On zero-sum subsequences of restricted size II

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Abstract

Let $G$ be a finite abelian group of exponent $m$, and $k$ a positive integer. Let $s_{km}(G)$ be the smallest integer $t$ such that every sequence of $t$ elements in $G$ contains a zero-sum subsequence of length $km$. In this paper, we determine $s_{km}(G)$ for some special groups $G$ and study the number of zero-sum subsequences of length $m$.

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1. Introduction

In 1961, Erdős et al. proved that every sequence of $2|G| - 1$ elements in a finite abelian group $G$ contains a zero-sum subsequence of length $|G|$, where $|G|$ denotes the order of $G$.

This result is one of the starting points of the recent study of zero-sum problems and has been generalized in various directions \cite{2,3,6–9,14,16,24}. Another starting point is the Davenport’s constant $D(G)$, which is the smallest integer $d$ such that every sequence of $d$ elements contains a nonempty subsequence with sum zero. This constant has been studied for about 30 years \cite{1,4,5,10,11,19,21,23,26,27} and people are still very interested in determining $D(G)$.

Let $G$ be a finite abelian group of order $n$, and let $m = \text{exp}(G)$, the exponent of $G$. For every positive integer $k$, let $s_{km}(G)$ be the smallest integer $t$ such that every
sequence of $t$ elements in $G$ contains a zero-sum subsequence of length $km$. Now the result of Erdős et al. mentioned above can be restated that $s_n(G) \leq 2n - 1$.

Recently, it has been shown that $s_n(G) = n + D(G) - 1$ [16]. Set $s(G) = s_m(G)$. Let $C_n$ denote the cyclic group of $n$ elements, and $C_n^k$ the direct product of $k$ copies of $C_n$. In 1973, Harborth [24] considered the problem to determine $s(G)$ and derived the following results:

$$2^k(mn - 1) + 1 \leq s(C_n^k) \leq m(s(C_n^k) - 1) + s(C_m^k).$$

Since then, many studies have been made on $s(G)$. Here we list some known results on $s(G)$.

**Theorem 1.1.** Let $p$ be a prime, and $G$ a finite abelian group of exponent $m$. Then,

1. $s(C_n) = 2n - 1$ [12].
2. $s(C_p^2) \leq 5p - 1$ [18].
3. $s(C_p^2) \leq 5p - 2$ for sufficiently large $p$ [2].
4. $s(C_m^2) = 4m - 3$ if $m = 2^x3^y5^z$ [25].
5. $s(C_m^2) = 4mn - 3$ if $n = 2^a3^b5^c7^d$ and $m \leq (4n/3)^{1/3}$ [14].
6. $s(C_m^2) \leq 6m - 5$ [2].
7. There exists an absolute constant $c$ such that $s(G) \leq (cd \log_2 d)^d m$, where $d$ is the rank of $G$ [3].
8. $s(G) \leq |G| + m - 1$ [22].
9. If $G = C_{p_1} \oplus \cdots \oplus C_{p_k}$ with $p^{\nu_i} \geq 1 + \sum_{i=1}^{k-1} (p^{\nu_i} - 1)$ then $s(G) \leq 4p^{\nu_k} - 3 + 2\sum_{i=1}^{k-1} (p^{\nu_i} - 1)$ [20].
10. $s(C_3^2) = 19$ and $s(C_7^2) = 41$ [25].

In Section 2 of this paper, we show that Theorem 1.1 (5) holds with the restriction of $m \leq (4n/3)^{1/3}$ replaced by $m \leq \frac{2}{3} \sqrt{10(n - 1)}$ (Theorem 2.5) and $s(G) = \rho(G) + \exp(G) - 1$ for $\exp(G) = 3, 4$ (Theorem 2.5, for the definition of $\rho(G)$ see Section 2). In Section 3 we determine $s_{km}(G)$ for some $p$-groups (Theorems 3.4 and 3.6). In Section 4 we study the number of zero-sum subsequences of length exactly $m$ (= $\exp(G)$) (Theorem 4.1).

2. On $s(G)$

Let $G$ be a finite abelian group. If $G$ is nontrivial, then $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$, with $1 < n_1 | \cdots | n_r$, where $n_r = \exp(G)$ is the exponent of $G$ and $r$ is called the rank of $G$. Set $M(G) = 1 + \sum_{i=1}^r (n_i - 1)$.

Let $S = (a_1, \ldots, a_k)$ be a sequence of elements in $G$. We define

$$\sigma(S) = \sum_{i=1}^k a_i$$

the sum (of the elements) of $S$. Let $\lambda$ be the empty sequence and adopt the convention that $\sigma(\lambda) = 0$. A sequence $T = (a_{i_1}, \ldots, a_{i_l})$ is called a subsequence of $S$, if
Let $1 \leq i_1 < \cdots < i_l \leq k$. Denote the index set $\{i_1, \ldots, i_l\}$ by $I_l$. We say that subsequences $T_1, \ldots, T_u$ of $S$ are disjoint if their index sets $I_{T_1}, \ldots, I_{T_u}$ are disjoint, and by $T_1 \cdots T_u$ we denote the subsequence with index $I_{T_1} \cup \cdots \cup I_{T_u}$. Sometimes, we denote $S$ also by $\prod_{i=1}^k a_i$ and denote the sequence $(a; \ldots; a_{m-1})$ by $a'$. For two subsequences $A, B$ of $S$, let $A \cap B$ be the subsequence with index set $I_A \cap I_B$, and $S \setminus A$ the subsequence with index set $\{1, \ldots, k\} \setminus I_A$. Furthermore, let

$$\Sigma(S) = \left\{ \sum_{i \in I} a_i \mid \emptyset \neq I \subset \{1, \ldots, k\} \right\}$$

$$= \{\sigma(T) \mid \lambda \neq T \text{ is a subsequence of } S\} \subset G$$

denote the set of sums of nonempty subsequences of $S$. We say, that the sequence $S$ is zero-free if $0 \notin \Sigma(S)$, a zero-sum sequence if $\sigma(S) = 0$, a minimal zero-sum sequence if it is a zero-sum sequence and each proper subsequence is zero-free.

For every group homomorphism $\phi: G \to H$, let $\phi(S) = (\phi(a_1), \ldots, \phi(a_k)) = \prod_{i=1}^k \phi(a_i)$.

For every $b \in G$, by $b + S$ we denote the sequence $(b + a_1, \ldots, b + a_k)$.

**Definition 2.1.** Let $G$ be a finite abelian group of exponent $m$. We define $\rho(G)$ to be the smallest integer $t$ such that every sequence of $t$ elements in $G$ contains a nonempty zero sum subsequence of length not exceeding $m$.

**Lemma 2.2.** If $G$ is a finite abelian group of exponent $m$, then

1. $s(G) \geq \rho(G) + m - 1$.
2. $\rho(C_m^2) = 3m - 2$.

**Proof.**

1. Let $S = (a_1, \ldots, a_{\rho(G)-1})$ be a sequence of $\rho(G) - 1$ elements in $G$ such that $S$ contains no nonempty zero sum subsequence of length not exceeding $m$. Set

$$T = \left( a_1, \ldots, a_{\rho(G)-1}, 0, \ldots, 0 \right)_{m-1}. $$

Clearly, $T$ contains no zero sum subsequence of length $m$. Therefore, $s(G) \geq \rho(G) + m - 1$.

2. It has been proved in [17,27].

So far, for all $G$ that $s(G)$ is known, $s(G) = \rho(G) + m - 1$. It is suggested that

**Conjecture 2.3.** $s(G) = \rho(G) + m - 1$ holds for every finite abelian group $G$, where $m$ is the exponent of $G$ [20].
Conjecture 2.3, if true, with Lemma 2.2 would imply the following well known conjecture by Kemnitz.

**Conjecture 2.4.** \( s(C_n^2) = 4n - 3 \) holds for every positive integer \( n \) [25].

The aim of this section is to prove the following result.

**Theorem 2.5.** Let \( G \) be a finite abelian group of exponent \( m \).

1. If \( m \in \{2, 3, 4\} \), then \( s(G) = \rho(G) + m - 1 \).
2. If \( s(C_n^2) = 4n - 3 \) and \( m \leq \frac{2}{5} \sqrt{10(n - 1)} \), then \( s(C_{nm}^2) = 4nm - 3 \).

For the proof of Theorem 2.5 we need the following lemmata.

**Lemma 2.6.** Let \( G \) be a finite abelian group of exponent \( m \), and let \( i \) be in the interval \( \{1, \ldots, \lceil \frac{m}{2} \rceil \} + 1 \). Let \( S \) be a sequence of \( \rho(G) + i - 1 \) elements in \( G \). Then, \( S \) contains a zero-sum subsequence \( T \) such that \( i \leq |T| \leq m \).

**Proof.** We proceed by induction on \( i \).

\( i = 1 \), trivial.

Assume the lemma is true for \( 1 \leq i \leq \lceil \frac{m}{2} \rceil \), we want to prove it is true also for \( i+1 \).

By the assumption of the induction, there is a zero-sum subsequence \( T_1 \) of \( S \) such that \( i \leq |T_1| \leq m \). If \( |T_1| \geq i+1 \) then we are done. Otherwise, \( |T_1| = i \), then \( |S \setminus T_1| \geq \rho(G) \).

Therefore, there is a zero-sum subsequence \( T_2 \) of \( S \setminus T_1 \) such that \( 1 \leq |T_2| \leq m \). If \( |T_2| \geq i+1 \) then we are done. Otherwise \( 1 \leq |T_2| \leq i \), put \( T = T_1T_2 \). Then, \( T \) is a zero-sum subsequence of \( S \) and \( i+1 \leq |T| = |T_1| + |T_2| \leq 2i \).

**Proposition 2.7.** Let \( S \) be a sequence of \( \rho(G) + m - 1 \) elements in \( G \). Suppose that \( S \) contains some element at least \( m - \lceil \frac{m}{2} \rceil - 1 \) times. Then, \( S \) contains a zero-sum subsequence of length \( m \).

**Proof.** Write \( S = a'S_1 \) with \( t \geq m - \lceil \frac{m}{2} \rceil - 1 \). Without loss of generality, we may assume that \( a=0 \) (otherwise we consider \( -a+S \) instead of \( S \)). Now \( S = 0'S_1 \). If \( t \geq m \), then \( 0^m \) is a zero-sum subsequence of \( S \) and we are done. Otherwise \( m - \lceil \frac{m}{2} \rceil - 1 \leq t \leq m - 1 \).

Since \( |S_1| = \rho(G) + m - t - 1 \) and \( 1 \leq m - t \leq \lceil \frac{m}{2} \rceil + 1 \), it follows from Lemma 2.6 that there is a zero-sum subsequence \( T \) of \( S \) such that \( m - t \leq |T| \leq m \). Now \( 0^{m-|T|}T \) is a zero-sum subsequence of length \( m \).

**Proof of Theorem 2.5.** 1. Since \( m - \lceil \frac{m}{2} \rceil - 1 \leq 1 \) for \( m = 2, 3 \) or 4, now the theorem follows from Proposition 2.7 and Lemma 2.2.

2. Let \( S = (a_1, \ldots, a_{4nm-3}) \) be a sequence of \( 4nm-3 \) elements in \( C_{nm}^2 \). It is sufficient to prove that \( S \) contains a zero-sum subsequence of length \( nm \). Let \( \phi \) be a homomorphism from \( C_{nm}^2 \) onto \( C_n^2 \) with \( \ker \phi = C_n^2 \) (up to isomorphism). For any \( g \in C_n^2 \), by \( S_g \) we denote the subsequence of \( S \) consisting of all terms \( a_i \) with \( \phi(a_i) = g \). Choose \( h \in C_n^2 \)
so that $|S_h| = \max\{|S_g|, g \in C^2_n\}$. Then,

$$|S_h| \geq \frac{|S|}{|C^2_n|} = \frac{4nm - 3}{m^2} \geq \frac{5m}{2} \geq 5(m - \lfloor m/2 \rfloor - 1).$$

Therefore

$$|S_h| \geq 5(m - \lfloor m/2 \rfloor - 1). \quad (1)$$

Now applying Theorem 1.1 (6) repeatedly to the sequence $S \setminus S_h$, one can find disjoint subsequences $T_1, \ldots, T_l$ such that $\phi(T_i)$ is a zero-sum sequence of length $m$ for $i = 1, \ldots, l$, and such that $\phi(S \setminus S_h T_1 \cdots T_l)$ contains no zero-sum subsequence of length $m$. Therefore

$$\sigma(T_i) \in C^2_n \text{ and } |T_i| = m \text{ for } i = 1, \ldots, l. \quad (2)$$

Set $W = S \setminus S_h T_1 \cdots T_l$, then $|W| = |\phi(W)| \leq 6m - 6$ follows from Theorem 1.1 (6). If $|W| \geq 4m - 3 - (m - \lfloor \frac{m}{2} \rfloor - 1)$, then by Proposition 2.7 and Lemma 2.2, there is a subsequence $U_1$ of $WS_h$ such that $|U_1| = m_1, |U_1 \cap S_h| \leq m - \lfloor \frac{m}{2} \rfloor - 1, |U_1 \cap W| \geq \lfloor \frac{m}{2} \rfloor + 1$ and $\phi(U_1)$ is zero-sum. Hence, $\sigma(U_1) \in C^2_n$. If $|W \setminus (U_1 \cap W)| \geq 4m - 3 - (m - \lfloor \frac{m}{2} \rfloor - 1)$, similarly one can get a subsequence $U_2$ of $WS_h \setminus U_1$ such that $|U_2| = m_2, |U_2 \cap S_h| \leq m - \lfloor \frac{m}{2} \rfloor - 1, |U_2 \cap W| \geq \lfloor \frac{m}{2} \rfloor + 1$ and $\phi(U_2)$ is zero-sum. Hence, $\sigma(U_2) \in C^2_n$ and $U_1$ and $U_2$ are disjoint. Continue the same process above and note that $|W| \leq 6m - 6$ and $|S_h| \geq 5(m - \lfloor \frac{m}{2} \rfloor - 1)$, one can find $t (\leq 5)$ disjoint subsequences

$$U_1, U_2, \ldots, U_t$$

such that $|U_i| = m_i, |U_i \cap S_h| \leq m - \lfloor \frac{m}{2} \rfloor - 1, |U_i \cap W| \geq \lfloor \frac{m}{2} \rfloor + 1$ and $\sigma(U_i) \in C^2_n$ for $i = 1, \ldots, t$, and $|W \setminus (W \cap (U_1 \cdots U_t))| \leq 4m - 4 - (m - \lfloor \frac{m}{2} \rfloor - 1)$. Let

$$V_1, \ldots, V_r$$

be some disjoint subsequences of $S_h \setminus (S_h \cap (U_1 \cdots U_t))$ such that $|(S_h \setminus (S_h \cap (U_1 \cdots U_t))) \setminus V_1 \cdots V_r| \leq m - 1$ and such that $|V_i| = \cdots = |V_r| = m$. Clearly, $\sigma(V_i) \in C^2_n$. Now we have $|WS_h \setminus (U_1 \cdots U_t V_1 \cdots V_r)| \leq 4m - 4 - (m - \lfloor \frac{m}{2} \rfloor - 1) + m - 1 = 4m - 4 + \lfloor \frac{m}{2} \rfloor$. We distinguish two cases.

Case 1: If $|WS_h \setminus (U_1 \cdots U_t V_1 \cdots V_r)| \leq 4m - 4$, then $l + t + r = |S \setminus (WS_h \setminus (U_1 \cdots U_t V_1 \cdots V_r))|/m \geq (4nm - 3 - (4m - 4))/m > 4n - 4$. Therefore, $l + t + r \geq 4n - 3$. It follows from $s(C^2_n) = 4n - 3$ that

$$\sigma(T_1) \cdots \sigma(T_l) \sigma(U_1) \cdots \sigma(U_t) \sigma(V_1) \cdots \sigma(V_r)$$

contains a zero-sum subsequence of length $n$ and hence, $S$ contains a zero-sum subsequence of length $mn$. 

Case 2: If $|WS_k\setminus(U_1 \cdots U_i V_1 \cdots V_r)| \geq 4m - 3$. Since $|W'(W \cap (U_1 \cdots U_i))| \leq 4m - 4 - (m - \lceil \frac{m}{p} \rceil - 1)$, $|(WS_k\setminus(U_1 \cdots U_i V_1 \cdots V_r)) \cap S_k| \geq (m - \lceil \frac{m}{p} \rceil - 1)$. By Proposition 2.7 and Lemma 2.2, $WS_k\setminus(U_1 \cdots U_i V_1 \cdots V_r)$ contains an $m$-term subsequence $M$ with $\sigma(M) \in C_n^2$. Note that $|WS_k\setminus(U_1 \cdots U_i V_1 \cdots V_r,M)| \leq 4m - 4 + \lceil \frac{m}{p} \rceil - m \leq 4m - 4$. Now, similarly to the proof of Case 1 one can prove the theorem.

Corollary 2.8. 1. If $n = 2a3^b5^c7^d$ and $m \leq \frac{2}{3} \sqrt{10(n-1)}$ then $s(C_{nm}^2) = 4nm - 3$.
2. If $s(C_{n}^2) = 4n - 3$ and if $p \leq \frac{2}{3} \sqrt{6(n-1)}$ then $s(C_{np}^2) = 4np - 3$, where $p > 7$ is a prime.
3. If $n = 2a3^b5^c7^d$ and $p \leq \frac{2}{3} \sqrt{6(n-1)}$ then $s(C_{np}^2) = 4np - 3$, where $p > 7$ is a prime.
4. If $m$ is a positive integer, then $s(C_{k}^2) = 4k - 3$ holds for $k = 2^m m$.

Proof. (1) It follows from Theorems 2.5 and 1.1 (4).
(2) The proof is similar to that of Theorem 2.5 but by using Theorem 1.1 (2) instead of Theorem 1.1 (6) in it.
(3) It follows from (2) and Theorem 1.1 (4).
(4) It follows from (1) with $n = 2^m$.

3. On $s_{km}(G)$

Theorem 3.1. Let $G$ be a finite abelian group of exponent $m$, and $k$ a positive integer. Then,
1. $s_{km}(G) \geq km + D(G) - 1$.
2. If $k < D(G)/m$ then $s_{km}(G) > km + D(G) - 1$.

Proof. 1. Let $a_1, \ldots, a_{D(G)-1}$ be a zero-free sequence of $D(G)-1$ elements in $G$. Set $T = 0^{km-1} \prod_{i=1}^{D(G)-1} a_i$. Clearly, $T$ contains no zero-sum subsequence of length $km$ and $|T| = km + D(G) - 2$. Hence, $s_{km}(G) \geq km + D(G) - 1$.
2. For $km < D(G)$, let $a_1, \ldots, a_{D(G)}$ be a minimal zero-sum sequence of $D(G)$ elements in $G$. Set $T = 0^{km-1} \prod_{i=1}^{D(G)} a_i$. Clearly, $T$ contains no zero-sum subsequence of length $km$ and $|T| = km + D(G) - 1$. Hence, $s_{km}(G) > km + D(G) - 1$.

Theorem 3.2. Let $G$ be a finite abelian group of exponent $m$, and $k$ a positive integer. Suppose that $k \geq n/m$. Then, $s_{km}(G) = km + D(G) - 1$ [13, 16].

Proof. In [16], it has been shown that $s_{km}(G) = km + D(G) - 1$ for $k = n/m$ and the method used there works also for $k \geq n/m$.

Lemma 3.3 (Olson [26]). If $G$ is finite abelian $p$-group then $D(G) = M(G)$, where $p$ is a prime.

Theorem 3.4. Let $p$ be a prime, and $G$ a finite abelian $p$-group. Let $l$ be a positive integer such that $p^l \geq M(G)$. Then, $s_{p^l k}(G) = p^l k + M(G) - 1$ holds for every positive integer $k$. 

Proof. Let $a_1, \ldots, a_{p^kM(G) - 1}$ be a sequence of $p^k + M(G) - 1$ elements in $G$. Let $b_i = (a_i, 1)$ with $1 \in C_{p^k}$ for $i = 1, \ldots, p^k + M(G) - 1$. Then, $b_1, \ldots, b_{p^k + M(G) - 1}$ is a sequence of $p^k + M(G) - 1$ elements in $G \oplus C_{p^k}$. By Lemma 3.3, $D(G \oplus C_{p^k}) = M(G \oplus C_{p^k}) = p^k + M(G) - 1$. Therefore, there exists a nonempty subset $I \subset \{1, \ldots, p^k + M(G) - 1\}$ such that $\prod_{i \in I} b_i$ is a zero-sum sequence of elements in $G \oplus C_{p^k}$. Hence, $\prod_{i \in I} a_i$ is a zero-sum sequence in $G$. By the making of $b_i$, we have $p^k \| |I|$, but $1 \leq |I| \leq p^k + M(G) - 1 < 2p^k$. This forces that $|I| = p^k$. Hence, $\prod_{i \in I} a_i$ is a zero-sum sequence of $p^k$ elements in $G$.

**Definition 3.5.** Let $G$ be a finite abelian group of exponent $m$. Define $l(G)$ to be the smallest integer $t$ such that for every $k \geq t$, $s_{km}(G) = km + D(G) - 1$.

**Theorem 3.6.** Let $p$ be a prime, and $G$ a finite abelian group of exponent $m$. Then,

1. $D(G)/m \leq l(G) \leq |G|/m$.
2. $s_{2kp^r}(C_{m^2}) = 2kp^r + 2p^r - 2$ holds for every positive integer $k$.
3. $l(C_{m^2}) \leq 4$.
4. $l(C_{m^2}) = 2$ for $p \in \{2, 3, 5, 7\}$.

To prove Theorem 3.6 we need the following

**Lemma 3.7.** If $D(C_{m^2}) = M(C_{m^2}) = 3m - 2$, then every sequence of $3m - 2$ elements in $C_{m^2}$ contains a zero-sum subsequence of length $m$ or $2m$ [18].

**Proof of Theorem 3.6.** (1) Follows from Theorem 3.1 and Theorem 3.2.

(2) For $k = 1$, let $S$ be a sequence of $4p^r - 2$ elements in $C_{m^2}$. By Lemmas 3.3 and 3.7, there is a zero-sum subsequence $S_1$ of $S$ such that either $|S_1| = 2p^r$ and we are done, or $|S_1| = p^r$. Again by Lemmas 3.3 and 3.7, there is a zero-sum subsequence $S_2$ of $S \setminus S_1$ such that either $|S_2| = 2p^r$ and we are done, or $|S_2| = p^r$. Therefore, $S_1S_2$ is a zero-sum subsequence of $S$ with $|S_1S_2| = 2p^r$. This shows that $s_{2p^r}(C_{m^2}) \leq 4p^r - 2$ and the equality follows from Theorem 3.1. For $k \geq 2$, it is easy to prove it by induction on $k$ and we omit the details here.

(3) Let $k \geq 4$, and let $S$ be a sequence of $kp^r + 2p^r - 2$ elements in $C_{m^2}$. If $k$ is even, by (2), there is a zero-sum subsequence of $S$ with length $kp^r$. If $k$ is odd, by Theorem 1.1 (6), there is a zero-sum subsequence $S_1$ of $S$ with $|S_1| = p^r$. Now apply (2) to $S \setminus S_1$, one can find a zero-sum subsequence $S_2$ of $S \setminus S_1$ with $|S_2| = (k - 1)p^r$. Hence, $S_1S_2$ is a zero-sum subsequence of $S$ with $|S_1S_2| = kp^r$. This shows that $s_{kp^r}(C_{m^2}) \leq kp^r + 2p^r - 2$ for $k \geq 4$ and the equality follows from Theorem 3.1. Hence, $l(C_{m^2}) \leq 4$.

(4) can be proved like (3) and we omit the details here.

**Conjecture 3.8.** If $k < l(G)$ then $s_{km}(G) > km + D(G) - 1$, where $G$ is a finite abelian group of exponent $m$.

**Conjecture 3.9.** If $G \not\in \{C_m, C_{m^2}\}$ then $l(G) < |G|/m$, where $m = \exp(G)$. 

4. On the number of zero-sum subsequences of length $\exp(G)$

Let $p$ be a prime, and $G$ a finite abelian $p$-group. Let $S$ be a sequence of elements in $G$. For every $g \in G$, by $r(S, g)$ we denote the number of subsequences $T$ of $S$ with $|T| = \exp(G)$ and $r(T) = g$. For $G = C_p$, it has been shown [15] that $r(S, 0) \equiv 1 \pmod{p}$ and $r(S, g) \equiv 0 \pmod{p}$ for $0 \neq g \in C_p$. Quite recently, Sury [28] gave a new proof of this result by using the Chevalley–Warning theorem. In this section we prove the following general result.

**Theorem 4.1.** Let $p$ be a prime, and let $G = C_{p^{e_1}} \oplus \cdots \oplus C_{p^{e_k}}$ with $1 \leq e_1 \leq \cdots \leq e_k$. Let $n$ be a positive integer such that $p^n \geq 1 + \sum_{i=1}^{k} (p^{e_i} - 1)$. Let $S$ be a sequence of elements in $G$ with $|S| \geq \exp(G)$. Then, $r(S, 0) \equiv 1 \pmod{p}$ and $r(S, g) \equiv 0 \pmod{p}$ for $0 \neq g \in G$.

Let $S$ be a sequence of elements in a finite abelian group $G$, and $g \in G$. Define $f_E(S, g)$ (resp. $f_O(S, g)$) to be the number of subsequences $T$ with $\sigma(T) = g$ and $2|T|$ (resp. $2/|T|$). Recall that $\emptyset$ is the empty sequence. Clearly, $f_E(S, 0) \equiv f_E(\emptyset, 0) = 1$ for every sequence $S$.

**Lemma 4.2** (Olson [26]). Let $p$ be a prime, and let $G = C_{p^{e_1}} \oplus \cdots \oplus C_{p^{e_k}}$ with $1 \leq e_1 \leq \cdots \leq e_k$. Let $S$ be a sequence of elements in $G$ with $|S| \geq 1 + \sum_{i=1}^{k} (p^{e_i} - 1)$. Then, $f_E(S, g) - f_O(S, g) \equiv 0 \pmod{p}$ for every $g \in G$.

**Proof of Theorem 4.1.** Suppose $S = (a_1, \ldots, a_t)$. Define $b_i = (a_i, 1)$ for $i = 1, \ldots, t$, where $1 \in C_{p^e}$. Clearly, $b_i \in C_{p^e} \oplus G$. Set $W = (b_1, \ldots, b_t)$. Since $t = |S| \geq p^n + \sum_{i=1}^{k} (p^{e_i} - 1)$, by Lemma 4.2 we have $f_E(W, h) - f_O(W, h) \equiv 0 \pmod{p}$ for every $h \in C_{p^e} \oplus G$. Now we distinguish two cases:

Case 1. $p > 2$. By the making of $W$ we clearly have $f_E(W, (0, 0)) = 1$ and $r(S, 0) = f_O(W, (0, 0)) = 1 \pmod{p}$, where $0 \in G$. Again by the making of $W$ we clearly have $f_E(W, (0, g)) = 0$ for every $0 \neq g \in G$ and $r(S, g) = f_O(W, (0, g)) \equiv 0 \pmod{p}$.

Case 2. $p = 2$. By the making of $W$ we clearly have $f_O(W, (0, 0)) = 0$ and $r(S, 0) = f_E(W, (0, 0)) - 1 \equiv -1 \equiv 1 \pmod{2}$, where $0 \in G$. Again by the making of $W$ we clearly have $f_O(W, (0, g)) = 0$ for every $0 \neq g \in G$ and $r(S, g) = f_E(W, (0, g)) \equiv 0 \pmod{2}$. This completes the proof. □

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**References**