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# On the condensed density of the generalized eigenvalues of pencils of Gaussian random matrices and applications

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ABSTRACT

Pencils of matrices whose elements have a joint noncentral Gaussian distribution with nonidentical covariance are considered. An approximation to the distribution of the squared modulus of their determinant is computed which allows to get a closed form approximation of the condensed density of the generalized eigenvalues of the pencils. Implications of this result for solving several moments problems are discussed and some numerical examples are provided.

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## 0. Introduction

Let $\mathbf{G}$ be a $p \times p$ random matrix and denote by $\{\boldsymbol{\xi}_j,\ j = 1, \ldots, p\}$ its eigenvalues which form a set of exchangeable random variables. Their marginal density $h(z),\ z \in \mathbb{C}$, also called condensed density [16] or normalized one-point correlation function [14], is the expected value of the (random) normalized counting measure on the zeros of $\mathbf{G}$ i.e.

$$h(z) = \frac{1}{p} E\left[\sum_{j=1}^{p} \delta(z - \boldsymbol{\xi}_j)\right]$$

or, equivalently, for all Borel sets $A \subset \mathbb{C}$

$$\int_A h(z)dz = \frac{1}{p}\sum_{j=1}^{p} \mathrm{Prob}(\boldsymbol{\xi}_j \in A).$$

It can be proved that (see e.g. [4]) $h(z) = \frac{1}{4\pi}\Delta u(z)$ where $\Delta$ denotes the Laplacian operator with respect to $x, y$ if $z = x + iy$ and $u(z) = \frac{1}{p}E\left\{\log(|\det(\mathbf{G} - zI_p)|^2)\right\}$ is the corresponding logarithmic potential where $I_p$ denotes the identity matrix of order $p$.

If a pencil $\mathbf{G} = (\mathbf{G}_1, \mathbf{G}_0)$ of random matrices is considered, the condensed density of its generalized eigenvalues can be computed by the formula above where now the logarithmic potential is given by $u(z) = \frac{1}{p}E\left\{\log(|\det(\mathbf{G}_1 - z\mathbf{G}_0)|^2)\right\}$.

In this paper a method to estimate the condensed density of the generalized eigenvalues of a pencil $\mathbf{G} = (\mathbf{G}_1, \mathbf{G}_0)$ of Gaussian random matrices is proposed. The main motivation is provided by the role that this function plays in solving some

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difficult moments problems such as the trigonometric, the complex, the Hausdorff ones. It was shown in [6,5,4,2,3,12,10,9,11,18,19] that all these problems can be reduced to the complex exponentials approximation problem (CEAP), which can be stated as follows.

Let us consider a uniformly sampled signal made up of a linear combination of complex exponentials

$$s_k = \sum_{j=1}^{p^*} c_j \xi_j^k \tag{1}$$

where $c_j, \xi_j \in \mathbb{C}$. Let us assume to know an even number $n = 2p,\ p \geq p^*$ of noisy samples

$$\mathbf{a}_k = s_k + \boldsymbol{\epsilon}_k, \quad k = 0, 1, 2, \ldots, n-1$$

where $\boldsymbol{\epsilon}_k$ is a complex Gaussian, zero mean, white noise, with finite known variance $\sigma^2$. We want to estimate $p^*, c_j, \xi_j,\ j = 1, \ldots, p^*$, which is a well known ill-posed inverse problem.

We notice that, in the noiseless case and when $p = p^*$, the parameters $\xi_j$ are the generalized eigenvalues of the pencil $(U_1, U_0)$ where $U_1$ and $U_0$ are Hankel matrices defined as

$$U_0 = \begin{bmatrix} s_0 & s_1 & \cdots & s_{p-1} \\ s_1 & s_2 & \cdots & s_p \\ \cdot & \cdot & \cdots & \cdot \\ s_{p-1} & s_p & \cdots & s_{n-2} \end{bmatrix}, \qquad U_1 = \begin{bmatrix} s_1 & s_2 & \cdots & s_p \\ s_2 & s_3 & \cdots & s_{p+1} \\ \cdot & \cdot & \cdots & \cdot \\ s_p & s_{p+1} & \cdots & s_{n-1} \end{bmatrix}. \tag{2}$$

If we define $\mathbf{U}_1$ and $\mathbf{U}_0$ as $U_0$ and $U_1$ but starting from $\mathbf{a}_k,\ k = 0, \ldots, n-1$ it is evident that the condensed density of the generalized eigenvalues of the pencil $(\mathbf{U}_1, \mathbf{U}_0)$ provides information about the location in the complex plane of the generalized eigenvalues $\xi_j,\ j = 1, \ldots, p$ whose estimation is the most difficult part of CEAP.

Unfortunately the computation of the condensed density starting from a single realization of the pencil is very difficult even in the Hankel case illustrated above.

The first difficulty is a computational one. The condensed density should be computed at least on a lattice covering the region of the complex plane which contains in its interior all the generalized eigenvalues. Therefore for each lattice point a determinant must be evaluated and this can be computationally expensive.

The main difficulty however is that, for each lattice point, an expected value w.r. to the joint probability measure of the elements of the matrices $(\mathbf{G}_1, \mathbf{G}_0)$ must be computed. As we assume to have a single realization of the pencil this estimation problem is critical. Two different approaches are reported in the literature to cope with it.

In [11] the CEAP problem stated above was considered. Noticing that the generalized eigenvalues of the pencil $(\mathbf{G}_1, \mathbf{G}_0)$ are the roots of a random polynomial, by applying the Delta method, the joint distribution of its coefficients can be approximated by a multivariate Gaussian distribution. Then an explicit expression of $h(z)$ proposed by Hammersley [16] can be used. Unfortunately the distribution of polynomial coefficients is not well approximated by a Gaussian distribution. It can be shown in fact in simple cases that it is close to a Cauchy distribution [8, Theorem 1.2, Corollary 1.3].

In [5] for the CEAP problem a stochastic perturbation method was proposed to compute the expectation. Many pseudo-samples were generated by adding i.i.d. zero mean complex Gaussian variables with variance small w.r. to $\sigma^2$ to the original observations $\mathbf{a}_k,\ k = 0, \ldots, n-1$. The generalized eigenvalues of the pencils associated to each pseudo-sample were evaluated and the required expectation was estimated by a sample mean. It was proved that the statistical properties of this estimator are good. However the computational burden was relevant.

Finally both estimates quoted above are usually not smooth functions. This can be a problem if the estimated condensed density is used to make inference on the generalized eigenvalues.

In the following a method to cope with these difficulties is proposed. A non-linear transformation of the pencil is considered which allows to solve in closed form the expectation evaluation problem. Moreover, for the CEAP problem, the computational burden can be substantially reduced and the noise contribution to $h(z)$ can be smoothed out to some extent simply acting on a parameter of the approximant.

The paper is organized as follows. In Section 1 some algebraic and statistical preliminaries are developed. In Section 2 the closed form approximation of $h(z)$ is defined in the general case. In Section 3 the smooth estimate of $h(z)$ is derived in the Hankel case. In Section 4 computational issues are discussed in the Hankel case. Finally in Section 5 some numerical examples are provided.

## 1. Preliminaries

Let us consider the $p \times p$ complex random pencil $\mathbf{G}(z) = \mathbf{G}_1 - z\mathbf{G}_0,\ \ z \in \mathbb{C}$ where the elements of $\Re\mathbf{G}_0, \Im\mathbf{G}_0, \Re\mathbf{G}_1, \Im\mathbf{G}_1$ have a joint Gaussian distribution and $\Re$ and $\Im$ denotes the real and imaginary parts. Dropping the dependence on $z$ for simplifying the notations, let us define

$$\mathbf{G} = [\underline{\mathbf{g}}_1, \ldots, \underline{\mathbf{g}}_p], \quad \underline{\mathbf{g}} = \mathrm{vec}(\mathbf{G}) = [\underline{\mathbf{g}}_1^T, \underline{\mathbf{g}}_2^T, \ldots, \underline{\mathbf{g}}_p^T]^T.$$

Moreover $\forall z$, let us define $\check{\underline{\mathbf{g}}}_k = [\Re \underline{\mathbf{g}}_k{}^T, \Im \underline{\mathbf{g}}_k{}^T]^T$ and

$$\check{\underline{\mathbf{g}}} = [\check{\underline{\mathbf{g}}}_1^T, \check{\underline{\mathbf{g}}}_2^T, \ldots, \check{\underline{\mathbf{g}}}_p^T]^T.$$

Then $\check{\underline{\mathbf{g}}}$ will have a multivariate Gaussian distribution with mean $\underline{\mu} = E[\check{\underline{\mathbf{g}}}] \in \mathbb{R}^{2p^2}$ and covariance $\Sigma \in \mathbb{R}^{2p^2 \times 2p^2}$. We notice that no independence assumption neither between elements of $\overline{\mathbf{G}}_0$ and $\overline{\mathbf{G}}_1$ nor between real and imaginary parts is made. Hence this is the most general hypothesis that can be done about the Gaussian distribution of the complex random vector $\underline{\mathbf{g}}$, (see [23] for a full discussion of this point).

Let us consider the $QR$ factorization of $\mathbf{G}$ where $\mathbf{Q}^H\mathbf{Q} = \mathbf{Q}\mathbf{Q}^H = I_p$ where $H$ denotes transposition plus conjugation, $\mathbf{R}$ is an upper triangular matrix and $I_p$ is the identity matrix of order $p$. We then have

$$|\det(\mathbf{G})|^2 = |\det(\mathbf{QR})|^2 = |\det(\mathbf{R})|^2 = \prod_{k=1,p} |\mathbf{R}_{kk}|^2.$$

We want to compute the condensed density of the generalized eigenvalues of the pencil $\mathbf{G}(z)$ which is given by [16,4]:

$$h(z) = \frac{1}{4\pi p} \Delta E\left\{\log(|\det[\mathbf{G}(z)]|^2)\right\} = \frac{1}{4\pi p} \Delta \sum_{k=1}^{p} E\left\{\log|\mathbf{R}_{kk}(z)|^2\right\}.$$

We are therefore interested on the distribution of $|\mathbf{R}_{kk}|^2$, $k = 1, \ldots, p$ in order to compute $E[\log|\mathbf{R}_{kk}|^2]$.

To perform the $QR$ factorization of the random matrix $\mathbf{G}$ we can use the Gram–Schmidt algorithm. In the following $[\underline{\mathbf{q}}_1, \ldots, \underline{\mathbf{q}}_p]$ denote the orthonormalized vectors obtained from $[\underline{\mathbf{g}}_1, \ldots, \underline{\mathbf{g}}_p]$.

We notice that $|\mathbf{R}_{kk}| = \mathbf{R}_{kk}$ and

$$\mathbf{R}_{kk}^2 = \begin{cases} \underline{\mathbf{g}}_k^H \underline{\mathbf{g}}_k, & \text{if } k = 1 \\ \underline{\mathbf{g}}_k^H \left(I_p - \sum_{i=1}^{k-1} \underline{\mathbf{q}}_i \underline{\mathbf{q}}_i^H\right) \underline{\mathbf{g}}_k, & \text{if } k > 1 \end{cases}$$

where $\underline{\mathbf{q}}_i$ are functions of $\underline{\mathbf{g}}_j$, $j = 1, \ldots, i$. Therefore, denoting by $\tilde{\underline{\mathbf{g}}}_k = \{\check{\underline{\mathbf{g}}}_1, \ldots, \check{\underline{\mathbf{g}}}_{k-1}\}$ we have that

$$\begin{cases} \mathbf{R}_{11}^2 & \text{is a quadratic form in Gaussian variables} \\ \mathbf{R}_{kk}^2, & k > 1, \text{ conditioned on } \tilde{\underline{\mathbf{g}}}_k, \text{ is a quadratic form in Gaussian variables.} \end{cases}$$

Moreover let us denote by $\underline{e}_k$ the $k$-th column of $I_p$ and let be $E_k = \underline{e}_k \otimes I_{2p}$ then $\underline{\mu}_k = E_k^T \underline{\mu}$, $\Sigma_k = E_k^T \Sigma E_k$ are the mean vector and covariance matrix of $\check{\underline{\mathbf{g}}}_k$. Then we have

**Lemma 1.** *For $k = 1$ and for $k > 1$, conditioned on $\tilde{\underline{\mathbf{g}}}_k$, $\mathbf{R}_{kk}^2$ is distributed as $\sum_{r=1}^{n} \lambda_r^{(k)} \chi_{\nu_r}^2(\delta_r)$, $n = 2p$, and $\chi_{\nu_r}^2(\delta_r)$ are independent, where $2(p-k+1) = \sum_{r=1}^{n} \nu_r$, $\lambda_r^{(k)}$ are the distinct eigenvalues of $\Sigma_k^{1/2} \mathcal{R}(\mathbf{A}_k) \Sigma_k^{1/2}$ with multiplicity $\nu_r$, $\underline{u}_i^{(k)}$, $i = 1, \ldots, n$ are the corresponding eigenvectors, $\delta_r = \sum_{(r)} ((\underline{u}_i^{(k)})^T \Sigma_k^{-1/2} \underline{\mu}_k)^2$, the summation being over all eigenvectors corresponding to eigenvalue $\lambda_r^{(k)}$,*

$$\mathbf{A}_k = \left(I_p - \sum_{i=1}^{k-1} \underline{\mathbf{q}}_i \underline{\mathbf{q}}_i^H\right)$$

*and*

$$\mathcal{R}(\mathbf{A}_k) = \begin{bmatrix} \Re(\mathbf{A}_k) & -\Im(\mathbf{A}_k) \\ \Im(\mathbf{A}_k) & \Re(\mathbf{A}_k) \end{bmatrix}$$

*with $-\Im(\mathbf{A}_k) = \Im(\mathbf{A}_k)^T$ is the real isomorph of $\mathbf{A}_k$.*

**Proof.** The proof follows by that of [7, Lemma 1] by noticing that $\text{rank}(\mathcal{R}(\mathbf{A}_k)) = 2(p-k+1)$ because the eigenvalues of $\mathcal{R}(\mathbf{A}_k)$ are those of $\mathbf{A}_k$ with multiplicity 2 and $\underline{\mathbf{g}}_k^H \mathbf{A}_k \underline{\mathbf{g}}_k = \check{\underline{\mathbf{g}}}_k^T \mathcal{R}(\mathbf{A}_k) \check{\underline{\mathbf{g}}}_k$. □

**Corollary 1.** *If $\Sigma = I_{2p^2}$ and $\underline{\mu} = 0$, $\mathbf{R}_{kk}^2$ is distributed as $\chi_{2(p-k+1)}^2$.*

**Proof.** As $\Sigma = I_{2p^2}$ the eigenvalues of $\Sigma_k^{1/2} \mathcal{R}(\mathbf{A}_k) \Sigma_k^{1/2}$ are those of $\mathcal{R}(\mathbf{A}_k)$ which are 1 with multiplicity $2(p-k+1)$ and 0 with multiplicity $2(k-1)$. As $\underline{\mu} = 0$, $\delta_i = 0$. Remembering that the $\chi_1^2(\delta_i)$ appearing in the previous lemma are independent, the corollary follows by the additivity property of $\chi^2$ distribution. □

**Remark.** The corollary follows also by Bartlett's decomposition of a i.i.d. zero mean Gaussian random matrix [13].

## 2. Closed form approximation of $h(z)$

Unfortunately we cannot use the easy result stated in the Corollary 1 because in the case of interest the matrix $\mathbf{G}(z)$ has a mean different from zero and a covariance structure depending on $z$. By Lemma 1 we know that $\mathbf{R}_{11}^2$ is distributed as a linear combination of non-central $\chi^2$ distributions. It is known that this distribution admits an expansion $\mathcal{L}(\alpha, \beta, \tau)$ in series of generalized Laguerre polynomials [17, Chapter 29, Section 6.3] and the series is uniformly convergent in $\mathbb{R}^+$. More specifically let us denote the generalized Laguerre polynomial of order $m$ by

$$L_m(x, \alpha) = \frac{x^{-(\alpha-1)}e^x}{m!}\frac{\partial^m}{\partial x^m}(x^{m+\alpha-1}e^{-x}) = \sum_{h=0}^{m} c_{hm}x^h$$

where

$$c_{hm} = \frac{(-1)^h\Gamma(\alpha+m)}{h!(m-h)!\Gamma(\alpha+h)}.$$

Then, following [22], we have

**Lemma 2.** *The density function of $\mathbf{R}_{11}^2$ is given by*

$$f_1(y) = b_0\frac{y^{\alpha-1}e^{-y/\beta}}{\beta^\alpha\Gamma(\alpha)} + \frac{y^{\alpha-1}e^{-y/\beta}}{\beta^\alpha\Gamma(\alpha)}\sum_{m=1}^{\infty} b_m L_m(y/\tau, \alpha) = \mathcal{L}(\alpha, \beta, \tau)$$

*where $\alpha$ and $\beta$ are such that the first two moments of $\mathbf{R}_{11}^2$ are identical to the first two moments of the gamma distribution representing the leading term of the expansion. Moreover the $b_m$ are uniquely determined by the moments and $\tau$ is a free parameter. If $\lambda_{\max}$ denotes the maximum eigenvalue of $\Sigma_1$, when $\tau^{-1} > 2(\beta^{-1} - (2\lambda_{\max})^{-1})$ the series $\mathcal{L}(\alpha, \beta, \tau)$ is uniformly convergent $\forall y \in \mathbb{R}^+$. If $\beta > \lambda_{\max}$ then $\tau = \beta$ makes the series to converge uniformly, $b_0 = 1$ and $b_m$ are determined by the first $m$ moments of $\mathbf{R}_{11}^2$.*

**Proof.** The proof follows by the results given in [22] for the distribution of quadratic forms in central normal variables which hold true also in the non-central case as can be easily checked. □

Denoting the logarithmic derivative of the gamma function by $\Psi(\cdot)$, we can compute $E[\log(\mathbf{R}_{11}^2)]$ by

**Lemma 3.**

$$E[\log(\mathbf{R}_{11}^2)] = b_0[\log\beta + \Psi(\alpha)] + \sum_{m=1}^{\infty} b_m \sum_{h=0}^{m} c_{hm}\frac{\Gamma(\alpha+h)}{\Gamma(\alpha)}\left(\frac{\beta}{\tau}\right)^h [\log\beta + \Psi(\alpha+h)].$$

**Proof.** By Lemma 2 the series $\mathcal{L}(\alpha, \beta, \tau)$ converges uniformly. Therefore term-by-term integration can be performed and the result follows by noticing that, for $h = 0, 1, \ldots$

$$\frac{1}{\beta^\alpha}\int_0^\infty \log(y)\left(\frac{y}{\tau}\right)^h y^{\alpha-1}e^{-y/\beta}dy = \Gamma(\alpha+h)\left(\frac{\beta}{\tau}\right)^h [\log\beta + \Psi(\alpha+h)]. \quad \square$$

We have then obtained a closed form expression for $E[\log(\mathbf{R}_{11}^2)]$ as a function of the moments of $\mathbf{R}_{11}^2$. By noticing that the same result holds true for the distribution of $\mathbf{R}_{kk}^2$ conditioned on $\underline{\tilde{\mathbf{g}}}_k$, we show now how to get an approximation of $E[\log(\mathbf{R}_{kk}^2)]$, $k > 1$.

**Theorem 1.** *The density function $f_k(y)$ of $\mathbf{R}_{kk}^2$ can be expanded in a uniformly convergent series of Laguerre functions*

$$f_k(y) = b_0^{(k)}\frac{y^{\alpha_k-1}e^{-y/\beta_k}}{\beta_k^{\alpha_k}\Gamma(\alpha_k)} + \frac{y^{\alpha_k-1}e^{-y/\beta_k}}{\beta_k^{\alpha_k}\Gamma(\alpha_k)}\sum_{m=1}^{\infty} b_m^{(k)} L_m(y/\tau_k, \alpha_k). \tag{3}$$

*When the parameter $\tau_k$, that controls the uniform convergence of the series, can be chosen equal to $\beta_k$, then $b_0^{(k)} = 1$ and $b_m^{(k)}$, $m = 0, \ldots, N$ depends on the first $N+1$ moments of $\mathbf{R}_{kk}^2$. Moreover*

$$E[\log(\mathbf{R}_{kk}^2)] = b_0^{(k)}[\log\beta_k + \Psi(\alpha_k)] + \sum_{m=1}^{\infty} b_m^{(k)} \sum_{h=0}^{m} c_{hm}\frac{\Gamma(\alpha_k+h)}{\Gamma(\alpha_k)}\left(\frac{\beta_k}{\tau_k}\right)^h [\log\beta_k + \Psi(\alpha_k+h)]. \tag{4}$$

**Proof.** The first part of the thesis is proved in [7, Theorem 4]. As the convergence of the series (3) is uniform we can integrate term-by-term and we get Eq. (4). □

**Proposition 1.** *If the series (3) is truncated after $N+1$ terms, the approximation error*

$$\eta_N^{(k)} = \left| E[\log(\mathbf{R}_{kk}^2)] - \left( b_0^{(k)}[\log\beta_k + \Psi(\alpha_k)] + \sum_{m=1}^{N} b_m^{(k)} \sum_{h=0}^{m} c_{hm} \frac{\Gamma(\alpha_k+h)}{\Gamma(\alpha_k)} \left(\frac{\beta_k}{\tau_k}\right)^h [\log\beta_k + \Psi(\alpha_k+h)] \right) \right|$$

*is bounded by*

$$K_1 \frac{\epsilon^{N+1}}{\alpha_k^2 \beta_k^{\alpha_k} \Gamma(\alpha_k)} \left( {}_2F_2\,(\alpha_k, \alpha_k; 1+\alpha_k, 1+\alpha_k; K_2) + G_{0,2}^{3,0}\left(-K_2 \left| \begin{matrix} 1-\alpha_k, 1-\alpha_k \\ 0, -\alpha_k, -\alpha_k \end{matrix} \right.\right)\right)$$

*where $K_1 > 0$, $K_2 > 0$ and $0 < \epsilon < 1$ are constants, ${}_2F_2$ is a generalized hypergeometric function and $G_{0,2}^{3,0}$ is a Meijer's G-function.*

**Proof.** The approximation error $\eta_N^{(k)}$ can be written as

$$\eta_N^{(k)} = \left| \int_0^\infty \log(y) e_N(y) dy \right|$$

where

$$e_N(y) = \frac{y^{\alpha_k - 1} e^{-y/\beta_k}}{\beta_k^{\alpha_k} \Gamma(\alpha_k)} \sum_{m=N+1}^{\infty} b_m^{(k)} L_m(y/\tau_k, \alpha_k).$$

In [22, Eq. (31)] the bound

$$|e_N(y)| \le K_1 \epsilon^{N+1} \frac{y^{\alpha_k-1} e^{K_2 y}}{\beta_k^{\alpha_k}\Gamma(\alpha_k)}, \quad 0 < \epsilon < 1, \qquad K_2 = -\beta_k^{-1} + \frac{R}{\tau_k(1+R)}, \quad \epsilon < R < 1$$

is given where $K_1 > 0$, $K_2$ are constants. But then

$$\begin{aligned}
\eta_N^{(k)} &\le \int_0^\infty |\log(y) e_N(y)|\, dy \le K_1 \frac{\epsilon^{N+1}}{\beta_k^{\alpha_k}\Gamma(\alpha_k)} \int_0^\infty |\log(y)| y^{\alpha_k-1} e^{K_2 y} dy \\
&= K_1 \frac{\epsilon^{N+1}}{\beta_k^{\alpha_k}\Gamma(\alpha_k)} \cdot \left( \int_1^\infty \log(y) y^{\alpha_k-1} e^{K_2 y} dy - \int_0^1 \log(y) y^{\alpha_k-1} e^{K_2 y} dy \right) \\
&= G_{0,2}^{3,0}\left(-K_2 \left| \begin{matrix} 1-\alpha_k, 1-\alpha_k \\ 0, -\alpha_k, -\alpha_k \end{matrix} \right.\right) + \frac{1}{\alpha_k^2}\, {}_2F_2\,(\alpha_k, \alpha_k; 1+\alpha_k, 1+\alpha_k; K_2)\,. \quad \square
\end{aligned}$$

**Remark.** The bound on the error given above is of little use in practice because the computation of the constants $K_1, K_2$ is quite involved as they depend on all moments. However, by Corollary 1 we know that when $\Sigma = I_{2p^2}$ and $\underline{\mu} = 0$ the expansion terminates after the first term and $b_0^{(k)} = 1$. Moreover from Theorem 1 we know that it can happen that the coefficients $b_k^{(k)}$, $k = 0, \ldots, N$ are determined by the first $N+1$ moments only. Therefore by continuity we can conjecture that the first term of the expansion provides most of the information in the general case and therefore the truncation error should be small. This conjecture is strongly supported by numerical evidence as discussed in Section 5.

We can now prove the main theorem

**Theorem 2.** *If $\mathbf{Q}(z)\mathbf{R}(z)$ is the QR factorization of $\mathbf{G}(z)$,*

$$\begin{aligned}
u(z) &= \frac{1}{p} E\{\log(|\det[\mathbf{G}_1 - z\mathbf{G}_0]|^2)\} = \frac{1}{p}\sum_{k=1}^{p} E\left\{\log|\mathbf{R}_{kk}(z)|^2\right\} \\
&= \frac{1}{p}\sum_{k=1}^{p} \Bigg\{ b_0^{(k)}(z)[\log\beta_k(z) + \Psi(\alpha_k(z))] + \sum_{m=1}^{\infty} b_m^{(k)}(z) \sum_{h=0}^{m} c_{hm} \frac{\Gamma(\alpha_k(z)+h)}{\Gamma(\alpha_k(z))} \left(\frac{\beta_k(z)}{\tau_k(z)}\right)^h \\
&\quad \times [\log\beta_k(z) + \Psi(\alpha_k(z)+h)] \Bigg\}.
\end{aligned}$$

*Moreover*

$$u(z) \approx \tilde{u}(z) = \frac{1}{p}\sum_{k=1}^{p} [\log\beta_k(z) + \Psi(\alpha_k(z))]$$

*and*

$$|u(z) - \tilde{u}(z)| \le \frac{1}{p}\sum_{k=1}^{p} \eta_1^{(k)}(z).$$

**Proof.** By Theorem 1 we can approximate the density function $f_k(y)$ of $\mathbf{R}_{kk}^2$ by the first term divided by $b_0^{(k)}$ of its Laguerre expansion i.e. by

$$\frac{y^{\alpha_k - 1} e^{-y/\beta_k}}{\beta_k^{\alpha_k}\Gamma(\alpha_k)}.$$

This is a consistent approximation because this normalized first term is a $\Gamma$ density with parameters $\alpha_k$, $\beta_k$. But then the corresponding approximation of $E[\log(\mathbf{R}_{kk}^2)]$ is $\log\beta_k + \Psi(\alpha_k)$ and then we have

$$\tilde{u}(z) = \frac{1}{p}\sum_{k=1}^{p}[\log\beta_k(z) + \Psi(\alpha_k(z))].$$

The other statements are obvious consequences of Theorem 1. □

In order to compute the condensed density $h(z)$ we have to take the Laplacian of $u(z)$. As differentiation can be a very unstable process, when we make use of the first order approximation of $u(z)$, we can expect that even a small approximation error in $u(z)$ can produce a large error in $h(z)$. However, in practice we have to approximate the Laplacian by finite differences by defining a square lattice over the region of $\mathbb{R}^2$ which the unknown complex numbers $\xi_j$ are supposed to belong to. This provides an implicit regularization method if the lattice size is properly chosen as a function of the approximation error of $u(z)$. We have

**Theorem 3.** *If* $\sup_{\mathbb{C}} \|u(z) - \tilde{u}(z)\| \le \varepsilon$ *and if* $z = x + iy$ *and the Laplacian operator is approximated by*

$$\hat{\Delta}u(x,y) = \frac{1}{\delta^2}\left[u(x-\delta,y) + u(x+\delta,y) + u(x,y-\delta) + u(x,y+\delta) - 4u(x,y)\right]$$

*on a square lattice with mesh size* $\delta$ *where* $\delta(\varepsilon) = C\varepsilon^{1/4}$, $C$ *constant, then*

$$\|\hat{\Delta}\tilde{u} - \Delta u\| = O(\varepsilon^{1/4})$$

*and this is the best possible approximation achievable.*

**Proof.** By Taylor expansion of $u(x \pm \delta, y \pm \delta)$ about $(x, y)$ we get $|\hat{\Delta}u(z) - \Delta u(z)| = O(\delta^2)$ and

$$\sup_{\mathbb{C}}|\hat{\Delta}u(z) - \Delta u(z)| = \|\hat{\Delta}u - \Delta u\| = O(\delta^2).$$

But $u(z) = \tilde{u}(z) + \eta(z)$ hence

$$\|\hat{\Delta}\tilde{u} - \Delta u\| = \|\hat{\Delta}u - \Delta u - \hat{\Delta}\eta\| \le O(\delta^2) + \frac{5\varepsilon}{\delta^2}.$$

For fixed $\varepsilon$ this error becomes unbounded as $\delta \to 0$. However by choosing $\delta(\varepsilon)$ such that $\delta(\varepsilon) \to 0$ and $\frac{\varepsilon}{\delta(\varepsilon)} \to 0$ for $\varepsilon \to 0$ we get

$$\|\hat{\Delta}\tilde{u} - \Delta u\| \to 0 \quad \text{as } \varepsilon \to 0.$$

Looking for a mesh size of the form $\delta(\varepsilon) = C\varepsilon^a$ such that the terms $O(\delta^2)$ and $\frac{5\varepsilon}{\delta^2}$ are balanced, we get

$$O(C^2\varepsilon^{2a}) = O\left(\frac{5\varepsilon}{C^2\varepsilon^{2a}}\right)$$

which implies $a = \frac{1}{4}$. In [15] it is proved that this bound is the best possible for all approximation errors $\eta(z)$ such that $\|\eta\| \le \varepsilon$. □

As a final remark we notice that for computing $h(z)$ we could start from the real isomorph

$$\mathcal{R}(\mathbf{G}) = \begin{bmatrix} \mathbf{V}_R & -\mathbf{V}_I \\ \mathbf{V}_I & \mathbf{V}_R \end{bmatrix} \in \mathbb{R}^{n\times n}$$

of $\mathbf{G}$ instead than from $\mathbf{G}$. The following proposition holds [16, Theorem 5.1]:

**Proposition 2.** *If* $\mathbf{G} = \mathbf{V}_R + i\mathbf{V}_I$, $\mathbf{V}_R, \mathbf{V}_I \in \mathbb{R}^{p\times p}$, *then*

$$|\det(\mathbf{G})|^2 = \det(\mathcal{R}(\mathbf{G})).$$

Let $\mathcal{R}(\mathbf{G}) = \check{\mathbf{Q}}\check{\mathbf{R}}$ be the $QR$ factorization of $\mathcal{R}(\mathbf{G})$. Then have

$$|\det(\mathbf{G})|^2 = \det \mathcal{R}(\mathbf{G}) = \prod_{k=1,n} \check{\mathbf{R}}_{kk}$$

and

$$\begin{aligned} h(z) &= \frac{1}{4\pi p} \Delta E\left\{\log(|\det[\mathbf{G}(z)]|^2)\right\} = \frac{1}{4\pi p} \Delta \sum_{k=1}^{n} E\left\{\log \check{\mathbf{R}}_{kk}(z)\right\} \\ &= \frac{1}{4\pi n} \Delta \sum_{k=1}^{n} E\left\{\log \check{\mathbf{R}}_{kk}^2(z)\right\}. \end{aligned}$$

It will be shown in Section 4 however that this expression of $h(z)$ is not convenient from the computational point of view.

## 3. Smooth estimate of the condensed density in the Hankel case

We want to show now that we can exploit the closed form expression of the condensed density to smooth out the noise contribution to $h(z)$. This allows us to get a good estimate of $p^*$ and $\xi_j$, $j = 1, \ldots, p^*$, – which is the nonlinear most difficult part of CEAP – from a single realization of the measured process $\{\mathbf{a}_k\}$.

We first notice that by approximating the density of $\mathbf{R}_{kk}^2$ by a $\Gamma$ density with parameters $\alpha_k$, $\beta_k$, the mean and variance of $\mathbf{R}_{kk}^2$ are approximated respectively by $\alpha_k\beta_k$ and $\alpha_k\beta_k^2$ and, if $b_0^{(k)} = 1$, we have exactly

$$\gamma_1 = \alpha_k\beta_k, \qquad \gamma_2 = \alpha_k\beta_k^2 + (\alpha_k\beta_k)^2.$$

However we know that [1, Theorem 4.1]

$$\gamma_m = \int_{\mathbb{R}^{2p(k-1)}} \gamma_m(\underline{\tilde{\mathbf{g}}}_k) h(\underline{\tilde{\mathbf{g}}}_k) d\underline{\tilde{\mathbf{g}}}_k, \quad m = 1, 2, \ldots \tag{5}$$

where $\gamma_m(\underline{\tilde{\mathbf{g}}}_k)$ are the moments of $\mathbf{R}_{kk}^2|\underline{\tilde{\mathbf{g}}}_k$. The first two of them are given by [20]

$$\begin{aligned} \gamma_1(\underline{\tilde{\mathbf{g}}}_k) &= \mathrm{tr}[(\Sigma_k + 2\underline{\mu}_k\underline{\mu}_k^T)\mathcal{R}(\mathbf{A}_k)] \\ \gamma_2(\underline{\tilde{\mathbf{g}}}_k) &= 2\mathrm{tr}[(\Sigma_k + 2\underline{\mu}_k\underline{\mu}_k^T)\mathcal{R}(\mathbf{A}_k)\Sigma_k\mathcal{R}(\mathbf{A}_k)] + \gamma_1^2(\underline{\tilde{\mathbf{g}}}_k) \end{aligned}$$

where $\underline{\mu}_k = E[\underline{\check{\mathbf{g}}}_k]$ and $\Sigma_k = \mathrm{cov}(\underline{\check{\mathbf{g}}}_k)$.

When $\mathbf{G} = \mathbf{G}_1 - z\mathbf{G}_0$ is an Hankel matrix and the elements of $\mathbf{G}_0$ and $\mathbf{G}_1$ are normally distributed with variance $\sigma^2$ as stated in the Introduction, it is easy to prove that the covariance matrix of $\underline{\mathbf{g}}_k$ does not depend on $k$ and it is a tridiagonal matrix $Z$ with $1 + |z|^2$ on the main diagonal and $-z$ and $\bar{z}$ on the secondary ones. If $z = x + iy$ it turns out that $\forall k$ the covariance matrix of $\underline{\check{\mathbf{g}}}_k$ is $\Sigma_k = \sigma^2\mathcal{R}(Z)$ where $\mathcal{R}(Z)$ is a $2 \times 2$ block tridiagonal matrix given by

$$\mathcal{R}(Z) = \begin{bmatrix} (-x, 1 + |z|^2, -x) & (y, 0, -y) \\ (-y, 0, y) & (-x, 1 + |z|^2, -x) \end{bmatrix}$$

where $(a, b, c)$ denotes a tridiagonal matrix with $b$ on the main diagonal, $a$ and $c$ on the lower and upper diagonals respectively. But then we have

$$\begin{aligned} \gamma_1(\underline{\tilde{\mathbf{g}}}_k) &= \sigma^2\mathrm{tr}[\mathcal{R}(Z)\mathcal{R}(\mathbf{A}_k)] + 2\mathrm{tr}[\underline{\mu}_k\underline{\mu}_k^T\mathcal{R}(\mathbf{A}_k)] \\ \gamma_2(\underline{\tilde{\mathbf{g}}}_k) &= 2\sigma^4\mathrm{tr}[\mathcal{R}(Z)\mathcal{R}(\mathbf{A}_k)\mathcal{R}(Z)\mathcal{R}(\mathbf{A}_k)] + 4\sigma^2\mathrm{tr}[(\underline{\mu}_k\underline{\mu}_k^T)\mathcal{R}(\mathbf{A}_k)\mathcal{R}(Z)\mathcal{R}(\mathbf{A}_k)] + \gamma_1^2(\underline{\tilde{\mathbf{g}}}_k) \end{aligned}$$

where the dependence on $\underline{\tilde{\mathbf{g}}}_k$ is only in $\mathcal{R}(\mathbf{A}_k)$. By performing the integration in Eq. (5) we get

$$\gamma_1 = \sigma^2 c + d, \qquad \gamma_2 = \sigma^4 a + \sigma^2 b + \gamma_1^2$$

where

$$\begin{aligned} a &= 2\int_{\mathbb{R}^{2p(k-1)}} \mathrm{tr}[\mathcal{R}(Z)\mathcal{R}(\mathbf{A}_k)\mathcal{R}(Z)\mathcal{R}(\mathbf{A}_k)] h(\underline{\tilde{\mathbf{g}}}_k) d\underline{\tilde{\mathbf{g}}}_k \\ b &= 4\int_{\mathbb{R}^{2p(k-1)}} \mathrm{tr}[(\underline{\mu}_k\underline{\mu}_k^T)\mathcal{R}(\mathbf{A}_k)\mathcal{R}(Z)\mathcal{R}(\mathbf{A}_k)] h(\underline{\tilde{\mathbf{g}}}_k) d\underline{\tilde{\mathbf{g}}}_k \\ c &= \int_{\mathbb{R}^{2p(k-1)}} \mathrm{tr}[\mathcal{R}(Z)\mathcal{R}(\mathbf{A}_k)] h(\underline{\tilde{\mathbf{g}}}_k) d\underline{\tilde{\mathbf{g}}}_k \\ d &= 2\int_{\mathbb{R}^{2p(k-1)}} \mathrm{tr}[\underline{\mu}_k\underline{\mu}_k^T\mathcal{R}(\mathbf{A}_k)] h(\underline{\tilde{\mathbf{g}}}_k) d\underline{\tilde{\mathbf{g}}}_k. \end{aligned}$$

But then we have

**Theorem 4.** *If the density of $\mathbf{R}_{kk}^2$ is approximated by a $\Gamma$ density with parameters $\alpha_k$, $\beta_k$ such that the first two moments of $\mathbf{R}_{kk}^2$ coincide with those of the approximant then $\beta_k$ is a nondecreasing function of $\sigma^2$ and $\alpha_k$ is a nondecreasing function of $\frac{1}{\sigma^2}$ if $\frac{\|\underline{\mu}_k\|_2^2}{\sigma^2} > \frac{E[\mathrm{tr}(\mathcal{R}(Z)\mathcal{R}(\mathbf{A}_k))]}{2E[\underline{\tilde{\mu}}_k^T\mathcal{R}(\mathbf{A}_k)\underline{\tilde{\mu}}_k]}$ where $\underline{\tilde{\mu}}_k = \frac{\underline{\mu}_k}{\|\underline{\mu}_k\|_2}$ and $E$ denotes expectation w.r. to $h(\underline{\tilde{\mathbf{g}}}_k)$.*

**Proof.**

$$\beta_k = \frac{\gamma_2 - \gamma_1^2}{\gamma_1} = \sigma^2\left[\frac{\sigma^2 a + b}{\sigma^2 c + d}\right], \qquad \alpha_k = \frac{\gamma_1^2}{\gamma_2 - \gamma_1^2} = \frac{\left(\sigma^2 c + d\right)^2}{\sigma^4 a + \sigma^2 b}.$$

Differentiating these expressions respectively with respect to $\sigma^2$ and to $\rho = \frac{d}{\sigma^2}$ where $d$ is assumed fixed and $\sigma^2$ is variable, we get

$$\frac{\partial\beta_k}{\partial\sigma^2} = \frac{bd + a\sigma^2(2d + c\sigma^2)}{(d + c\sigma^2)^2}, \qquad \frac{\partial\alpha_k}{\partial\rho} = \frac{2ad^2(\rho + c) + bd(\rho^2 - c^2)}{(ad + b\rho)^2}.$$

But $Z$ and $\mathbf{A}_k$ are positive semidefinite matrices. Therefore $\mathcal{R}(Z)$ and $\mathcal{R}(\mathbf{A}_k)$ are also positive semidefinite because their eigenvalues are the same of those of $Z$ and $\mathbf{A}_k$ with multiplicity 2. Remembering that if $X$, $Y$ are positive semidefinite matrices $\mathrm{tr}(XY)^n \geq 0,\ n = 1, 2, \ldots$, it follows that $a \geq 0, b \geq 0, c \geq 0, d \geq 0$ because the expectation of a nonnegative quantity is nonnegative. It follows that $\frac{\partial\beta_k}{\partial\sigma^2} \geq 0$. Moreover $\frac{\partial\alpha_k}{\partial\rho} \geq 0$ if

$$\rho^2 - c^2 = \frac{4}{\sigma^4}E[\mathrm{tr}(\underline{\mu}_k\underline{\mu}_k^T\mathcal{R}(\mathbf{A}_k))]^2 - c^2 = \frac{4(\underline{\mu}_k^T\underline{\mu}_k)^2}{\sigma^4}E[\underline{\tilde{\mu}}_k^T\mathcal{R}(\mathbf{A}_k)\underline{\tilde{\mu}}_k]^2 - c^2 > 0.$$

The theorem follows by noticing that $\mathrm{prob}\{\underline{\tilde{\mu}}_k^T\mathcal{R}(\mathbf{A}_k)\underline{\tilde{\mu}}_k > 0\} = 1$. In fact $\mathcal{R}(\mathbf{A}_k)$ is a random projector and the random quadratic form in the deterministic nonzero vector $\underline{\tilde{\mu}}_k$ can be zero only if $\underline{\tilde{\mu}}_k$ is orthogonal to the random eigenvectors $\underline{\mathbf{v}}_{2k-1}, \ldots, \underline{\mathbf{v}}_{2p}$ of $\mathcal{R}(\mathbf{A}_k)$ corresponding to nonzero eigenvalues. As this event has probability zero, $E[\underline{\tilde{\mu}}_k^T\mathcal{R}(\mathbf{A}_k)\underline{\tilde{\mu}}_k] > 0$. □

The idea is then to use the parameters $\beta_k$ as smoothing parameters and $\alpha_k$ as signal-related parameters. By fixing $\beta_k = \sigma^2\beta,\ \forall k$ and taking $\alpha_k = \frac{\gamma_{1k}}{\sigma^2\beta}$ the variance of $\mathbf{R}_{kk}^2(z)$ is controlled by $\beta$ and $h(z)$ can be estimated by

$$\hat{h}(z) \propto \sum_{k=1}^{p}\hat{\Delta}\left(\Psi\left[\frac{\hat{\gamma}_{1k}(z)}{\sigma^2\beta}\right]\right) \tag{6}$$

where $\hat{\Delta}$ is the discrete Laplacian and $\hat{\gamma}_{1k}(z)$ is an estimate of $\gamma_{1k}(z)$. In the following we assume that the value $\hat{R}_{kk}^2(z)$ – obtained by the $QR$ factorization of a realization of $\mathbf{G}(z)$ corresponding to a given set of observations $\{a_k\}$ – is an estimate of the mode of $\mathbf{R}_{kk}^2(z)$ and therefore

$$\hat{R}_{kk}^2(z) = \beta_k(\alpha_k - 1)$$

(see e.g [17, Chapter 17]). Then we get

$$\frac{\hat{\gamma}_{1k}(z)}{\sigma^2\beta} = \left(\frac{\hat{R}_{kk}^2(z)}{\sigma^2\beta} + 1\right).$$

From a qualitative point of view, increasing $\beta$ has the effect to make larger the support of all modes of $h(z)$ and to lower their value because $h(z)$ is a probability density. Hence the noise-related modes are likely to be smoothed out by a sufficiently large $\beta$. However a value of $\beta$ too large can result in a low resolution spectral estimate.

## 4. Computational issues in the Hankel case

To estimate $h(z)$ on a lattice we must compute the $QR$ factorization of $\mathbf{G}(z)$ for all values $z$ in the lattice. This requires $O(m^2p^3)$ flops if the lattice is square of size $m$. However we notice that $\mathbf{G}(z) = \mathbf{U}_1 - z\mathbf{U}_0 = \mathbf{U}(\mathbf{E}_1 - z\mathbf{E}_0)$ where

$$\mathbf{U} = \begin{bmatrix} \mathbf{a}_0 & \mathbf{a}_1 & \cdots & \mathbf{a}_p \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_{p+1} \\ \cdot & \cdot & \cdots & \cdot \\ \mathbf{a}_{p-1} & \mathbf{a}_p & \cdots & \mathbf{a}_{2p-1} \end{bmatrix} \in \mathbb{C}^{p\times p+1} \tag{7}$$

does not depend on $z$ and

$$\mathbf{E}_0 = [\underline{\mathbf{e}}_1 \cdots \underline{\mathbf{e}}_p] \in \mathbb{C}^{p+1\times p}, \qquad \mathbf{E}_1 = [\underline{\mathbf{e}}_2 \cdots \underline{\mathbf{e}}_{p+1}] \in \mathbb{C}^{p+1\times p}$$

and $\underline{\mathbf{e}}_k$ is the $k$-th column of the identity matrix of order $p+1$. If $\mathbf{U} = \mathbf{QR}$ is the $QR$ factorization of $\mathbf{U}$ where $\mathbf{Q} \in \mathbb{C}^{p\times p}$ is unitary and $\mathbf{R} \in \mathbb{C}^{p\times p+1}$ is upper trapezoidal, then the $QR$ factorization of $\mathbf{G}(z)$ can be obtained simply by reducing to upper triangular form by unitary transformations the Hessemberg matrix

$$\mathbf{C}(z) = \mathbf{R}(\mathbf{E}_1 - z\mathbf{E}_0) \in \mathbb{C}^{p\times p}.$$

This is the only task that must be performed for each $z$. By using Givens rotations this can be performed in $O(p)$ flops. The total cost of the $QR$ factorization of $\mathbf{G}(z)$ in the lattice reduces then to $O(m^2p + p^3)$ flops.

Finally we notice that if we start from $\mathcal{R}(\mathbf{G}) = \check{\mathbf{Q}}\check{\mathbf{R}}$, $\mathbf{C}(z)$ is a $2\times 2$ block matrix with Hessemberg diagonal blocks and triangular off-diagonal ones. Therefore it cannot be transformed to triangular form in $O(2p)$ flops.

## 5. Numerical results

In this section some experimental evidence of the claims made in the previous sections is given. In the first subsection the goodness of the approximation to the density of $\mathbf{R}^2_{kk}$ provided by the truncated Laguerre expansion is shown. In the second subsection the advantage of the closed form estimate $\hat{h}(z)$ with respect to an estimate of the condensed density obtained by MonteCarlo simulation is shown. In the third subsection the quality of the estimates of the parameters $p^*$, $\underline{\xi}$ and $\underline{c}$ which can be obtained from $\hat{h}(z)$ is assessed by a MonteCarlo simulation. Finally, in the last subsection, the reconstruction of a piecewise constant function from noisy Fourier coefficients is solved by using the proposed estimate of the condensed density and the results are compared with those obtained in [19].

### 5.1. Performance of the truncated Laguerre expansion

To appreciate the goodness of the approximation to the density of $\mathbf{R}^2_{kk}$ provided by the truncated Laguerre expansion, $N = 4\cdot 10^6$ independent realizations $a_k^{(r)}$, $k = 1,\ldots,n$, $r = 1,\ldots,N$ of the r.v. $\mathbf{a}_k$ were generated from the complex exponentials model with $p^* = 5$ components given by

$$\underline{\xi} = \left[e^{-0.1-i2\pi 0.3}, e^{-0.05-i2\pi 0.28}, e^{-0.0001+i2\pi 0.2}, e^{-0.0001+i2\pi 0.21}, e^{-0.3-i2\pi 0.35}\right]$$
$$\underline{c} = [6, 3, 1, 1, 20], \qquad n = 74, \qquad p = 37, \qquad \sigma = 0.5.$$

The matrices $U_0^{(r)}, U_1^{(r)}$ based on $a_k^{(r)}$ were computed. The matrix $U_1^{(r)} - zU_0^{(r)}$ with $z = \cos(1) + i0.8$ was formed, its $QR$ factorization and the first 10 empirical moments $\hat{\gamma}_j$ were computed. Estimates of the first 10 coefficients of the Laguerre expansion were then computed by [21]

$$\hat{\alpha}_k = \frac{\hat{\gamma}_1^2}{\hat{\gamma}_2 - \hat{\gamma}_1^2}, \qquad \hat{\beta}_k = \frac{\hat{\gamma}_2 - \hat{\gamma}_1^2}{\hat{\gamma}_1}$$
$$\hat{b}_h^{(k)} = (-1)^h\Gamma(\hat{\alpha}_k)\sum_{j=0}^{h}(-1)^j\binom{h}{j}\frac{\hat{\gamma}_{h-j}}{\Gamma(\hat{\alpha}_{k+h-j})}, \quad \hat{\gamma}_0 = 1,\ h = 1,\ldots,10.$$

The one term and ten terms approximations of the density were then computed and compared with the empirical density of $\mathbf{R}^2_{kk}$ for $k = 1,\ldots,p$. The results are given in Fig. 1. In the top left part the real part of the signal and of the data are plotted. In the top right part the $L_2$ norm of the difference between the empirical density of $\mathbf{R}^2_{kk}$, $k = 1,\ldots,p$ computed by MonteCarlo simulation and its approximation obtained by truncating the series expansion of the density after the first term and after the first 10 terms is given. In the bottom left part the density of $\mathbf{R}^2_{kk}$, $k = 36$, approximated by the first term of its series expansion and the empirical density are plotted. In the bottom right part the density of $\mathbf{R}^2_{kk}$, $k = 36$, approximated by the first 10 terms of its series expansion and the empirical density are plotted. It can be noticed that the first order approximation is quite good even if it becomes worse for large $k$. The choice $\sigma = 0.5$ is justified by the fact that this value is in the range of values used in the examples below. However the same kind of conclusions can be drawn for every signal-to-noise ratio (SNR).

### 5.2. Advantage of the closed form estimate of the condensed density

To appreciate the advantage of the closed form estimate $\hat{h}(z)$ with respect to an estimate of the condensed density obtained by MonteCarlo simulation an experiment was performed. $N = 100$ independent realizations of the r.v. generated above were considered. We notice that the frequencies of the 3rd and 4th components are closer than the Nyquist frequency ($0.21 - 0.20 = 0.01 < 1/n = 0.0135$). Hence a super-resolution problem is involved in this case. Two values of the noise s.d. $\sigma$ were used

$$\sigma = 0.2,\ 0.8$$

which give rise to different qualitative behavior of the condensed density estimate as shown below.

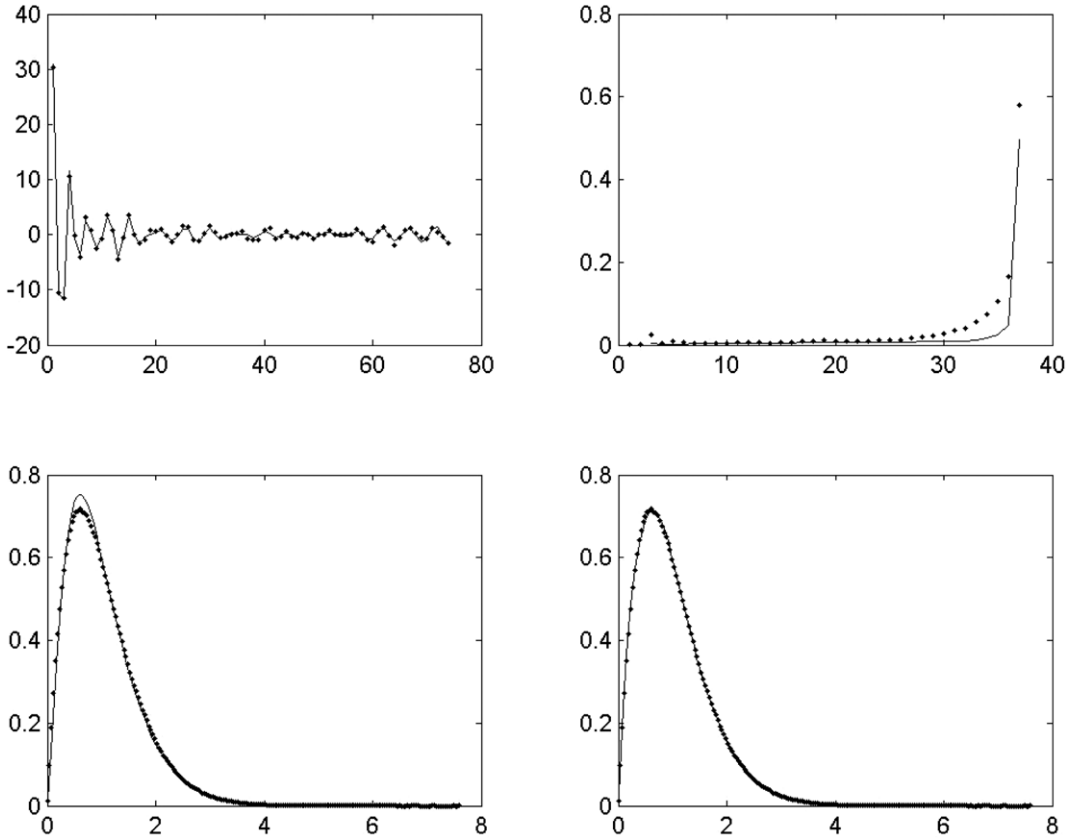


**Fig. 1.** Top left: real part of the signal (solid) and data (dotted) with $\sigma = 0.5$; top right: $L_2$ norm of the difference between the empirical density of $\mathbf{R}^2_{kk},\ k = 1, \ldots, 36$ computed by MonteCarlo simulation with $4 \cdot 10^6$ samples and its approximation obtained by truncating the series expansion of the density after the first term (dotted) and after the first 10 terms (solid); bottom left: density of $\mathbf{R}^2_{kk},\ k = 36$, approximated by the first term of its series expansion (solid), empirical density (dotted); bottom right: density of $\mathbf{R}^2_{kk},\ k = 36$, approximated by the first 10 terms of its series expansion (solid), empirical density (dotted).

An estimate of $h(z)$ was computed on a square lattice of dimension $m = 100$ by

$$\hat{h}(z) \propto \sum_{r=1}^{N} \sum_{k=1}^{p} \hat{\Delta} \left\{ \Psi \left[ \left( \frac{R^{(r)}_{kk}(z)^2}{\sigma^2 \beta} + 1 \right) \right] \right\}$$

where $R^{(r)}(z)$ is obtained by the QR factorization of the matrix $U^{(r)}_1 - zU^{(r)}_0$. In the top left part of Fig. 2 the estimate of $h(z)$ obtained by Monte Carlo simulation is plotted. In the top right part the smoothed estimates $\hat{h}(z)$ for $\sigma = 0.2$ and $\beta = 5n$ based on a single realization was plotted. In the bottom parts of Fig. 2 the results obtained with $\sigma = 0.8$ and $\beta = 5n$ are shown.

We notice that by the proposed method we get an improved qualitative information with respect to that obtained by replicated measures. This is an important feature for applications where usually only one data set is measured. We also notice that when $\sigma = 0.2$ the probability to find a root of $P(z)$ in a neighbor of $\xi_j$ is larger than the probability to find it elsewhere. This is no longer true when $\sigma = 0.8$ even if the signal-related complex exponentials are well separated.

In the following we will say that the complex exponential model is identifiable if this last case occurs and it is strongly identifiable if the first case occurs. Therefore if the model is identifiable the signal-related complex exponentials are well separated but the relative importance of some of them – measured by the value of the local maxima of $h(z)$ – is not larger than the relative importance of some noise-related complex exponentials. Therefore in this case we need some a-priori information about the location of the $\xi_j$ in order to separate signal-related components from the noise-related ones.

### 5.3. Estimates of the complex exponentials parameters

We want now to show by means of a small simulation study the quality of the estimates of the parameters $p^*$, $\underline{\xi}$ and $\underline{c}$ which can be obtained from $\hat{h}(z)$. To this aim the following estimation procedure was used:

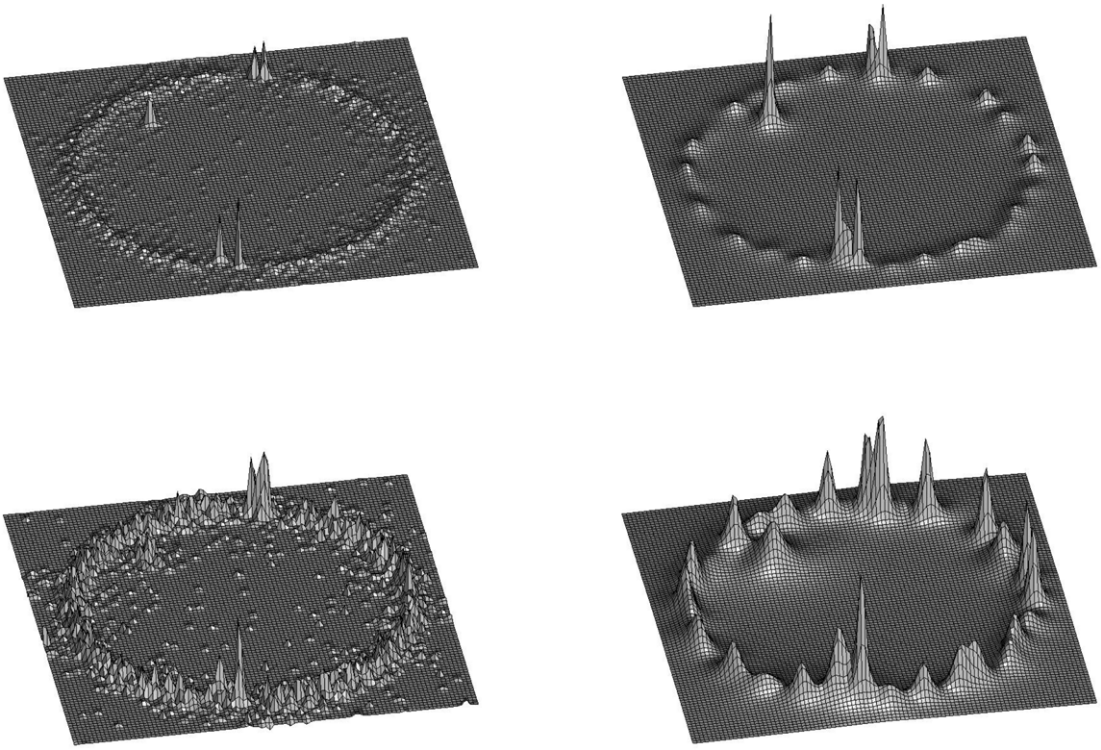

**Fig. 2.** Top left: Monte Carlo estimate of the condensed density when $\sigma = 0.2$; top right: estimate of the condensed density by the closed form approximation with $\beta = 14.8$. Bottom left: Monte Carlo estimate of the condensed density when $\sigma = 0.8$; bottom right: estimate of the condensed density by the closed form approximation with $\beta = 237$.

- the local maxima of $\hat{h}(z)$ are computed and sorted in decreasing magnitude
- a clustering method is used to group the local maxima into two groups. If the model is strongly identifiable the signal-related maxima are larger than the noise-related ones, therefore a simple thresholding is enough to separate the two groups. A good threshold is the one that produces an estimate of $s_k$ which best fits the data $a_k$ in $L_2$ norm as the noise is assumed to be Gaussian
- the cardinality $\hat{p}$ of the class with largest average value is an estimate of $p^*$
- the local maxima $\hat{\xi}_j,\ j = 1, \ldots, \hat{p}$ of the class with largest average value are estimates of $\xi_j,\ j = 1, \ldots, p^*$. Of course if $\hat{p} \neq p^*$ some $\xi_j$ are not estimated or viceversa some spurious complex exponentials are found
- $\underline{c}$ is estimated by solving the linear least squares problem

$$\hat{\underline{c}} = \underset{\underline{x}}{\operatorname{argmin}} \|V\underline{x} - \underline{a}\|_2^2, \quad \underline{a} = [a_0, \ldots, a_{n-1}]^T$$

where $V \in \mathbb{C}^{n\times\hat{p}}$ is the Vandermonde matrix based on $\hat{\xi}_j,\ j = 1, \ldots, \hat{p}$.

The bias, variance and mean squared error (MSE) of each parameter separately were estimated. $N = 500$ independent data sets $\underline{a}^{(r)}$ of length $n$ were generated by using the model parameters given above and $\sigma = 0.2$. For $r = 1, \ldots, N$ the condensed density estimate $\hat{h}^{(r)}(z)$ was computed on a square lattice of dimension $m = 100$. The estimation procedure is then applied to each of the $\hat{h}^{(r)}(z),\ r = 1, \ldots, N$ and the corresponding estimates $\hat{\xi}_j^{(r)}, \hat{c}_j^{(r)},\ j = 1, \ldots, \hat{p}^{(r)}$ of the unknown parameters were obtained. If the estimate $\hat{p}^{(r)}$ was less than the true value $p^*$, the corresponding data set $\underline{a}^{(r)}$ was discarded.

In Table 1 the bias, variance and MSE of each parameter including $p^*$ is reported. They were computed by choosing among the $\hat{\xi}_j^{(r)},\ j = 1, \ldots, \hat{p}^{(r)} \geq p^*$ the one closest to each $\xi_k,\ k = 1, \ldots, p^*$ and the corresponding $\hat{c}_j^{(r)}$. If more than one $\xi_k$ is estimated by the same $\hat{\xi}_j^{(r)}$ the $r$-th data set $\underline{a}^{(r)}$ was discarded. In the case considered all the data sets were accepted.

### 5.4. Reconstruction of a piecewise constant function from noisy Fourier coefficients

As a second example the reconstruction of a piecewise constant function from noisy Fourier coefficients is considered. The problem is stated as follows. Given a real interval $[-\pi, \pi]$ and $N + 1$ numbers $-\pi \leq l_1 < l_2 \cdots < l_{N+1} \leq \pi$, let $\mathcal{F}$ be

**Table 1**
Statistics of the parameters $\hat{p}$, $\hat{\xi}_j$, $j = 1, \ldots, p^*$ and $\hat{c}_j$, $j = 1, \ldots, p^*$.

| | $p^*$ | bias($\hat{p}$) | s.d. ($\hat{p}$) | MSE($\hat{p}$) |
|---|---|---|---|---|
| | 5 | 0.0000 | 0.0000 | 0.0000 |
| | $\xi_j$ | bias($\hat{\xi}_j$) | s.d. $\hat{\xi}_j$ | MSE($\hat{\xi}_j$) |
| $j = 1$ | $-0.2796 - 0.8606i$ | $-0.0008 + 0.0001i$ | 0.0000 | 0.0000 |
| $j = 2$ | $-0.1782 - 0.9344i$ | $0.0036 - 0.0010i$ | 0.0000 | 0.0000 |
| $j = 3$ | $0.3090 + 0.9510i$ | $0.0057 - 0.0064i$ | 0.0031 | 0.0001 |
| $j = 4$ | $0.2487 + 0.9685i$ | $-0.0058 + 0.0110$ | 0.0019 | 0.0002 |
| $j = 5$ | $-0.4354 + 0.5993i$ | $-0.0047 + 0.0054i$ | 0.0108 | 0.0002 |
| | $c_j$ | bias($\hat{c}_j$) | s.d. ($\hat{c}_j$) | MSE($\hat{c}_j$) |
| $j = 1$ | 6.0000 | 0.0440 | 0.1238 | 0.0173 |
| $j = 2$ | 3.0000 | −0.0407 | 0.0688 | 0.0064 |
| $j = 3$ | 1.0000 | 0.0441 | 0.0736 | 0.0074 |
| $j = 4$ | 1.0000 | −0.6767 | 0.0808 | 0.4644 |
| $j = 5$ | 20.0000 | −0.1007 | 0.2574 | 0.0764 |

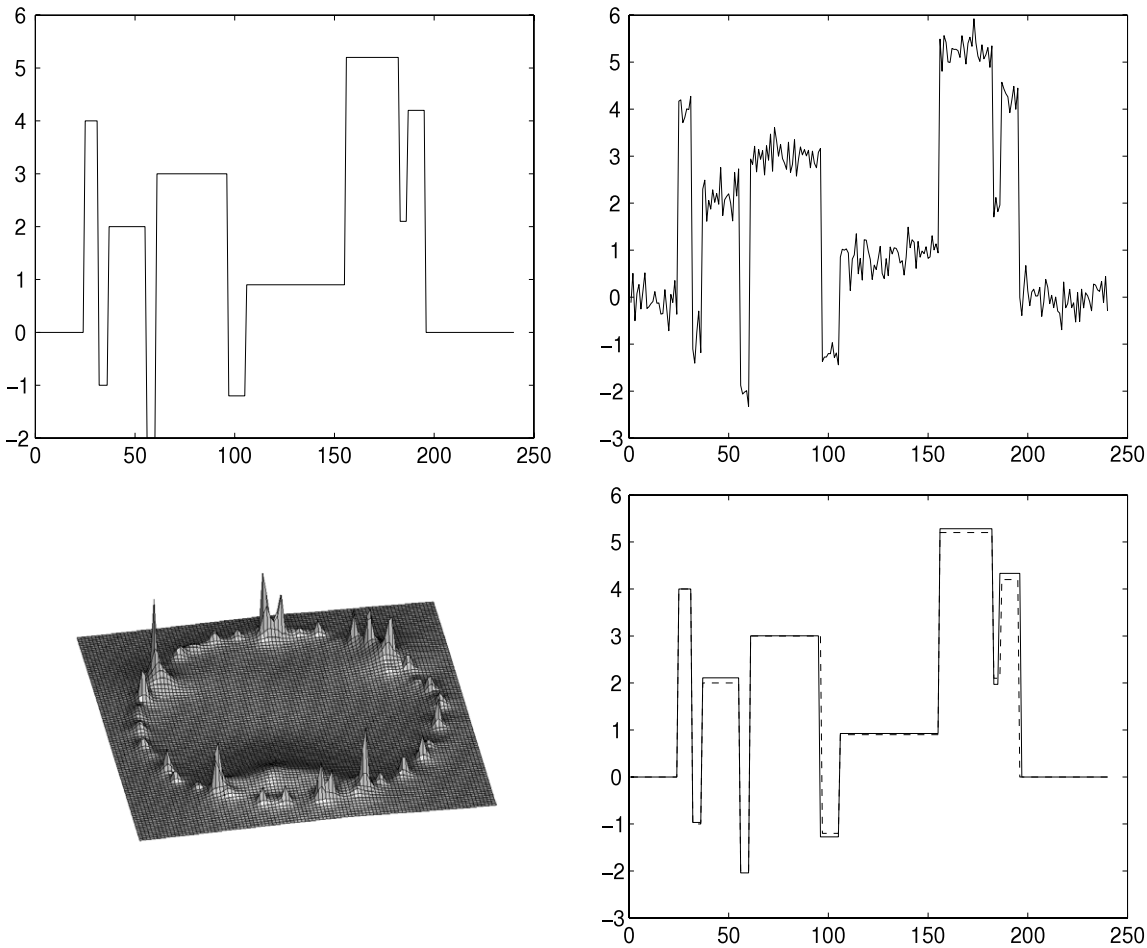

**Fig. 3.** Top left: original function; top right: rough estimate of $F(t)$ when the moments are affected by a Gaussian noise with SNR $= 7$. Bottom left: estimate of the condensed density by the closed form approximation; bottom right: reconstruction of the original function.

the class of functions defined as

$$F(t) = \sum_{j=1}^{N} w_j \chi_j(t),$$

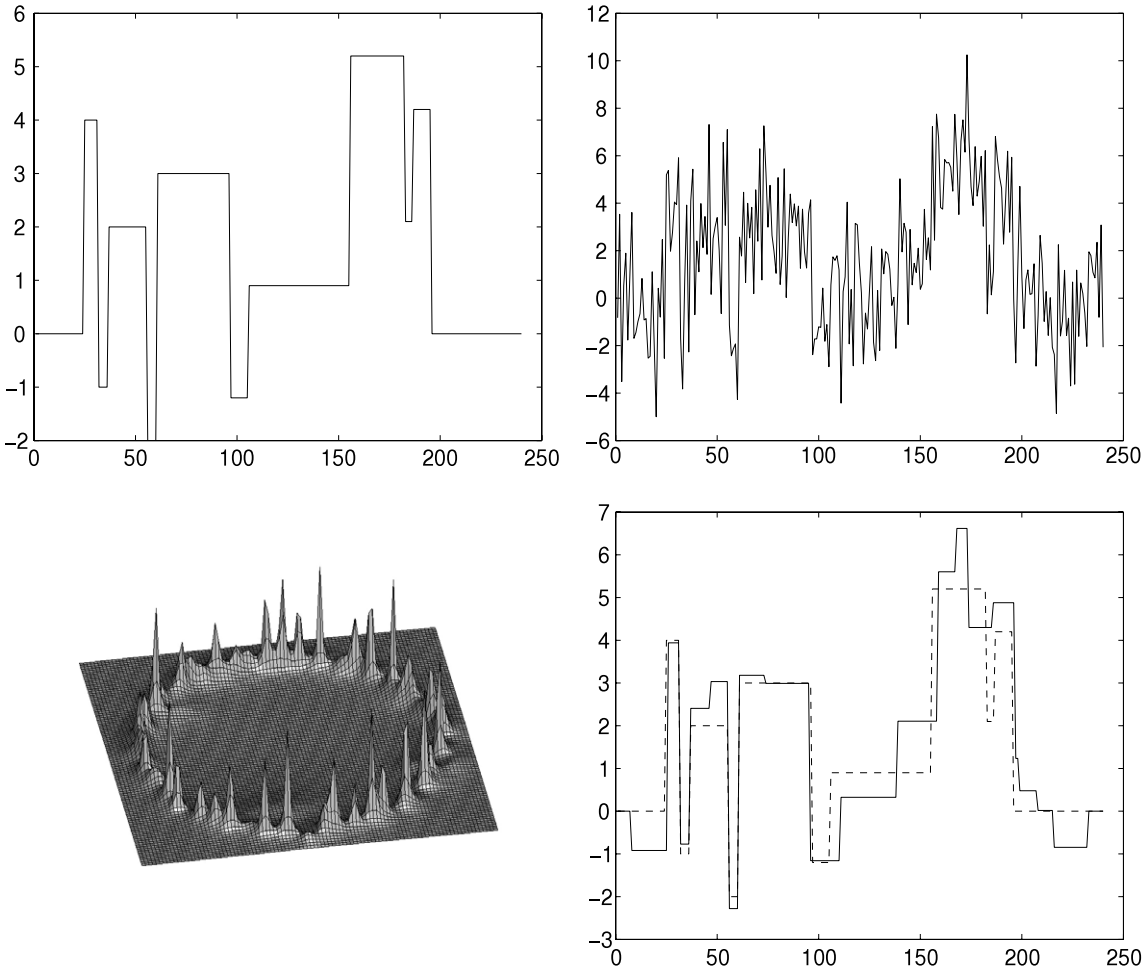

**Fig. 4.** Top left: original function; top right: rough estimate of $F(t)$ when the moments are affected by a Gaussian noise with SNR $= 1$. Bottom left: estimate of the condensed density by the closed form approximation; bottom right: reconstruction of the original function.

where

$$\chi_j(t) = \begin{cases} 1 & \text{if } t \in [l_j, l_{j+1}] \\ 0 & \text{otherwise,} \end{cases}$$

and the $w_j$ are real weights. The problem consists of reconstructing a function $F(t) \in \mathcal{F}$ from a finite number of its noisy Fourier coefficients

$$\mathbf{a}_k = \frac{1}{2}\int_{-\pi}^{\pi} F(t)e^{itk}dt + \boldsymbol{\epsilon}_k = s_k + \boldsymbol{\epsilon}_k, \quad k = 0, \ldots, n-1,$$

where $\boldsymbol{\epsilon}_k$ is a complex Gaussian, zero mean, white noise, with variance $\sigma^2$. We are looking for a solution which is not affected by Gibbs artifact and can cope, stably, with the noise.

The basic observation is the following. The unperturbed moments $s_k$ are given by

$$s_k = \frac{1}{2}\int_{-\pi}^{\pi} F(t)e^{itk}dt = \sum_{j=1}^{N} w_j \frac{\sin(\beta_j k)}{k} \exp(i\lambda_j k),$$

where

$$\beta_j = \frac{l_{j+1} - l_j}{2}, \qquad \lambda_j = \frac{l_{j+1} + l_j}{2}.$$

Then consider the $Z$-transform of the sequence $\{s_k\}$

$$s(z) = \sum_{j=1}^{N} w_j \left(\beta_j + \frac{1}{2i}\log\frac{z - e^{il_j}}{z - e^{il_{j+1}}}\right)$$

which converges if $|z| > 1$ and is defined by analytic continuation if $|z| \leq 1$. We notice that $s(z)$ has a branch point at $\xi_j = e^{il_j},\ j = 1, \ldots, N+1$ where $l_j$ are the discontinuity points of $F(t)$.

It was proved in [18,19] that the $\xi_j$ are strong attractors of the poles of the Padé approximants $[q, r]_f(z)$ to the noisy $Z$-transform

$$f(z) = \sum_{k=0}^{\infty} a_k z^{-k}$$

when $q, r \to \infty$ and $q/r \to 1$. It is easy to show that the poles of $[q, r]_f(z)$ are the generalized eigenvalues of the pencil $(U_1, U_0)$ built from the data $a_k,\ k = 0, \ldots, n-1$ whose condensed density is $h(z)$. Therefore, as shown in [18,19], the local maxima of $h(z)$ are concentrated along a set of arcs which ends in the branch points $\xi_j$ and on a set of arcs close to the unit circle.

As the branch points are strong attractors for the Padé poles, the probability to find a pole in a neighbor of a branch point is larger than elsewhere, therefore it can be expected that the branch points correspond to the largest local maxima of $h(z)$, as far as the SNR is sufficiently large.

In order to compute estimates $\hat{l}_j$ of $l_j$, it is sufficient to compute the arguments of the main local maxima of $\hat{h}(z)$. The $w_j$ are then estimated by taking the median in each interval $[\hat{l}_j, \hat{l}_{j+1}]$ of the rough estimate of $F(t)$ obtained by taking the discrete Fourier transform of $a_k,\ k = 0, \ldots, n-1$. The median is in fact robust with respect to errors affecting $\hat{l}_j$.

The method was applied to an example considered in [19] where comparisons with other methods were also reported. In the top left part of Fig. 3 the original function $F(t)$ is plotted. In the top right the rough estimate of $F(t)$ when SNR $= 7$ is reported where the SNR is measured as the ratio of the standard deviations of $\{s_k\}$ and $\{\epsilon_k\}$. In the bottom parts the condensed density and the reconstructed function $\hat{F}(t)$ are plotted. Looking at the condensed density we notice that the model is strongly identifiable, therefore the estimation procedure outlined above was applied. In Fig. 4 the same quantities as above but with SNR $= 1$ are plotted. In this case the model is identifiable but not strongly therefore the clustering step does not work. The number of complex exponentials used to get the reconstruction plotted in Fig. 4 is $\hat{p} = 20$ and was found by trial and errors.

We notice that when SNR $= 7$ we get an almost perfect reconstruction, better than that reported in [19]. When SNR $= 1$ the reconstruction quality is worse as expected but still comparable with the one reported in [19].

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