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Hawking radiation as tunneling derived from black hole thermodynamics through the quantum horizon

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ABSTRACT

We show that the first law of the black hole thermodynamics can lead to the tunneling probability through the quantum horizon by calculating the change of entropy with the quantum gravity correction and the change of surface gravity is presented clearly in the calculation. The method is also applicable to the general situation which is independent on the form of black hole entropy and this verifies the connection of black hole tunneling with thermodynamics further. In the end we discuss the crucial role of the relation between the radiation temperature and surface gravity in this derivation.

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1. Introduction

About 30 years ago, Hawking discovered [1] that when considering quantum effect black holes could radiate particles as if they were hot bodies with the temperature $\kappa/2\pi$ where κ was the surface gravity of the black hole and explained [2] the particles of radiation as stemming from vacuum fluctuations tunneling through the horizon of the black hole with Hartle together. But the semiclassical derivation of Hawking based on the Bogoliubov transformation didn't have the directly connection with the view of tunneling. Parikh and Wilczek [3] calculated directly the particle flux from the tunneling picture and made the tunneling physical explanation holds firm basis. In their consideration the energy conservation played a fundamental role and the outgoing particle itself created the barrier [4]. After this, there have been some works which have extended the Parikh–Wilczek tunneling framework to different cases [5,6] and the question of information loss has been discussed in this framework [7,8]. Recently the general approach has been suggested [9] for the tunneling of matter from the horizon by using the first law of thermodynamics or the con-

servation of energy. On the other side the tunneling probability has also been calculated [10] directly through the change of the entropy that is proportional to area by the first law of thermodynamics, which verifies the connection of black hole radiation with thermodynamics [11] further.

We have noticed that when the quantum gravity effect is considered the tunneling formula can also be obtained by Parikh–Wilczek method and the Hawking temperature relation [12–14]. In this Letter we will proceed this kind of consideration by using the same method as in Ref. [10] but for the entropy which is modified by the logarithmic term caused by quantum gravity effect as in Ref. [12]. In the new method we show clearly the necessary change of the surface gravity when considering the quantum gravity effect and the crucial role which the Hawking temperature relation plays. We note that the method could be extended to general situation where the tunneling probability is obtained by calculating the change of entropy, independent on the form of the entropy, from the first law of black hole thermodynamics. The generalization verifies the connection of black hole tunneling with thermodynamics further.

In this Letter we take the unit convention $k = \hbar = c = G = 1$.

2. The first law of black hole thermodynamics and entropy

The first law of black hole thermodynamics [15] states:

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If one throws a small amount of mass into a static non-charged and non-rotated black hole, it will settle down to a new static black hole.¹ This change can be described as $dM = \frac{\kappa}{8\pi} dA$, which is analogue to the usual first law of thermodynamics $dM = T dS$. The case is the same for radiating a small amount of mass from black hole [11].

According to Hawking, the temperature of black hole is taken as $T = \frac{\kappa}{2\pi}$, so the entropy can be obtained as $S = \frac{1}{4}A$. It has been shown [10] that the tunneling formulas for static, spherically symmetric black hole radiation are obtained by the first law of thermodynamics and the area-entropy relation, even if the radiation temperature is different from the Hawking temperature. From the first law of black hole thermodynamics, we can see that if the black hole temperature is changed, the area-entropy relation will also be changed. Note that in Ref. [10] the author calculated the tunneling probability by using the entropy being proportional to horizon area and so the temperature was also proportional to the surface gravity. But when considering the entropy which is modified by the logarithmic term due to quantum gravity effect [12], it looks as if the black hole temperature were not proportional to the black hole surface gravity. Then in such situation, could the tunneling probability be obtained by calculating the change of entropy with log-area term modification when considering the quantum gravity effect in the same way as in Ref. [10]? The answer is positive! Before discussing this problem, we will first present the method proposed by Pilling.

3. Thermodynamics and tunneling

In this section we will review the method, presented in Ref. [10], which is used to obtain the tunneling probability directly from black hole thermodynamics. Let us start by writing the metric for a general spherically symmetric system in ADM form [16],

$$ds^2 = -N_t(t, r)^2 dt^2 + L(t, r)^2 [dr + N_r(t, r) dt]^2 + R(t, r)^2 d\Omega^2. \quad (1)$$

The metric is used for the situation where the geometry is spherically symmetric and has a Killing vector which is timelike outside the horizon. Specially one can consider the case of a massless particle and fix the gauge appropriately ($L = 1, R = r$) which is particularly useful to study across horizon phenomena. So the metric becomes

$$ds^2 = -N_t(r)^2 dt^2 + [dr + N_r(r) dt]^2 + r^2 d\Omega^2. \quad (2)$$

The metric is well behaved on the horizon and for a four-dimensional spherically Schwarzschild solution, $N_t = 1, N_r = \sqrt{\frac{2M}{r}}$ (M is the mass of the black hole), for a four-dimensional Reissner-Nordstrom solution, $N_t = 1, N_r = \sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}}$ (M is the mass and Q is the charge of the black hole). And we also note that for $N_t = \sqrt{\frac{f(r)}{g(r)}, N_r = f(r)\sqrt{\frac{1-g(r)}{f(r)g(r)}}$, the metric (2) becomes the same as that in Ref. [10].

Now let us consider the Parikh-Wilczek tunneling [3]. Supposed the mass of the black hole is fixed and the mass is allowed to fluctuate, then the shell of energy E travels on the geodesics given by the line element (2). Taking into account self-gravitation effects, the outgoing radial null geodesics near the horizon are given approximately by

$$\dot{r} = N_t(r) - N_r(r) \simeq (N'_t(R) - N'_r(R))(r - R) + O((r - R)^2), \quad (3)$$

where the horizon, $r = R$, is determined from the condition $N_t(R) - N_r(R) = 0$ and the last formula is the expansion of the radial geodesics in power of $r - R$.

According to the definition of a time-like Killing vector the surface gravity of the black hole near the horizon is obtained as

$$\kappa \simeq N'_t(R) - N'_r(R). \quad (4)$$

So the radiation temperature is

$$T = \frac{\kappa}{2\pi} = \frac{N'_t(R) - N'_r(R)}{2\pi}. \quad (5)$$

Now we consider the black hole thermodynamics in the region near the horizon. The change of the Bekenstein-Hawking entropy, if the mass of black hole changes from M_i to M_f , is given as

$$\Delta S = \int_{M_i}^{M_f} \frac{dS}{dM} dM = \int_{M_i}^{M_f} 2\pi R \frac{dR}{dM} dM. \quad (6)$$

Considering the small path near R , we can insert the mathematical identity $\text{Im} \int_{r_i}^{r_f} \frac{1}{r-R} dr = -\pi$ in the formula (6). Thus we obtain

$$\Delta S = -2 \text{Im} \int_{M_i}^{M_f} \int_{r_i}^{r_f} \frac{R}{r-R} \frac{dR}{dM} dM. \quad (7)$$

Using (5) and the expression of the temperature in thermodynamics $\frac{1}{T} = \frac{\partial S}{\partial E}$, we attain

$$R \frac{dR}{dM} = \frac{1}{N'_t(R) - N'_r(R)}. \quad (8)$$

Then Eqs. (3) and (8) give the final form of the change of entropy (7) as

$$\Delta S = -2 \text{Im} \int_{M_i}^{M_f} \int_{r_i}^{r_f} \frac{dR}{\dot{r}} dM = -2 \text{Im} I, \quad (9)$$

where I is the action for an s-wave outgoing positive particle in WKB approximation.

So the tunneling probability is given as

$$\Gamma \sim e^{\Delta S} = e^{-2 \text{Im} I}. \quad (10)$$

Thus we obtain the tunneling probability from the change of entropy as a direct consequence of the first law of black hole thermodynamics by using the same method as that in Ref. [10]. Let us emphasize that in the original method the author uses the general radiation temperature different from the Hawking temperature in order to discuss the factor of 2 problem. However the new temperature is still proportional to the surface gravity like the Hawking temperature and only the proportional relation is crucial for the discussed problem in this Letter. So we take the Hawking temperature as the black hole temperature without loss of generality.

4. The tunneling through the quantum horizon

We note that for spherically symmetric black holes a generalized treatment [9] has been suggested, in which the tunneling probability is gotten directly from the principle of conservation of energy by calculating the imaginary part of the action in WKB approximation and the method is independent on the form of black hole entropy. For the entropy which is proportional to area [3] or contains the logarithmic modification caused by the presence of quantum gravity [12], we know that the tunneling probability has been obtained by calculating the imaginary part of the action in WKB approximation. Recently Pilling has suggested that the tunneling probability is obtained directly from the first law of thermodynamics by calculating the change of the entropy being proportional to area, even if the radiation temperature is different

¹ The law is also applicable to the general situation such as that of charged and rotating black holes.

from the Hawking temperature [10]. Then could the Pilling method be applied to the situation where the entropy is modified by logarithmic term when considering quantum gravity effect? In the following we will discuss the problem.

First we take into account the modification of entropy caused by the presence of quantum gravity effect which gives a leading order correction with a logarithmic dependence on the area besides reproducing the familiar Bekenstein–Hawking linear relation [17–19]

$$S_{\text{QG}} = \frac{A}{4L_p^2} + \alpha \ln \frac{A}{L_p^2} + O\left(\frac{L_p^2}{A}\right), \quad (11)$$

where A is the area of black hole horizon and L_p is the Planck length. The relation exists in string theory and loop quantum gravity. The difference is that α is negative in the case of loop quantum gravity [20], but in String Theory the sign of α depends on the number of field species appearing in the low energy approximation [21]. It is noted that there is an interesting phenomenon that this log-area correction is closely related to black hole remnant when the coefficient α is negative [22].

Along Pilling's line we calculate the tunneling probability by using the entropy modified by quantum gravity effect. For brevity, we write

$$S_{\text{QG}} = \frac{1}{4}A + \alpha \ln A = \pi R^2 + \alpha \ln(4\pi R^2), \quad (12)$$

where the logarithmic correction can also be obtained by considering the one-loop effects of quantum matter fields near a black hole [17,19]. Whatever consideration we take, the spacetime will change. If we continue to use the spacetime represented by (2), the wrong result will be gotten. We can see this point clearly from the following calculation.

When the mass of the black hole changes from M_i to M_f , we have

$$\Delta S = \int_{M_i}^{M_f} \frac{dS}{dM} dM = \int_{M_i}^{M_f} \left(2\pi R + \frac{2\alpha}{R}\right) \frac{dR}{dM} dM = \Delta S_1 + \Delta S_2, \quad (13)$$

where $\Delta S_1 = \int_{M_i}^{M_f} 2\pi R \frac{dR}{dM} dM$, $\Delta S_2 = \int_{M_i}^{M_f} \frac{2\alpha}{R} \frac{dR}{dM} dM$. It is noted that if we continue to calculate according to the same method presented in the section above, it will be found that $\Delta S_1 = -2\text{Im} \int_{M_i}^{M_f} \int_{r_i}^{r_f} \frac{dR}{r} dM$ by using the surface gravity (4). It seems that ΔS_2 is not related to the action of the black hole and so is not related to the tunneling probability. This is inconsistent with the result obtained in Ref. [12] where

$$\Gamma(E) \sim e^{-2\text{Im}I} = \left(1 - \frac{E}{M}\right)^{2\alpha} e^{(-8\pi ME(1 - \frac{E}{2M}))}. \quad (14)$$

Consequently, we see that the imaginary part of the action is expressed as the change of the whole entropy but not that of partial entropy. The calculation above shows that when considering the entropy with logarithmic correction, the spacetime will change and carry the quantum gravity effect. On the other side we note that in Ref. [10] the author uses the entropy being proportional to area, so the radiation temperature is obviously proportional to the surface gravity. But here we take the entropy $S_{\text{QG}} = \frac{1}{4}A + \alpha \ln A$, it looks as if formally the temperature were not proportional to the surface gravity according to the first law of black hole thermodynamics. A straight way to contain the quantum gravity effect is to use the thermodynamic relation to get the surface gravity afresh. From thermodynamics, the temperature can be given as

$$\frac{1}{T} = \frac{dS}{dM} = \left(2\pi R + \frac{2\alpha}{R}\right) \frac{dR}{dM} \equiv \frac{2\pi}{\kappa}. \quad (15)$$

Thus we can get the surface gravity in the entropy with logarithmic modification as

$$\kappa \equiv 2\pi / \frac{dS_{\text{QM}}}{dM} = \frac{2\pi}{\left(2\pi R + \frac{2\alpha}{R}\right) \frac{dR}{dM}}, \quad (16)$$

where the surface gravity is not only dependent on the mass of the black hole but also dependent on the coefficient α which accords with the consideration that the surface gravity carries the quantum gravity correction.

We note that if we want to obtain the relation $\Delta S = -2\text{Im}I$ as that in the section above when considering the entropy with logarithmic correction, we have to find the method to calculate the radial null geodesic trajectory which is difficult to be calculated because we do not know the property of such spacetime clearly. At the same time it has been pointed out that the quantum entropy comes from counting states in a quantum theory, whereas geodesics make sense in a classical spacetime. So the concept of geodesics has to be managed carefully when the logarithmic correction of entropy is explained as quantum gravity effect [12–14]. However, the attained result is consistent with the explanation of the tunneling probability of quantum mechanics. Thus the feasibility of using the concept of geodesics means that there maybe exist the physical reason to explain the mathematical consistency. We note that the logarithmic correction of the black hole entropy can be obtained from the purely quantum gravity effect and can also be obtained from the one-loop effects of quantum matter fields near a black hole [19]. The difference lies in the value of the parameter α , but the problem here is not concerned about it. So we can calculate the geodesics by considering the one-loop effects of quantum matter fields near a black hole. It is noted that in such consideration the expression of spacetime presented in (2) could still be used [17,23], but some quantities, such as the mass, the temperature, the surface gravity and so on, has to be changed. On the other hand we also note that here the modification of surface gravity (16) is consistent with the result obtain by considering the one-loop correction as in Refs. [23,24]. For example, for Schwarzschild spacetime, the classical surface gravity is expressed as $\kappa_0 = \frac{1}{4M}$ and the radius is $R = 2M$, so by Eq. (16) the modified surface gravity can be gotten as $\kappa \simeq \kappa_0(1 - \frac{\alpha}{4\pi M^2})$ which accords with the modified surface gravity due to one loop back reaction effects [23,24].

In the following we will show that the tunneling probability can be recovered by calculating the change of the entropy with logarithmic modification. By (16), we can write the change of entropy as

$$\Delta S = \int_{M_i}^{M_f} \left(2\pi R + \frac{2\alpha}{R}\right) \frac{dR}{dM} dM = \int_{M_i}^{M_f} \frac{2\pi dM}{\kappa}. \quad (17)$$

Because the spacetime (2) can still be used, so the radial null geodesic trajectory is written as [25,26],

$$\dot{r} \simeq \kappa(r - R), \quad (18)$$

where the formula can be gotten by using Eqs. (3) and (4) and it must be stressed that here the surface gravity and the event horizon have been changed and are different from that appeared in the section above. Thus we replace the surface gravity in Eq. (17) by Eq. (18), insert the mathematical identity $\text{Im} \int_{r_i}^{r_f} \frac{1}{r-R} dr = -\pi$ and have

$$\Delta S = -2\text{Im} \int_{M_i}^{M_f} \int_{r_i}^{r_f} \frac{1}{\dot{r}} dr dM = -2\text{Im}I. \quad (19)$$

So the tunneling probability is gotten as

$$\Gamma \sim e^{\Delta S} = e^{-2\text{Im} \int_{r_i}^{r_f} p_r dr}. \quad (20)$$

In this way we have finished the calculation of black hole tunneling probability from the first law of thermodynamics when considering the quantum gravity effect. In the derivation the introduction of the new surface gravity is a crucial step because this maintains the general expression of the relation between radiation temperature and surface gravity. We can see that the Hawking temperature relation $T = \frac{\kappa}{2\pi}$ or the proportional relation between the radiation temperature and surface gravity is key in the calculation here as that in [9,10,12]. In general, when we discuss the connection of the black hole radiation as tunneling with thermodynamics, only if we accept the Hawking temperature relation or the proportional relation between the temperature and surface gravity, can we obtain the tunneling probability directly from the first law of the black hole thermodynamics, not dependent on the form of entropy of the black hole, which is seen by writing the change of the entropy as

$$\Delta S = \int_{M_i}^{M_f} \frac{dS}{dM} dM = \int_{M_i}^{M_f} \frac{1}{T} dM = \int_{M_i}^{M_f} \frac{2\pi dM}{\kappa}. \quad (21)$$

Thus we can conclude that it is the relation between black hole temperature and surface gravity that plays the crucial role that relates the black hole thermodynamics with the tunneling picture of the black hole. At the same time the concept of geodesics has to be managed carefully.

After the Hawking temperature was discovered, there have also been several other methods [2,25] to derive the same result as that obtained by Hawking [1]. Recently, however, it has been pointed out [27] that the tunneling approach produces a temperature that is double the original Hawking temperature, which is used to question either the tunneling methods or the value of Hawking temperature. This problem is discussed again in [28] where the authors consider the incoming solution besides the outgoing solution and uses the ratio of the outgoing and incoming probabilities to recover the Hawking temperature. The factor of 2 problem about black hole temperature is also discussed generally in Ref. [10].

5. Conclusion

We have showed that the tunneling probability can be obtained from the first law of thermodynamics by using the entropy with logarithmic modification which contains the quantum gravity effect and the change of the surface gravity has been presented

clearly in the calculation. We have also showed the important role that the relation between the radiation temperature and the surface gravity plays. One can note that our derivation can be generalized only by starting from the first law of thermodynamics $dM = T dS$ and the relation $T = \frac{\kappa}{2\pi}$ instead of considering the form of the black hole entropy. The generalization verifies the connection of black hole radiation with thermodynamics further.

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