Discrete Applied Mathematics 160 (2012) 1501–1506

Contents lists available at SciVerse ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On a problem of Erdős, Herzog and Schönheim☆

Yong-Gao Chen*, Cui-Ying Hu

School of Mathematical Sciences and Institute of Mathematics, Nanjing Normal University, Nanjing 210046, PR China

ARTICLE INFO

Article history:
Received 1 August 2011
Received in revised form 31 January 2012
Accepted 9 February 2012
Available online 4 March 2012

Keywords:
Intersection theorems
Divisors
Extremal sets

ABSTRACT

Let $p_1, p_2, \dots, p_n$ be distinct primes. In 1970, Erdős, Herzog and Schönheim proved that if $\mathcal{D}, |\mathcal{D}| = m$, is a set of divisors of $N = p_1^{\alpha_1} \cdots p_n^{\alpha_n}, \alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_n$, no two members of the set being coprime and if no additional member may be included in $\mathcal{D}$ without contradicting this requirement then $m \ge \alpha_n \prod_{i=1}^{n-1}(\alpha_i + 1)$. They asked to determine all sets $\mathcal{D}$ such that the equality holds. In this paper we solve this problem. We also pose several open problems for further research.

1. Introduction

Many theorems on intersections of sets have been established. One of the intersection theorems is the next theorem of Erdős, Ko and Rado.

Theorem A ([2, *Erdős–Ko–Rado*]). *If $\mathcal{A} = \{A_1, A_2, \dots, A_m\}$ is a family of (different) subsets of a given set M, $|M| = n$, such that $A_i \cap A_j \neq \emptyset$ for every i, j, then*

(a) *$m \le 2^{n-1}$ and for every n there are $m = 2^{n-1}$ such subsets;*
(b) *if $m < 2^{n-1}$ then additional members may be included in $\mathcal{A}$, the enlarged family still satisfying $A_i \cap A_j \neq \emptyset$ for every i, j.*

Theorem A is equivalent to the following theorem.

Theorem B. *If $\mathcal{A} = \{d_1, d_2, \dots, d_m\}$ is a set of (different) divisors of a given positive integer N, $N = p_1p_2 \cdots p_n$, where $p_1, p_2, \dots, p_n$ are distinct primes, such that $(d_i, d_j) > 1$ for every i, j, then*

(a) *$m \le 2^{n-1}$ and for every n there are $m = 2^{n-1}$ such divisors;*
(b) *if $m < 2^{n-1}$ then additional members may be included in $\mathcal{A}$, the enlarged set still satisfying $(d_i, d_j) > 1$ for every i, j.*

This means that if $\mathcal{A}$ is a maximal set with the property $(d_i, d_j) > 1$ for every i, j, then $|\mathcal{A}| = 2^{n-1}$. If we allow repetitions in M (resp. N is not squarefree), it is more convenient to state results with the language of divisors (see [1,3,4]).

In this paper, $p_1, p_2, \dots, p_n$ are always distinct primes. Erdős et al. [1] proved the following theorem.

Theorem C ([1, *Erdős–Herzog–Schönheim*]). *If $\mathcal{D}, |\mathcal{D}| = m$, is a set of divisors of $N = p_1^{\alpha_1} \cdots p_n^{\alpha_n}, \alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_n$, no two members of the set being coprime and if no additional member may be included in $\mathcal{D}$ without contradicting this requirement then*

$$m \ge \alpha_n \prod_{i=1}^{n-1}(\alpha_i + 1).$$

☆ This work was supported by the National Natural Science Foundation of China, Grant No. 11071121.
* Corresponding author. Tel.: +86 2583598129.
E-mail address: ygchen@njnu.edu.cn (Y.-G. Chen).

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doi:10.1016/j.dam.2012.02.013

If $\mathcal{D}$ is the set of all positive divisors of N which are divisible by p_n, then $\mathcal{D}$ satisfies the assumptions of Theorem C and has the minimum size, that is,

$$|\mathcal{D}| = \alpha_n \prod_{i=1}^{n-1} (\alpha_i + 1). \tag{1}$$

In [1, Final remark], Erdős et al. remarked that it would be of interest to determine all sets $\mathcal{D}$ satisfying the assumptions of Theorem C with (1).

In this paper we solve this problem. For convenience, we introduce the following definitions.

Definition 1. A set $\mathcal{D}$ of positive divisors of N is an N-set if no two elements of the set are coprime. An N-set $\mathcal{D}$ is maximal if no additional divisor of N may be included.

Definition 2. For a set $\mathcal{D}$ of positive divisors of $N = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$, an element d of $\mathcal{D}$ is a divisible minimal element if d is not divisible by any other element of $\mathcal{D}$. Denote by $d(\mathcal{D})$ the set of all divisible minimal elements of $\mathcal{D}$.

It is clear that if $\mathcal{D}$ is a maximal N-set and $d \in \mathcal{D}$, then $l \in \mathcal{D}$ for all $l \mid N$ with $d \mid l$. Now the Erdős–Herzog–Schönheim problem above can be restated as follows.

Problem 1. Let $N = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$, where $\alpha_1 \ge \cdots \ge \alpha_n > 0$. Determine all the maximal N-sets $\mathcal{D}$ with the minimum size.

First we find some maximal N-sets $\mathcal{D}$ with the minimum size. Let

$$\alpha_1 \ge \cdots \ge \alpha_u > \alpha_{u+1} = \cdots = \alpha_n.$$

If $\alpha_1 = \cdots = \alpha_n$, let $u = 0$. For any v with $1 \le v \le n$, let

$$\mathcal{D}(p_v) = \{d : d \mid N, p_v \mid d\}.$$

Then all $\mathcal{D}(p_v) (1 \le v \le n)$ are maximal N-sets. For $u + 1 \le v \le n$ we have

$$|\mathcal{D}(p_v)| = \alpha_v \prod_{i=1, i \ne v}^{n} (\alpha_i + 1) = \alpha_n \prod_{i=1}^{n-1} (\alpha_i + 1).$$

For $v \le u$ we have

$$|\mathcal{D}(p_v)| = \alpha_v \prod_{i=1, i \ne v}^{n} (\alpha_i + 1) > \alpha_n \prod_{i=1}^{n-1} (\alpha_i + 1).$$

Now we consider the special case $\alpha_n = 1$. Let $\mathcal{D}'$ be a maximal $p_{u+1} \cdots p_n$-set. By Theorem B we have $|\mathcal{D}'| = 2^{n-u-1}$. Let

$$\mathcal{D} = \left\{ dd' : d \mid \frac{N}{p_{u+1} \cdots p_n}, d' \in \mathcal{D}' \right\}.$$

Since $\mathcal{D}'$ is a $p_{u+1} \cdots p_n$-set, we have $\mathcal{D}$ is an N-set. For $l \mid N$ and $l \notin \mathcal{D}$, let $l = l_1 l_1'$, where

$$l_1 \mid \frac{N}{p_{u+1} \cdots p_n}, \qquad l_1' \mid p_{u+1} \cdots p_n.$$

By $l \notin \mathcal{D}$ we have $l_1' \notin \mathcal{D}'$. Since $\mathcal{D}'$ is a maximal $p_{u+1} \cdots p_n$-set, there exists $d' \in \mathcal{D}'$ such that $(l_1', d') = 1$. Thus $(l, d') = 1$ and $d' \in \mathcal{D}$. Thus we have proved that $\mathcal{D}$ is a maximal N-set. We have

$$|\mathcal{D}| = |\mathcal{D}'| \prod_{i=1}^{u} (\alpha_i + 1) = 2^{n-u-1} \prod_{i=1}^{u} (\alpha_i + 1) = \alpha_n \prod_{i=1}^{n-1} (\alpha_i + 1).$$

In this paper we show that these are all the maximal N-sets $\mathcal{D}$ with the minimum size.

Theorem 1. *Let $N = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ with $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_u > \alpha_{u+1} = \cdots = \alpha_n \ge 2$. Then the following statements are equivalent to each other.*

(a) *$\mathcal{D}$ is a maximal N-set with the minimum size.*
(b) *$\mathcal{D}$ is a maximal N-set with $d(\mathcal{D}) = \{p_v\}$ for some $u + 1 \le v \le n$.*
(c) *$\mathcal{D} = \{d : d \mid N, p_v \mid d\}$ for some $u + 1 \le v \le n$.*

Theorem 2. *Let $N = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_u > \alpha_{u+1} = \cdots = \alpha_n = 1$. Then the following statements are equivalent to each other.*

(a) *$\mathcal{D}$ is a maximal N-set with the minimum size.*
(b) *$\mathcal{D}$ is a maximal N-set with $d(\mathcal{D}) \subseteq \{d : d \mid p_{u+1} \cdots p_n\}$.*
(c)

$$\mathcal{D} = \left\{ dd' : d \mid \frac{N}{p_{u+1} \cdots p_n}, d' \in \mathcal{D}' \right\}$$

for a maximal $p_{u+1} \cdots p_n$-set $\mathcal{D}'$.

For a set $\mathcal{T}$ of positive divisors of N, let $R(\mathcal{T}, N)$ be the set of all positive divisors of N which are divisible by at least one of the elements of $\mathcal{T}$. It is easy to see that $R(\mathcal{T}, N)$ is an N-set if and only if $\mathcal{T}$ is an N-set.

With these notations, we have the following theorems.

Theorem 3. *Let $N = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_u > \alpha_{u+1} = \cdots = \alpha_n = 1$, and let $\mathcal{T}_1, \ldots, \mathcal{T}_k$ be all sets of positive divisors of $p_{u+1} \cdots p_n$ such that for each i,*

(a) *no two elements of $\mathcal{T}_i$ are coprime;*
(b) *no element of $\mathcal{T}_i$ is divisible by another element of $\mathcal{T}_i$;*
(c) *any divisor of $p_{u+1} \cdots p_n$ is either coprime to some element of $\mathcal{T}_i$ or divisible by one element of $\mathcal{T}_i$.*

Then $R(\mathcal{T}_1, N), \ldots, R(\mathcal{T}_k, N)$ are all the maximal N-sets $\mathcal{D}$ with the minimum size.

Example. Let $N = 420 = 2^2 \cdot 3 \cdot 5 \cdot 7$. Then $p_{u+1} \cdots p_n = 3 \cdot 5 \cdot 7$ and the sets satisfying (a)–(c) are

$$\mathcal{T}_1 = \{3\}, \qquad \mathcal{T}_2 = \{5\}, \qquad \mathcal{T}_3 = \{7\}, \qquad \mathcal{T}_4 = \{3 \cdot 5, 3 \cdot 7, 5 \cdot 7\}.$$

Thus there are exactly four maximal 420-sets $R(\mathcal{T}_1, N), R(\mathcal{T}_2, N), R(\mathcal{T}_3, N), R(\mathcal{T}_4, N)$ with the minimum size.

Theorem 4. *Let $N = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_u > \alpha_{u+1} = \cdots = \alpha_n \geq 2$. Then $R(\{p_{u+1}\}, N), \ldots, R(\{p_n\}, N)$ are all the maximal N-sets $\mathcal{D}$ with the minimum size.*

Theorem 4 follows from Theorem 1 immediately. We pose the following problem.

Problem 2. Determine the number $H(N)$ of maximal N-sets $\mathcal{D}$ with the minimum size.

Remark. If $N = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_u > \alpha_{u+1} = \cdots = \alpha_n > 1$, then by Theorem 4 we have $H(N) = n - u$. For the case $N = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_u > \alpha_{u+1} = \cdots = \alpha_n = 1$, then $H(N)$ is the number of sets with (a)–(c) in Theorem 3.

2. Preliminary lemmas

Let $N = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$, where $\alpha_1 \geq \cdots \geq \alpha_n > 0$. Let $N' = p_1 \cdots p_n$. For $d \mid N'$, define

$$\alpha(d) = \prod_{p_i \mid d} \alpha_i, \qquad \bar{d} = \frac{N'}{d}.$$

Let

$$\mathcal{A} = \{d : d \mid N', d \in \mathcal{D}\}$$

and

$$\mathcal{A}_n = \{d : d \in \mathcal{A}, p_n \mid d\}, \qquad \mathcal{A}'_n = \{d : d \in \mathcal{A}, p_n \nmid d\}.$$

In this section we always assume that $\mathcal{D}$ is a maximal N-set. Then $\mathcal{A}$ is a maximal N'-set.

Lemma 1. *Let $p_v \in \mathcal{D}$ for some $u + 1 \leq v \leq n$. Then $d(\mathcal{D}) = \{p_v\}$ for some $u + 1 \leq v \leq n$.*

Proof. Since $\mathcal{D}$ is an N-set, we have $p_v \mid d$ for all $d \in \mathcal{D}$. Hence $d(\mathcal{D}) = \{p_v\}$. This completes the proof of Lemma 1. □

Lemma 2. *Let $d \mid N'$. Then exactly one of d and $\bar{d}$ is in $\mathcal{A}$.*

Proof. Since $(d, \bar{d}) = 1$ and $\mathcal{A}$ is the N'-set, we know that at most one of d and $\bar{d}$ is in $\mathcal{A}$.

Suppose that $d \notin \mathcal{A}$. By the maximality of $\mathcal{A}$ there exists $d' \in \mathcal{A}$ such that $(d, d') = 1$. Hence $d' \mid \bar{d}$. Again, by the maximality of $\mathcal{A}$ and $d' \mid \bar{d}$ we have $\bar{d} \in \mathcal{A}$. This completes the proof of Lemma 2. □

Lemma 3. *We have*

$$\mathcal{A}_n \cup \{\bar{d} : d \in \mathcal{A}'_n\} = \{lp_n : l \mid p_1 \cdots p_{n-1}\}.$$

Proof. It is clear that $\mathcal{A}_n \cup \{\bar{d} : d \in \mathcal{A}'_n\} \subseteq \{lp_n : l \mid p_1 \cdots p_{n-1}\}$. Now let $l \mid p_1 \cdots p_{n-1}$. Suppose that $lp_n \notin \mathcal{A}_n$. Then $lp_n \notin \mathcal{A}$. By Lemma 2 we have $\overline{lp_n} \in \mathcal{A}$. Thus $\overline{lp_n} \in \mathcal{A}'_n$. So $lp_n = \overline{\overline{lp_n}} \in \{\bar{d} : d \in \mathcal{A}'_n\}$. This completes the proof of Lemma 3. □

Lemma 4. *Let $\mathcal{D}$ be a maximal N-set with the minimum size and $\mathcal{A}'_n = \{d_1, d_2, \ldots, d_s\}$. Then there exists a permutation $i_1, i_2, \ldots, i_s$ of $1, 2, \ldots, s$ such that*

$$\bar{d}_{i_j} \mid d_j p_n, \qquad \alpha(d_j) = \alpha(\bar{d}_{i_j}), \quad j = 1, 2, \ldots, s.$$

Proof. For $d = p_{i_1}^{\beta_1} \cdots p_{i_k}^{\beta_k}$ with $0 < \beta_j \le \alpha_{i_j} (1 \le j \le k)$, by the maximality of $\mathcal{D}$, we have $d \in \mathcal{D}$ if and only if $p_{i_1} \cdots p_{i_k} \in \mathcal{A}$. So

$$|\mathcal{D}| = \sum_{d \in \mathcal{A}} \alpha(d) = \sum_{d \in \mathcal{A}_n} \alpha(d) + \sum_{d \in \mathcal{A}'_n} \alpha(d). \tag{2}$$

By Lemma 3 we have ($\alpha(1) = 1$)

$$\alpha_n \prod_{i=1}^{n-1} (\alpha_i + 1) = \sum_{l \mid p_1 \cdots p_{n-1}} \alpha(lp_n) = \sum_{d \in \mathcal{A}_n} \alpha(d) + \sum_{d \in \mathcal{A}'_n} \alpha(\bar{d}). \tag{3}$$

Since $\mathcal{D}$ is a maximal N-set with the minimum size, we have

$$|\mathcal{D}| = \alpha_n \prod_{i=1}^{n-1} (\alpha_i + 1). \tag{4}$$

By (2)–(4) we have

$$\sum_{d \in \mathcal{A}'_n} \alpha(d) = \sum_{d \in \mathcal{A}'_n} \alpha(\bar{d}). \tag{5}$$

In order to prove Theorem C, Erdős, Herzog and Schönheim proved a combinatorial theorem [1, Theorem 3]. We will employ its following equivalent form to prove Lemma 4.

Theorem D. *Let M be a squarefree integer. Denote by $\bar{d}' = M/d$ for $d \mid M$. If $F = \{d_1, d_2, \ldots, d_s\}$ is a set of divisors of M such that $d_i \mid d \mid M \Rightarrow d \in F$, then there exists a permutation $i_1, i_2, \ldots, i_s$ of $1, 2, \ldots, s$ such that $\bar{d}_{i_j}{}' \mid d_j$ $(1 \le j \le s)$.*

In order to employ Theorem D, let $M = p_1 \cdots p_{n-1}$ and $F = \mathcal{A}'_n$. If $d_i \mid d \mid M$, then by the maximality of $\mathcal{A}$ we have $d \in \mathcal{A}'_n$. Noting that

$$\bar{d}_i{}' = \frac{M}{d_i} = \frac{N'/d_i}{p_n} = \frac{\bar{d}_i}{p_n},$$

by Theorem D there exists a permutation $i_1, i_2, \ldots, i_s$ of $1, 2, \ldots, s$ such that

$$\frac{\bar{d}_{i_j}}{p_n} \mid d_j, \quad 1 \le j \le s.$$

That is, $\bar{d}_{i_j} \mid d_j p_n$. Let $d_j p_n = \bar{d}_{i_j} e_j$ $(1 \le j \le s)$. Since $d_{i_j} \in \mathcal{A}$, by Lemma 2 we have $\bar{d}_{i_j} \notin \mathcal{A}$. Thus $\bar{d}_{i_j}/p_n \notin \mathcal{A} (1 \le j \le s)$ by the maximality of $\mathcal{A}$. So $e_j > 1$ $(1 \le j \le s)$, otherwise, $\bar{d}_{i_j}/p_n = d_j \in \mathcal{A}$, a contradiction. Thus, for $1 \le j \le s$, we have

$$\alpha(d_j)\alpha(p_n) = \alpha(d_j p_n) = \alpha(\bar{d}_{i_j} e_j) = \alpha(\bar{d}_{i_j})\alpha(e_j) \ge \alpha(\bar{d}_{i_j})\alpha(p_n).$$

Hence

$$\alpha(d_j) \ge \alpha(\bar{d}_{i_j}), \quad 1 \le j \le s. \tag{6}$$

By (5) and (6) we have

$$\alpha(d_j) = \alpha(\bar{d}_{i_j}), \quad 1 \le j \le s.$$

This completes the proof of Lemma 4. □

Lemma 5. *We have $\mathcal{D} = R(d(\mathcal{D}), N)$.*

Proof. By the maximality of $\mathcal{D}$ and $d(\mathcal{D}) \subseteq \mathcal{D}$ we have $R(d(\mathcal{D}), N) \subseteq \mathcal{D}$. By the definition of $d(\mathcal{D})$ and $R(d(\mathcal{D}), N)$ we have $\mathcal{D} \subseteq R(d(\mathcal{D}), N)$. So $\mathcal{D} = R(d(\mathcal{D}), N)$. This completes the proof of Lemma 5. □

3. Proof of Theorems

Proof of Theorem 1. (a) $\Rightarrow$ (b): By Lemma 1 we may assume that $\{p_{u+1}, \ldots, p_n\} \cap \mathcal{D} = \emptyset$. Then $p_n \notin \mathcal{A}$. By Lemma 2 we have $\bar{p_n} \in \mathcal{A}$. That is, $\bar{p_n} \in \mathcal{A}'_n$. Let $\mathcal{A}'_n = \{d_1, d_2, \ldots, d_s\}$. By Lemma 4 there exists a permutation $i_1, i_2, \ldots, i_s$ of $1, 2, \ldots, s$ such that

$$\bar{d_{i_j}} \mid d_j p_n, \qquad \alpha(d_j) = \alpha(\bar{d_{i_j}}).$$

Without loss of generality, we may assume that $d_{i_1} = \bar{p_n}$. Then $\alpha(d_1) = \alpha(\bar{d_{i_1}}) = \alpha(p_n) = \alpha_n$. Since $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_u > \alpha_{u+1} = \cdots = \alpha_n \geq 2$, we have $d_1 \in \{p_{u+1}, \ldots, p_n\}$, a contradiction with $\{p_{u+1}, \ldots, p_n\} \cap \mathcal{D} = \emptyset$.

(b) $\Rightarrow$ (c): It follows from Lemma 5.

(c) $\Rightarrow$ (a): It follows from the arguments before Theorem 1.

This completes the proof of Theorem 1. $\square$

Proof of Theorem 2. (a) $\Rightarrow$ (b): By Lemma 1 we may assume that $\{p_{u+1}, \ldots, p_n\} \cap \mathcal{D} = \emptyset$. Then $p_n \notin \mathcal{A}$. By Lemma 2 we have $\bar{p_n} \in \mathcal{A}$. That is, $\bar{p_n} \in \mathcal{A}'_n$. Let $\mathcal{A}'_n = \{d_1, d_2, \ldots, d_s\}$. By Lemma 4 there exists a permutation $i_1, i_2, \ldots, i_s$ of $1, 2, \ldots, s$ such that $\bar{d_{i_j}} \mid d_j p_n$, $\alpha(d_j) = \alpha(\bar{d_{i_j}})$. As in Lemma 4, let $d_j p_n = \bar{d_{i_j}} e_j (1 \leq j \leq s)$. Since $\alpha_n = 1$ and $\alpha(d_j) = \alpha(\bar{d_{i_j}})$, we have $\alpha(e_j) = 1 (1 \leq j \leq s)$. Hence, for $1 \leq v \leq u$ and $1 \leq j \leq s$ we have $p_v \nmid e_j$ and

$$p_v \mid d_j \Leftrightarrow p_v \mid \bar{d_{i_j}} \Leftrightarrow p_v \nmid d_{i_j}.$$

Thus, for $1 \leq v \leq u$ we have

$$|\{j : p_v \mid d_j\}| = |\{j : p_v \nmid d_{i_j}\}| = |\{j : p_v \nmid d_j\}|.$$

So, for $1 \leq v \leq u$ we have

$$|\{j : p_v \mid d_j\}| = |\{j : p_v \nmid d_j\}| = \frac{1}{2}|\mathcal{A}'_n|. \tag{7}$$

Let $d(\mathcal{D}) = \{h_1, h_2, \ldots, h_t\}$. Then $h_i \nmid h_j$ for all $i \neq j$. Without loss of generality, we may assume that $p_n \nmid h_i$ $(1 \leq i \leq r)$ and $p_n \mid h_j$ $(r + 1 \leq j \leq t)$. Then each $d_i \in \mathcal{A}'_n$ is divisible by at least one of $h_1, h_2, \ldots, h_r$. Since $\mathcal{D}$ is a maximal N-set, we have $d(\mathcal{D}) \subseteq \mathcal{A}$. So $h_1, h_2, \ldots, h_r \in \mathcal{A}'_n$. Fix $1 \leq v \leq u$. Without loss of generality, we may assume that $h_1, h_2, \ldots, h_w$ are all h_i with $p_v \nmid h_i$ and $p_n \nmid h_i$.

Let $\mathcal{B} = \{d : p_v \nmid d, d \in \mathcal{A}'_n\}$. By (7) we have

$$|\{p_v d : d \in \mathcal{B}\}| = |\mathcal{B}| = \frac{1}{2}|\mathcal{A}'_n|.$$

Since $\mathcal{B} \cap \{p_v d : d \in \mathcal{B}\} = \emptyset$, we have $\mathcal{A}'_n = \mathcal{B} \cup \{p_v d : d \in \mathcal{B}\}$. Let $d \in \mathcal{B}$. If $w < i \leq r$, then by $p_v \mid h_i$ we have $h_i \nmid d$. If $r < i \leq t$, then by $p_n \mid h_i$ and $d \in \mathcal{A}'_n$ we have $h_i \nmid d$. That is, d is not divisible by any h_i with $i > w$. So d is divisible by one of $h_1, h_2, \ldots, h_w$. Thus each $d' \in \mathcal{A}'_n$ is divisible by one of $h_1, h_2, \ldots, h_w$. Since $w \leq r$ and $h_1, h_2, \ldots, h_r \in \mathcal{A}'_n$ and $h_i \nmid h_j$ for all $i \neq j$, we have $w = r$. Thus, we have proved that for all $1 \leq v \leq u$ we have $p_v \nmid h_i$ $(1 \leq i \leq r)$.

Now we have proved that for any given i with $1 \leq i \leq t$, if $p_n \nmid h_i$, then $p_v \nmid h_i$ for any $1 \leq v \leq u$. Since $\alpha_{u+1} = \cdots = \alpha_n = 1$, the primes $p_{u+1}, \ldots, p_n$ are in the same position. Hence, for any given i, j with $1 \leq i \leq t$ and $u + 1 \leq j \leq n$, if $p_j \nmid h_i$, then $p_v \nmid h_i$ for any $1 \leq v \leq u$. This means that for $1 \leq i \leq t$, if $p_{u+1} \cdots p_n \nmid h_i$, then $(p_1 \cdots p_u, h_i) = 1$, i.e., $h_i \mid p_{u+1} \cdots p_n$. So, for each $1 \leq i \leq t$, either $p_{u+1} \cdots p_n \mid h_i$ or $h_i \mid p_{u+1} \cdots p_n$. Since $h_i \nmid h_j$ for all $i \neq j$, we have either $p_{u+1} \cdots p_n \mid h_i$ for all $1 \leq i \leq t$ or $h_i \mid p_{u+1} \cdots p_n$ for all $1 \leq i \leq t$. If $p_{u+1} \cdots p_n \mid h_i$ for all $1 \leq i \leq t$, then $p_n \mid h_i$ for all $1 \leq i \leq t$. Thus $p_n \mid d$ for all $d \in \mathcal{A}$, a contradiction with $\bar{p_n} \in \mathcal{A}$ and $p_n \nmid \bar{p_n}$. Hence $h_i \mid p_{u+1} \cdots p_n$ for all $1 \leq i \leq t$. That is,

$$d(\mathcal{D}) \subseteq \{d : d \mid p_{u+1} \cdots p_n\}.$$

(b) $\Rightarrow$ (c): Let $\mathcal{D}' = \mathcal{D} \cap \{d : d \mid p_{u+1} \cdots p_n\}$. Since $\mathcal{D}$ is an N-set, $\mathcal{D}'$ is a $p_{u+1} \cdots p_n$-set. For $d \mid p_{u+1} \cdots p_n$, if $d \notin \mathcal{D}'$, then $d \notin \mathcal{D}$. Since $\mathcal{D}$ is a maximal N-set, there exists $l \in \mathcal{D}$ such that $(d, l) = 1$. By the definition of $d(\mathcal{D})$, l is divisible by an element l' of $d(\mathcal{D})$. So $(d, l') = 1$. By $d(\mathcal{D}) \subseteq \{d : d \mid p_{u+1} \cdots p_n\}$ we have $l' \in \mathcal{D}'$. Thus we have proved that $\mathcal{D}'$ is a maximal $p_{u+1} \cdots p_n$-set. By $d(\mathcal{D}) \subseteq \{d : d \mid p_{u+1} \cdots p_n\}$ we have $d(\mathcal{D}') = d(\mathcal{D})$. By Lemma 5 we have $\mathcal{D}' = R(d(\mathcal{D}'), p_{u+1} \cdots p_n) = R(d(\mathcal{D}), p_{u+1} \cdots p_n)$. Again, by Lemma 5 and $d(\mathcal{D}) \subseteq \{d : d \mid p_{u+1} \cdots p_n\}$ we have

$$\mathcal{D} = R(d(\mathcal{D}), N) = \left\{ dd' : d \mid \frac{N}{p_{u+1} \cdots p_n}, d' \in R(d(\mathcal{D}), p_{u+1} \cdots p_n) \right\}$$
$$= \left\{ dd' : d \mid \frac{N}{p_{u+1} \cdots p_n}, d' \in \mathcal{D}' \right\}.$$

(c) $\Rightarrow$ (a): It follows from the arguments before Theorem 1.

This completes the proof of Theorem 2. $\square$

Proof of Theorem 3. Suppose that $\mathcal{D}$ is a maximal N-set with the minimum size. By Theorem 2 we have

$$d(\mathcal{D}) \subseteq \{d : d \mid p_{u+1} \cdots p_n\}.$$

Since no two elements of $\mathcal{D}$ are coprime, we know that no two elements of $d(\mathcal{D})$ are coprime. That is (a). By the definition of $d(\mathcal{D})$ we know that no element of $d(\mathcal{D})$ is divisible by another element of $d(\mathcal{D})$. That is (b). Let $l \mid p_{u+1} \cdots p_n$. If $l \in \mathcal{D}$, then l is divisible by an element of $d(\mathcal{D})$. If $l \notin \mathcal{D}$, then, by the maximality of $\mathcal{D}$, there exists $d_1 \in \mathcal{D}$ with $(d_1, l) = 1$. Since $d_1 \in \mathcal{D}$, there exists $d \in d(\mathcal{D})$ with $d \mid d_1$. Hence $(d, l) = 1$. That is (c). Hence $d(\mathcal{D})$ is one of $\mathcal{T}_1, \ldots, \mathcal{T}_k$. By Lemma 5 we have $\mathcal{D} = R(d(\mathcal{D}), N)$. Hence $\mathcal{D}$ is one of $R(\mathcal{T}_1, N), \ldots, R(\mathcal{T}_k, N)$.

Now we show that each $R(\mathcal{T}_i, N)$ is a maximal N-set with the minimum size.

Since no two elements of $\mathcal{T}_i$ are coprime, we know that no two elements of $R(\mathcal{T}_i, N)$ are coprime. That is, $R(\mathcal{T}_i, N)$ is an N-set. In order to prove that $R(\mathcal{T}_i, N)$ is maximal, it is enough to prove that for any $l > 1$ with $l \mid N$ and $l \notin R(\mathcal{T}_i, N)$ there exists $d \in R(\mathcal{T}_i, N)$ with $(d, l) = 1$. It is enough to prove that there exists $d \in \mathcal{T}_i$ with $(d, l) = 1$. Let $l_1 = (l, p_{u+1} \cdots p_n)$. Noting that $\mathcal{T}_i$ is a set of positive divisors of $p_{u+1} \cdots p_n$, it is enough to prove that there exists $d \in \mathcal{T}_i$ with $(d, l_1) = 1$. Since $l \notin R(\mathcal{T}_i, N)$, we know that l is not divisible by any element of $\mathcal{T}_i$. So l_1 is not divisible by any element of $\mathcal{T}_i$. By the definition of $\mathcal{T}_i$ (i.e. (c) of Theorem 3), there exists $d \in \mathcal{T}_i$ with $(d, l_1) = 1$. Thus we have proved that $R(\mathcal{T}_i, N)$ is a maximal N-set. Noting that no element of $\mathcal{T}_i$ is divisible by another element of $\mathcal{T}_i$, we have $d(R(\mathcal{T}_i, N)) = \mathcal{T}_i$. Since $\mathcal{T}_i \subseteq \{d : d \mid p_{u+1} \cdots p_n\}$, by Theorem 2 we have $R(\mathcal{T}_i, N)$ has the minimum size. This completes the proof of Theorem 3. □

4. Final remarks

Finally we pose the following problems for further research.

Problem 3. Fix $t \geq 2$ and $N = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. Let $\mathcal{D}$ be a set of positive divisors d of N which have exactly t distinct prime factors (i.e. $\omega(d) = t$) such that no two members of the set being coprime and no additional member may be included in $\mathcal{D}$ without contradicting this requirement. Determine $m(N, t) = \min |\mathcal{D}|$.

Problem 4. Fix $t \geq 2$ and $N = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. Let $\mathcal{D}$ be a set of positive divisors d of N which have exactly t prime factors (i.e. $\Omega(d) = t$) such that no two members of the set being coprime and no additional member may be included in $\mathcal{D}$ without contradicting this requirement. Determine $M(N, t) = \min |\mathcal{D}|$.

Acknowledgments

We are grateful to the referees for their valuable comments.

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