Letting $\varepsilon \to 0$, we have
\[
0 < s \leq \varphi(s) < s,
\]
a contradiction. Hence $s = 0$ and $d_k(f^n x_0, f^n w) \to 0$ as $n \to \infty$.

We now prove that $d(f^n x_0, f^n w) \to 0$ as $n \to \infty$. Choose any $\varepsilon > 0$. Determine $p \in N$ with $\sum_{k=p+1}^{\infty} 1/2^k < \frac{1}{2}\varepsilon$. Whereas
\[
\sum_{k=1}^{p} \frac{1}{2^k} \cdot \frac{d_k(f^n x_0, f^n w)}{1 + d_k(f^n x_0, f^n w)} \to 0 \text{ as } n \to \infty,
\]
we can find some $M \in N$ such that for $n \geq M$, 
\[
\frac{p}{2^k} \cdot \frac{d_k(f^n x_0, f^n w)}{1 + d_k(f^n x_0, f^n w)} < \frac{1}{2} \varepsilon.
\]
This gives that for $n \geq M$,
\[
d(f^n x_0, f^n w) = \sum_{k=1}^{p} \frac{1}{2^k} \cdot \frac{d_k(f^n x_0, f^n w)}{1 + d_k(f^n x_0, f^n w)} \]
\[
+ \sum_{k=p+1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(f^n x_0, f^n w)}{1 + d_k(f^n x_0, f^n w)} \]
\[
< \frac{1}{2} \varepsilon + \frac{1}{2} \varepsilon = \varepsilon.
\]
That is, $d(f^n x_0, f^n w) \to 0$ as $n \to \infty$. Note that $f^n x_0 \to w$ as $n \to \infty$. It follows that $f^n w \to w$ as $n \to \infty$.

We next prove that for any $i \in N$,
\[
d_{k_0}(w) = d_{k_1}(w) = \cdots = d_{k_i}(w). \tag{2.2}
\]
Suppose that $d_{k_0}(w) > d_{k_i}(w)$. Note that
\[
d_{k_0}(w) = \max\{\sup\{d_k(f^n w, w) : n \in N\}, d_{k_1}(w)\}
\[
= \sup\{d_k(f^n w, w) : n \in N\}.
\]
So, for any $n > m \geq 1$ we have
\[
d_k(w, f^m w) \leq d_k(w, f^n w) + d_k(f^m w, f^n w)
\]
\[
\leq d_k(w, f^n w) + d_{k_1}(w).
\]
Letting $n \to \infty$, we get
\[
d_k(w, f^m w) \leq d_{k_1}(w)
\]
for all $m \in N$. This gives that
\[
d_{k_0}(w) \leq d_{k_1}(w),
\]
a contradiction. Hence $d_{k_0}(w) \leq d_k(w)$. Whereas $\{d_k(w)\}_{k \in N}$ is nonincreasing, so $d_{k_0}(w) = d_{k_1}(w)$. That is, (2.2) holds for $i = 1$. Suppose that (2.2) holds for some $i = j$. Now for $i = j + 1$, if
\[
d_{k_0}(w) = d_{k_1}(w) = \cdots = d_{k_j}(w) > d_{k_{j+1}}(w),
\]
then as in the preceding proof we have

$$d_{k_j}(w) = \max\{\sup\{d_k(f^n w, f^{j+i} w) : n > j\}, d_{k_{j+1}}(w)\}$$

$$= \sup\{d_k(f^n w, f^{j+i} w) : n > j\}. \quad (2.3)$$

It follows from (a2) that for all $i \in N$,

$$d_k(f^i w, f^{j+i} w) \leq \varphi(\delta_k(O_j(f^{j+i-1} w))) = \varphi(d_{k_{j-1}}(w)). \quad (2.4)$$

From (2.3) and (2.4) we have

$$0 < d_{k_j}(w) \leq \varphi(d_{k_{j-1}}(w)) < d_{k_{j-1}}(w),$$

a contradiction. Therefore, (2.2) holds for $i = j + 1$. Thus, by induction we conclude that (2.2) holds for all $i \in N$.

(2.1) and (2.2) ensure that $d_{k_0}(w) = 0$. It follows that

$$d(w, fw) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(w, fw)}{1 + d_k(w, fw)} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_{k_0}(w)}{1 + d_{k_0}(w)} = 0.$$

Therefore, $w = fw$. That is, $w$ is a fixed point of $f$. If $f$ has another fixed point $v \in X$, by (a2) we have

$$d_k(w, v) = d_k(fw, fv) \leq \varphi(\delta_k(O_j(w, v))) = \varphi(d_k(w, v)),$$

which implies that $d_k(w, v) = 0$. Hence

$$d(w, v) = \sum_{p=1}^{\infty} \frac{1}{2^p} \cdot \frac{d_k(w, v)}{1 + d_k(w, v)} = 0.$$

That is, $w = v$. Thus, $f$ has a unique fixed point $w \in X$. This completes the proof.

Remark 2.1. Theorem 2.1 generalizes Theorem 2.1 of Bhakta and Choudhury [6] and Theorem 1.1 of Bhakta and Mitra [7].

Theorem 2.2. Suppose that the following condition holds.

(a3) There exists $\varphi \in \Phi_2$ such that

$$d_k(f x, f y) \leq \varphi(\max\{d_k(x, y), d_k(x, f x), d_k(y, f y), d_k(x, f y), d_k(y, f x)\})$$

for any $k \in N$ and $x, y \in X$. 
Then

(i) \( f \) has a unique fixed point \( w \) in \( X \) and the iterative sequence \( \{f^n x\}_{n \in N} \) converges to \( w \) for each \( x \in X \).

(ii) \( d_k(f^n x_0, w) \leq \varphi^n(c(d_k(x_0, f x_0))) \) for all \( k, n \in N \).

Proof. Let \( k \) be in \( N \) and let \( x_0 \) be an arbitrary point of \( X \). Define \( x_n = f^n x_0 \) for all \( n \in N \). We first prove that for each \( n \in N \), there exists \( p \in \{1, 2, \ldots, n\} \) satisfying

\[
d_k(x_0, x_p) = \delta_k(O_f(x_0, n)).
\]  

(2.5)

For any \( i, j \in \{1, 2, \ldots, n\} \), by (a3) we have

\[
d_k(x_i, x_j) = d_k(f x_{i-1}, f x_{j-1})
\]

\[\leq \varphi\left(\max\{d_k(x_{i-1}, x_{j-1}), d_k(x_{i-1}, f x_{j-1}), d_k(x_{j-1}, f x_{j-1}),
\]

\[d_k(x_{i-1}, f x_{j-1}), d_k(x_{j-1}, f x_{j-1})\}\right)

\[\leq \varphi(\delta_k(O_f(x_0, n))).
\]

This means that there exists \( p \in \{1, 2, \ldots, n\} \) satisfying (2.5). Thus, by (a3) and (2.5) we have for each \( n \in N \),

\[
d_k(O_f(x_0, n)) = d_k(x_0, x_p) \leq d_k(x_0, x_1) + d_k(x_1, x_p)
\]

\[\leq d_k(x_0, x_1) + \varphi(\delta_k(O_f(x_0, n))).
\]  

(2.6)

Take \( a = d_k(x_0, x_1) \). Condition (\( \ast \)) and (2.6) ensure that there is \( c = c(a) \) \( \in R_+ \) such that \( \delta_k(O_f(x_0, n)) \leq c \) for all \( n \in N \). This gives

\[
\delta_k(O_f(x_0)) \leq c.
\]  

(2.7)

We now prove that \( \{x_n\}_{n \in N} \) is a Cauchy sequence. It follows from (a3), (2.5), and (2.7) that for any \( m > n \geq 1 \), there exist positive integers \( p_1, p_2, \ldots, p_n \) such that \( p_i \leq m - n + i \), \( \delta_k(O_f(x_{n-i}, m-n+i)) = d_k(x_{n-i}, f^p x_{n-i}) \) for \( i \in \{1, 2, \ldots, n\} \) and that

\[
d_k(x_n, x_m) = d_k(x_n, f^{m-n} x_n) = d_k(f x_{n-1}, f x_{m-1})
\]

\[\leq \varphi\left(\max\{d_k(x_{n-1}, x_{m-1}), d_k(x_{n-1}, x_n), d_k(x_{m-1}, x_m),
\]

\[d_k(x_{n-1}, x_m), d_k(x_{m-1}, x_n)\}\right)

\[\leq \varphi(\delta_k(O_f(x_{n-1}, m-n+1))) = \varphi(d_k(x_{n-1}, f^p x_{n-1}))
\]

\[\leq \varphi^2(\delta_k(O_f(x_{n-2}, m-n+2))) \leq \cdots
\]

\[\leq \varphi^n(\delta_k(O_f(x_0, m))) \leq \varphi^n(c).
\]  

(2.8)
Note that $\varphi^n(c) \to 0$ as $n \to \infty$. Therefore, $d_k(x_n, x_m) \to 0$ as $m, n \to \infty$. That is, $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence and it converges to some $w \in X$ because $(X, d)$ is $f$-orbitally complete.

We next prove that $w$ is a fixed point of $f$. It follows from (a3) that

$$d_k(w, f w) \leq d_k(w, f^{n+1}x_0) + d_k(f x_n, f w)$$
$$\leq d_k(w, x_{n+1}) + \varphi(\max\{d_k(x_n, w), d_k(x_n, x_{n+1}), d_k(w, f w), d_k(x_n, f w), d_k(w, x_{n+1})\})$$
$$\leq d_k(w, x_{n+1}) + \varphi(d_k(x_n, w) + d_k(x_n, x_{n+1}) + d_k(w, f w)).$$

Letting $n \to \infty$, by the right continuity of $\varphi$ we have

$$d_k(w, f w) \leq \varphi(d_k(w, f w)),$$

which implies that $d_k(w, f w) = 0$. So

$$d(w, f w) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(w, f w)}{1 + d_k(w, f w)} = 0.$$

Hence $w = f w$. That is, $w$ is a fixed point of $f$. The uniqueness of fixed points of $f$ follows from (a3).

It follows from (2.8) that for any $m > n \geq 1$,

$$d_k(x_n, w) \leq d_k(x_n, x_m) + d_k(x_m, w)$$
$$\leq \varphi^n(c(d_k(x_0, f x_0) + d_k(x_m, w)).$$

Letting $m \to \infty$, we have

$$d_k(f^n x_0, w) \leq \varphi^n(c(d_k(x_0, f x_0)))$$

for all $n \in \mathbb{N}$. This completes the proof.

**Remark 2.2.** Theorem 2.2 extends Theorem 2.1 of Bhakta and Choudhury [6], Theorem 1.1 of Bhakta and Mitra [7], and Theorem 1 of Ciric [10].

### 3. Existence and Uniqueness of Solutions

Throughout this section we assume that $(X, \| \cdot \|)$ and $(Y, \| \cdot \|')$ are real Banach spaces, $S \subset X$ is the state space, and $D \subset Y$ is the decision space. We denote by $\mathcal{B}(S)$ the set of all real-value mappings on $S$ that are bounded on bounded subsets of $S$. Clearly $\mathcal{B}(S)$ is a linear space over $\mathbb{R}$.
under usual definitions of addition and multiplication by scalars. For any
\( k \in N \) and \( a, b \in \text{BB}(S) \), let
\[
d_k(a, b) = \sup \{|a(x) - b(x)| : x \in \overline{B}(0, k)\},
\]
\[
d(a, b) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(a, b)}{1 + d_k(a, b)},
\]
where \( \overline{B}(0, k) = \{x : x \in S \text{ and } \|x\| \leq k\} \). It is easily verified that \( \{d_k\}_{k \in N} \)
is a countable family of pseudometrics on \( \text{BB}(S) \) and that \( \text{BB}(S), d \) is a complete metric space.

Let \( H \) be a mapping of \( S \times D \times \text{BB}(S) \) into \( R \), let \( T \) be a mapping of \( S \times D \) into \( S \), and let \( u \) be a mapping of \( S \times D \) into \( R \).

**Theorem 3.1.** Suppose that the following conditions hold.

(b1) For any \( k \in N \) and \( a \in \text{BB}(S) \), there exists \( p(k, a) > 0 \) such that for all \( (x, y) \in \overline{B}(0, k) \times D \),
\[
\max\{|u(x, y)| + |H(x, y, a(T(x, y)))|, \delta_k(O_f(a))\} \leq p(k, a),
\]
where the mapping \( f \) is defined as
\[
a(x, y) = \text{opt} \{u(x, y) + H(x, y, a(T(x, y)))\}, \quad x \in S, \ a \in \text{BB}(S).
\]
(3.1)

(b2) There exists \( \varphi \in \Phi_1 \) such that
\[
|H(x, y, a(t)) - H(x, y, b(t))| \leq \varphi(\delta_k(O_f(a, b)))
\]
for any \( k \in N, (x, y) \in \overline{B}(0, k) \times D, t \in S, \) and \( a, b \in \text{BB}(S) \).

Then the functional equation
\[
f(x) = \text{opt} \{u(x, y) + H(x, y, f(T(x, y)))\}, \quad x \in S, \quad (3.2)
\]
possesses a unique solution \( w \in \text{BB}(S) \) and \( \{f^n a\}_{n \in N} \) converges to \( w \) for each \( a \in \text{BB}(S) \).

**Proof.** Let \( k \) be in \( N \) and let \( a \) be in \( \text{BB}(S) \). It follows from (b1) that
\[
|u(x, y)| + |H(x, y, a(T(x, y)))| \leq p(k, a)
\]
for all \( (x, y) \in \overline{B}(0, k) \times D \). By (3.1) we immediately conclude that \( f \) maps \( \text{BB}(S) \) into itself.
Let $k$ be in $N$, let $a, b$ be in $BB(S)$, and let $x$ be in $S$. We now have to consider two cases:

**Case 1.** Suppose that $\text{opt} = \inf$. Then for any $\varepsilon > 0$, there exist $y, z \in D$ such that

$$fa(x) > u(x, y) + H(x, y, a(T(x, y))) - \varepsilon, \quad (3.3)$$

$$fb(x) > u(x, z) + H(x, z, a(T(x, z))) - \varepsilon. \quad (3.4)$$

Note that

$$fa(x) \leq u(x, z) + H(x, z, a(T(x, z))), \quad (3.5)$$

$$fb(x) \leq u(x, y) + H(x, y, a(T(x, y))). \quad (3.6)$$

By (3.4), (3.5), and (b2) we infer that

$$fa(x) - fb(x) < H(x, z, a(T(x, z))) - H(x, z, b(T(x, z))) + \varepsilon$$

$$\leq |H(x, z, a(T(x, z))) - H(x, z, b(T(x, z)))| + \varepsilon$$

$$\leq \varepsilon + \varphi\left(\delta_k(O_f(a, b))\right). \quad (3.7)$$

In view of (3.3), (3.6), and (b2), we get that

$$fa(x) - fb(x) > H(x, y, a(T(x, y))) - H(x, y, b(T(x, y))) - \varepsilon$$

$$\geq -|H(x, y, a(T(x, y))) - H(x, y, b(T(x, y)))| - \varepsilon$$

$$\geq -\varepsilon - \varphi\left(\delta_k(O_f(a, b))\right). \quad (3.8)$$

It follows from (3.7) and (3.8) that

$$|fa(x) - f(b(x))| \leq \varepsilon + \varphi\left(\delta_k(O_f(a, b))\right),$$

which implies that

$$d_k(fa, fb) \leq \varepsilon + \varphi\left(\delta_k(O_f(a, b))\right).$$

Letting $\varepsilon \to 0$, we obtain that

$$d_k(fa, fb) \leq \varphi\left(\delta_k(O_f(a, b))\right). \quad (3.9)$$

**Case 2.** Suppose that $\text{opt} = \sup$. As in the proof of Case 1, we can conclude that (3.9) holds also. Therefore, by Theorem 2.1, $f$ has a unique fixed point $w \in BB(S)$ and $\{f^n a\}_{n \in N}$ converges to $w$ for each $a \in BB(S)$. That is, $w(x)$ is a unique solution of the functional equation (3.2). This completes the proof.
THEOREM 3.2. Suppose that the following conditions hold.

(b3) For any \( k \in \mathbb{N} \) and \( a \in \mathbb{B}(S) \), there exists \( p(k, a) > 0 \) such that for all \( (x, y) \in \overline{B}(0, k) \times D \),
\[
|u(x, y)| + |H(x, y, a(T(x, y)))| \leq p(k, a).
\]

(b4) There exists \( \varphi \in \Phi_k \) such that
\[
|H(x, y, a(t)) - H(x, y, b(t))| \leq \varphi(\max\{d_k(a, b), d_k(a, fa), d_k(b, fb), d_k(a, fb), d_k(b, fa)\})
\]
for any \( k \in \mathbb{N} \), \( (x, y) \in \overline{B}(0, k) \times D \), \( t \in S \), and \( a, b \in \mathbb{B}(S) \), where the mapping \( f \) is defined by (3.1).

Then the functional equation (3.2) possesses a unique solution \( w \in \mathbb{B}(S) \) and \( \{f^n a\}_{n \in \mathbb{N}} \) converges to \( w \) for each \( a \in \mathbb{B}(S) \). Furthermore,
\[
d_k(f^n a, w) \leq \varphi^n(c(d_k(a, fa)))
\]
for all \( k, n \in \mathbb{N} \) and \( a \in \mathbb{B}(S) \).

Proof. As in the proof of Theorem 3.1, we similarly obtain that \( f \) maps \( \mathbb{B}(S) \) into itself and that
\[
d_k(f(a, fb)) \leq \varphi(\max\{d_k(a, b), d_k(a, fa), d_k(b, fb), d_k(a, fb), d_k(b, fa)\})
\]
for any \( k \in \mathbb{N} \) and \( a, b \in \mathbb{B}(S) \). Note that each fixed point of \( f \) is a solution of the functional equation (3.2) in \( \mathbb{B}(S) \) and, conversely, each solution of the functional equation (3.2) in \( \mathbb{B}(S) \) is a fixed point of \( f \).

Thus Theorem 3.2 follows immediately from Theorem 2.2. This completes the proof.

Remark 3.2. Theorems 3.1 and 3.2 extend both Theorem 3.1 of Bhakta and Choudhury [6] and Theorem 2.1 and Corollary 2.1 of Bhakta and Mitra [7].

THEOREM 3.3. Let \( h_i, v_i : S \times D \to R \) and \( T_i : S \times D \to S \) for \( i \in \{1, 2, \ldots, m\} \). Suppose that:

(b5) Given \( k \in \mathbb{N} \), there exists \( p(k) > 0 \) such that
\[
\max\{|u_i(x, y)|, |v_i(x, y)| : i \in \{1, 2, \ldots, m\}\} \leq p(k)
\]
for all \( (x, y) \in \overline{B}(0, k) \times D \);
(b6) \[
\max\{|T_i(x, y)| : i \in \{1, 2, \ldots, m\}\} \leq \|x\|
\]
for all \((x, y) \in S \times D\);
\[
\sum_{i=1}^{m} |h_i(x, y)| \leq r < 1
\]
for all \((x, y) \in S \times D\).

Then the functional equation
\[
f(x) = \operatorname{opt}_{y \in D} \left\{ u(x, y) + \sum_{i=1}^{m} h_i(x, y) \cdot \operatorname{opt}_{v \in D} \{ v_i(x, y), f(T_i(x, y)) \} \right\},
\]
\[
x \in S, \hspace{1cm} (3.10)
\]
possesses a unique solution \(w\) in \(BB(S)\) and the sequence \(\{a_n\}_{n \in \mathbb{N}}\) converges to \(w\) for each \(a_0 \in BB(S)\), where the sequence \(\{a_n\}_{n \in \mathbb{N}}\) is defined as
\[
a_n(x) = \operatorname{opt}_{y \in D} \left\{ u(x, y) + \sum_{i=1}^{m} h_i(x, y) \cdot \operatorname{opt}_{v \in D} \{ v_i(x, y), a_{n-1}(T_i(x, y)) \} \right\},
\]
\[
x \in S, \hspace{1cm} n \in \mathbb{N}.
\]
Furthermore,
\[
d_k(a_n, w) \leq \frac{r^n}{1 - r} \cdot d_k(a_0, a_1),
\]
for all \(k, n \in \mathbb{N}\).

Proof. Put
\[
H(x, y, a(T_i(x, y))) = \sum_{i=1}^{m} h_i(x, y) \cdot \operatorname{opt}_{v \in D} \{ v_i(x, y), a(T_i(x, y)) \},
\]
\[
f_a(x) = \operatorname{opt}_{y \in D} \left\{ u(x, y) + H(x, y, a(T_i(x, y))) \right\}
\]
for all \((x, y, a) \in S \times D \times BB(S)\). Taking \(k \in \mathbb{N}\), \((x, y) \in \overline{B}(0, k) \times D\), and \(a \in BB(S)\), by (b5), (b6), and (b7), we have
\[
|u(x, y)| + |H(x, y, a(T_i(x, y)))| \leq p(k) + \sum_{i=1}^{m} |h_i(x, y)| \cdot \operatorname{opt}_{v \in D} \{ |v_i(x, y)|, |a(T_i(x, y))| \}
\]
\[
\leq p(k) + r \left[ p(k) + \sup_{s \in \overline{B}(0, k)} |a(s)| \right].
\]
This means that
\[
\begin{align*}
\text{opt} \left\{ |u(x, y)| + |H(x, y, a(T_i(x, y)))| \right\} & \leq 2p(k) + \sup \{ |a(s)| : s \in \overline{B}(0, k) \}
\end{align*}
\]
for all \( k \in N, x \in \overline{B}(0, k), \) and \( a \in \text{BB}(S); \) that is, \( f \) maps \( \text{BB}(S) \) into itself. For \( k \in N, (x, y) \in \overline{B}(0, k) \times D, \) and \( a, b \in \text{BB}(S), \) by Lemma 1.1, (b6), and (b7), we get that
\[
|H(x, y, a(T_i(x, y))) - H(x, y, b(T_i(x, y)))| \leq \sum_{i=1}^{m} |h_i(x, y)| \cdot \text{opt} \{ v_i(x, y), a(T_i(x, y)) \}
\]
for all \( x, y \in \overline{B}(0, k), \) where \( \varphi(t) = rt \in \Phi_S. \) Thus Theorem 3.3 follows from Theorem 3.2. This completes the proof.

From Theorem 3.3 we have

**THEOREM 3.4.** Let \( h_i : S \times D \to R \) and \( T_i : S \times D \to S \) for \( i \in \{1, 2, \ldots, m\}. \) Suppose that (b6), (b7), and the following condition (b8) hold.

(b8) Given \( k \in N, \) there exists \( p(k) > 0 \) such that
\[
|u(x, y)| \leq p(k)
\]
for all \( (x, y) \in \overline{B}(0, k) \times D. \)

Then the functional equation
\[
f(x) = \text{opt} \left\{ u(x, y) + \sum_{i=1}^{m} h_i(x, y) \cdot f(T_i(x, y)) \right\}, \quad x \in S, \quad (3.11)
\]
possesses a unique solution \( w \) in \( \text{BB}(S) \) and the sequence \( \{a_n\}_{n \in N} \) converges to \( w \) for each \( a_0 \in \text{BB}(S), \) where the sequence \( \{a_n\}_{n \in N} \) is defined as
\[
a_n(x) = \text{opt} \left\{ u(x, y) + \sum_{i=1}^{m} h_i(x, y) \cdot a_{n-1}(T_i(x, y)) \right\}, \quad x \in S, \quad n \in N.
\]
Furthermore,
\[ d_k(a_n, w) \leq \frac{r^n}{1-r} \cdot d_k(a_0, a_1) \]
for all \( k, n \in \mathbb{N} \).

Remark 3.3. Theorem 3.4 extends and improves Theorem 3.3 of Bhakta and Choudhury [6] and the results of Bellman [2, p. 124] and Bellman and Roosta [5, p. 545].

**Theorem 3.5.** Suppose that the following conditions hold.

\[(9)\]
\[ |u(x, y)| \leq M|x| \]
for all \((x, y) \in S \times D\), where \( M \) is a positive constant.

\[(10)\]
\[ \|T(x, y)\| \leq \varphi(\|x\|) \]
for all \((x, y) \in S \times D\), where \( \varphi \in \Phi_2 \).

Then the functional equation
\[ f(x) = \inf_{y \in D} \max\{u(x, y), f(T(x, y))\}, \quad x \in S, \quad (3.12) \]
possesses a solution \( w \in BB(S) \) that satisfies the following conditions:

\[(11)\]
The sequence \( \{a_n\}_{n \in \mathbb{N}} \) converges to \( w \), where the sequence \( \{a_n\}_{n \in \mathbb{N}} \) is defined as
\[ a_0(x) = \inf_{y \in D} u(x, y), \]
\[ a_n(x) = \inf_{y \in D} \max\{u(x, y), a_{n-1}(T(x, y))\}, \quad n \in \mathbb{N}, \ x \in S. \quad (3.13) \]

\[(12)\]
If \( x_0 \in S, \ (y_n)_{n \in \mathbb{N}} \subset D \), and \( x_n = T(x_{n-1}, y_n) \) for all \( n \in \mathbb{N} \), then
\[ \lim_{n \to \infty} w(x_n) = 0. \quad (b12) \]

\[(13)\]
For all \( x \in S \),
\[ w(x) \geq 0. \quad (b13) \]

Furthermore, the solution \( w \) in \( BB(S) \) of the functional equation (3.12) is also unique with respect to condition (b12).

**Proof.** For \((x, y, a) \in S \times D \times BB(S)\). Set
\[ H(x, y, a) = \max\{u(x, y), a(T(x, y))\}, \]
\[ fa(x) = \inf_{y \in D} H(x, y, a). \]

\[ \]
It is easy to see that \( f \) maps \( \mathbb{B}B(S) \) into itself. Let \( k \in \mathbb{N}, (x, y) \in \bar{B}(0, k) \times D \), and \( a, b \in \mathbb{B}B(S) \). Then proceeding as in Theorem 3.1, we infer that
\[
d_k(fa, fb) \leq d_k(a, b),
\]
which implies that
\[
d(fa, fb) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(fa, fb)}{1 + d_k(fa, fb)} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(a, b)}{1 + d_k(a, b)} = d(a, b). \quad (3.14)
\]

It follows from (3.13), (b9), and (b10) that for \( x \in S \) and \( n \in \omega \),
\[
|a_0(x)| \leq \inf_{y \in D} |u(x, y)| \leq M\|x\|,
\]
\[
|a_1(x)| = |fa_0(x)| \leq \inf_{y \in D} |H(x, y, a_0)| \leq \inf_{y \in D} \max\{M\|x\|, M\|T(x, y)\|\} \leq \inf_{y \in D} \max\{M\|x\|, M\varphi(\|x\|)\} \leq M\|x\|,
\]

\[
|a_n(x)| = |fa_{n-1}(x)| \leq \inf_{y \in D} \{M\|x\|, M\|T(x, y)\|\} \leq \inf_{y \in D} \{M\|x\|, M\varphi(\|x\|)\} = M\|x\|;
\]

that is,
\[
|a_n(x)| \leq M\|x\| \quad (3.15)
\]
for \( x \in S \) and \( n \in \omega \).

We now show by induction that for \( x \in S \),
\[
a_0(x) \leq a_1(x) \leq a_2(x) \leq \cdots \leq a_n(x) \leq a_{n+1}(x) \leq \cdots \quad (3.16)
\]
Clearly,
\[
a_0(x) = \inf_{y \in D} u(x, y) \leq \inf_{y \in D} H(x, y, a_0) = a_1(x).
\]
Assume that for some \( n \in N \) and all \( x \in S \),
\[
a_{n-1}(x) \leq a_n(x).
\]

Note that for fixed \((x, y) \in S \times D\), \( H(x, y, \cdot) \) is nondecreasing. Thus we infer easily that
\[
a_n(x) = \inf_{y \in D} H(x, y, a_{n-1}) \leq \inf_{y \in D} H(x, y, a_n) = a_{n+1}(x).
\]

Hence, by the principle of finite induction, (3.16) holds.

We next show that \( \{a_n\}_{n \in N} \) is a Cauchy sequence in \( (BB(S), d) \). Take \( k \in N \) and \( x \in B(0, k) \). For any \( \varepsilon > 0 \) and \( n, p \in N \), by (3.13), we can find \( y_1 \in D \) and \( x_1 = T(x, y_1) \in S \) such that
\[
a_n(x) > H(x, y_1, a_{n-1}) - \frac{1}{2} \varepsilon, \quad a_{n+p}(x) \leq H(x, y_1, a_{n+p-1}). \tag{3.17}
\]

It follows from Lemma 1.1, (3.16) and (3.17) that
\[
0 \leq a_{n+p}(x) - a_n(x) < H(x, y_1, a_{n+p-1}) - H(x, y_1, a_{n-1}) + \frac{1}{2} \varepsilon
\]
\[
= \max\{u(x, y_1), a_{n+p-1}(x_1)\} - \max\{u(x, y_1), a_{n-1}(x_1)\} + \frac{1}{2} \varepsilon
\]
\[
\leq a_{n+p-1}(x_1) - a_{n-1}(x_1) + \frac{1}{2} \varepsilon;
\]
that is,
\[
0 \leq a_{n+p}(x) - a_n(x) < a_{n+p-1}(x_1) - a_{n-1}(x_1) + \frac{1}{2} \varepsilon. \tag{3.18}
\]

Proceeding in this way, we can find \( y_i \in D \), \( x_i = T(x_{i-1}, y_i) \in S \), \( i \in \{2, 3, \ldots, n\} \) such that
\[
0 \leq a_{n+p-1}(x_1) - a_{n-1}(x_1) < a_{n+p-2}(x_2) - a_{n-2}(x_2) + \frac{1}{2^2} \varepsilon,
\]
\[
0 \leq a_{n+p-2}(x_2) - a_{n-2}(x_2) < a_{n+p-3}(x_3) - a_{n-3}(x_3) + \frac{1}{2^3} \varepsilon,
\]
\[
\vdots
\]
\[
0 \leq a_{p+1}(x_{n-1}) - a_1(x_{n-1}) < a_p(x_n) - a_0(x_n) + \frac{1}{2^n} \varepsilon.
\tag{3.19}
\]

In view of (3.18) and (3.19), we immediately infer that
\[
0 \leq a_{n+p}(x) - a_n(x) < a_p(x_n) - a_0(x_n) + \varepsilon. \tag{3.20}
\]
By virtue of (3.15), (3.20), and (b10), we have

\[
0 \leq a_{n+p}(x) - a_n(x) \leq |a_p(x_n)| + |a_0(x_n)| + \varepsilon \\
\leq 2M\|x_n\| + \varepsilon = 2M\|T(x_{n-1}, y_n)\| + \varepsilon \\
\leq 2M\varphi(\|x_{n-1}\|) + \varepsilon \leq 2M\varphi^2(\|x_{n-2}\|) + \varepsilon \\
\vdots \\
\leq 2M\varphi^n(\|x\|) + \varepsilon \leq 2M\varphi^n(k) + \varepsilon.
\]

This gives that

\[
d_k(a_{n+p}, a_n) \leq 2M\varphi^n(k) + \varepsilon.
\]

Letting \( \varepsilon \to 0 \), we have

\[
d_k(a_{n+p}, a_n) \leq 2M\varphi^n(k) \to 0 \quad \text{as } n \to \infty.
\]

Thus, \( \{a_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \((\mathbb{B}(S), d)\) and it converges to some \( w \in \mathbb{B}(S) \) because \((\mathbb{B}(S), d)\) is complete. It follows from (3.14) that for any \( n \in \mathbb{N} \),

\[
d( fw, w) \leq d( fw, fa_n) + d( a_{n+1}, w) \\
\leq d( w, a_n) + d( a_{n+1}, w) \to 0 \quad \text{as } n \to \infty.
\]

Therefore \( w = fw \); that is, the functional equation (3.12) possesses a solution \( w \in \mathbb{B}(S) \).

We now show that (b12) holds. Let \( x_0 \in S \), \( \{y_n\}_{n \in \mathbb{N}} \) and \( x_n = T(x_{n-1}, y_n) \) for all \( n \in \mathbb{N} \). It is easy to show that

\[
\|x_n\| \leq \varphi^n(\|x_0\|) \to 0 \quad \text{as } n \to \infty. \tag{3.21}
\]

Let \( \varepsilon > 0 \). Take \( k = \lceil\|x_0\|\rceil + 1 \), where \( \lceil t \rceil \) denotes the largest integers not exceeding \( t \). Then \( x_n \in \overline{B}(0, k) \) for all \( n \in \omega \). By (3.15), we have

\[
|w(x_n)| \leq |w(x_n) - a_i(x_n)| + |a_i(x_n)| \leq d_k(w, a_i) + M\|x_n\|. \tag{3.22}
\]

It follows from \( \lim_{n \to \infty} d_k(w, a_i) = 0 \) that there is \( i \in N \) with \( d_k(w, a_i) < \frac{1}{2}\varepsilon \). Using (3.21), we can find \( m \in \mathbb{N} \) such that \( \|x_n\| \leq \frac{1}{2}\varepsilon \) for all \( n \geq m \). In view of (3.22), we infer that \( |w(x_n)| < \varepsilon \) for all \( n \geq m \). That is, \( \lim_{n \to \infty} w(x_n) = 0 \).

We next show that (b13) holds. Let \( x \in S \) and \( \varepsilon > 0 \). Whereas \( w(x) = \inf_{y \in D} \max(\mu(x, y), w(T(x, y))) \), so there are \( y_1 \in D \) and \( x_1 = T(x, y_1) \in D \).
such that
\[ w(x) > \max\{u(x, y_1), w(x_1)\} - \frac{1}{2} \varepsilon. \]  (3.23)

Similarly, there are also \( y_2, y_3, \ldots, y_n \in D \) and \( x_2 = T(x_1, y_2), x_3 = T(x_2, y_3), \ldots, x_n = T(x_{n-1}, y_n) \in S \) such that
\[ w(x_1) > \max\{u(x_1, y_2), w(x_2)\} - \frac{1}{2^n} \varepsilon, \]
\[ w(x_2) > \max\{u(x_2, y_3), w(x_3)\} - \frac{1}{2^n} \varepsilon, \]  (3.24)
\[ \vdots \]
\[ w(x_{n-1}) > \max\{u(x_{n-1}, y_n), w(x_n)\} - \frac{1}{2^n} \varepsilon. \]

Substituting (3.24) into (3.23), we have
\[ w(x) > w(x_1) - \frac{1}{2} \varepsilon > \max\{u(x_1, y_2), w(x_2)\} - \left( \frac{1}{2} + \frac{1}{2^2} \right) \varepsilon \]
\[ \geq w(x_2) - \left( \frac{1}{2} + \frac{1}{2^2} \right) \varepsilon > \cdots > w(x_n) - \left( \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \right) \varepsilon \]
\[ = w(x_n) - \left( 1 - \frac{1}{2^n} \right) \varepsilon. \]

Letting \( n \to \infty \), by (b12) we infer that
\[ w(x) \geq -\varepsilon. \]

Whereas \( \varepsilon > 0 \) is arbitrary, it follows that
\[ w(x) \geq 0. \]

We last show that \( w \) is a unique solution of the functional equation (3.12) in \( BB(S) \) satisfying condition (b12). Suppose that \( v \) is another solution of the functional equation (3.12) in \( BB(S) \) satisfying condition (b12). Let \( x \in S \) and \( \varepsilon > 0 \). Using the same argument as before, we can find sequences \( \{y_n\}_{n \in N}, \{z_n\}_{n \in N} \subset D, \{x_n\}_{n \in \omega^*}, \) and \( \{t_n\}_{n \in \omega} \subset S \), where \( x_n = T(x_{n-1}, y_n), t_n = T(t_{n-1}, z_n) \) for \( n \in \omega \) and \( x_0 = t_0 = x \), such that for any \( i \in \omega, \)
\[ w(x_i) > \max\{u(x_i, y_{i+1}), w(x_{i+1})\} - \frac{1}{2^{i+1}} \varepsilon, \]  (3.25)
\[ v(t_i) > \max\{u(t_i, z_{i+1}), v(t_{i+1})\} - \frac{1}{2^{i+1}} \varepsilon. \]  (3.26)
Clearly, for each $i \in \omega$ we have

$$w(t_i) \leq \max\{u(t_i, z_{i+1}), w(t_{i+1})\}, \quad (3.27)$$

$$v(x_i) \leq \max\{u(x_i, y_{i+1}), v(x_{i+1})\}. \quad (3.28)$$

It follows from (3.25) that

$$w(x_0) > \max\left\{u(x_0, y_1), \max\{u(x_1, y_2), w(x_2)\} - \frac{1}{2^2} \epsilon \right\} - \frac{1}{2} \epsilon$$

$$\geq \max\{u(x_0, y_1), u(x_1, y_2), w(x_2)\} - \left(\frac{1}{2} + \frac{1}{2^2}\right) \epsilon$$

$$\geq \max\{u(x_0, y_1), u(x_1, y_2), \max\{u(x_2, y_3), w(x_3)\} - \frac{1}{2^3}\right\}$$

$$- \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}\right) \epsilon$$

$$\vdots$$

$$\geq \max\{u(x_0, y_1), u(x_1, y_2), \ldots, u(x_{n-1}, y_n), w(x_n)\}$$

$$- \left(\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}\right) \epsilon$$

$$= \max\{u(x_0, y_1), u(x_1, y_2), \ldots, u(x_{n-1}, y_n), w(x_n)\}$$

$$- \left(1 - \frac{1}{2^n}\right) \epsilon;$$

that is,

$$w(x_0) > \max\{u(x_0, y_1), u(x_1, y_2), \ldots, u(x_{n-1}, y_n), w(x_n)\} - \left(1 - \frac{1}{2^n}\right) \epsilon. \quad (3.29)$$
In view of (3.26), (3.27), and (3.28), we similarly conclude that

\[ v(t_0) > \max \{ u(t_0, z_1), u(t_1, z_2), \ldots, u(t_{n-1}, z_n), v(t_n) \} - \left( 1 - \frac{1}{2^n} \right) \epsilon, \]

(3.30)

\[ w(t_0) \leq \max \{ u(t_0, z_1), u(t_1, z_2), \ldots, u(t_{n-1}, z_n), w(t_n) \}, \]

(3.31)

\[ v(x_0) \leq \max \{ u(x_0, y_1), u(x_1, y_2), \ldots, u(x_{n-1}, y_n), v(x_n) \}. \]

(3.32)

Subtracting (3.29) and (3.32), by Lemma 1.1 we obtain that

\[
w(x_0) - v(x_0) \\
> \max \{ u(x_0, y_1), u(x_1, y_2), \ldots, u(x_{n-1}, y_n), w(x_n) \} - \left( 1 - \frac{1}{2^n} \right) \epsilon \\
- \max \{ u(x_0, y_1), u(x_1, y_2), \ldots, u(x_{n-1}, y_n), v(x_n) \} \\
\geq - |w(x_n) - v(x_n)| - \left( 1 - \frac{1}{2^n} \right) \epsilon \\
\geq - |w(x_n) - v(x_n)| - |w(t_n) - v(t_n)| - \left( 1 - \frac{1}{2^n} \right) \epsilon.
\]

(3.33)

From Lemma 1.1, (3.30), and (3.31) we have

\[
w(t_0) - v(t_0) \\
< \max \{ u(t_0, z_1), u(t_1, z_2), \ldots, u(t_{n-1}, z_n), w(t_n) \} + \left( 1 - \frac{1}{2^n} \right) \epsilon \\
- \max \{ u(t_0, z_1), u(t_1, z_2), \ldots, u(t_{n-1}, z_n), v(t_n) \} \\
\leq |w(t_n) - v(t_n)| + \left( 1 - \frac{1}{2^n} \right) \epsilon \\
\leq |w(x_n) - v(x_n)| + |w(t_n) - v(t_n)| + \left( 1 - \frac{1}{2^n} \right) \epsilon.
\]

(3.34)

(3.33) and (3.34) ensure that

\[ |w(x) - v(x)| < |w(x_n) - v(x_n)| + |w(t_n) - v(t_n)| + \left( 1 - \frac{1}{2^n} \right) \epsilon. \]

Letting \( n \to \infty \), we get that

\[ |w(x) - v(x)| \leq \epsilon. \]
As $ε → 0$, we immediately infer that

$$w(x) = v(x).$$

This completes the proof.

Remark 3.4. Theorem 3.5 generalizes and improves Theorem 3.5 of Bhakta and Choudhury [6] and a result of Bellman [2, p. 149].

Question 3.1. If the functional equation (3.12) is replaced by the functional equation

$$f(x) = \inf_{y \in D} \min\{u(x, y), f(T(x, y))\}, \quad x \in S, \quad (3.35)$$

and the other assumptions of Theorem 3.5 do not change, do the conclusions of Theorem 3.5 hold?

4. Existence theorems of coincidence solutions

In this section we assume that $(X, \| \cdot \|)$ and $(Y, \| \cdot \|)$ are real Banach spaces, $S \subset X$ is the state space, and $D \subset Y$ is the decision space. Let $M, T: S \times D \to S$, $F, G: S \times D \times R \to R$, and $u, v: S \times D \to R$. We denote by $B(S)$ the set of all real-value bounded mappings on $S$. For any $a, b \in B(S)$, let

$$d(a, b) = \sup\{|a(x) - b(x)|: x \in S\}.$$

It is easy to verify that $(B(S), d)$ is a complete metric space. $\hat{f}, \hat{g}$ are called coincidence solutions of the system of functional equations (4.1),

\begin{align*}
\hat{f}(x) &= \text{opt}_{y \in D} \{v(x, y) + G(x, y, g(M(x, y)))\}, \quad x \in S, \\
\hat{g}(x) &= \text{opt}_{y \in D} \{u(x, y) + F(x, y, f(T(x, y)))\}, \quad x \in S, \quad (4.1)
\end{align*}

if the following conditions are satisfied:

\begin{align*}
\hat{f}(x) &= \text{opt}_{y \in D} \{v(x, y) + G(x, y, \hat{g}(M(x, y)))\}, \quad x \in S, \\
\hat{g}(x) &= \text{opt}_{y \in D} \{u(x, y) + F(x, y, \hat{f}(T(x, y)))\}, \quad x \in S.
\end{align*}
Theorem 4.1. Suppose that the following conditions hold.

1. \( \max(\|M(x, y)\|, \|T(x, y)\|) \leq \varphi(\|x\|) \) for all \((x, y) \in S \times D\),
2. \( \max(|u(x, y)|, |v(x, y)|) \leq \|x\| \) for all \((x, y) \in S \times D\).
3. \( 0 \leq F(x, y, z) \leq |z|, \|G(x, y, z)\| \leq |z| \) for all \((x, y, z) \in S \times D \times R\).
4. For fixed \((x, y) \in S \times D\), \(G(x, y, \cdot)\) and \(F(x, y, \cdot)\) are non-increasing, \(G(x, y, \cdot)\) is left continuous, and \(F(x, y, \cdot)\) is right continuous on \(R\).

Then the system of functional equations

\[
f(x) = \inf_{y \in D} \{v(x, y) + G(x, y, g(M(x, y)))\}, \quad x \in S,
\]

\[
g(x) = \sup_{y \in D} \{u(x, y) + F(x, y, f(T(x, y)))\}, \quad x \in S,
\]

possesses coincidence solutions in \(B(S)\).

Proof. For any \(x \in S\), put

\[
g_0(x) = \sup_{y \in D} u(x, y),
\]

\[
g_{2n}(x) = \sup_{y \in D} \{u(x, y) + F(x, y, f_{2n-1}(T(x, y)))\}, \quad n \in N, \quad (4.3)
\]

\[
f_{2n+1}(x) = \inf_{y \in D} \{v(x, y) + G(x, y, g_{2n}(M(x, y)))\}, \quad n \in \omega.
\]

From (c3) and (c4) we easily conclude that for all \(x \in S\) and \(n \in \omega\),

\[
g_0(x) \leq g_2(x) \leq \cdots \leq g_{2n}(x) \leq g_{2n+2}(x) \leq \cdots, 
\]

\[
f_1(x) \geq f_3(x) \geq \cdots \geq f_{2n+1}(x) \geq f_{2n+3}(x) \geq \cdots. \quad (4.4)
\]

We now claim that for each \(x \in S\), the sequences \(\{g_{2n}(x)\}_{n \in \omega}\) and \(\{f_{2n+1}(x)\}_{n \in \omega}\) are bounded. Let \(x \in S\) and take \(k \in N\) with \(\|x\| \leq k\). (c2) ensures that \(|g_0(x)| \leq \|x\|\). Thus, by (4.3), (c1), (c2), and (c3) we have

\[
|f_{1}(x)| \leq |v(x, y)| + |G(x, y, g_0(M(x, y)))| 
\]

\[
\leq \|x\| + |g_0(M(x, y))| \leq \|x\| + \|M(x, y)\| 
\]

\[
\leq \sum_{i=0}^{1} \varphi'(\|x\|).
\]
Whereas
\[ |u(x, y)| + |F(x, y, f_i(T(x, y)))| \leq \|x\| + |f_i(T(x, y))| \leq \|x\| + \sum_{i=0}^{1} \varphi^i(\|T(x, y)\|) \leq \sum_{i=0}^{2} \varphi^i(\|x\|), \]
it implies that
\[ |g_2(x)| \leq \sum_{i=0}^{2} \varphi^i(\|x\|). \]

Proceeding in this way we obtain that
\[ |g_{2n}(x)| \leq \sum_{i=0}^{2n} \varphi^i(\|x\|) \leq \sum_{i=0}^{\infty} \varphi^i(k), \quad n \in \mathbb{N}, \]
\[ |f_{2n+1}(x)| \leq \sum_{i=0}^{2n+1} \varphi^i(\|x\|) \leq \sum_{i=0}^{\infty} \varphi^i(k), \quad n \in \omega. \]

Therefore, \( (g_{2n}(x))_{n \in \omega} \) and \( (f_{2n+1}(x))_{n \in \omega} \) are bounded. It follows from (4.4) and (4.5) that \( (g_{2n}(x))_{n \in \omega} \) and \( (f_{2n+1}(x))_{n \in \omega} \) converge to \( g(x) \) and \( f(x) \), respectively. In view of (4.5), we infer that \( \max ||g(x)||, |f(x)|| \leq \sum_{i=0}^{\infty} \varphi^i(k) \); that is, \( g, f \in B(S) \). Define
\[ A(x) = \inf_{y \in D} \{ \nu(x, y) + G(x, y, g(M(x, y))) \}, \quad x \in S, \]
\[ B(x) = \sup_{y \in D} \{ \nu(x, y) + F(x, y, f(T(x, y))) \}, \quad x \in S. \] (4.6)

By virtue of (4.3), (4.4), and (4.6), we conclude that for all \( n \in \mathbb{N} \) and \( x \in S \),
\[ u(x, y) + F(x, y, f_{2n-1}(T(x, y))) \leq g_{2n}(x) \leq g(x), \]
\[ \nu(x, y) + G(x, y, g_{2n}(M(x, y))) \geq f_{2n+1}(x) \geq f(x). \]

Letting \( n \to \infty \), by the left and right continuity of \( G \) and \( F \), respectively, we infer that for all \( x \in S \),
\[ u(x, y) + F(x, y, f(T(x, y))) \leq g(x), \]
\[ \nu(x, y) + G(x, y, g(M(x, y))) \geq f(x), \]
which imply that for all \( x \in S \),
\[ B(x) \leq g(x) \quad \text{and} \quad A(x) \geq f(x). \] (4.7)
In view of (4.4), (4.6), and (c4), we obtain that for all \((x, y) \in S \times D\) and \(n \in N\),

\[
A(x) \leq v(x, y) + G(x, y, g(M(x, y))) \\
\leq v(x, y) + G(x, y, g_{2n}(M(x, y))),
\]

which implies that

\[
A(x) \leq \inf_{y \in D} \{v(x, y) + G(x, y, g_{2n}(M(x, y)))\} = f_{2n+1}(x)
\]

for all \(n \in N\) and \(x \in S\). As \(n \to \infty\), we have

\[A(x) \leq f(x), \quad x \in S.\]  (4.8)

Note that for all \((x, y) \in S \times D\) and \(n \in N\),

\[
B(x) \geq u(x, y) + F(x, y, f(T(x, y))) \\
\geq u(x, y) + F(x, y, f_{2n-1}(T(x, y))).
\]

This gives that

\[
B(x) \geq \sup_{y \in D} \{u(x, y) + F(x, y, f_{2n-1}(T(x, y)))\} = g_{2n}(x), \quad x \in S.
\]

Letting \(n \to \infty\), we get that

\[B(x) \geq g(x), \quad x \in S.\]  (4.9)

By virtue of (4.7), (4.8), and (4.9), we immediately infer that

\[B(x) = g(x) \quad \text{and} \quad A(x) = f(x), \quad x \in S.\]

It follows from (4.6) that \(f\) and \(g\) are coincidence solutions of the system of functional equations (4.2). This completes the proof.

By virtue of the method used to prove Theorem 4.1, we can similarly conclude the following result.

**Theorem 4.2.** Suppose that conditions (c1), (c2), and (c3) in Theorem 4.1 and the following condition (c5) hold.

(c5) For fixed \((x, y) \in S \times D\), \(G(x, y, \cdot)\) and \(F(x, y, \cdot)\) are both non-decreasing and left continuous on \(R\).
Then the system of functional equations

\[
\begin{align*}
    f(x) &= \sup_{y \in D} \{ u(x, y) + G(x, y, g(M(x, y))) \}, \\
    g(x) &= \sup_{y \in D} \{ u(x, y) + F(x, y, f(T(x, y))) \},
\end{align*}
\]

possesses coincidence solutions in \( B(S) \).

Remark 4.1. Theorem 4.2 extends and improves Theorem 2.3 of Bhakta and Mitra [7] and Theorem 4.1 of Chang [8].

REFERENCES