q-Analog of the Möbius function and the cyclotomic identity associated to a profinite group

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Abstract

Let $G$ be a profinite group and $q$ an indeterminate. In this paper, we introduce and study a $q$-analog of the Möbius function and the cyclotomic identity arising from the lattice of open subgroups of $G$. When $q$ is any integer, we show that they have close connections with the functors $\mathbb{W}_q^G$, $\mathbb{N}_q^G$, and $\hat{\mathbb{N}}_q^G$ introduced in [Y.-T. Oh, $q$-Deformation of Witt–Burnside rings, Math. Z. 257 (2007) 151–191]. In particular, we interpret the multiplicative property of the inverse of the table of marks and the Möbius function of $G$ as a composition property of certain functors. Classification of $\mathbb{W}_q^G$, $\mathbb{N}_q^G$, and $\hat{\mathbb{N}}_q^G$ up to strict natural isomorphism as $q$ varies over the set of integers and its application will be dealt with, too.

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1. Introduction

The Möbius function of a partially ordered set has occupied a central position within algebraic combinatorics and has deep connections with other areas of mathematics. It was Weisner and Hall that first applied it to the theory of groups. For example, Weisner [23] found the number of $k$-tuples of elements of a finite $p$-group, say $G$, which generate $G$ by computing the Möbius function of the lattice of subgroups of $G$. In [3], Delsarte considered the Möbius function asso-
associated to a finite abelian group and exploited it to derive a formula for the number of subgroups of a given type of a prime power abelian group.

In 1983, Metropolis and Rota [12] introduced necklace rings to study the structure of the universal ring of Witt vectors and showed there implicitly that the classical Möbius inversion function arises naturally from the natural transformation

$$\varphi : \text{Nr} \rightarrow \text{gh}$$

from the functor of necklace rings to that of ghost rings. Indeed, \(\varphi\) is given by the \(\mathbb{N} \times \mathbb{N}\) matrix \(\zeta\) defined by

$$\zeta(i, j) = \begin{cases} 1 & \text{if } j|i, \\ 0 & \text{otherwise.} \end{cases}$$

Dress and Siebeneicher [6] showed that \(\text{Nr}(\mathbb{Z})\), the necklace ring over \(\mathbb{Z}\), is isomorphic to the Burnside ring \(\hat{\Omega}(C)\) of the infinite cyclic group \(C\). On the other hand, \(\text{Nr}\) can be naturally recovered from \(\hat{\Omega}(C)\) since its operations are determined by a family of polynomials with integer coefficients. This approach made it possible to explain the construction due to Metropolis and Rota in a group-theoretic setting. To be more precise, given any profinite group \(G\), let \(\hat{\Omega}(G)\) be the Burnside–Grothendieck ring of \(G\) of isomorphism classes of almost finite \(G\)-spaces introduced in [5]. As in the case of \(C\), we obtain a covariant functor \(\hat{\text{Nr}}_G\) from the category of commutative rings with identity into itself from the operations of \(\hat{\Omega}(G)\). By construction it follows that

(i) \(\hat{\text{Nr}}_G(\mathbb{Z}) \cong \hat{\Omega}(G)\) and (ii) \(\hat{\text{Nr}}_{\hat{C}} = \text{Nr}\).

Here, \(\hat{C}\) denotes the profinite completion of \(C\). Generalizing this construction further, in [15,16], it has been shown that there exists a unique covariant functor \(\text{Nr}_G\), from the category of special \(\lambda\)-rings to the category of commutative rings with identity. In fact, \(\text{Nr}_G\) is more suitable in studying the Witt–Burnside ring functor, denoted by \(\mathbb{W}_G\), of a profinite group \(G\) introduced by Dress and Siebeneicher [5]. Also, it should be mentioned that \(\text{Nr}_G\) is naturally isomorphic to \(\hat{\text{Nr}}_G\) when they are regarded as a functor from the category of binomial rings to the category of commutative rings with identity.

Quite surprisingly, \(\hat{\text{Nr}}_G\) turns out to have a \(q\)-deformation. It follows from [17] that given any profinite group \(G\) and any integer \(q\), there exists a covariant functor \(\hat{\text{Nr}}_G^q\) from the category of commutative rings (not necessarily unital) into itself. When \(q = 1\), it coincides with \(\hat{\text{Nr}}_G\) if we restrict the domain category to the category of commutative rings with identity. From this point of view, it would be quite natural to expect the theory of the \(q\)-deformed Möbius function associated to \(G\). Our expectation can be illustrated well in the following diagram:
In this paper, we study the inverse of the table of marks, the Möbius function, and their $q$-deformations intensively. It is quite noteworthy that they have a multiplicative property when $q = 1$, $0$, $-1$, and which plays a fundamental role in our computation when $G$ is a pronilpotent group since in that case $G$ is isomorphic to the cartesian product of its Sylow $p$-subgroups. If $G$ is a $p$-group with $p \neq 2$, then the inverse of the table of marks and the Möbius function of $G$ at $q = -1$ coincide with those at $q = 1$. However, if $p = 2$, this is not the case any longer. For more information, refer to Sections 4 and 5.

As for the cyclotomic identity, it arises from isomorphisms among the universal ring of Witt vectors, the necklace ring, and the Grothendieck ring of formal power series ring with constant term 1, respectively (see [12]). To be more precise, it is given by

$$\frac{1}{1 - xt} = \prod_{n \geq 1} \left( \frac{1}{1 - t^n} \right)^{M(x, n)}$$

and was first introduced by Gauss [8]. Here, the exponent $M(x, n)$ denotes the polynomial in $x$

$$\frac{1}{n} \sum_{d | n} \mu(d) x^{\frac{n}{d}}, \quad n \geq 1,$$

and $\mu$ denotes the classical Möbius inversion function. It appears in many different enumeration problems related to necklaces, cyclic sets, Lyndon words, combinatorial species, and so on. Several noteworthy combinatorial proofs are available now. For instance, in 1984, Metropolis and Rota gave a completely set-theoretic proof by counting $n!$ in a remarkable way (see [13]). On the contrary, the proofs due to Varadarajan and Wehrhahn [21], Talyor [20], Virgil and Schmitt [22] are based on the theory of combinatorial species.

It follows from [6] that given any profinite group $G$, there exists a unique covariant functor $\mathbb{W}_G$ on the category of commutative rings with identity, satisfying

(i) $\mathbb{W}_G(\mathbb{Z}) \cong \hat{\Omega}(G)$ and (ii) $\mathbb{W}_c = \mathbb{W}$.

Here, $\mathbb{W}$ represents the functor of the universal ring of Witt vectors. In view of the relation between the cyclotomic identity and the universal ring of Witt vectors, one can naturally expect the existence of an analog of the cyclotomic identity arising from the context of $\mathbb{W}_G$, $Nr_G$, and $\hat{Nr}_G$. In 1990, Nelson [14] succeeded in providing such an identity, called a generalized cyclotomic identity,

$$\sharp G\text{-SET}(xt) = \prod_{[V] \in \mathcal{O}(G)} \sharp V\text{-SET}(t^{(G; V)})^{M_G(x, [V])},$$

for each group $G$ satisfying suitable finiteness conditions. Here, the notation $\sharp G\text{-SET}(t)$ denotes the exponential generating function of the species $G\text{-SET}$ of finite $G$-sets and $M_G(x, [V])$ the polynomial in $x$ such that if $x$ is a positive integer, then it counts $G$-orbits isomorphic to $G/V$ in $\text{Hom}(G, \{1, 2, \ldots, x\})$. However, his proof is completely species-theoretical and does not reveal any link with $\mathbb{W}_G$, $Nr_G$, and $\hat{Nr}_G$.

In this paper, we will introduce $q$-deformations of Nelson’s identity from the ring-theoretic point of view. To do this, we observe that given any profinite group $G$ and any integer $q$, there
exists a unique covariant functor $W^q_G$, on the category of commutative rings (not necessarily unital), which coincides with $W_G$ at $q = 1$ (see [17]). Now, from a certain commutativity condition among $(W^q_G, Nr^q_G, gh_G)$ (respectively $(\hat{W}^q_G, \hat{Nr}^q_G, gh_G)$), we derive a $q$-deformed Nelson’s identity Eq. (6.3) (respectively Eqs. (6.4) and (6.5)). For instance, Eq. (6.5) takes on the following form:

$$\sharp_G\text{-SET}(qxt) = \prod_{[V] \in O(G)} \sharp_V\text{-SET}(q t^{(G;V)})^M_G(x,[V]).$$  \hspace{1cm} (1.1)

As expected, Nelson’s identity follows as a special case at $q = 1$. In particular, if $G = \hat{C}$ and $q = -1$, Eq. (1.1) gives rise to the *cocyclotomic identity*,

$$1 + xt = \prod_{n \geq 1} (1 + t^n)^{M^{-1}(x,n)},$$

which was due to Labelle and Leroux [11].

This paper is structured as follows: In Section 2, we lay down notations and terminologies which will be used throughout this paper. In particular, $W^q_G$, $Nr^q_G$, and $\hat{Nr}^q_G$ will be introduced. In Section 3, we deal with the classification of the above functors up to strict natural isomorphism as $q$ varies over the set of integers for profinite groups satisfying a suitable condition. In particular, in the case of $Nr^q_G$ and $\hat{Nr}^q_G$, the explicit form of the strict natural isomorphism will be provided (Corollary 3.7, Theorems 3.12 and 3.13). As a significant byproduct, we will derive a relation between $\mu^q_G$ and $\mu^r_G$ (Corollary 3.14). In Section 4, when $q = 1, 0, -1$, we will show that the inverse of the table of marks $\mu^q_G$ has the multiplicative property, and interpret it as a certain composition formula arising from $Nr^q_G$ (Theorem 4.2 and Corollary 4.3). In Section 5, the $q$-deformed Möbius function $\mu^q_G$ of the lattice of open subgroups of $G$ will be introduced. When $q = 1, 0, -1$, it has the multiplicative property (Theorem 5.2). Connection between $\mu^q_G$ and $\mu^r_G$ will be dealt with extensively (Theorem 5.5). The final section is devoted to the study of the generalization of Nelson’s identity and its ring-theoretic meaning associated to $W^q_G$, $Nr^q_G$, and $\hat{Nr}^q_G$.

2. Notations and preliminaries

2.1. Functors associated to a profinite group

Let $G$ be a profinite group. Given open subgroups $U$ and $V$ of $G$, we say that $U$ is subconjugate to $V$ if $U$ is a subgroup of some conjugates of $V$. This gives rise to a partial ordering on the set of the conjugacy classes of open subgroups of $G$, and will be denoted by $[V] \preceq [U]$. Denote this poset by $O(G)$. Throughout this paper, we fix an enumeration of $O(G)$ subject to the following condition:

$$\text{If } [V] \preceq [U], \text{ then } [V] \text{ precedes } [U].$$
For instance, if $G$ is abelian or Hamiltonian,\footnote{In this case, the bracket notation is usually omitted.} then $O(G)$ is just the set of open subgroups of $G$ equipped with the ordering

$$V \preceq U \iff U \subseteq V.$$  

For any $G$-space $X$ and any subgroup $U$ of $G$ define $\phi_U(X)$ to be the cardinality of the set $X^U$ of $U$-invariant elements of $X$ and let $G/U$ denote the $G$-space of left cosets of $U$ in $G$. A $G$-space $X$ is called \textit{almost finite} if it is discrete and $\phi_U(X)$ is finite for every open subgroup $U$ of $G$. For any $G$-spaces $S$ and $T$, denote by $\text{Hom}(S, T)$ the set of all continuous maps from $S$ into $T$ with the $G$-action defined by

$$(g \cdot f)(s) := g \cdot f(g^{-1} \cdot s) \quad (g \in G, \ f \in \text{Hom}(S, T), \ s \in S).$$

It is well known that $\text{Hom}(S, T)$ is almost finite in case where $T$ is a finite $G$-space and $S$ a $G$-space such that the number $\#(G\backslash S)$ of $G$-orbits in $S$ is finite (see [5]).

With the above notation, we will introduce covariant functors $\mathcal{W}^q_G$, $\mathcal{N}_q^G$, and $\hat{\mathcal{N}}_q^G$ which were recently introduced in [17]. To do this, we first introduce \textit{the ghost ring functor of} $G$, denoted by $gh_G$, which is the functor from the category of commutative rings with identity into itself such that

1. as a set, $gh_G(A) = A^{O(G)}$,
2. addition and multiplication are defined componentwise, and
3. for every ring homomorphism $f : A \to B$ and every $\alpha \in gh_G(A)$ one has $gh_G(f)(\alpha) = f \circ \alpha$.

Let $q$ be any integer. We define $\mathcal{W}_q^G$ by the unique covariant functor from the category of commutative rings with identity into the category of commutative rings (not necessarily unital) subject to the conditions:

1. As a set, $\mathcal{W}_q^G(A) = A^{O(G)}$.
2. For every ring homomorphism $f : A \to B$ and every $\alpha \in \mathcal{W}_q^G(A)$ one has $\mathcal{W}_q^G(f)(\alpha) = f \circ \alpha$.
3. The map,

$$\Phi_q^G : \mathcal{W}_q^G(A) \to gh_G(A),$$

$$\alpha \mapsto \left( \sum_{[G] \subseteq [V], [V] \subseteq [U]} \phi_U(G/V)q^{(V:U)-1} \alpha([V])^{(V:U)} \right)_{[U] \in O(G)},$$

is a ring homomorphism. Here, $(V : U)$ represents $(G : U)/(G : V)$.

It should be mentioned that $\mathcal{W}_q^G(A)$ fails to be unital in general. This functor may be viewed as a $q$-deformation of the Witt–Burnside ring functor $\mathcal{W}_G$ due to Dress and Siebeneicher [5] in the sense that they turn out to be identical at $q = 1$. As in the case of $\mathcal{W}_G$, algebraic structure of $\mathcal{W}_q^G(\mathbb{Z})$ is quite complicated. This led us to consider a functor whose value at $\mathbb{Z}$ is isomorphic to
\( \mathbb{W}_q^G(\mathbb{Z}) \), but which is much easier to describe. To be more precise, we define \( Nr^q_G \) by the unique covariant functor from the category of special \( \lambda \)-rings into the category of commutative rings (not necessarily unital) satisfying the following conditions:

1. As a set, \( Nr^q_G(A) = A^{O(G)} \).
2. For every special \( \lambda \)-ring homomorphism \( f : A \to B \) and every \( \alpha \in Nr^q_G(A) \) one has \( Nr^q_G(f)(\alpha) = f \circ \alpha \).
3. (The map, \( \widetilde{\phi}^q_G: Nr^q_G(A) \to gh_G(A) \),
\[ \alpha \mapsto \left( \sum_{[G] \subseteq [V] \subseteq [U]} \phi_U(G/V) q^{[V]} \psi(U/V)(\alpha([V])) \right)_{[U] \in O(G)}, \]

is a ring homomorphism.

Indeed, \( Nr_G(\mathbb{Z}) \overset{\text{def}}{=} Nr^1_G(\mathbb{Z}) \) is isomorphic to \( \hat{\Omega}(G) \), the Burnside–Grothendieck ring of \( G \) of isomorphism classes of almost finite \( G \)-spaces introduced in [5]. For a special \( \lambda \)-ring \( A \), the abelian group structure of \( Nr^q_G(A) \) is very simple since the addition is defined componentwise. However, its multiplication is somewhat complicated. To describe the multiplication rule we need the following definition.

**Definition 2.1.** Let \( x_1, x_2, \ldots, x_n \) be indeterminates and assume that

\[ f \in \mathbb{Q}[x_1, x_2, \ldots, x_n]. \]

Under this assumption, \( f \) is called **numerical** if it takes integer values for integer arguments.

**Theorem 2.2.** (See [17, Eq. (7.4)].) Let \( A \) be a special \( \lambda \)-ring, and let \( x, y \in Nr^q_G(A) \). Then, given \( [U] \in O(G) \), one has

\[ x \cdot y([U]) = \sum_{[V], [W] \subseteq [U]} P_{V, W}^U(q) \psi(V/U)(x([V])) \psi(W/U)(y([W])) \]

for some numerical polynomials \( P_{V, W}^U(q) \in \mathbb{Q}[q] \).

Finally, we introduce \( \widehat{Nr}_G^q \), which is the unique covariant functor from the category of commutative rings with identity into the category of commutative rings (not necessarily unital) satisfying the following conditions:

1. As a set, \( \widehat{Nr}_G^q(A) = A^{O(G)} \).
2. For every ring homomorphism \( f : A \to B \) and every \( \alpha \in \widehat{Nr}_G^q(A) \) one has \( \widehat{Nr}_G^q(f)(\alpha) = f \circ \alpha \).
(3) The map,

$$\varphi^q_G : \hat{N}r^q_G (A) \rightarrow \text{gh}_G (A),$$

$$\alpha \mapsto \left( \sum_{[G] \preceq [V] \preceq [U]} \phi_U (G / V) q^{(V : U) - 1} \alpha ([V]) \right)_{[U] \in O(G)},$$

is a ring homomorphism.

Note that $\hat{N}r^q_G$ can be naturally induced from $\hat{\Omega}(G)$ in the sense that their operation rule is exactly same. The addition of $\hat{N}r^q_G (A)$ is defined componentwise, whereas the multiplication is given by

$$x \cdot y ([U]) = \sum_{[V] \preceq [W] \preceq [U]} P^U_{V, W} (q) x ([V]) y ([W])$$

for those $P^U_{V, W} (q)$'s in Eq. (2.1). Although $\hat{N}r^q_G$ and $N r^q_G$ are not naturally equivalent, they come to be identical if we view them as a functor from the category of binomial rings to the category of commutative rings. Here, a binomial ring means a special $\lambda$-ring in which $\Psi^n = \text{id}$ for all $n \geq 1$. For more information, see [7,10].

Remark 2.3. (a) Let $G$ be a profinite group whose order is given by $\prod_{p \text{ prime}} p^{k(p)}$ with $k(p) \in \mathbb{N} \cup \{ \infty \}$. Denote by $D^\text{pr}(G)$ the set of prime divisors dividing the order of $G$ (equivalently, $p \in D^\text{pr}(G)$ if and only if $k(p) \geq 1$). Now, let us assume that $2 \notin D^\text{pr}(G)$. Then $(V : U)$ is odd for all $[G] \preceq [V] \preceq [U]$. It follows that

$$\Phi^q_G = \Phi^{-q}_G, \quad \tilde{\varphi}^q_G = \tilde{\varphi}^{-q}_G, \quad \varphi^q_G = \varphi^{-q}_G,$$

and thus

$$W^q_G = W^{-q}_G, \quad N r^q_G = N r^{-q}_G, \quad \hat{N}r^q_G = \tilde{N}r^{-q}_G.$$

(b) Note that $\Phi^0_G = \varphi^0_G$. It follows that $W^0_G = \tilde{N}r^0_G$.

2.2. Natural transformation between $W^q_G$ and $N r^q_G$ when restricted to the category of special $\lambda$-rings

Let $q$ be an indeterminate, $G$ a profinite group, and $X$ an alphabet, that is, a set of commuting variables $\{ x_1, x_2, \ldots, x_m \}$. Also, denote by $\Psi^n (n \geq 1)$, the $n$th Adams operation on $\mathbb{Q}[q][x_i : 1 \leq i \leq m]$ defined as follows:

$$\Psi^n (x_i) = x_i^n, \quad 1 \leq i \leq m, \quad n \geq 1,$$

$$\Psi^n (c) = c, \quad c \in \mathbb{Q}[q].$$

As usual, $p_n$ will denote the $n$th power sum symmetric polynomial in $x_1, x_2, \ldots, x_m$, that is, $p_n (X) = x_1^n + x_2^n + \cdots + x_m^n$. Then it is obvious that $\Psi^k (p_n (X)) = p_{kn} (X)$ for all $k, n \geq 1$. Note
that for a symmetric polynomial \( f \) in \( x_1, x_2, \ldots, x_m \), \( \Psi^k(f) \) coincides with the plethysm of \( f \) by \( p_k \), i.e., \( f(x^k_1, x^k_2, \ldots, x^k_m) \).

With this preparation, we define a \( \mathcal{O}(G) \times \mathcal{O}(G) \) matrix \( \breve{\zeta}^q_G \) by

\[
\breve{\zeta}^q_G([V], [W]) = \begin{cases} 
\phi_V(G/W)q^{(W:V)} - 1 & \text{if } [W] \preceq [V], \\
0 & \text{otherwise.}
\end{cases}
\]

It is obvious that \( \breve{\zeta}^q_G \in \text{End}_{\mathbb{Q}[q]}(\mathbb{Q}[q][x_i: 1 \leq i \leq m]^{\mathcal{O}(G)}) \). We also define a \( \mathcal{O}(G) \times \mathcal{O}(G) \) matrix \( \zeta^q_G \) by

\[
\zeta^q_G([V], [W]) = \begin{cases} 
\phi_V(G/W)q^{(W:V)} - 1 & \text{if } [W] \preceq [V], \\
0 & \text{otherwise.}
\end{cases}
\]

Let \( \tilde{\mu}^q_G \) (respectively \( \mu^q_G \)) be the inverse of \( \breve{\zeta}^q_G \) (respectively \( \zeta^q_G \)). Then it is not difficult to show that for every \([W], [U] \in \mathcal{O}(G)\) satisfying \([W] \preceq [U]\),

\[
\tilde{\mu}^q_G([U], [W]) = \mu^q_G([U], [W]) \Psi^{(W:U)}. \tag{2.2}
\]

**Convention.** It should be remarked that \( \zeta^{-1}_G, \breve{\zeta}^{-1}_G, \mu^{-1}_G, \tilde{\mu}^{-1}_G \) do not denote the inverse of \( \zeta_G, \breve{\zeta}_G, \mu_G, \tilde{\mu}_G \), respectively. To denote its inverse we will use the notation \( (\zeta_G)^{-1}, (\breve{\zeta}_G)^{-1}, (\mu_G)^{-1}, (\tilde{\mu}_G)^{-1} \).

**Remark 2.4.** Let \( q = 1 \) and let \( G \) be a finite group. In the literature, \( \zeta_G \) frequently appears as the zeta function defined on the lattice of subgroups, that is,

\[
\zeta_G(V, W) = \begin{cases} 
1 & \text{if } V \leq W, \\
0 & \text{otherwise,}
\end{cases}
\]

and \( \mu_G \) as its inverse, called the Möbius function of \( G \). But, in the present paper, \( \zeta_G \) will denote the table of marks\(^2\) of \( G \) and \( \mu_G \) its inverse.

**Definition 2.5.** (See [17].) With the above notation, for \([U] \in \mathcal{O}(G)\), let us define

\[
M^q_G(X, [U]) = \sum_{[W] \leq [U]} \mu^q_G([U], [W])q^{(G:W) - 1}p^{(W:U)}(X)^{(G:W)},
\]

\[
M^q_G(x, [U]) = \sum_{[W] \leq [U]} \mu^q_G([U], [W])q^{(G:W) - 1}x^{(G:W)}.
\]

Notice that \( M^q_G(X, [U]) \) is a polynomial in \( x_1, x_2, \ldots, x_m \) subject to the relation (if we use the column notation):

---

\(^2\) The concept of a table of marks of a finite group was first introduced by William Burnside in the second edition of his classical book “Theory of groups of finite order.”
\[
\begin{pmatrix}
q^{(G:U)-1}p_1(X)^{(G:U)} \\
\vdots \\
q^{(G:U)-1}X^{(G:U)} \\
\end{pmatrix}
\begin{pmatrix}
[U] \in \mathcal{O}(G)
\end{pmatrix}
= \tilde{\zeta}_G^q
\begin{pmatrix}
\cdots \\
\cdots \\
\cdots \\
\end{pmatrix}
\begin{pmatrix}
M^q_G(X, [U]) \\
\vdots \\
M^q_G(x, [U]) \\
\end{pmatrix}
\begin{pmatrix}
[U] \in \mathcal{O}(G)
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
q^{(G:U)-1}x^{(G:U)} \\
\vdots \\
q^{(G:U)-1}X^{(G:U)} \\
\end{pmatrix}
\begin{pmatrix}
[U] \in \mathcal{O}(G)
\end{pmatrix}
= \tilde{\zeta}_G^q
\begin{pmatrix}
\cdots \\
\cdots \\
\cdots \\
\end{pmatrix}
\begin{pmatrix}
M^q_G(x, [U]) \\
\vdots \\
M^q_G(x, [U]) \\
\end{pmatrix}
\begin{pmatrix}
[U] \in \mathcal{O}(G)
\end{pmatrix}
\]

Equivalently, for every \([U] \in \mathcal{O}(G)\), we can write

\[
q^{(G:U)-1}p_1(X)^{(G:U)} = \sum_{[V] \in [U]} \phi_{U/G/V} q^{(V:U)-1} \Psi^{(V:U)}(M^q_G(X, [V])), \quad (2.3)
\]

\[
q^{(G:U)-1}x^{(G:U)} = \sum_{[V] \in [U]} \phi_{U/G/V} q^{(V:U)-1} M^q_G(x, [V]). \quad (2.4)
\]

**Example 2.6.** Let \(G = \hat{C}\) and \(n\) be a prime. Then

\[
M^q(X, n) \overset{\text{def}}{=} M^q_C(X, \hat{C}^n) = \frac{q^n - 1}{n} (p_1(X)^n - p_n(X)),
\]

\[
M^q(x, n) \overset{\text{def}}{=} M^q_C(x, \hat{C}^n) = \frac{q^n - 1}{n} (x^n - x).
\]

**Remark 2.7.**

(a) A combinatorial interpretation of \(M^q_G(X, [U])\) and \(M^q_G(x, [U])\) can be found in [17, Section 3]. Indeed, the former is given by a sum of \(q\)-aperiodic monomials in \(x_1, x_2, \ldots, x_m\) satisfying suitable conditions, and the latter counts \(G\)-orbits isomorphic to \(G/U\) in \(\text{Hom}(G, \{1, 2, \ldots, n\} \times \mathbb{Z}/q\mathbb{Z})/\sim\).

(b) Let \(A_m\) denote the ring of symmetric polynomials in \(x_1, x_2, \ldots, x_m\). Then, for every \([U] \in \mathcal{O}(G)\) and for every integer \(q\), one has

\[
M^q_G(X, [U]) \in A_m.
\]

For example, in case where \(X = \{x_1, x_2\},

\[
M^q(X, p) = \frac{q^n - 1}{p} \sum_{i=1}^{p-1} \binom{p}{i} x_1^{p-i} x_2^i,
\]

where \(p\) is a prime.

(c) When \(q = 1\), it is our convention to omit the superfix. For example, \(\mathbb{W}_G = \mathbb{W}^1_G, \mathbb{N}r_G = \mathbb{N}r^1_G, \hat{\mathbb{N}}r_G = \hat{\mathbb{N}}r^1_G, \text{ and } M_G(X, [U]) = M^1_G(X, [U]).\)
Although $\mathbb{W}_G^q$ and $\text{Nr}_G^q$ are not naturally equivalent, they turn out to be intimately related. Before stating the concrete relation, it should be remarked that in case where $A$ has two different $\lambda$-ring structures, say $\lambda_{(1)}$, $\lambda_{(2)}$, $(A, \lambda_{(1)})$ and $(A, \lambda_{(2)})$ denote different rings. Under this assumption, viewed as a functor from the category of special $\lambda$-rings to the category of commutative rings, $\mathbb{W}_G^q$ and $\text{Nr}_G^q$ turn out to be naturally equivalent. Let me review the explicit form of the natural equivalence. Let $A$ be a special $\lambda$-ring. Given an element $r \in A$, set

$$M_G^q(r, [V]) = \sum_{[W] \in [V]} \mu_G^q([V], [W])q^{(G:W) - 1}\Psi^{(W:V)}(r(G:W)),$$

(2.5)

We now define a map $\tau_G^q : A \rightarrow \text{Nr}_G^q(A)$ by

$$\tau_G^q(r)([V]) = M_G^q(r, [V]), \quad \forall [V] \in \mathcal{O}(G).$$

And, for each $[U] \in \mathcal{O}(G)$, let us introduce induction

$$\text{Ind}_U^G : \text{Nr}_U^q(A) \rightarrow \text{Nr}_G^q(A), \quad x \mapsto \text{Ind}_U^G(x)$$

where

$$\text{Ind}_U^G(x)([W]) = \sum_{[V] \in \mathcal{O}(U)} x([V]).$$

Obviously $\text{Ind}_U^G$ is additive for every $[U] \in \mathcal{O}(G)$. Finally, we set

$$\tau_G^q : \mathbb{W}_G^q(A) \rightarrow \text{Nr}_G^q(A),
\quad f \mapsto \sum_{[U] \in \mathcal{O}(G)} \text{Ind}_U^G \circ \tau_U^q(f([U])).$$

It is known that $\tau_G^q$ is bijective and for a special $\lambda$-ring homomorphism $f : A \rightarrow B$ we have the commutative diagram:

$$\begin{array}{ccc}
\mathbb{W}_G^q(A) & \xrightarrow{\tau_G^q} & \text{Nr}_G^q(A) \\
\mathbb{W}_G^q(f) \downarrow & & \downarrow \text{Nr}_G^q(f) \\
\mathbb{W}_G^q(A) & \xrightarrow{\tau_G^q} & \text{Nr}_G^q(A).
\end{array}$$

**Theorem 2.8.** (See [17].) If we view $\mathbb{W}_G^q$ as a functor from the category of special $\lambda$-rings to the category of commutative rings, then $\tau_G^q$ is a natural isomorphism satisfying $\Phi_G^q = \tilde{\varphi}_G^q \circ \tau_G^q$. Also, if we view $\mathbb{W}_G^q$, $\text{Nr}_G^q$, and $\text{Nr}_G^q$ as a functor from the category of binomial rings to the category of commutative rings, then they are naturally isomorphic.
3. Classification of $\mathbb{W}_G^q, \mathbb{N}_G^q,$ and $\widehat{\mathbb{N}}_G^q$ up to strict natural isomorphism

In the theory of profinite groups, it is well known that the order of a profinite group can be expressed as a Steinitz number (or super natural number), say

$$\prod_{p: \text{prime}} p^{k(p)} \quad \text{with } k(p) \in \mathbb{N} \cup \{\infty\}.$$  

Throughout this section, $G$ is assumed to be nontrivial and subject to the following condition. The index set corresponding to 1 is assumed to the empty set.

**Condition ($\star$).** For every prime $p$ dividing the order of $G$ (equivalently, $k(p) \geq 1$), there exists a maximal open subgroup $U$ of $G$ such that $p|(G:U)$.

For example, pronilpotent groups or finite solvable groups satisfy this requirement. Under this condition, this section will be devoted to the classification of $\mathbb{W}_G^q, \mathbb{N}_G^q,$ and $\widehat{\mathbb{N}}_G^q$ up to strict natural isomorphism as $q$ ranges over the set of integers.

**Definition 3.1.** Let $U, V$ be open subgroups of $G$.

(a) $n_G(U, V)$ is defined by the cardinality of $G$-conjugates of $V$ containing $U$.

(b) $\nu_G(U, V)$ is defined by the cardinality of the set

$$\{ [W] \in \mathcal{O}(V): [W] = [U] \text{ in } \mathcal{O}(G) \}.$$

The following lemma will play a crucial role throughout this paper.

**Lemma 3.2.** (See [17].) Given open subgroups $U, V$ of $G$, we have

$$\phi_U(G/V) = \left[ N_G(V): V \right] n_G(U, V) = \sum_{[U_i] \in \mathcal{O}(V)} \left[ N_G(U_i): N_V(U_i) \right], \quad (3.1)$$

where $N_G(\bullet)$ (respectively $N_V(\bullet)$) denotes the normalizer of $\bullet$ in $G$ (respectively $V$). In particular, in case $U \subseteq V$, we have

$$\phi_U(G/V) = \left[ N_G(U): N_V(U) \right] \nu_G(U, V). \quad (3.2)$$

3.1. Classification of $\mathbb{W}_G^q$ up to strict natural isomorphism

Given an integer $q$, we denote by $D(q)$ the set of divisors of $q$, and by $D^{pr}(q)$ the set of prime divisors of $q$, respectively. Conventionally, $D(0)$ will denote the set of positive integers $\mathbb{N}$, and $D^{pr}(0)$ the set of all primes in $\mathbb{N}$.

**Definition 3.3.** Let $q$ and $r$ be arbitrary integers.

(a) Given a commutative ring $A$, $\mathbb{W}_G^q(A)$ is said to be strictly-isomorphic to $\mathbb{W}_G^r(A)$ if there exists a ring isomorphism, say $\omega_q^r : \mathbb{W}_G^q(A) \to \mathbb{W}_G^r(A)$, satisfying $\Phi^q_G = \Phi^r_G \circ \omega_q^r$. In this case, $\omega_q^r$ is called a strict-isomorphism.
(b) \( \mathbb{W}_G^q \) is said to be strictly-isomorphic to \( \mathbb{W}_G^r \) if there exists a natural isomorphism, say \( \omega^q_r : \mathbb{W}_G^q \rightarrow \mathbb{W}_G^r \), satisfying \( \Phi_G^q = \Phi_G^r \circ \omega^q_r \). In this case, \( \omega^q_r \) is called a strict natural isomorphism.

Recall that \( D^{pr}(G) \) is the set of prime divisors dividing the order of \( G \) (see Remark 2.3). Let

\[
D^{pr}(q) \cap D^{pr}(G) = \{ p_1, \ldots, p_k \}; \quad c_1, \ldots, c_s
\]

\[
D^{pr}(r) \cap D^{pr}(G) = \{ p_1, \ldots, p_k \}; \quad d_1, \ldots, d_t
\]

That is, \( p_i \)'s are primes in \( D^{pr}(q) \cap D^{pr}(r) \cap D^{pr}(G) \).

**Theorem 3.4.** Let \( A \) be a commutative ring with identity, and let \( q, r \) be arbitrary integers. Then, there exists a unique strict-isomorphism between \( \mathbb{W}_G^q(A) \) and \( \mathbb{W}_G^r(A) \) if and only if \( A \) is a \( \mathbb{Z}[\frac{1}{c_i}, \frac{1}{d_j} : 1 \leq i \leq s, 1 \leq j \leq t] \)-algebra.

In the following, we will introduce two auxiliary lemmas to prove Theorem 3.4. Let \( q, r \) be indeterminates and let

\[
R = \mathbb{Q}[X_U, Y_U] : [U] \in \mathcal{O}(G).
\]

We also let \( X \) and \( Y \) be the vectors \( (X_U)_{[U] \in \mathcal{O}(G)} \) and \( (Y_U)_{[U] \in \mathcal{O}(G)} \), respectively. Assume that

\[
\Phi^q_G(X) = \Phi^r_G(Y). \tag{3.3}
\]

Then the set of equations arising from Eq. (3.3) consists of the following identities: For every \( [U] \in \mathcal{O}(G) \),

\[
\sum_{[G] \leq [V] \leq [U]} \phi_U(G/V)q^{(V:U)-1}X^{(V:U)}_{[V]} = \sum_{[G] \leq [V] \leq [U]} \phi_U(G/V)r^{(V:U)-1}Y^{(V:U)}_{[V]} \tag{3.4}
\]

From now on, we will adopt the following convention for the simplicity of notation.

**Convention (**)**.

Whenever we meet the summation \( \sum_{[G] \leq [V] \leq [U]} \), we assume that \( U \subseteq V \). \tag{3.5}

For instance, under this convention,

\[
\sum_{[G] \leq [V] \leq [U]} \phi_U(G/V)q^{(V:U)-1}X^{(V:U)}_{[V]} = \sum_{[G] \leq [V] \leq [U]} \left(N_G(U) : N_V(U)\right)v_G(U, V)q^{(V:U)-1}X^{(V:U)}_{[V]}.
\]
Denote by $\mathbb{Z}_G$ the commutative ring $\mathbb{Z}[\frac{1}{p}: p \in \text{D}^\text{pr}(G)]$. With this preparation, the first lemma can be stated as follows:

**Lemma 3.5.** Let $G$ be an arbitrary profinite group. Then, for every $[U] \in \mathcal{O}(G)$, it holds that

$$Y_{[U]} - X_{[U]} \in \left( \mathbb{Z}_G \cap \mathbb{Z}[\frac{1}{q}, \frac{1}{r}] \right) [X_S]: [G] \preceq [S] \preceq [U]].$$

**Proof.** Given $[U] \in \mathcal{O}(G)$ let us express $Y_{[U]}$ as a polynomial in $X_{[V]}$’s with $[V] \preceq [U]$ in an inductive way. First, note that $Y_{[G]} = X_{[G]}$ which follows from Eq. (3.4). Now, we assume that $Y_{[V]} - X_{[V]} \in \mathbb{Z}_G[X_S]: [G] \preceq [S] \preceq [V]]$ for all $[G] \preceq [V] \preceq [U]$. Due to the convention (3.5) and Eq. (3.2), we can transform Eq. (3.4) as follows:

$$Y_{[U]} - X_{[U]} = \sum_{[G] \preceq [V] \preceq [U]} \frac{(N_G(U) : N_V(U))v_G(U, V)(q^{(V:U) - 1}X_{[V]} - r^{(V:U) - 1}Y_{[V]})}{(N_G(U) : U)}.$$  (3.6)

Our induction hypothesis immediately implies

$$Y_{[U]} - X_{[U]} \in \mathbb{Z}_G[X_S]: [G] \preceq [S] \preceq [U]].$$  (3.7)

Next, let us show that the coefficients are in $\mathbb{Z}[\frac{1}{q}, \frac{1}{r}]$. Let us first assume that $q$ and $r$ are nonzero, and then multiply $qr/(q, r)$ to both sides of Eq. (3.4) to obtain the identity

$$\Phi_G((qX_{[U]})_{[U]} \in \mathcal{O}(G)) = \Phi_G\left( \frac{q}{(q, r)} \cdot (rY_{[U]})_{[U]} \in \mathcal{O}(G) \right),$$

where $\Phi_G$ denotes $\Phi^1_{G}$. It follows from the injectiveness of $\Phi_G$ that

$$\frac{r}{(q, r)} \cdot (qX_{[U]})_{[U]} \in \mathcal{O}(G) = \frac{q}{(q, r)} \cdot (rY_{[U]})_{[U]} \in \mathcal{O}(G).$$

Then it is not difficult to show that, for all $[U] \in [G]$, the $[U]$th coordinate of $r/(q, r) \cdot (qY_{[U]})_{[U]} \in \mathcal{O}(G)$ is of the form

$$\frac{rq}{(q, r)} Y_{[U]} + \text{a polynomial contained in } \mathbb{Z}[qY_S]: [G] \preceq [S] \preceq [U]] \quad (3.8)$$

and that of $q/(q, r) \cdot (rX_{[U]})_{[U]} \in \mathcal{O}(G)$ is of the form

$$\frac{rq}{(q, r)} X_{[U]} + \text{a polynomial contained in } \mathbb{Z}[qX_S]: [G] \preceq [S] \preceq [U]]. \quad (3.9)$$
Assume that
\[
Y[V] - X[V] \in \mathbb{Z}\left[\frac{1}{q}, \frac{1}{r}\right]\left[X[S] : [G] \preceq [S] \preceq [V]\right]
\]
for all \([G] \preceq [V] \preceq [U]\). By applying the above induction hypothesis to Eqs. (3.8) and (3.9), we can immediately show that
\[
Y[U] - X[U] \in \mathbb{Z}\left[\frac{1}{q}, \frac{1}{r}\right]\left[X[S] : [G] \preceq [S] \preceq [U]\right].
\tag{3.10}
\]
Putting Eqs. (3.7) and (3.10) together yields the desired result. □

Based on Lemma 3.5 we can state a much stronger lemma.

**Lemma 3.6.** Let \(q, r\) be arbitrary integers. Then, for every open subgroup \(U\) of \(G\), it holds that
\[
Y[U] - X[U] \in \mathbb{Z}\left[\frac{1}{c_i}, \frac{1}{d_j} : 1 \leq i \leq s, 1 \leq j \leq t\right]\left[X[S] : [G] \preceq [S] \preceq [U]\right].
\]

**Proof.** Let us first assume that \(q\) and \(r\) are nonzero. Note that \(Y[G] = X[G]\). Now, we assume that \(Y[V] - X[V] \in \mathbb{Z}\left[\frac{1}{q}, \frac{1}{r}\right]\left[X[S] : [G] \preceq [S] \preceq [V]\right]\)
for all \([G] \preceq [V] \preceq [U]\). On the other hand, we showed in Eq. (3.6) that
\[
Y[U] - X[U] = \sum_{[G] \preceq [V] \preceq [U]} \frac{v_G(U, V)(q(V:U) - 1)X(V:U) - r(V:U) - 1 Y(V:U))}{(N_V(U) : U)}.
\]

If a prime \(p\) divides \((N_V(U) : U)\) and \(q\), then it cannot be a divisor of the denominator of the irreducible fraction of \(q(V:U) - 1\) since \((N_V(U) : U)||V : U)\). Also, it cannot be a divisor of the denominator of the irreducible fraction of \(r(V:U) - 1\) if it divides \(r\).

Now, our assertion follows from the induction hypothesis.

Next, we assume that \(q\) is zero and \(r\) is nonzero. In this case, Eq. (3.4) reduces to
\[
Y[U] - X[U] = -\sum_{[G] \preceq [V] \preceq [U]} \frac{v_G(U, V)r(V:U) - 1 Y(V:U))}{(N_V(U) : U)}.
\]

Since a prime \(p\) dividing both \((N_V(U) : U)\) and \(r\) cannot be a divisor of the denominator of the irreducible fraction of
\[
\frac{v_G(U, V)r(V:U) - 1 Y(V:U)}{(N_V(U) : U)},
\]
we obtain the desired result by induction hypothesis. So we are done. □
Up to now, we have shown that $Y_\{U\}$ is a polynomial in $X_\{V\}$’s with $[G] \preceq [V] \preceq [U]$ for each $[U] \in \mathcal{O}(G)$. Let us denote by $\mathcal{P}_\{U\}$ this polynomial. Now, we are ready to prove Theorem 3.4.

**Proof of Theorem 3.4.** ($\iff$) The “if” part follows from Lemma 3.6. To be more precise, we define

$$\omega^\prime_r : \mathbb{W}^\prime_G(A) \to \mathbb{W}^\prime_G(A), \quad (X_\{U\})_\{U\} \in \mathcal{O}(G) \to (\mathcal{P}_\{U\})_\{U\} \in \mathcal{O}(G).$$

In view of Lemma 3.6 this map is well defined and $\Phi^q_G = \Phi^r_G \circ \omega^\prime_q$ by construction. The bijection can be easily verified since the coefficient of $X_\{U\}$ in $P_\{U\}$ is given by 1. In order to show that $\omega^\prime_q$ is a ring homomorphism, put $A = \mathbb{Q}[X_\{U\}, Y_\{U\}] : [U] \in \mathcal{O}(G)$.

From the bijectiveness of $\Phi^q_G$ and $\Phi^r_G$ it follows that

$$\omega^\prime_q(X + Y) = \omega^\prime_q(X) + \omega^\prime_q(Y). \quad (3.11)$$

Moreover, for each $[U] \in \mathcal{O}(G)$, the $[U]$th component of the left-hand side and that of the right-hand side equals as a polynomial in $X_\{V\}$, $Y_\{V\}$’s with $[G] \preceq [V] \preceq [U]$ over $\mathbb{Z}_G \cap \mathbb{Z}[\frac{1}{q}, \frac{1}{r}]$. This implies that Eq. (3.11) is valid for all $\mathbb{Z}_G \cap \mathbb{Z}[\frac{1}{q}, \frac{1}{r}]$-algebras. In the same way one can show that

$$\omega^\prime_q(XY) = \omega^\prime_q(X) \cdot \omega^\prime_q(Y).$$

So, we are done.

($\Rightarrow$) For the “only if” part, let us assume that there exists a unique strict-isomorphism, say $\omega^\prime_r : \mathbb{W}^\prime_G(A) \to \mathbb{W}^\prime_G(A)$. Hence, we have $\Phi^q_G = \Phi^r_G \circ \omega^\prime_q$. Let $p$ be a prime satisfying $p \in D^\text{pr}(q) \cap D^\text{pr}(G) \setminus D^\text{pr}(r)$. Fix a maximal open subgroup $U$ of $G$ such that $p|(G : U)$. Letting

$$(a[U])_\{U\} \in \mathcal{O}(G) = \omega^\prime_q(\{G\text{th}\}) \mathbb{1}, 0, 0, \ldots)$$

we have

$$\Phi^q_G((a[U])_\{U\} \in \mathcal{O}(G)) = \Phi^r_G \circ \omega^\prime_q(1, 0, 0, \ldots).$$

Comparing the $[U]$th component of both sides gives

$$q^{(G : U)-1} = (G : U)a[U] + r^{(G : U)-1},$$

equivalently,

$$-r^{(G : U)-1} = (G : U)a[U] - q^{(G : U)-1}. \quad (3.12)$$
Since \( p \) divides the right-hand side of Eq. (3.12), but does not divide its left-hand side. This implies that \( p \cdot 1 \) must be a unit in \( A \). Similarly, a prime
\[
p \in D^\text{pt}(r) \cap D^\text{pt}(G) \setminus D^\text{pt}(q)
\]
must be a unit. This implies that \( A \) is a \( \mathbb{Z}[\frac{1}{c_i}, \frac{1}{d_j} : 1 \leq i \leq s, 1 \leq j \leq t] \)-algebra. \( \square \)

From Theorem 3.4 it is immediate that \( \mathbb{W}_G^d(\mathbb{Z}) \) is classified up to strict-isomorphism by the set of prime divisors of \( q \) contained in \( D^\text{pt}(G) \). Thus, we have proved:

**Corollary 3.7.** Let \( q \) vary over the set of integers. Then, \( \mathbb{W}_G^d \) is classified up to strict natural isomorphism by the set of prime divisors of \( q \) contained in \( D^\text{pt}(G) \).

### 3.2. Classification of \( \text{Nr}_G^d \) and \( \text{Hnr}_G^d \) up to strict natural isomorphism

In the previous section, we have classified \( \mathbb{W}_G^d \) up to strict natural isomorphism, but unfortunately we could not provide the explicit form of the natural isomorphism. However, in the case of \( \text{Nr}_G^d \) and \( \text{Hnr}_G^d \), we can classify \( \text{Nr}_G^d \) and \( \text{Hnr}_G^d \) up to strict natural isomorphism and also compute the explicit form of the natural isomorphism. Let us first recall the definition of strict natural isomorphism. For the time being, \( G \) will be an arbitrary profinite group.

**Definition 3.8.** (See [17].) Let \( q \) and \( r \) be arbitrary integers.

- (a) Given a special \( \lambda \)-ring \( A \), \( \text{Nr}_G^d(A) \) is said to be strictly-isomorphic to \( \text{Nr}_G^r(A) \) if there exists a ring isomorphism, say \( n_q^r : \text{Nr}_G^d(A) \to \text{Nr}_G^r(A) \), satisfying \( \tilde{\varphi}_G^d = \tilde{\varphi}_G^r \circ n_q^r \). In this case, \( n_q^r \) is called a strict isomorphism.
- (b) \( \text{Nr}_G^d \) is said to be strictly-isomorphic to \( \text{Nr}_G^r \) if there exists a natural isomorphism, say \( n_q^r : \text{Nr}_G^d \to \text{Nr}_G^r \), satisfying \( \tilde{\varphi}_G^d = \tilde{\varphi}_G^r \circ n_q^r \). In this case, \( n_q^r \) is called a strict natural isomorphism.
- (c) Given a commutative ring \( A \), \( \text{Hnr}_G^d(A) \) is said to be strictly-isomorphic to \( \text{Hnr}_G^r(A) \) if there exists a ring isomorphism, say \( \hat{n}_q^r : \text{Hnr}_G^d(A) \to \text{Hnr}_G^r(A) \), satisfying \( \tilde{\varphi}_G^d = \tilde{\varphi}_G^r \circ \hat{n}_q^r \). In this case, \( \hat{n}_q^r \) is called a strict isomorphism.
- (d) \( \text{Hnr}_G^d \) is said to be strictly-isomorphic to \( \text{Hnr}_G^r \) if there exists a natural isomorphism, say \( \hat{n}_q^r : \text{Hnr}_G^d \to \text{Hnr}_G^r \), satisfying \( \tilde{\varphi}_G^d = \tilde{\varphi}_G^r \circ \hat{n}_q^r \). In this case, \( \hat{n}_q^r \) is called a strict natural isomorphism.

Since \( \tilde{\varphi}_G^d \) (respectively \( \tilde{\varphi}_G^r \)) is the left multiplication by \( \tilde{\zeta}_G^d \) (respectively \( \tilde{\zeta}_G^r \)), \( n_q^r \) should be defined by \( (\tilde{\zeta}_G^r)^{-1} \tilde{\zeta}_G^d \) if each matrix is invertible. Letting \( q \) and \( r \) be indeterminates, it is not difficult to show that each entry of \( (\tilde{\zeta}_G^r)^{-1} \tilde{\zeta}_G^d \) is contained in \( \mathbb{Q}(q, r) \). The key point here is that it takes integer values for suitable pairs of \( q \) and \( r \).

Let us first compute its explicit form. To begin with, let \( \Psi^n(n \geq 1) \), be denote the \( n \)-th Adams operation on \( \mathbb{Q}(q, r)[x_i : 1 \leq i \leq m] \) defined as follows:
\[
\Psi^n(x_i) = x_i^n, \quad 1 \leq i \leq m, \ n \geq 1, \\
\Psi^n(c) = c, \quad c \in \mathbb{Q}(q, r).
\]
Lemma 3.9. For \([U], [W] \in \mathcal{O}(G)\)

\[
(\tilde{\zeta}_G^r)^{-1} \zeta_G^q([U], [W]) = (\zeta_G^r)^{-1} \zeta_G^q([U], [W]) \Psi^{(W:U)}.
\]

Proof. This follows from Eq. (2.2). To be more precise,

\[
(\tilde{\zeta}_G^r)^{-1} \zeta_G^q([U], [W]) = \sum_{[W] \ll [S] \ll [U]} \mu_G^r([U], [S]) \tilde{\zeta}_G^q([S], [W]) \Psi^{(S:U)} \zeta_G^q([S], [W]) \Psi^{(W:S)}
\]

\[
= (\zeta_G^r)^{-1} \zeta_G^q([U], [W]) \Psi^{(W:U)}. \quad \Box
\]

By Lemma 3.9 the problem of computing \((\tilde{\zeta}_G^r)^{-1} \zeta_G^q\) reduces to that of computing \((\zeta_G^r)^{-1} \zeta_G^q\).

Now, let \(A = \mathbb{Q}(q, r)[x]\). It should be remarked that \(q, r\) are being regarded as indeterminates during this computation. For every \([U] \in \mathcal{O}(G)\) put

\[
A_{[U]} = q M_G^q(r x, [U]) = \sum_{[W] \ll [U]} \mu_G^q([U], [W]) (qr)^{G:W} x^{G:W},
\]

\[
B_{[U]} = r M_G^r(q x, [U]) = \sum_{[W] \ll [U]} \mu_G^r([U], [W]) (qr)^{G:W} x^{G:W}.
\]

Since

\[
\varphi_G^q(A_{[U]})_{[U] \in \mathcal{O}(G)} = \varphi_G^r(B_{[U]})_{[U] \in \mathcal{O}(G)} = ((qr)^{G:U} x^{G:U})_{[U] \in \mathcal{O}(G)},
\]

we have

\[
(B_{[U]})_{[U] \in \mathcal{O}(G)} = (\zeta_G^r)^{-1} \zeta_G^q(A_{[U]})_{[U] \in \mathcal{O}(G)}.
\]

Note that we are using the column notation. Set

\[
g_{[U], [W]}(q, r) = (\zeta_G^r)^{-1} \zeta_G^q([U], [W]),
\]

equivalently,

\[
B_{[U]} = \sum_{[G] \ll [W] \ll [U]} g_{[U], [W]}(q, r) A_{[W]}.
\]

Comparing the coefficient of \(A_{[W]} \) \(([G] \ll [W] \ll [U])\), in

\[
\sum_{[G] \ll [V] \ll [U]} \phi_U(G/V) q^{(V:U)-1} A_{[V]} = \sum_{[G] \ll [V] \ll [U]} \phi_U(G/V) r^{(V:U)-1} B_{[V]}, \quad \forall [U] \in \mathcal{O}(G),
\]

yields
\[ \phi_U(G/W) q(W:U)^{-1} = \sum_{[W] \leq [V] \leq [U]} \phi_U(G/V) r(V:U)^{-1} g_{[V],[W]}(q, r). \] (3.13)

Denote by \( e_{[W]}(q/r) \) the element in \( \mathbb{W}_G(A) \) defined by
\[
e_{[W]}(q/r)([Z]) = \frac{q}{r} \delta_{[W],[Z]}, \quad \forall [Z] \in \mathcal{O}(G).
\]

Here, \( \delta \) means Kronecker’s delta. Then, by Eq. (3.13)
\[
\Phi^r_G(e_{[W]}(q/r)) = \left( \phi_U(G/W) q(W:U)^{-1} \right)_{[U] \in \mathcal{O}(G)}
\]
\[
= \left( \sum_{[W] \leq [V] \leq [U]} \phi_U(G/V) r(V:U)^{-1} g_{[V],[W]}(q, r) \right)_{[U] \in \mathcal{O}(G)}.
\]

Finally, \( \Phi^r_G = \phi^r_G \circ \tau^r_G \) implies
\[
g_{[U],[W]}(q, r) = \frac{r}{q} \text{Ind}_{W}^{G} \left( \tau_{W}^{r} \left( \frac{q}{r} \right) \right)([U]) = \frac{r}{q} \sum_{[S] \in \mathcal{O}(W), [S] = [U]} M_{W}^{r} \left( \frac{q}{r}, [S] \right). \] (3.14)

From now on, we will require that \( G \) should satisfy Condition \( (\star) \).

**Lemma 3.10.** If \( q \) and \( r \) are integers satisfying the condition
\[
D^{pr}(q) \cap D^{pr}(G) = D^{pr}(r) \cap D^{pr}(G),
\]
then for every \([U], [W] \in \mathcal{O}(G)\), \( g_{[U],[W]}(q, r) \) takes an integer value.

**Proof.** By Corollary 3.7 we have the following commutative diagram:
\[
\begin{array}{ccc}
\mathbb{W}_G^q(\mathbb{Z}) & \xrightarrow{\omega^r_q} & \mathbb{W}_G^r(\mathbb{Z}) \\
\tau^q_G \downarrow \cong & & \tau^r_G \downarrow \cong \\
N_{r}^q_G(\mathbb{Z}) & \xrightarrow{n^r_q} & N_{r}^r_G(\mathbb{Z}).
\end{array}
\]

From the commutativity of the above diagram it follows that \( n^r_q \) should be an isomorphism. Thus every entry of \( n^r_q \) should be an integer. \( \square \)

**Remark 3.11.** In case where \( q = 0 \) or \( r = 0 \), formula (3.14) may not make sense. To avoid it one has to proceed specializations after canceling common factors out first. For instance, when \( q = r \), cancelation gives \( g_{[U],[W]}(r, r) = \delta_{[U],[W]} \). Thus, we have \( g_{[U],[W]}(0, 0) = \delta_{[U],[W]} \).

Lemma 3.10 provides the following criteria for classification.
Theorem 3.12.

(a) Let \( q \) vary over the set of integers. Then, \( \text{Nr}_G^q \) is classified up to strict natural isomorphism by the set of prime divisors of \( q \) contained in \( D^\text{pr}(G) \).
(b) Suppose that \( q \) and \( r \) satisfy the condition
\[
D^\text{pr}(q) \cap D^\text{pr}(G) = D^\text{pr}(r) \cap D^\text{pr}(G).
\]
Then the strict natural isomorphism \( n^q_r \) is given by the left multiplication by the matrix
\[
n^q_r([U], [W]) = \begin{cases} 
\frac{r}{q} \left( \sum_{[S] \in \mathcal{O}(W), [S]=[U]} \sum_{[S] \in \mathcal{O}(G)} M^r_W(q, [S]) \right) \Psi(W:U) & \text{if } [W] \preceq [U], \\
0 & \text{otherwise}.
\end{cases}
\]

Theorem 3.13.

(a) Let \( q \) vary over the set of integers. Then, \( \hat{\text{Nr}}_G^q \) is classified up to strict natural isomorphism by the set of prime divisors of \( q \) contained in \( D^\text{pr}(G) \).
(b) Suppose that \( q \) and \( r \) satisfy the condition
\[
D^\text{pr}(q) \cap D^\text{pr}(G) = D^\text{pr}(r) \cap D^\text{pr}(G).
\]
Then the strict natural isomorphism \( \hat{n}^q_r \) is given by the left multiplication by the matrix
\[
\hat{n}^q_r([U], [W]) = \begin{cases} 
\frac{r}{q} \left( \sum_{[S] \in \mathcal{O}(W), [S]=[U]} \sum_{[S] \in \mathcal{O}(G)} M^r_W(q, [S]) \right) \Psi(W:U) & \text{if } [W] \preceq [U], \\
0 & \text{otherwise}.
\end{cases}
\]

As an immediate corollary of Theorem 3.13(b) we obtain a very interesting relation between \( \mu^q_G \) and \( \mu^r_G \).

Corollary 3.14. Let \( G \) be any profinite group and \( q, r \) be indeterminates. Then, for \( [U] \in \mathcal{O}(G) \) one has the formula
\[
\mu^r_G([U], [G]) = \sum_{[W] \preceq [U]} \frac{r}{q} \left( \sum_{[S] \in \mathcal{O}(W), [S]=[U]} \sum_{[S] \in \mathcal{O}(G)} M^r_W(q, [S]) \right) \mu^q_G([W], [G]). \tag{3.15}
\]

Proof. The desired formula can be established by comparing the coefficient of \( x \) from both sides of the identity
\[
rM^r_G(qx, [U]) = \sum_{[W] \preceq [U]} g_{[U],[W]}(q, r)qM^r_G(rx, [W]). \quad \square
\]

If \( G \) is abelian or Hamiltonian, Eq. (3.15) reduces to
\[
\mu^r_G(U, G) = \frac{r}{q} \sum_{U \subseteq W} M^r_W(q, U) \mu^q_G(W, G).
\]
In addition, considering the case where $q = 1$ or $r = 1$, we obtain

$$
\mu^q_G(U, G) = q \sum_{U \subseteq W} M^q_W \left( \frac{1}{q}, U \right) \mu_G(W, G), \quad \text{and}
$$

$$
\mu_G(U, G) = \frac{1}{q} \sum_{U \subseteq W} M_W(q, U) \mu^q_G(W, G),
$$

(3.16)

where $q$ is an indeterminate.

Example 3.15. Let $G = \hat{C}$. Then Eq. (3.16) implies

$$
\mu^{-1}_\hat{C}(\hat{C}^n, \hat{C}) = - \sum_{d|n} M^{-1}_{\hat{C}d}(-1, \hat{C}^n) \mu_{\hat{C}}(\hat{C}^d, \hat{C}), \quad \text{and}
$$

$$
\mu_{\hat{C}}(\hat{C}^n, \hat{C}) = - \sum_{d|n} M_{\hat{C}d}(-1, \hat{C}^n) \mu^{-1}_{\hat{C}}(\hat{C}^d, \hat{C}).
$$

Put $n = 2^k n'$ with $(n', 2) = 1$. Applying the formulae

$$
M^{-1}_{\hat{C}d}(-1, \hat{C}^n) = \begin{cases} 
-1 & \text{if } n \text{ is of the form } 2^m d, \\
0 & \text{otherwise},
\end{cases}
$$

$$
M_{\hat{C}d}(-1, \hat{C}^n) = \begin{cases} 
1 & \text{if } n = d, \\
0 & \text{otherwise,}
\end{cases}
$$

to the above identity, we finally obtain

$$
\mu^{-1}_\hat{C}(\hat{C}^n, \hat{C}) = \sum_{m=0}^k \mu_{\hat{C}}(\hat{C}^{2^m n'}, \hat{C}), \quad \text{and}
$$

$$
\mu_{\hat{C}}(\hat{C}^n, \hat{C}) = \begin{cases} 
\mu^{-1}_\hat{C}(\hat{C}^n, \hat{C}) - \mu^{-1}_\hat{C}(\hat{C}^{2^n}, \hat{C}) & \text{if } n \text{ is even,} \\
\mu^{-1}_\hat{C}(\hat{C}^n, \hat{C}) & \text{otherwise.}
\end{cases}
$$

4. The functorial meaning of the multiplicativity of $\mu^q_G$ and $\tilde{\mu}^q_G$ when $q = 1, 0, -1$

4.1. The multiplicativity of $\mu^q_G$ and $\tilde{\mu}^q_G$ when $q = 1, 0, -1$

One of the most essential properties of the classical Möbius function may be its multiplicativity. In this section, we show that $\mu^q_G$ and $\tilde{\mu}^q_G$ also have the multiplicative property when $q = 1, 0, -1$. Indeed, it is intimately related with a composition property of $Nr^q_G$ and $\tilde{Nr}^q_G$. 
Definition 4.1.

(a) Two Steinitz numbers, say
\[ \prod_{p \in I} p^{k(p)} \quad \text{and} \quad \prod_{p \in J} p^{k'(p)}, \]
where \( k(p), k'(p) \geq 1 \), are called coprime if \( I \cap J = \emptyset \). Here, \( I \) and \( J \) denote index sets of primes and \( k(p) \geq 1 \) for all \( p \in I \cup J \).

(b) Given two profinite group \( G \) and \( H \), we say that they are coprime if their orders are coprime.

Suppose that \( G, H \) are profinite groups which are coprime. Although easy, the following statement is of great importance.

As in the case of finite groups, one can easily show that every open subgroup of \( G \times H \) arises uniquely as \( U \times V \), where \( U \) is an open subgroup of \( G \) and \( V \) an open subgroup of \( H \), respectively.

Let us induce an enumeration of \( \mathcal{O}(G) \times \mathcal{O}(H) \) from that of \( \mathcal{O}(G \times H) \) under the obvious identification \( ([U],[V]) \leftrightarrow [U \times V] \). With this preparation, let \( A \) be a commutative ring with identity and \( q = 1, 0, -1 \). Applying the functoriality of \( \varphi^q_G \) to the ring homomorphism \( \varphi^q_G(A) : \hat{N}r^q_G(A) \to \text{gh}_G(A) \), we have the commutative diagram

\[
\begin{array}{ccc}
\hat{N}r^q_H(\hat{N}r^q_G(A)) & \xrightarrow{\varphi^q_H(\hat{N}r^q_G(A))} & \text{gh}_H(\hat{N}r^q_G(A)) \\
\downarrow & & \downarrow \\
\hat{N}r^q_H(\text{gh}_G(A)) & \xrightarrow{\varphi^q_H(\text{gh}_G(A))} & \text{gh}_H(\text{gh}_G(A)).
\end{array}
\]

Note that \( \text{gh}_H(\text{gh}_G(A)) \) can be identified with \( \text{gh}_{G \times H}(A) \) in the obvious way. The resulting ring homomorphism

\[
\varphi^q_{G,H}(A) : \hat{N}r^q_H(\hat{N}r^q_G(A)) \to \text{gh}_{G \times H}(A)
\]
sends \( X = (X_{[U],[V]}|U \in \mathcal{O}(G), [V] \in \mathcal{O}(H)) \) to \( Y = (Y_{[U],[V]}|U \in \mathcal{O}(G), [V] \in \mathcal{O}(H)) \), where

\[
Y_{[U],[V]} = \sum_{[H] \ll [T]} q^{(T : V) - 2} \phi_V(H/T) \left( \sum_{[S] \ll [U]} \phi_U(G/S)X_{[S],[T]} \right).
\]

The following functorial property is almost straightforward.

Theorem 4.2. Suppose that \( G, H \) are profinite groups which are coprime and \( q = 1, 0, -1 \). Then there is a unique functorial isomorphism,

\[
id_{G,H} : \hat{N}r^q_H \circ \hat{N}r^q_G \to \hat{N}r^q_{G \times H}, \quad X \mapsto X,
\]
satisfying \( \varphi^q_{G,H} = \varphi^q_{G \times H} \circ \text{id}_{G,H} \).
**Proof.** Let

\[ R = Z[X_{[U],[V]}]: [U] \in \mathcal{O}(G), [V] \in \mathcal{O}(H) \]

and let \( X = (X_{[U],[V]})_{[U] \in \mathcal{O}(G), [V] \in \mathcal{O}(H)} \). We also let

\[ Z = (Z_{[U],[V]})_{[U] \in \mathcal{O}(G), [V] \in \mathcal{O}(H)} := (\varphi_{G,H}^{q})^{-1}(\varphi_{G,H}^{q}(X)) \in \widehat{N_{rG\times H}^{q}}(\mathbb{Q} \otimes R). \]

For our purpose it suffices to show that \( Z \in \widehat{N_{rG\times H}^{q}}(R) \). Indeed, we will show that

\[ Z_{[U],[V]} = X_{[U],[V]} \]

for all \([U] \in \mathcal{O}(G), [V] \in \mathcal{O}(H)\). To do this, let me first observe that

\[ \varphi_{G,H}^{q}(X) = \varphi_{G,H}^{q}(Z) \]

equivalently,

\[
\sum_{[H] \leq [T] \leq [V]} q^{(T:V)-1} \varphi_{V}(H/T) \left( \sum_{[G] \leq [S] \leq [U]} \varphi_{U}(G/S)q^{(S:U)-1}X_{[S],[T]} \right) \\
= \sum_{[G] \leq [S] \leq [U]} q^{(T \times S:V \times U)-1} \varphi_{U}(G \times H / S \times T)Z_{[S],[T]}.
\]

However, since \( G \) and \( H \) are coprime, we have

\[
\varphi_{U}(G/S)\varphi_{V}(H/T) = \varphi_{U \times V}(G \times H / S \times T), \\
(S \times T : U \times V) = (S : U)(T : V), \\
q^{(T:V)+(S:U)-2} = q^{(T \times S:V \times U)-1} \tag{4.1}
\]

for all \([G] \leq [S] \leq [U], [H] \leq [T] \leq [V]\). In view of Eq. (4.1), it follows that \( Z_{[U],[V]} = X_{[U],[V]} \). This implies that \( \text{id}_{G,H}(X) = X \) and \( \varphi_{G,H}^{q} = \varphi_{G\times H}^{q} \circ \text{id}_{G,H} \), as required. \( \square \)

**Corollary 4.3.** Suppose that \( G, H \) are profinite groups which are coprime and \( q = 1, 0, -1 \). Then, for \([U], [S] \in \mathcal{O}(G)\) and \([V], [T] \in \mathcal{O}(H)\), we have

\[
\mu_{G \times H}^{q}([U \times V], [S \times T]) = \mu_{G}^{q}([U], [S]) \mu_{H}^{q}([V], [T]). \tag{4.2}
\]

**Proof.** For simplicity of notation, we let

\[ X_{[V]} = \left( \begin{array}{c} \vdots \\ X_{[U],[V]} \\ \vdots \end{array} \right)_{[U] \in \mathcal{O}(G)} \quad \text{and} \quad Y_{[V]} = \left( \begin{array}{c} \vdots \\ Y_{[U],[V]} \\ \vdots \end{array} \right)_{[U] \in \mathcal{O}(G)} \]
for every \([V] \in \mathcal{O}(H)\). Note that
\[
\varphi^q_{G,H} \left( \begin{array}{c}
\vdots \\
X_{[V]} \\
\vdots
\end{array} \right)_{[V] \in \mathcal{O}(H)} = \left( \begin{array}{c}
\vdots \\
Y_{[V]} \\
\vdots
\end{array} \right)_{[V] \in \mathcal{O}(H)}
\]
is equivalent to
\[
\zeta^q_{H} \left( \begin{array}{c}
\vdots \\
\zeta^q_{G}X_{[V]} \\
\vdots
\end{array} \right)_{[V] \in \mathcal{O}(H)} = \left( \begin{array}{c}
\vdots \\
Y_{[V]} \\
\vdots
\end{array} \right)_{[V] \in \mathcal{O}(H)},
\]
which is also equivalent to
\[
\left( \begin{array}{c}
\vdots \\
\zeta^q_{G}X_{[V]} \\
\vdots
\end{array} \right)_{[V] \in \mathcal{O}(H)} = \left( \begin{array}{c}
\vdots \\
Y_{[V]} \\
\vdots
\end{array} \right)_{[V] \in \mathcal{O}(H)}
\]
Comparing the \([V]\)th component of both sides in the above equality gives rise to
\[
\zeta^q_{G}X_{[V]} = \sum_{[T] \leq [V]} \mu^q_{H}([V],[T])Y_{[T]}.
\]
Consequently, we have
\[
\left( \begin{array}{c}
\vdots \\
X_{[V]} \\
\vdots
\end{array} \right)_{[V] \in \mathcal{O}(H)} = \mu^q_{G} \left( \sum_{[T] \leq [V]} \mu^q_{H}([V],[T])Y_{[T]} \right).
\]
Note that this is equivalent to
\[
X_{[U],[V]} = \sum_{[S] \leq [U], [T] \leq [V]} \mu^q_{G}([U],[S])\mu^q_{H}([V],[T])Y_{[S],[T]}
\tag{4.3}
\]
for all \([U] \in \mathcal{O}(G), [V] \in \mathcal{O}(H)\). On the other hand, Theorem 4.2 says that
\[
\left( \begin{array}{c}
\vdots \\
X_{[V]} \\
\vdots
\end{array} \right)_{[V] \in \mathcal{O}(H)} = \mu^q_{G \times H} \left( \begin{array}{c}
\vdots \\
Y_{[U \times V]} \\
\vdots
\end{array} \right)_{[U] \in \mathcal{O}(G), [V] \in \mathcal{O}(H)}
\]
under the obvious identification \(Y_{[U],[V]} = Y_{[U \times V]}\). Thus,
\[ X_{[U],[V]} = \sum_{[S \times T] \leq [U \times V]} \mu^q_{G \times H}([U \times V], [S \times T]) Y_{[S \times T]} \]
\[ = \sum_{[S] \leq [U]} \sum_{[T] \leq [V]} \mu^q_{G \times H}([U \times V], [S \times T]) Y_{[S],[T]}. \tag{4.4} \]

Now, our result can be obtained by combining Eq. (4.3) with Eq. (4.4). \(\square\)

We can derive an analogous result for \(\tilde{\mu}_G\). To do this, let us first recall Eq. (2.2), which says that
\[ \tilde{\mu}^q_G([U],[W]) = \mu^q_G([U],[W]) \Psi^{(W:U)}. \]

**Corollary 4.4.** Suppose that \(G, H\) are profinite groups which are coprime and \(q = 1, 0, -1.\) Then for \([U],[S] \in \mathcal{O}(G)\) and \([V],[T] \in \mathcal{O}(H),\) we have
\[ \tilde{\mu}^q_{G \times H}([U \times V],[S \times T]) = \tilde{\mu}^q_G([U],[S]) \tilde{\mu}^q_H([V],[T]). \]

**Proof.** Let us apply Eq. (2.2) to Corollary 4.3. Then
\[ \tilde{\mu}^q_{G \times H}([U \times V],[S \times T]) = \mu^q_{G \times H}([U \times V],[S \times T]) \Psi^{(S \times T:U \times V)} \]
and
\[ \tilde{\mu}^q_{G \times H}([U \times V],[S \times T]) = \mu^q_G([U],[S]) \mu^q_H([V],[T]) \Psi^{(S:U)} \Psi^{(T:V)} \Psi \]

Now, the desired result follows from the fact that \(\Psi^m \Psi^n = \Psi^{mn}\) for all \(m, n \in \mathbb{N}.\) \(\square\)

The functorial meaning of Corollary 4.4 can be explained as follows: Let \(q = 1, 0, -1,\) and let \(A\) be any special \(\lambda\)-ring. Introduce a map
\[ \tilde{\varphi}^q_{G,H}(A) : \text{gh}_{G \times H}(A) \rightarrow \text{gh}_{G \times H}(A), \]
which is defined by sending
\[ \mathbf{X} = (X_{[U],[V]})_{[U] \in \mathcal{O}(G), \ [V] \in \mathcal{O}(H)} \quad \text{to} \quad \mathbf{Y} = (Y_{[U],[V]})_{[U] \in \mathcal{O}(G), \ [V] \in \mathcal{O}(H)}, \]
where \(Y_{[U],[V]}\) is given by
\[ \sum_{[H] \leq [T] \leq [V]} q^{(T:V)+(S:U) - 2} \phi_V(H/T) \Psi^{(T:V)} \left( \sum_{[G] \leq [S] \leq [U]} \phi_U(G/S) \Psi^{(S:U)} (X_{[S],[T]}) \right). \]

**Theorem 4.5.** Suppose that \(G, H\) are profinite groups which are coprime and \(q = 1, 0, -1.\) Then there is a unique bijection,
\[ \text{id}_{G,H} : \text{gh}_{G \times H}(A) \rightarrow N_{r^q_{G \times H}}, \quad \mathbf{X} \mapsto \mathbf{X}, \]
satisfying \(\tilde{\varphi}^q_{G,H} = \varphi^q_{G \times H} \circ \text{id}_{G,H}.\)
Proof. Combining Eq. (4.1) with the property

$$\Psi^m \circ \Psi^n = \Psi^{mn}, \quad \forall m, n \in \mathbb{N},$$

$$\Psi^n(a + b) = \Psi^n(a) + \Psi^n(b), \quad \forall n \in \mathbb{N}, \forall a, b \in A,$$

yields the desired result. □

**Remark 4.6.**

(a) It should be remarked that the notation $N r^q_H(N r^q_G(A))$ does not always make sense. For it to have a meaning, $N r^q_G(A)$ should have a special $\lambda$-ring structure.

(b) Assume that $N r^q_G(A)$ has a special $\lambda$-ring structure. And let us make $g h_G(A)$ into a $\Psi$-ring by defining

$$\Psi^n((a[U])[U] \in O(G)) = (\Psi^n(a[U]))_{[U] \in O(G)}, \quad \forall n \geq 1.$$

Then we can consider the following composition of ring homomorphisms:

$$N r^q_H(N r^q_G(A)) \xrightarrow{N r^q_H(\overline{\psi}^q_G(A))} N r^q_H(g h_G(A)) \xrightarrow{\overline{\psi}^q_H(g h_G(A))} g h_H(g h_G(A)) \cong g h_{G \times H}(A).$$

An easy computation shows that

$$\overline{\varphi}^q_{G,H} = \overline{\varphi}^q_H(g h_G(A)) \circ N r^q_H(\overline{\psi}^q_G(A)).$$

We can also consider the composition

$$N r^q_H(N r^q_G(A)) \xrightarrow{\overline{\omega}^q_H(N r^q_G(A))} g h_H(N r^q_G(A)) \xrightarrow{\overline{\omega}^q_H(g h_G(A))} g h_H(g h_G(A)) \cong g h_{G \times H}(A).$$

In this case the resulting composition map $g h_H(\overline{\varphi}^q_G(A)) \circ \overline{\varphi}^q_H(N r^q_G(A))$ sends $X = (X_{[U],[V])}[U] \in O(G), \quad [V] \in O(H)$ to $Y = (Y_{[U],[V])}[U] \in O(G), \quad [V] \in O(H)$, where $Y_{[U],[V]}$ is given by

$$\sum_{[H] \leq [T] \leq [V]} q^{(T : V)+(S : U)-2} \overline{\psi}_V(H/T) \overline{\psi}^{(T : V)} \left( \sum_{[G] \leq [S] \leq [U]} \overline{\omega}_U(G/S) \overline{\psi}^{(S : U)}(X_{[S],[T]}) \right).$$

But, in general, we cannot argue that $\overline{\varphi}^q_H(g h_G(A)) \circ N r^q_H(\overline{\psi}^q_G(A))$ coincides with $g h_H(\overline{\varphi}^q_G(A)) \circ \overline{\varphi}^q_H(N r^q_G(A))$ since the bold-faced $\overline{\psi}^{(T : V)}$ means the $(T : V)$th Adams operation of $N r^q_G(A)$, not that of $A$.

(c) If $2 \notin D^G_{pr}$, then $\overline{\zeta}^q_G = \overline{\zeta}^{-q}_G$ and $\zeta^q_G = \zeta^{-q}_G$ for every $q \in \mathbb{Z}$. So, we can say that $\overline{\mu}^q_G = \overline{\mu}^{-q}_G$ and $\mu^q_G = \mu^{-q}_G$ if $2 \notin D^G_{pr}$ (see Remark 2.3).
4.2. The case where \( G \) is an abelian profinite \( p \)-group

As in the case of finite groups, any pronilpotent group \( G \) is isomorphic to the cartesian product of its Sylow subgroups. In view of Corollary 4.3, the problem of computing \( \mu^q_G \) reduces to that of computing \( \mu^q_{G_p} \) for each Sylow \( p \)-subgroup \( G_p \) of \( G \). In this section, we investigate the value of \( \mu^q_G \) (\( q = 1, 0, -1 \)) in case where \( G \) is an abelian profinite \( p \)-group.

Case (1): \( q = 1 \).

Let \( G \) be an abelian profinite \( p \)-group and \( U, V \) open subgroups of \( G \). Combining Theorem 5.5 together with Examples 5.3 and 5.7, we can deduce that \( \mu_G(U, V) = 0 \) if \( U \) is not a subgroup of \( V \) or \( U \) is a subgroup of \( V \) such that \( V/U \) is not of type \((1, 1, \ldots, 1)\). In all other cases

\[
\mu_G(U, V) = \frac{1}{(G:U)}(-1)^s p^{s(s-1)/2} \quad \text{where} \quad (V:U) = p^s.
\]

Case (2): \( q = 0 \).

Given any profinite group \( G \) it is not difficult to show that

\[
\mu^0_G([U], [V]) = \begin{cases} \frac{1}{(N_G(U):U)} & \text{if} \quad [U] = [V], \\ 0 & \text{otherwise}. \end{cases}
\]

If \( G \) is abelian, then the above formula reduces to

\[
\mu^0_G(U, V) = \begin{cases} \frac{1}{(G:U)} & \text{if} \quad U = V, \\ 0 & \text{otherwise}. \end{cases}
\]

Case (3): \( q = -1 \).

(i) \( p \neq 2 \):

Let \( G \) be any profinite \( p \)-group. From Remark 4.6 it follows that

\[
\mu^{-1}_G = \mu_G.
\]

(ii) \( p = 2 \):

To begin with, we start with an arbitrary profinite 2-group \( G \). From Eq. (3.15) it follows that

\[
\mu^{-1}_G([U], [G]) = - \sum_{[G] \leq [W] \leq [U]} \left( \sum_{[S] \in \mathcal{O}(W)} M^{-1}_W(-1, [S]) \right) \mu_G([W], [G]). \quad (4.5)
\]

To compute the coefficients of the right-hand side we recall the relation

\[
\sum_{[W] \leq [Z] \leq [S] \text{ (in } \mathcal{O}(W))} \phi_S(W/Z)(-1)^{(Z:S)-1}M^{-1}_W(-1, [Z]) = -1,
\]
which follows from the definition of $M_w^{-1}(-1, [Z])$. For simplicity, we will use the notation $x_w([S])$ instead of $M_w^{-1}(-1, [S])$ for $[S] \in \mathcal{O}(W)$. Note that $x_w([W]) = -1$. Since $(Z : S)$ is even unless $[Z] = [S]$, the above equation can be written as

$$\phi_S(W/S)x_w([S]) = \sum_{[W] \leq [Z] \geq [S]} \phi_S(W/Z)x_w([Z]) - 1.$$  

Dividing both sides by $\phi_S(W/S)$ yields

$$x_w([S]) = \frac{1}{\phi_S(W/S)} \sum_{[W] \leq [Z] \geq [S]} \phi_S(W/Z)x_w([Z]) - 1 = \sum_{[W] \leq [Z] \geq [S]} \frac{v_w(S, Z)}{(N_Z(S) : S)} x_w([Z]) - \frac{1}{(N_W(S); S)} \quad \text{(by Eq. (3.2))}. \quad (4.6)$$

From now on, we assume that $G$ is an abelian 2-group. Then $W$ is also an abelian 2-group. Then formula (4.6) reduces to

$$x_w(S) = \begin{cases} -1 & \text{if } S = W, \\ \sum_{S \subseteq Z \subseteq W} \frac{1}{(Z : S)} x_w(Z) - \frac{1}{(W : S)} & \text{otherwise}, \end{cases} \quad (4.7)$$

and formula (4.5) reduces to

$$\mu^{-1}_G(U, G) = - \sum_{U \leq W \leq G} x_w(S)\mu_G(W, G).$$

Applying Theorem 5.5(c) and Example 5.3 to the above formula, we can conclude that

$$\mu^{-1}_G(U, G) = - \sum_{U \leq W \leq G} \frac{1}{2^s} x_w(U)(-1)^s 2^{\frac{1}{2^s}(s-1)}. \quad (4.8)$$

**Example 4.7.** Let $G = \mathbb{Z}/2^s\mathbb{Z}$ and denote by $G_n$ the unique subgroup of index $2^n$. Note that $W = G$, $G_{s-1}$ are the only subgroups of $G$ such that $G/W$ is of type $(1, \ldots, 1)$. And we can deduce from Eq. (4.7) that $x_w(S) = -1$ for all $S \in \mathcal{O}(W)$. Finally, Eq. (4.8) implies

$$\mu^{-1}_G(G_n, G) = \mu_G(G, G) + \mu_G(G_{s-1}, G) = \begin{cases} 1 - \frac{1}{2} = \frac{1}{2} & \text{if } n \leq s - 1, \\ 1 & \text{otherwise}. \end{cases}$$

4.3. Some remarks on the functorial property

In this section, we show that for every integer $q$, $\mathbb{W}_G^q$ and $\hat{N}_H^q$ also have the composition property such as Theorems 4.2 and 4.5 for some pairs of profinite groups. To do this, let us briefly introduce some prerequisite definitions and notation.

Let $\mathbb{N}$ be the set of positive integers, and let $\emptyset \neq N \subseteq \mathbb{N}$ be a truncation set, i.e., $N$ contains every positive divisor of each of its elements. In particular, the submonoid of $(\mathbb{N}, \cdot)$ generated
by any set of prime numbers is a truncation set. We call these sets monoidal. For a prime \( p \)
 denote by \( M(p) \) the monoidal set generated by \( p \), that is, \( M(p) = \{1, p, p^2, \ldots\} \). If \( M \) and \( N \)
 are truncation sets, then \( MN = \{mn \mid m \in M, n \in N\} \) is also a truncation set. For instance, if \( M \)
 is monoidal, then \( M = \prod_{p: \text{prime} \in M} M(p) \).

Given a truncation set \( N \) and an integer \( q \), we define \( \mathbb{W}_N^q \) to be the unique covariant functor
from the category of commutative rings into itself characterized by the following conditions:

1. As a set, \( \mathbb{W}_N^q(A) \) equals \( A^N \).
2. For any ring homomorphism \( f: A \to B \), the map \( \mathbb{W}_N^q(f): (X_n)_{n \in N} \mapsto (f(X_n))_{n \in N} \) is a ring homomorphism.
3. The map

\[
\Phi_N^q : \mathbb{W}_N^q(A) \to gh_N(A), \quad (X_n)_{n \in N} \mapsto \left( \sum_{d \mid n} dq^{n-1} \frac{X_d}{d} \right)_{n \in N}
\]

is a ring homomorphism.

**Example 4.8.**

(a) If \( [n] \) denotes the set of all divisors of \( n \in \mathbb{N} \), then \( \mathbb{W}_{[n]} \) is isomorphic to \( \mathbb{W}_{C(n)} \). Here, \( C(n) \)
means the cyclic group of order \( n \).

(b) If \( N = \{1, p, p^2, \ldots\} \) for some prime \( p \), then \( \mathbb{W}_N \) is isomorphic to \( \mathbb{W}_{\hat{C}_p} \). Here, \( \hat{C}_p \) represents
the pro-\( p \)-completion of the infinite cyclic group \( C \). In the literature, this functor has been
denoted by \( \mathbb{W}_p \).

Let \( \Phi_{M,N}^q : \mathbb{W}_N^q \circ \mathbb{W}_M^q \to gh_{MN} \) be a natural transformation such that for any commutative
ring \( A \)

\[
\Phi_{M,N}^q(A) : \mathbb{W}_N^q(\mathbb{W}_M^q(A)) \to gh_{MN}(A)
\]

sends \( X = (X_{m,n})_{m \in M, n \in N} \) to \( (X^q_{(m,n)})_{m \in M, n \in N} \), where

\[
X^q_{(m,n)} = \sum_{d \mid n} dq^{n-1} \left( \sum_{c \mid m} cq^{m-1} X_c^{\frac{m}{d}} \right)^{\frac{n}{d}}.
\]

In a similar way as above let us obtain a natural transformation \( \varphi_{M,N}^q : \hat{N}_N^q \circ \hat{N}_M^q \to gh_{MN} \),
where given a commutative ring \( A \)

\[
\varphi_{M,N}^q(A) : \hat{N}_N^q(\hat{N}_M^q(A)) \to gh_N(gh_M(A))
\]

sends \( X = (X_{m,n})_{m \in M, n \in N} \) to \( (X^q_{(m,n)})_{m \in M, n \in N} \), where

\[
X^q_{(m,n)} = \sum_{d \mid n} dq^{n-1} \left( \sum_{c \mid m} cq^{m-1} X_c^{\frac{m}{d}} \right).
\]
Theorem 4.9. (See [1,19].) Let $q$ be any integer, and let $M, N$ be truncation sets with $M \cap N = \{1\}$. Then we have

(a) There is a unique functorial isomorphism

$$\omega^q_{M,N} : \mathcal{W}^q_N \circ \mathcal{W}^q_M \to \mathcal{W}^q_{MN}$$

satisfying $\Phi^q_{M,N} = \Phi^q_{MN} \circ \omega^q_{M,N}$.

(b) There is a unique functorial isomorphism

$$n^q_{M,N} : \hat{\mathcal{N}}^q_N \circ \hat{\mathcal{N}}^q_M \to \hat{\mathcal{N}}^q_{MN}$$

satisfying $\varphi^q_{M,N} = \varphi^q_{MN} \circ n^q_{M,N}$.

Consider the following cases:

1. $G, H$ are finite cyclic groups which are coprime;
2. $G = \hat{C}_p$ and $H = \hat{C}_{p'}$ for different primes $p$, $p'$;
3. $G$ is a finite cyclic group of order $n$ and $H = \hat{C}_p$ with $(n, p) = 1$;
4. $G = \hat{C}_p$ and $H$ is a finite cyclic group of order $n$ with $(n, p) = 1$.

In view of Theorem 4.9 and Example 4.8, there is a unique functorial isomorphism,

$$\omega^q_{G,H} : \mathcal{W}^q_H \circ \mathcal{W}^q_G \to \mathcal{W}^q_{G \times H}, \quad (4.9)$$

satisfying $\Phi^q_{G,H} = \Phi^q_{G \times H} \circ \omega^q_{G,H}$ in the above four cases (1)–(4). Applying Theorem 4.9 to (1)–(4) repeatedly yields the following composition property.

Corollary 4.10. Let $q$ be any integer, and let $I = \{p_1, p_2, \ldots, p_k\}$ be a finite set of distinct primes. Assume that $G_{p_i}$ denotes $C(p_i)$ or $\hat{C}_{p_i}$ for $1 \leq i \leq k$. Then we have

$$\mathcal{W}^q_{\prod_{i \in I} G_i} \cong \mathcal{W}^q_{G_{p_1}} \circ \mathcal{W}^q_{G_{p_2}} \circ \cdots \circ \mathcal{W}^q_{G_{p_k}},$$

$$\hat{\mathcal{N}}^q_{\prod_{i \in I} G_i} \cong \hat{\mathcal{N}}^q_{G_{p_1}} \circ \hat{\mathcal{N}}^q_{G_{p_2}} \circ \cdots \circ \hat{\mathcal{N}}^q_{G_{p_k}}.$$

We expect that Theorem 4.9 and Corollary 4.10 can be extended to arbitrary profinite groups $G, H$ which are coprime. In particular, we strongly believe that this is the case for at least abelian groups based on an amount of computation and diagram (4.10). So, we would like to propose the following conjecture.

Conjecture. Let $G, H$ be (abelian) profinite groups which are coprime. Then, for any integer $q$ we have

(i) $\mathcal{W}^q_H \circ \mathcal{W}^q_G \cong \mathcal{W}^q_{G \times H}$ and

(ii) $\hat{\mathcal{N}}^q_H \circ \hat{\mathcal{N}}^q_G \to \hat{\mathcal{N}}^q_{G \times H}.$
In case \( q = 0 \), the above conjecture is obvious since \( \mathbb{W}_G^0 = \hat{N}_G^0 \) for any profinite group \( G \). In case where \( q = 1, 0, -1 \), (ii) was proved in Theorem 4.2. But, in all other cases it has not been known up to now. Nevertheless, the following diagram may be a strong evidence supporting our expectation. Here, \( G \) and \( H \) denote any profinite groups which are coprime.

\[
\begin{array}{ccc}
\mathbb{W}_{G \times H}(\mathbb{Z}) & \xrightarrow{\exists \text{ isom?}} & \mathbb{W}_G \circ \mathbb{W}_H(\mathbb{Z}) \\
\cong \text{(by Th. 2.8)} & & \cong \text{(by Th. 2.8)} \\
\hat{N}_{G \times H}(\mathbb{Z}) & \xrightarrow{\exists \text{ isom?}} & \hat{N}_G \circ \hat{N}_H(\mathbb{Z})
\end{array}
\]

(4.10)

It shows that

\[
\mathbb{W}_{G \times H}(\mathbb{Z}) \cong \mathbb{W}_G \circ \mathbb{W}_H(\mathbb{Z}) \iff \mathbb{W}_G \circ \hat{N}_H(\mathbb{Z}) \cong \hat{N}_G \circ \hat{N}_H(\mathbb{Z}) \\
\iff \mathbb{W}_G \circ \hat{N}_H(\mathbb{Z}) \cong \hat{N}_G \circ \hat{N}_H(\mathbb{Z}).
\]

Note that if \( \hat{N}_H(\mathbb{Z}) \) has a special \( \lambda \)-ring structure, then

\[
\mathbb{W}_G \circ \mathbb{W}_H(\mathbb{Z}) \cong \hat{N}_G \circ \hat{N}_H(\mathbb{Z})
\]

due to Theorem 2.8. In the case of finite groups, it is known that abelian, Hamiltonian, and \( \Psi_p \)-groups for a prime \( p \) have this property (refer to [4,9]).

**Remark 4.11.** It would be very challenging to characterize pairs \((G, H)\) possessing functorial isomorphisms appearing in the above conjecture. And, to investigate relations among \( \mathbb{W}_{G \times H} \), \( \mathbb{W}_G \circ \hat{N}_H \), and \( \hat{N}_G \circ \mathbb{W}_H \) (or, their \( q \)-deformations) would be quite worthwhile.

### 5. \( q \)-Deformed Möbius function and its multiplicativity

As before, \( G \) will denote a profinite group. Denote \( S(G) \) by the lattice of open subgroups of \( G \). Fix an enumeration of \( S(G) \) subject to the condition:

If \( U \subseteq V \), then \( V \) precedes \( U \).

In particular, \( S(G) = \mathcal{O}(G) \) if \( G \) is abelian or Hamiltonian. Let \( q \) be an indeterminate and define a \( S(G) \times S(G) \) matrix \( \zeta^q_G \) by

\[
\zeta^q_G(V, W) = \begin{cases} 
q^{W:V-1} & \text{if } V \subseteq W, \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( \mu^q_G \) be the inverse of \( \zeta^q_G \).

**Remark 5.1.** If \( 2 \notin D_G^{pr} \), then \( \zeta^q_G = \zeta^{-q}_G \) for every \( q \in \mathbb{Z} \). So, we can say that \( \mu^q_G = \mu^{-q}_G \) if \( 2 \notin D_G^{pr} \) (see Remark 2.3).
Since \( \mu_G^{\text{def}} = \mu_G^1 \) denotes the Möbius function of the lattice of open subgroups of \( G \) at \( q = 1 \), we may regard \( \mu_G^q \) as a \( q \)-deformation of \( \mu_G \). From \( \mu_G^q \epsilon_G^q = \delta \) it follows that

\[
\mu_G^q(U, U) = 1, \quad \text{for all } U \in S(G),
\]

\[
\mu_G^q(U, V) = - \sum_{U \subseteq W \subset V} q^{(V:W)-1} \mu_G^q(U, W), \quad \text{for all } U \subset V.
\]

(5.1)

In view of Eq. (5.1) it is obvious that \( \mu_G^q(U, V) \) depends on \( U \) and \( V \), but not on \( G \). In particular, if \( U \) is a normal subgroup of \( V \), then

\[
\mu_G^q(U, V) = \mu_V^q(1, V/U) = \mu_{V/U}^q(1, V/U).
\]

As in the case of \( \mu_G^q \), \( \mu_G^q \) also has the multiplicative property. To verify this we will introduce \( M^q_G \) (\( q \in \mathbb{Z} \)), which is the unique covariant functor from the category of commutative rings with identity into itself satisfying the following conditions:

1. As a set, \( M^q_G(A) = A^{S(G)} \).
2. For every ring homomorphism \( f : A \to B \) and every \( \alpha \in M^q_G(A) \) one has \( M^q_G(f)(\alpha) = f \circ \alpha \).
3. The map,

\[
\eta^q \equiv \eta_G^q : M^q_G(A) \to gh_G(A),
\]

\[
\alpha \mapsto \left( \sum_{U \subseteq V \subseteq G} q^{(V:U)-1} \alpha(V) \right)_{U \in S(G)},
\]

is a ring homomorphism.

The existence of \( M^q_G \) follows from the fact that each entry of \( \mu_G^q \) has its value in \( \mathbb{Z} \). Also, its uniqueness is immediate in view of the third condition. The addition of \( M^q_G(A) \) is defined componentwise, whereas the multiplication is given by

\[
x \cdot y = \mu_G^q(\zeta_G^q x \cdot \zeta_G^q y)
\]

if we understand \( x, y \) as column vectors.

Let \( A \) be a commutative ring with identity and \( q = 1, 0, -1 \). Applying the functoriality of \( \eta_G^q \) to the ring homomorphism \( \eta_G^q : M^q_G(A) \to gh_G(A) \), we have the commutative diagram

\[
\begin{array}{ccc}
M^q_H(M^q_G(A)) & \xrightarrow{\eta^q_H(M^q_G(A))} & gh_H(M^q_G(A)) \\
\downarrow \eta^q_H(M^q_G(A)) & & \downarrow gh_H(\eta^q_G(A)) \\
M^q_H(gh_G(A)) & \xrightarrow{\eta^q_H(gh_G(A))} & gh_H(gh_G(A)).
\end{array}
\]
Note that $\text{gh}_H(\text{gh}_G(A))$ can be identified with $\text{gh}_{G\times H}(A)$ in the obvious way. The resulting ring homomorphism

$$\eta^q_{G,H}(A) : M^q_H(M^q_G(A)) \to \text{gh}_{G\times H}(A)$$

sends $X = (X_{U,V})_{U \in \mathcal{S}(G), \ V \in \mathcal{S}(H)}$ to $Y = (Y_{U,V})_{U \in \mathcal{S}(G), \ V \in \mathcal{S}(H)}$, where

$$Y_{U,V} = \sum_{V \subseteq T \subseteq H} q^{(T:V)+(S:U)-2} \left( \sum_{U \subseteq S \subseteq G} X_{S,T} \right).$$

**Theorem 5.2.** Suppose that $G$, $H$ are profinite groups whose orders are coprime and $q = 1, 0, -1$. Then we have

(a) There is a unique functorial isomorphism,

$$\text{id}_{G,H} : M^q_H \circ M^q_G \to M^q_{G\times H}, \quad X \mapsto X,$$

satisfying $\eta^q_{G,H} = \eta^q_{G\times H} \circ \text{id}_{G,H}$.

(b) For $U, S \in \mathcal{S}(G)$ and $V, T \in \mathcal{S}(H)$, we have

$$\mu^q_{G\times H}(U \times V, S \times T) = \mu^q_G(U, S)\mu^q_H(V, T).$$

**Proof.** We omit the proof since it can be done by the trivial modification of that of Theorem 4.2 and Corollary 4.3. $\square$

Theorem 5.2 is quite useful in computing $\mu^q_G$ in case where $G$ is pronilpotent since in that case $G$ is isomorphic to the cartesian product of its Sylow subgroups. Assume that $G$ is a $p$-group. If $p \neq 2$, then Remark 5.1 says that $\mu^1_G = \mu_G$. In this sense, it is quite significant to compute $\mu^1_G$ when $p = 2$. The case where $G$ is a finite $p$-group and $q = 1$ is due to Weisner [23]. See the following example.

**Example 5.3.** (See [23].) Let $G$ be a finite group of order $p^g$, where $p$ is a prime. Let $S < T$ be subgroups of $G$ of order $p^s$ and $p^t$ respectively. Then $\mu_G(S, T) = 0$ if $S$ is not a normal subgroup of $T$ or $S$ is a normal subgroup of $T$ but the quotient group $T/S$ is not abelian of type $(1, 1, \ldots, 1)$. In all other cases,

$$\mu_G(S, T) = (-1)^{t-s} p^{\frac{(t-s)(q-1)}{2}}.$$

Finally, we investigate connection between $\mu_G$ and $\mu_G$. Exploiting this, we investigate the case where $G$ is abelian or Hamiltonian in more detail when $q = -1$. Let $q$ be an indeterminate and let $G$ be any profinite group. For open subgroups $U, V$ of $G$ we have

$$\phi_U(G/V) = [N_G(V) : V] n_G(U, V),$$

where $n_G(U, V)$ equals the number of $G$-conjugates of $V$ containing $U$ (see Eq. (3.1)). Now, we define a $O(G) \times O(G)$ matrix $\lambda^q_G$ by
\[
\lambda^q_G([U], [V]) = \begin{cases} 
\frac{n_G(U, V)q^{(V:U) - 1}}{n_G(e, V)} & \text{if } [V] \preceq [U], \\
0 & \text{otherwise,}
\end{cases}
\]

and let \(\xi^q_G\) be its inverse. In case where \(G\) is finite, we introduce another \(O(G) \times O(G)\) matrix \(\bar{\lambda}^q_G\), which is defined by

\[
\bar{\lambda}^q_G([U], [V]) = \begin{cases} 
\frac{n_G(U, V)n_G(e, U)}{n_G(e, V)}q^{(V:U) - 1} & \text{if } [V] \preceq [U], \\
0 & \text{otherwise.}
\end{cases}
\]

Here, \(e\) denotes the identity of \(G\). It is easy to show that

\[
a_G([U], [V]) \overset{\text{def}}{=} \frac{n_G(U, V)n_G(e, U)}{n_G(e, V)}
\]

counts the number of subgroups conjugate to \(U\) contained in \(V\). Denote by \(\bar{\xi}^q_G\) the inverse of \(\bar{\lambda}^q_G\).

**Example 5.4.** Let \(G\) be an abelian or Hamiltonian profinite group. Then

(a) We have

\[
\lambda^q_G([U], [V]) = \begin{cases} 
n_G(U, V) & \text{if } [V] \preceq [U], \\
0 & \text{otherwise.}
\end{cases}
\]

Since \(\xi_G \overset{\text{def}}{=} \xi^1_G\) denotes the Möbius function of \(O(G)\), we may regard \(\xi^q_G\) as a \(q\)-deformation of \(\xi_G\).

(b) Also, we have

\[
\bar{\lambda}^q_G(U, V) = \begin{cases} 
(V : U)q^{(V:U) - 1} & \text{if } U \subseteq V, \\
0 & \text{otherwise.}
\end{cases}
\]

From \(\lambda^q_G\xi^q_G = \delta\) and \(\bar{\xi}^q_G\bar{\lambda}^q_G = \delta\) it follows that

\[
\delta_{[U], [V]} = \sum_{[V] \preceq [W] \preceq [U]} \nu_G(U, W)q^{(W:U) - 1}\xi^q_G([W], [V])
\]

and

\[
\delta_{[U], [V]} = \sum_{[V] \preceq [W] \preceq [U]} \bar{\xi}^q_G([U], [W])a_G([W], [V])q^{(V:W) - 1}
\]

for all \([V] \preceq [U]\).

**Theorem 5.5.** Let \(q\) be an indeterminate, \(G\) a profinite group, and \(U, V\) open subgroups of \(G\). Then

(a) \(\mu^q_G(U, G) = \xi^q_G([U], [G])\).
(b) If $G$ is finite, then $\mu^q_G(e, V) = \bar{\xi}^q_G(e, [V])$. Here, $e$ denotes the identity of $G$.

(c) If $G$ is abelian or Hamiltonian, then $\mu^q_G(U, V) = (G : U)\mu^q_G(U, V)$.

**Proof.** (a) Given $[U] \in \mathcal{O}(G)$

\[
\delta_{[U], [G]} = \sum_{[G] \leq [W] \leq [U]} \nu_G(U, W)q^{(W : U) - 1}\bar{\xi}^q_G([W], [G]) = \sum_{U \leq W} q^{(W : U) - 1}\bar{\xi}^q_G([W], [G]).
\]

Since $\mu^q_G(U, G)$ is completely determined by the same recursive formula, that is, by

\[
\delta_{U, G} = \sum_{U \leq W} q^{(W : U) - 1}\mu^q_G(W, G),
\]

we can conclude that $\mu^q_G(U, G) = \bar{\xi}^q_G([U], [G])$.

(b) Given $[V] \in \mathcal{O}(G)$

\[
\delta_{[V]} = \sum_{[V] \leq [W]} \bar{\xi}^q_G(e, [W])a_G([W], [V])q^{(V : W) - 1} = \sum_{W \leq V} q^{(V : W) - 1}\bar{\xi}^q_G(e, [W]).
\]

Since $\mu^q_G(e, V)$ is completely determined by the same recursive formula, that is, by

\[
\delta_{e, V} = \sum_{W \leq V} q^{(V : W) - 1}\mu^q_G(e, W),
\]

we can conclude that $\mu^q_G(e, V) = \bar{\xi}^q_G(e, [V])$.

(3) It follows from [17].

**Definition 5.6.** Let $q = 1, 0, -1$. Given a finite abelian or Hamiltonian group, say $H$, we define $\mu^q(H)$ by

\[
\mu^q_H(e, H).
\]

It is obvious that $\mu^q(H) = \mu^q_G(U, V)$ if $V / U$ isomorphic to $H$. By virtue of Eq. (5.1) one has the following recursive form:

\[
\mu^q(e) = 1,
\]

\[
\mu^q(H) = -\sum_{W \leq H} q^{(H : W) - 1}\mu^q(W), \quad \text{for all } W \leq H.
\]
Note that
\[ \mu^q(H) = \mu^q_H(e, H) = |H| \cdot \mu^q_H(e, H), \]
and \( \mu^q \) has the multiplicative property, that is,
\[ \mu^q(H \times K) = \mu^q(H)\mu^q(K) \]
for two finite abelian or Hamiltonian groups \( H, K \) such that \((|H|, |K|) = 1 \). The following fact follows from Example 5.3.

**Example 5.7.** Let \( q = 1 \) and \( H \) be a finite prime power abelian group. Then
\[ \mu(H) = \begin{cases} (-1)^{s} p^{\frac{1}{2}(s-1)} & \text{if } H \text{ is of type } (1, \ldots, 1) \text{ and of order } p^s, \\ 0 & \text{otherwise}. \end{cases} \]

More generally, if \( H \) is a finite abelian group, then it is isomorphic to
\[ H_1 \oplus H_2 \oplus \cdots \oplus H_k \]
for some finite prime power abelian groups \( H_i \)'s such that \((|H_i|, |H_j|) = 1 \) if \( i \neq j \). Therefore,
\[ \mu(H) = \prod_{i=1}^{k} \mu(H_i). \]

**Theorem 5.8.** Let \( G \) be an abelian or Hamiltonian profinite group. Then

(a) For an open subgroup \( U \) of \( G \) we have
\[ \mu^{-1}(G/U) = - \sum_{U \subseteq W} (W : U)M_W^{-1}(-1, U)\mu(G/W). \]

(b) Put \( n = 2^k n' \) with \( n' \) odd. Then
\[ \mu^{-1}(\hat{C}/\hat{C}^n) = \begin{cases} \mu(n) & \text{if } k = 0, \\ 2^{k-1}\mu(n') & \text{if } k \geq 1, \end{cases} \]
where \( \mu \) appearing in \( \mu(n), \mu(n') \) denotes the classical Möbius function.

**Proof.** (a) It can be obtained by combining Eq. (3.16) and Theorem 5.5(c).
(b) Put \( G = \hat{C}, U = \hat{C}^n, \) and \( W = \hat{C}^d \) where \( d|n \). Notice the identity
\[ M^{-1}(-1, n) = \begin{cases} -1 & \text{if } n = 2^m \text{ with } m \geq 0, \\ 0 & \text{otherwise}. \end{cases} \]

Therefore
\[ M_{\hat{C}^d}^{-1}(-1, \hat{C}^n) = \begin{cases} -1 & \text{if } n \text{ is of the form } 2^m d, \\ 0 & \text{otherwise.} \end{cases} \]

Since \( \mu(\hat{C}/\hat{C}^d) = \mu(d) \), (a) implies
\[
\mu^{-1}(\hat{C}/\hat{C}^n) = \sum_{d=2^m p'}^{2^k \mu(d) = 2^k \mu(n') + 2^{k-1} \mu(2n')} 2^{-m} 0 \leq m \leq k 
\]

On the other hand, due to the multiplicativity of the classical Möbius function, it follows that
\[
2^k \mu(n') + 2^{k-1} \mu(2n') = 2^k \mu(n') + 2^{k-1} \mu(2) \mu(n') = 2^{k-1} \mu(n').
\]

This completes the proof. \( \square \)

**Remark 5.9.** Set
\[
\tilde{\mu} : \mathbb{N} \to \mathbb{Z}, \quad n \mapsto \mu^{-1}(\hat{C}/\hat{C}^n), \quad \forall n \in \mathbb{N}.
\]

Due to Theorem 5.2 \( \tilde{\mu} \) is multiplicative, that is, \( \tilde{\mu}(mn) = \tilde{\mu}(m) \tilde{\mu}(n) \) if \( m, n \) are coprime. Nevertheless, this is not the case in general. To be more precise, given an integer \( q \), set
\[
\tilde{\mu}^q : \mathbb{N} \to \mathbb{Z}, \quad n \mapsto \mu^{-1}(\hat{C}/\hat{C}^n), \quad \forall n \in \mathbb{N}.
\]

It is easy to show that if \( p, p' \) are distinct primes, then \( \tilde{\mu}^q(pp') = -q^{pp'-1} + 2q^{p+p'-2} \), but \( \tilde{\mu}^q(p) \tilde{\mu}^q(p') = q^{p+p'-2} \). Thus, \( \tilde{\mu}^q(pp') \) and \( \tilde{\mu}^q(p) \tilde{\mu}^q(p') \) are not equal unless \( q = 1, 0, -1 \).

### 6. The cyclotomic identity associated to a profinite group

#### 6.1. \( q \)-deformed generalized cyclotomic identity of a profinite group

In [14], Nelson introduced the generalized cyclotomic identity of a group satisfying suitable finiteness condition in the context of algebraic combinatorics, more precisely \( G \)-SET species. The present section is devoted to the \( q \)-version of his identity associated to a profinite group. Compared with his method, ours is purely algebraic and is not dependent on the theory of species.

Throughout this section, we assume that \( G \) has only finitely many open subgroups of finite index \( m \) for each positive integer \( m \).\(^3\) For example, \( G \) may be a topologically finitely generated profinite group. Dividing both sides of Eq. (2.3) by \( (N_G(U) : U) \) and then applying Eq. (3.1) to the equation thus obtained, we can derive the identity
\[
\frac{q^{(G:U)-1} p_1(X)^{(G:U)}}{(N_G(U) : U)} = \sum_{[V] \leq [U]} \sum_{[W] \in \mathcal{O}(V)} \frac{q^{(V:U)-1} \Psi^{(V:U)}(M^q_G(X, [V]))}{(N_V(W) : W)}.
\]

Letting \( t_n (n \geq 1) \), be commuting variables, we have

\(^3\) This requirement is necessary for the species \( G \)-SET to be well-defined.
\[
\left( \sum_{[U] \in \mathcal{O}(G) \ (G:U) = n} \frac{1}{(N_G(U) : U)} \right) q^{n-1} p_1(X)^n t_n
\]

\[
= \sum_{[U] \in \mathcal{O}(G) \ (G:U) = n} \sum_{[V] \in [U]} \sum_{[W] \in \mathcal{O}(V) \ [W]=[U] \text{ in } \mathcal{O}(G)} \frac{q^{(V:U)-1} \psi^{n} (M_G(X, [V]))}{(N_V(W) : W)} t_n
\]

\[
= \sum_{[V] \in \mathcal{O}(G) \ (G:V) = n} q^{-n} \psi^{n} (M_G(X, [V])) \left( \sum_{[W] \in \mathcal{O}(V) \ (V:W) = \frac{n}{(G:V)}} \frac{1}{(N_V(W) : W)} \right) t_n. \quad (6.1)
\]

For simplicity of notation, set
\[
\text{Ind}_G(n) \overset{\text{def}}{=} n \sum_{[U] \in \mathcal{O}(G) \ (G:U) = n} \frac{1}{(N_G(U) : U)}, \quad n \geq 1.
\]

It is easy to show that it represents the number of open subgroups of \(G\) whose index in \(G\) is equal to \(n\).

**Example 6.1.**

(a) If \(G = \hat{\mathbb{C}}\), then \(\text{Ind}_G(n) = 1\) for every \(n \in \mathbb{N}\).

(b) If \(G\) is the finite cyclic group of order \(r\), then \(\text{Ind}_G(n) = 1\) if \(n|r\), and 0 otherwise.

(c) If \(G\) is the profinite completion of \(\mathbb{Z}^n\), then
\[
\text{Ind}_G(m) = \sum_{d_1d_2 \cdots d_n = m} \prod_{i=1}^n d_i^{i-1}.
\]

Summing up the first and the third term of Eq. (6.1), respectively, over the set of positive integers, we obtain
\[
\sum_{n \geq 1} \frac{1}{n} \text{Ind}_G(n) q^{n-1} p_1(X)^n t_n
\]

\[
= \sum_{n \geq 1} \sum_{[V] \in \mathcal{O}(G) \ (G:V) = n} q^{-n} \psi^{n} (M_G(X, [V])) \left( \sum_{[W] \in \mathcal{O}(V) \ (V:W) = \frac{n}{(G:V)}} \frac{1}{(N_V(W) : W)} \right) \text{Ind}_V \left( \frac{n}{(G:V)} \right) t_n
\]

\[
= \sum_{[V] \in \mathcal{O}(G) \ k \geq 1} \frac{1}{k} q^{k-1} \psi^{k} (M_G(X, [V])) \text{Ind}_V (k) t_{(G:V)k}. \quad (6.2)
\]

Taking the exponential of the first and the third term of Eq. (6.2) gives rise to the following formula.
**Theorem 6.2.** Let $q$ be an indeterminate and $X$ an alphabet $\{x_1, x_2, \ldots, x_m\}$. Assume that $G$ has only finitely many open subgroups of finite index $n$ for each positive integer $n$. Then the following equality

$$
\exp\left(\sum_{n \geq 1} \frac{1}{n} \text{Ind}_G(n) q^{n-1} p_1(X)^n t_n\right)
= \prod_{[V] \in \mathcal{O}(G)} \exp\left(\sum_{k \geq 1} \frac{1}{k} \text{Ind}_V(k) q^{k-1} \Psi^k (M^q_G(X,[V])) t(G;V)_k\right) \quad (6.3)
$$

holds.

**Corollary 6.3.** Let $q, x$ be indeterminates, and assume that $G$ has only finitely many open subgroups of finite index $n$ for each positive integer $n$. Then, we have

(a)

$$
\exp\left(\sum_{n \geq 1} \frac{1}{n} \text{Ind}_G(n) q^{n-1} x^n t_n\right)
= \prod_{[V] \in \mathcal{O}(G)} \exp\left(\sum_{k \geq 1} \frac{1}{k} \text{Ind}_V(k) q^{k-1} t(G;V)_k\right) \quad (6.4)
$$

(b)

$$
\sharp G\text{-SET}(qxt) = \prod_{[V] \in \mathcal{O}(G)} \sharp V\text{-SET}(q t(G;V)) M^q_G(x,[V]) \quad (6.5)
$$

**Proof.** (a) can be obtained from Eq. (2.4) by applying the same method that yielded Eq. (6.3). In order to establish (b) let us multiply each side of Eq. (2.4) by $q$. To the identity thus obtained let us apply the same process we used to deduce Eq. (6.3). It gives rise to the following identity

$$
\exp\left(\sum_{n \geq 1} \frac{1}{n} \text{Ind}_G(n) q^n x^n t_n\right)
= \prod_{[V] \in \mathcal{O}(G)} \exp\left(\sum_{k \geq 1} \frac{1}{k} \text{Ind}_V(k) q^k t(G;V)_k\right) \quad .
$$

Finally, the specialization $t_n = t^n$ for every $n \geq 1$ yields the desired result since $\sharp G\text{-SET}(t)$, the exponential generating function of the species $G$-SET, is given by

$$
\exp\left(\sum_{n \geq 1} \frac{1}{n} \text{Ind}_G(n) x^n t^n\right)
$$

(see [14, Proposition 2.2]). \(\square\)
Example 6.4. (a) If \( G = \hat{C} \), the profinite completion of the multiplicative infinite cyclic group \( C \), then the conjugacy classes of open subgroups are parametrized naturally by their index in \( \hat{C} \). Thus, Eq. (6.4) reduces to

\[
\exp \left( \sum_{n \geq 1} \frac{1}{n} q^{n-1} x^n t_n \right) = \prod_{n \geq 1} \exp \left( \sum_{k \geq 1} \frac{1}{k} q^{k-1} t_{nk} \right) M^q_{\hat{C}}(x,n).
\]

If there arises no confusion, we will omit the suffix from \( M^q_{\hat{C}}(x,n) \) from now on. In case where \( G = \hat{C} \) and \( t_n = t^n \), the resulting identity was first observed by Labelle and Leroux [11]. If we additionally assume that \( q = 1 \), the corresponding identity appears in the context of weighted combinatorial species, which was due to Bergeron [2].

(b) If \( G = \hat{C} \), then Eq. (6.5) reduces to

\[
\frac{1}{1 - qxt} = \prod_{n \geq 1} \left( \frac{1}{1 - qtn} \right)^{M^q(x,n)},
\]

which was observed in [18].

(c) Let \( G \) be the finite cyclic group of order \( r \). For \( d \mid r \) let \( G_d \) be the unique subgroup of order \( d \). Then, \( M^q_{G_d}(x, G_d) \) equals \( M^q(x, d) \) if \( d \mid r \) and 0 otherwise. Hence, Eq. (6.5) reduces to

\[
\exp \left( \sum_{d \mid r} \frac{1}{n} (qxt)^d \right) = \prod_{d \mid r} \exp \left( \sum_{e \mid d} \frac{1}{e} (qt^d)^e \right)^{M^q(x,d)}.
\]

(d) Eq. (6.4) at \( q = 1 \) reduces to

\[
\exp \left( \sum_{n \geq 1} \frac{1}{n} \text{Ind}_G(n)x^n t_n \right) = \prod_{[V] \in \mathcal{O}(G)} \exp \left( \sum_{k \geq 1} \frac{1}{k} \text{Ind}_V(k)t(G;V)k \right)^{M_G(x,[V])}.
\]

By the specialization \( t_n = t^n \) \( (n \geq 1) \), we can recover Nelson’s generalized cyclotomic identity:

\[
\sharp_G \text{-SET}(xt) = \prod_{[V] \in \mathcal{O}(G)} \sharp_V \text{-SET}(t(G;V))^{M_G(x,[V])}.
\]

6.2. Ring-theoretic interpretation of Eq. (2.3)

In the previous section, we have shown that Eq. (2.3) induces a \( q \)-deformed generalized cyclotomic identity Eq. (6.3). Taking logarithm on both sides of Eq. (6.3) we can also show that the latter induces Eq. (2.3). So, they are equivalent. In this section, we show that they are equivalent to the commutativity relation \( \Phi^q_G = \tilde{\phi}^q_G \circ \tau^q_G \) (see Theorem 2.8).

First, let us show that \( \Phi^q_G = \tilde{\phi}^q_G \circ \tau^q_G \) implies Eq. (2.3). Let \( G \) be a profinite group, \( X \) an alphabet \( \{x_1, x_2, \ldots, x_m\} \), and \( q \) an indeterminate. Suppose that \( A \) is the special \( \lambda \)-ring

\[
\mathbb{Q}[q][x_i: 1 \leq i \leq m]
\]
whose $\lambda$-ring structure is determined by the Adams operations given in Section 2.2. In Section 2.2, we have shown that Eq. (2.3) implies Eq. (6.3). The converse is also true. Indeed, the latter can be obtained by taking logarithm on both sides of the former. Now, consider the element

$$e_G(p_1(X)) := (p_1(X), 0, 0, \ldots) \in \mathbb{W}_G^q(A),$$

that is

$$e_G(p_1(X))([U]) = \begin{cases} p_1(X) & \text{if } [U] = [G], \\ 0 & \text{otherwise}. \end{cases}$$

One can see immediately that $\Phi_G^q(e_G(p_1(X))) = \psi_G^q \circ \tau_G^q(e_G(p_1(X)))$ is exactly same to Eq. (2.3).

For the converse, let $A$ be any special $\lambda$-ring. Let us first observe that Eq. (2.3) implies

$$\Phi_G^q(e_G(r)) = \psi_G^q \circ \tau_G^q(e_G(r)), \quad \forall r \in A.$$

Let $U, V$ be open subgroups of $W$. Plugging $W$ (an open subgroup of $G$) into Eq. (2.3) instead of $G$ and letting $p_1(X) = r$, the following emerges:

$$q^{(W:U)}-1_{r^{(W:U)}} = \sum_{[V] \leq [U] \text{ (in } \mathcal{O}(W))} \psi_U(W/V)q^{(V:U)-1}\Psi(V)(M^q_W(r, [V])) \quad (6.6)$$

(refer to Eq. (2.5)). From Lemma 3.2 it follows that for every $[U] \in \mathcal{O}(G)$,

$$\phi_U(G/W)q^{(W:U)-1_{r^{(W:U)}}}$$

$$= (N_G(U) : N_W(U)) \sum_{[S] \in \mathcal{O}(W)} q^{(W:S)-1_{r^{(W:S)}}}$$

$$= (N_G(U) : N_W(U)) \sum_{[S] \in \mathcal{O}(W)} \sum_{[V] \in \mathcal{O}(W)} \phi_S(W/V)q^{(V:S)-1}\Psi(V)(M^q_W(r, [V])).$$

Note that in the summation of the last term of the above equation,

$$\phi_S(W/V) = (N_W(S) : N_V(S))\nu_W(S, V),$$

and $[V]$ appears $n(S, V, G)/\nu_W(S, V)$-times. Thus, the coefficient of

$$\Psi^{(V:S)}(M^q_W(r, [V]))$$

is given by
\[
(N_G(U) : N_W(U))(N_W(S) : N_V(S))v_W(S, V)v_G(S, V)/v_W(S, V)
= (N_G(U) : N_V(U))v_G(S, V)
= \phi_S(G/V)
= \phi_U(G/V).
\]

Consequently,
\[
\phi_U(G/W)q^{(W:U)−1}r^{(W:U)}
= \sum_{[V] \in \mathcal{O}(G)} \phi_U(G/V)q^{(V:U)−1} \sum_{[T] \in \mathcal{O}(W) \text{ in } \mathcal{O}(G)} \Psi(V:U)(M^q_W(r, [T])).
\]

Thus, we have
\[
\Phi^q_G(e_W(r)) = \tilde{\varphi}^q(\text{Ind}_W^G \circ \tau^q_W(r)),
\]
where
\[
e_W(r)([U]) = \begin{cases} r & \text{if } [U] = [W], \\ 0 & \text{otherwise}. \end{cases}
\]

Now, \( \Phi^q_G = \tilde{\varphi}^q \circ \tau^q_G \) follows from the additivity of \( \tilde{\varphi}^q \) and \( \Phi^q_G \).

**Remark 6.5.** In order to deduce the relation between \( \Phi^q_G = \tilde{\varphi}^q_G \circ \tau^q_G \) and Eq. (2.4), we have only to consider the special \( \lambda \)-ring
\[
\mathbb{Q}[q][x]
\]
whose \( \lambda \)-ring structure is determined by the Adams operations \( \Psi^n = \text{id} \) for all \( n \geq 1 \).

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**References**