NOTE

Fissions of Classical Self-Dual Association Schemes

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We describe fission schemes of most known classical self-dual association schemes, such as the Hamming scheme $H(n, q)$ when $q$ is a prime power. These fission schemes are themselves self-dual, with the exception of certain quadratic forms schemes in even characteristic.

1. THE HAMMING SCHEME

The Hamming scheme $H(n, q)$ is defined on vertex set $X^n$ of words of length $n$ from an alphabet $X$ of size $q$. Two words are in relation $R_i$ if and only if they differ in precisely $i$ positions (we also say the words have distance $i$). If $q$ is a prime power, then we can put extra structure on the set $X$, and this allows us to refine one of the relations of the Hamming scheme. For some background in the theory of association schemes, we refer the reader to [1, Chap. 2].

THEOREM 1. Let $q$ be a prime power. Then the Hamming scheme $H(n, q)$ on vertex set $GF(q)^n$ has a self-dual fission scheme obtained by splitting the distance-$n$ relation into $q-1$ relations according to the value of $\prod_{i=1}^n (b_i - a_i)$, for $a, b \in GF(q)^n$ such that $wt(b-a) = n$.

Proof. Relations $R_i, i = 0, \ldots, n-1$ from the Hamming scheme remain the same, while $R_n$ is split into $S_x, x \in GF(q)^n$, where $(a, b) \in S_x$ if and only if $\prod_{i=1}^n (b_i - a_i) = x$. To prove that this defines an association scheme we show that the intersection parameters are well-defined; these come in six
types $p^*_0, p^*_1, p^*_2, p^*_3, p^*_4, p^*_5, p^*_6$, where $i, j$ and $k$ correspond to relations $R_i, R_j$, and $R_k$, $i, j, k = 0, \ldots, n - 1$, and similarly $x, y, z$ correspond to $S_x, S_y, S_z$.

The intersection parameters of type $p^*_0$ are of course well-defined, as they come from the Hamming scheme. Note also that $p^*_0$ is well-defined, and in fact equals the intersection parameter $p^*_0$ of the Hamming scheme, for all $x \in GF(q)^*$. Furthermore, it is not hard to show that $p^*_0 = (q - 2)^{i+k-n} (q-1)^{n-k-1}$ if $i + k \geq n$, and else equals zero; that $p^*_0 = (q - 2)^{i+1} (q-1)^{n-k-1}$.

In order to prove that $p^*_x$ is well-defined we consider two words whose difference has product $x$. Since all relations are translation-invariant, we can take the zero word for one of these words, and hence the other, say $a$, has product $\prod_{i=1}^{n} a_i = x$. Now we want to show that the number of words $b$ with $\prod_{i=1}^{n} b_i = y$ and $\prod_{i=1}^{n} (b_i - a_i) = z$ is independent of the choice of $a$, but depends only on $x, y$ and $z$. If $a'$ is any word with $\prod_{i=1}^{n} a'_i = x'$, then the map $b \mapsto b'$ defined by $b'_i = b_i a'_i / a_i$ is a bijection between words $b$ such that $\prod_{i=1}^{n} b_i = y$ and $\prod_{i=1}^{n} (b_i - a_i) = z$, and words $b'$ such that $\prod_{i=1}^{n} b'_i = y x' / x$ and $\prod_{i=1}^{n} (b'_i - a'_i) = z x' / x$. This shows that $p^*_x$ is well-defined, and moreover that $p^*_x = p^*_{u x u}$ for all $u \in GF(q)^*$. Hence we have a fission scheme of the Hamming scheme.

Since this new scheme is a translation scheme, the additive characters of $GF(q)^*$ are eigenvectors of all relations in the scheme (cf. [1, Sect. 2.10]), and it follows that the eigenvalues of $S_x$ are given by

$$\sum_{b : \langle b, a \rangle \in S_x} \chi(\langle a, b \rangle), \quad a \in GF(q)^*$$

(and those of $R_i$ are obtained by replacing $S_x$ by $R_i$ in the above expression) where $\chi$ is some fixed non-trivial character of $GF(q)$, and $\langle \cdot, \cdot \rangle$ denotes the standard inner product. By similar arguments used to prove that the intersection parameters are well-defined, it follows that the eigenvalues corresponding to $a$ only depend on the weight $wt(a)$ and the product $\prod_{i=1}^{n} a_i$ of $a$. This is enough to prove that the scheme is self-dual. Moreover, labelling the eigenvalues of $R_i$ by $P_{ij}$ and $P_{ji}$, and those of $S_x$ by $P_{ij}$ and $P_{ji}$, $i, j = 0, \ldots, n - 1$, $x, y \in GF(q)^*$, one finds that $P_{ij}$ and $P_{ji}$ are independent of $x$, and $P_{ij}$, $i, j = 0, \ldots, n - 1$, $x, y \in GF(q)^*$, only depends on the product $xy$.

Note that if both $n$ and $q$ are odd, then the fission scheme is non-symmetric. If $n$ is a multiple of $q - 1$, then the fission scheme is a linear scheme, and hence its eigenvalues are integers. (For a good introduction to linear association schemes, see [7].) Finding a closed expression for the eigenvalues corresponds to a hard number-theoretic problem of finding the number of solutions to certain equations over finite fields. In specific cases,
we can compute the eigenvalues directly from the character sums. For example, if \( n = 3 \) and \( q = 4 \), we find that

\[
P = \begin{pmatrix} 1 & 9 & 27 & 9 & 9 \\ 1 & 5 & 3 & -3 & -3 \\ 1 & 1 & -5 & 1 & 1 \\ 1 & -3 & 3 & 5 & -3 \\ 1 & -3 & 3 & -3 & 5 \\ 1 & -3 & 3 & -3 & 5 \end{pmatrix}.
\]

Incidentally, in this case, all relations \( S_n \) are isomorphic to the Hamming graph \( H(3, 4) \). Furthermore, this scheme has an interesting fusion scheme: merge \( R_1 \) with any of the \( S_n \), and merge the remaining two \( S_n \). This gives an amorphic three-class association scheme, i.e. all three relations are strongly regular graphs. Since this is a linear scheme, there is a corresponding partition of the points of the projective plane \( PG(2, 4) \). This is a partition into two hyperovals and a unital.

If \( n \) is a divisor of \( q - 1 \), then the fission scheme has a linear fusion scheme (which is still a fission scheme of the Hamming scheme) by merging all \( S_n \) which have equal \( x^{(q-1)/n} \). This one can, for example, do in the case of \( n = 2, q = 5 \), where the fission scheme has eigenmatrix

\[
P = \begin{pmatrix} 1 & 8 & 4 & 4 & 4 & 4 \\ 1 & 3 & -1 & -1 & -1 & -1 \\ 1 & -2 & \frac{1}{2}(3-\sqrt{5}) & -1-\sqrt{5} & -1+\sqrt{5} & \frac{1}{2}(3+\sqrt{5}) \\ 1 & -2 & -1+\sqrt{5} & \frac{1}{2}(3+\sqrt{5}) & \frac{1}{2}(3-\sqrt{5}) & -1+\sqrt{5} \\ 1 & -2 & \frac{1}{2}(3+\sqrt{5}) & -1+\sqrt{5} & -1-\sqrt{5} & \frac{1}{2}(3-\sqrt{5}) \end{pmatrix}.
\]

This example also shows that in general the eigenvalues need not be integers.

The Hamming schemes are among the metric association schemes with classical parameters, in the sense of [1, Sect. 6.1]. We have found that almost all known self-dual schemes of this type (cf. [1, Tables 6.1, 6.2]) have fission schemes, which are analogous to the above.

2. THE BILINEAR FORMS SCHEME

The bilinear forms scheme \( Bil(m \times n, q) \) has as vertices all \( m \times n \) matrices with entries from the field \( GF(q) \), where two matrices are in \( R_i \) if and only if their difference has rank \( i \). For more on this scheme, see [1, Sect. 9.5.A; 3].
Theorem 2. The bilinear forms scheme $\text{Bil}(n \times n, q)$ has a self-dual fission scheme obtained by splitting the $n$th relation into $q-1$ relations according to the (non-zero) value of the determinant of the difference of two matrices.

Proof. We keep the relations $R_i$, $i = 0, \ldots, n-1$ of $\text{Bil}(n \times n, q)$, but replace $R_n$ by the $q-1$ relations $S_x$, $x \in GF(q)^*$, where $(A, B) \in S_x$ if and only if $\det(B - A) = x$. We wish to show that the intersection parameters are well-defined; these come in six types as before, and also here those of types $p^k_{ij}$ and $p^*_{ij}$ are obviously well-defined.

In order to prove that $p^k_{iz}$ is well-defined, we consider a matrix $A$ of rank $k$ (note that also here we have translation-invariant relations). We will show that the number of matrices $B$ of rank $i$ such that $\det(B - A) = z$ only depends on $i$, $k$ and $z$, but not on $A$. If $A'$ is any matrix of rank $k$, then there exists an invertible matrix $U$ such that $A' = UA$; note that we can take $U$ with any prescribed determinant. The map $B \mapsto B' = UB$ is a bijection between matrices $B$ of rank $i$ with $\det(B - A) = z$, and matrices $B'$ of rank $i$ with $\det(B' - A') = z \det(U)$. This shows that $p^k_{iz}$ is well-defined, and is in fact independent of $z$. Similarly, all other intersection parameters are well-defined, and it easily follows that $p^*_{uv} = p^*_{vu}$, $p^*_{i} = p^*_{i}$, and $p^*_{uv} = p^*_{uv}$ for all $u \in GF(q)^*$. Hence we have a fission scheme of the bilinear forms scheme. The proof that this scheme is self-dual is similar to the proof in the case of the Hamming scheme: in this case the appropriate inner product is given by $\langle A, B \rangle = \text{tr}(A^T B)$; the details are left to the reader.

Note that our fission of the Hamming scheme $H(n, q)$ is isomorphic to a subscheme of this fission scheme of the bilinear forms scheme (i.e., the subscheme on the set of diagonal matrices).

3. THE ALTERNATING FORMS SCHEME

The alternating forms scheme $\text{Alt}(n, q)$ has as vertices all $n \times n$ skew-symmetric matrices with zero diagonal and entries from $GF(q)$. Two matrices are in $R_i$ if and only if their difference has rank $2i$. Note that a skew-symmetric matrix has even rank. For more on this scheme, see [1, Sect. 9.5.B; 4].

Theorem 3. Let $n$ be even. The alternating forms scheme $\text{Alt}(n, q)$ has a self-dual fission scheme obtained by splitting the $(n/2)$th relation into $q-1$ relations according to the (non-zero) value of the Pfaffian of the difference of two matrices.

Proof. All arguments are approximately the same as before. Here the essential part in proving that all intersection parameters are well-defined, is that if $A$ and $A'$ are skew-symmetric matrices of the same rank $k$, then
there is an invertible matrix $U$ such that $A' = UAU^T$ (cf. [9, pp. 57, 68]); and if $k < n$, then we can choose $U$ with any prescribed determinant. Also the Pfaffians of $A$ and $A'$ are related, i.e., by the equation $\text{Pf}(A') = \det(U) \text{Pf}(A)$. (For some background on Pfaffians, see [6, Chap. 7].)

4. THE HERMITIAN FORMS SCHEME

The Hermitian forms scheme $\text{Her}(n, q^2)$ has as vertices the $n \times n$ Hermitian matrices with entries in $GF(q^2)$, i.e. matrices $H$ such that $H = H^*$, where $(H^*)_{ij} = (H_{ji})^q$. Two matrices are in relation $R_i$ if and only if their difference has rank $i$ (cf. [1, Sect. 9.5.C]).

**Theorem 4.** The Hermitian forms scheme $\text{Her}(n, q^2)$ has a self-dual fission scheme obtained by splitting the $n$th relation into $q - 1$ relations according to the value of the determinant of the difference of two matrices.

**Proof.** Note that a Hermitian matrix $H$ has $\det(H) \in GF(q)$. The arguments are again about the same as before. In proving that all intersection parameters are well-defined, the essential part is that if $A$ and $A'$ have the same rank, then there is an invertible matrix $U$ such that $A' = UAU^*$ (cf. [8, p. 325]); and if $k < n$ then we can choose $U$ with any determinant we want.

5. THE QUADRATIC FORMS SCHEME

The distance-regular quadratic forms graph, and its scheme, are quite different from the previous ones. First we shall give what in our opinion is the “natural” scheme on quadratic forms. This scheme is already a fission scheme of the very interesting metric scheme of the quadratic forms graph. Then we shall further fission the scheme, more like our previous constructions.

The quadratic forms graph $\text{Qua}(n, q)$ has as vertices the quadratic forms in $n$ variables over $GF(q)$. In the quadratic forms graph two forms are at distance $i$ if and only if the rank of their difference equals $2i - 1$ or $2i$. For more on the quadratic forms graph, see [1, Sect. 9.6; 5]. The first fission of this scheme relies on the following result.

Under the group of invertible linear transformations of variables, the quadratic forms fall into $2n + 1$ ($q$ odd) or $\binom{3n + 1}{2}$ ($q$ even) orbits: each form of rank $k \neq 0$ is of one of two types (denoted by $\varepsilon$). For even rank there is the well-known distinction between hyperbolic ($\varepsilon = +$) and elliptic ($\varepsilon = -$) forms; in the case of odd rank, a (parabolic) form is equivalent to $x_1 x_2 + \ldots + x_{k-2} x_{k-1} + c x_k^2$, for some $c$, and the type depends on
whether \(c\) is a square \((c = +)\) or not \((c = -)\) (cf. [9, Chapter IV]). If \(q\) is even then each field element is a square, hence there is no distinction for odd rank.

**Theorem 5.** The quadratic forms scheme Qua\((n, q)\) has the following fission scheme. The non-trivial relations are labelled by \(i = 1, \ldots, n\) and \(\varepsilon = \pm\), and two forms are in relation \(R_{i, \varepsilon}\) if their difference has rank \(i\) and type \(\varepsilon\).

If \(q\) is odd, then the fission scheme is self-dual.

**Proof.** The proof that this defines a scheme relies mainly on the observation that an invertible linear transformation of variables preserves the rank and type of a form.

Now let \(q\) be odd. In this case there is a one-one correspondence between quadratic forms and symmetric matrices. The eigenvalues are given by appropriate character sums, the form of which shows that the scheme is self-dual (as in the proof of Theorem 1).

For \(n = 2\) and \(q = 3\) the eigenmatrix of the fission scheme is given by

\[
P = \begin{pmatrix}
1 & 4 & 4 & 6 & 12 \\
1 & \frac{1}{2}(1 + 3i \sqrt{3}) & \frac{1}{2}(1 - 3i \sqrt{3}) & -3 & 3 \\
1 & \frac{1}{2}(1 - 3i \sqrt{3}) & \frac{1}{2}(1 + 3i \sqrt{3}) & -3 & 3 \\
1 & -2 & -2 & 3 & 0 \\
1 & 1 & 1 & 0 & -3
\end{pmatrix}.
\]

This example shows that, in general, just looking at the rank of the difference of two forms does not give an association scheme (since we cannot merge relations 1 with 2, and 3 with 4). Incidentally, merging the non-symmetric relations in this example gives the Hamming scheme \(H(3, 3)\).

For \(q\) even we do not, in general, get a self-dual scheme. For example, if \(n = 2\) we get a scheme with eigenmatrix \(P\) and dual eigenmatrix \(Q\), where

\[
P = \begin{pmatrix}
1 & q^2 - 1 & \frac{1}{2}q(q - 1)^2 & \frac{1}{2}q(q^2 - 1) \\
1 & q^2 - 1 & -\frac{1}{2}q(q - 1) & -\frac{1}{2}q(q + 1) \\
1 & -1 & -\frac{1}{2}q(q - 1) & \frac{1}{2}q(q - 1) \\
1 & -1 & \frac{1}{2}q & -\frac{1}{2}q
\end{pmatrix},
\]

\[
Q = \begin{pmatrix}
1 & q - 1 & q^2 - 1 & (q - 1)(q^2 - 1) \\
1 & q - 1 & -1 & -q + 1 \\
1 & -1 & -q - 1 & q + 1 \\
1 & -1 & q - 1 & -q + 1
\end{pmatrix}.
\]
This scheme is a scheme of linked symmetric designs (cf. [2]). Since it is linear, it has a dual scheme, and this dual is metric: it is the scheme of an antipodal symplectic cover of the complete graph. This graph can be described as follows on vertex set $GF(q)^3$. Two distinct vertices are adjacent if their difference $(x, y, z)$ satisfies the equation $xy = z^2$. This graph was first constructed by Thas (cf. [1, p. 385]).

Next, we shall further fission the scheme of Theorem 5. Essential here is that to each quadratic form, there corresponds a unique symmetric bilinear form, and hence a corresponding symmetric matrix. Note that for $q$ even this correspondence between forms and matrices is not one-one, while it is for odd $q$.

**Theorem 6.** The fission scheme of Theorem 5 has the following fission scheme. The rank $n$ relations are further split according to the value of the determinant of the symmetric matrix corresponding to the difference of two forms. If $q$ is odd, then the fission scheme is self-dual.

**Proof.** An invertible linear transformation of variables corresponds to an invertible matrix $U$. If one form is mapped by this transformation of variables to a second form, then the corresponding matrices $A$ and $A'$ are related by $A' = UAU^T$. Now essentially the same arguments as before apply. 

Note that the rank $n$ relation was already split according to type; for odd $q$ this corresponds to the value of the determinant being a square or not.

Note also that the dual scheme is fissioned along with the primal scheme (as the fission scheme is again a translation scheme). In the case of $q$ even and $n = 2$, where the dual scheme of Theorem 5 is the scheme of a symplectic cover, the distance two relation is fissioned according to the difference of $xy$ and $z^2$ (which is nonzero), and the distance three relation (which is a disjoint union of cliques) is fissioned into matchings.

### 6. CONCLUDING REMARKS

The remaining series of known self-dual classical association schemes are the Doob schemes and the affine $E_6(q)$ schemes. The latter schemes have three classes, and the distance three relation naturally splits according to the value of $D(x)$ (see [1, Sect. 10.8] for the notation), giving a fission scheme with $q + 1$ classes; we omit the details.

The distance-regular Doob graphs are obtained as direct products of, say $k$, Shrikhande graphs and a Hamming graph $H(n - 2k, 4)$ (cf. [1, Sect. 9.2.B]). These graphs have the same parameters as the Hamming...
graph $H(n, 4)$. Notice that the Shrikhande scheme itself has a fission scheme with the same parameters as our fission scheme of $H(2, 4)$. This fission can be described as follows on vertex set $\mathbb{Z}_2^4$ where two distinct pairs are in relation $S_i$, $i = 1, 2$ if they agree in the $i$-th coordinate; distinct pairs $(a_1, a_2)$ and $(b_1, b_2)$ are in relation $S_3$ if $a_1 + a_2 = b_1 + b_2$; and relation $R_i$ is the remainder (which is the Shrikhande graph). This fission scheme, and our fission scheme of $H(n - 2k, 4)$ allows us to fission the distance-$n$ relation in the Doob scheme. On each pair of coordinate positions where we have a Shrikhande graph, we put the structure of its fission scheme, and we arbitrarily label the distance-2 relations $S_i$ by the nonzero elements of $GF(4)$. On the remaining positions we put the structure of the fissioned Hamming scheme. Now we can split the distance-$n$ relation of the Doob graph according to product of labels, like in the Hamming scheme. Showing that the intersection parameters are well-defined is really a matter of the intersection parameters of the smaller schemes being what they should be; if, instead of fissioned Shrikhande schemes, fissioned $H(2, 4)$ were used, we know that the result is the fission scheme of Theorem 1. Since the fissioned Shrikhande scheme and the fissioned Hamming scheme $H(2, 4)$ have the same parameters, we thus have a fission scheme of the Doob scheme with the same parameters as the fission scheme of the Hamming scheme $H(n, 4)$. Note that for $n = 3$, the distance-three graph of the Doob graph is fissioned into three isomorphic copies of the Doob graph, similar to what happened with the Hamming graph $H(3, 4)$.

For the remaining (non-self-dual) classical association schemes there seem to be no analogous fission schemes.

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