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Two-Scale Homogenization of Non-Linear Degenerate Evolution Equations

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Using the notion of two-scale convergence developed by Allaire, the homogenization of a degenerate non-linear evolution equation with periodically oscillating coefficients is presented. A two-scale homogenized system is obtained as the limit of the periodic problem. Monotone operator methods and two-scale convergence are employed to show that the solutions of the periodic problem converge to the unique solution of the homogenized system. Homogenized initial conditions are also obtained and the sense in which they hold for the homogenized initial value problem is made specific. © 1999 Academic Press

1. INTRODUCTION

We are concerned with the homogenization of non-linear degenerate evolution equations of the abstract form,

$$\frac{\partial}{\partial t}\mathscr{B}^{\varepsilon}u^{\varepsilon}(t) + \mathscr{A}^{\varepsilon}(t, u^{\varepsilon}(t)) = f(t), \tag{1.1}$$

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where $\mathscr{B}^\varepsilon$ is a linear operator that is non-negative and symmetric, but *degenerate*, $\mathscr{A}^\varepsilon$ is a *non-linear* monotone operator, and both operators are ε-periodic in the spatial variables. Degenerate evolution equations of this form arise in models of electrolytic microcapacitors. These are spongelike structures (see [15] for a detailed account) and the highly singular nature of their microscopic geometry necessitates some form of continuous approximation, such as homogenization, to model their behavior. In the distributed capacitance model described in [8], the structure is modeled as microcapacitor cells embedded in a macroscopic conducting medium. The operator, $\mathscr{B}$, is then necessarily degenerate since it involves capacitance effects which are localized in the microcapacitor cells. Non-linear resistance effects, incorporated in modeling the interface between the cells and the conducting medium, give rise to nonlinearities in $\mathscr{A}$. Similar non-linear degenerate evolution equations also arise in problems of fluid flow and diffusion of heat. See, for example, [16, Chap. 3, and the references therein].

Well posedness and other results concerning such equations, as well as a list of applications and a comprehensive bibliography are given in [14, 5]. Homogenization of non-linear second-order *elliptic* equations was considered as an example of the application of two-scale convergence in [1]. For an overview of homogenization of diffusion equations as well as an extensive bibliography see [9, 10]. Homogenization of systems of *non-degenerate* parabolic equations modeling fluid flow in a porous medium has been considered in [2, 4, 3, 6]. In [7] we considered the homogenization of linear implicit and degenerate evolution equations using the method of two-scale convergence. Here we show how the two-scale method can be applied to non-linear degenerate evolution equations. In particular, we consider a model of the form above in which the operator $\mathscr{B}^\varepsilon$ is non-negative, but not necessarily strictly positive, and so the equation is actually of elliptic−parabolic type. The combination of the nonlinearity of the second-order term with this degeneracy is an obstacle in the homogenization process. In addition the degenerate term forces us to carefully keep track of the effect of homogenization on the initial condition, particularly on the spaces involved since we wish to allow a relatively large set of possible initial conditions. We see that the initial condition satisfied by the resulting homogenized problem is directly influenced by the positivity of $\mathscr{B}^\varepsilon(x)$. The results of this paper are applied in [12] to give an alternative derivation of the non-linear microstructure models presented in [8].

2. TWO-SCALE CONVERGENCE WITH A PARAMETER

Suppose that $2 \leq p < \infty$ and that Ω is a bounded open subset of $\mathbf{R}^N$ with $\partial\Omega$ a C^1 manifold, $Y = [0, 1]^N$, and G is a subset of $\mathbf{R}^M$. (In what follows, G represents the domain of a parameter. Typically $G = [0, T]$.) We denote spaces of Y-periodic functions by a subscript $\#$. For example, $C_\#(\Omega)$ is the space of functions which are continuous and Y-periodic on Ω. We first quote some definitions and theorems from [1]. Since the situation we consider here involves homogenization with respect to some, but not all, variables, we have modified Allaire's results as in [7] to allow for homogenization with a parameter (which we denote by t). These changes do not affect the proofs, which can be found in [1], in any essential way.

DEFINITION 2.1. A function, $\psi(x, t, y)$, which is Y-periodic in y and which satisfies

$$\lim_{\varepsilon \to 0} \int_{\Omega \times G} \psi\left(x, t, \frac{x}{\varepsilon}\right)^{p'} dx\,dt = \int_{\Omega \times G} \int_Y \psi(x, t, y)^{p'}\, dy\,dx\,dt, \quad (2.1)$$

is called an *admissible* test function. Here p' is the conjugate of p, that is, $1/p + 1/p' = 1$.

DEFINITION 2.2. A sequence $\{u^\varepsilon\} \subseteq L^p(\Omega \times G)$ *two-scale* converges to $u_0(x, t, y) \in L^p(\Omega \times G \times Y)$ if for any admissible test function $\psi(x, t, y)$,

$$\lim_{\varepsilon \to 0} \int_\Omega \int_G u^\varepsilon(x, t)\,\psi\left(x, t, \frac{x}{\varepsilon}\right) dt\,dx = \int_\Omega \int_G \int_Y u_0(x, t, y)\,\psi(x, t, y)\, dy\,dt\,dx. \quad (2.2)$$

THEOREM 2.1. *If $\{u^\varepsilon\}$ is a bounded sequence in $L^p(\Omega \times G)$, then there exists a function $u_0(x, t, y)$ in $L^p(\Omega \times G \times Y)$ and a subsequence of $\{u^\varepsilon\}$ which two-scale converges to u_0. Moreover, the subsequence $\{u^\varepsilon\}$ converges weakly in $L^p(\Omega \times G)$ to $u(x, t) = \int_Y u_0(x, t, y)\, dy$.*

When the sequence, $\{u^\varepsilon\}$, is $W^{1,p}$-bounded, we get:

THEOREM 2.2. *Let $\{u^\varepsilon\}$ be a bounded sequence in $L^p(G; W^{1,p}(\Omega))$ that converges weakly to u in $L^p(G; W^{1,p}(\Omega))$. Then $\{u^\varepsilon\}$ two-scale converges to u, and there is a function $u_1(x, t, y)$ in $L^p(\Omega \times G; W^{1,p}_\#(Y)/\mathbf{R})$ such that, up to a subsequence, $\nabla_x u^\varepsilon$ two-scale converges to $\nabla_x u(x, t) + \nabla_y u_1(x, t, y)$.*

3. A DEGENERATE NON-LINEAR EVOLUTION EQUATION

We propose to extend our study of the two-scale homogenization of evolution equations from [7] to *non-linear* evolution equations. Let V be a separable, reflexive Banach space and suppose that we have $w_0 \in V$, and that for all $\varepsilon > 0$ we are given a monotone, continuous, symmetric linear operator $\mathscr{B}^\varepsilon\colon V \to V'$ and a family of monotone operators $\mathscr{A}^\varepsilon(t, \cdot)$:

$V \to V'$. For $p \geq 2$ and $f \in L^{p'}(0,T;V')$, we consider the problem of finding $u^\varepsilon \in L^p(0,T;V)$ such that

$$\frac{\partial}{\partial t}\mathscr{B}^\varepsilon u^\varepsilon(t) + \mathscr{A}^\varepsilon(t, u^\varepsilon(t)) = f(t) \quad \text{in } V', \tag{3.1}$$

at a.e. $t \in [0,T]$, and such that $\mathscr{B}^\varepsilon u^\varepsilon(0) = \mathscr{B}^\varepsilon w_0$ in some sense. We then seek to identify the limiting behavior of the problem as $\varepsilon \to 0$.

Assume that for some $p \geq 2$ and each i and j, $1 \leq i, j \leq N$, we are given a function a_{ij}: $\Omega \times (0,T) \times \mathbf{R} \to \mathbf{R}$ such that

$$a_{ij}(y,t,\xi) \text{ is measurable in } t, \text{ continuous in } y \text{ and } \xi \text{ and } Y\text{-periodic in } y, \tag{3.2}$$

$$\left|a_{ij}(y,t,\xi)\right| \leq c|\xi|^{p-1} + k(y,t) \quad \text{for } y \in Y, 0 \leq t \leq T, \xi \in \mathbf{R}, \tag{3.3}$$

$$\left(a_{ij}(y,t,\xi_j) - a_{ij}(y,t,\eta_j)\right)(\xi_i - \eta_i) > 0 \quad \text{for all } \vec{\xi} \neq \vec{\eta} \text{ in } \mathbf{R}^N, \tag{3.4}$$

$$a_{ij}(y,t,\xi_j)\xi_i \geq \alpha|\vec{\xi}|^p - k(y,t) \quad \text{for all } \vec{\xi} \in \mathbf{R}^N, \tag{3.5}$$

where $\alpha > 0$, $k \in L^{p'}(\Omega \times (0,T))$ and such that $k(y,t)$ is Y-periodic in y. (Here and in what follows, we employ an extended summation convention, summing (from 1 to N) on all repeated indices, including those present in the argument of the nonlinear functions a_{ij}). Also, let $F \in L^{p'}(\Omega \times (0,T))$ and define $V = W_0^{1,p}(\Omega)$ and $\mathscr{V} = L^p(0,T;V)$. Let $b(x,y) \geq 0$ be continuous and Y-periodic in y such that $|b(x,y)| \leq b^*(x) \in L^{q^*}(\Omega)$ where $q^* \equiv p^*/(p^* - 2)$ and p^* is the Sobolev conjugate of p. Recall that the imbedding $V \hookrightarrow L^{p^*}(\Omega)$ is continuous. For u and v in V define

$$\mathscr{A}^\varepsilon(t,u)(v) = \int_\Omega \left[a_{ij}\left(\frac{x}{\varepsilon}, t, \frac{\partial u}{\partial x_j}(x)\right)\frac{\partial v}{\partial x_i}(x)\right] dx,$$

$$\mathscr{B}^\varepsilon u(v) = \int_\Omega b\left(x, \frac{x}{\varepsilon}\right) u(x) v(x)\, dx,$$

and

$$f(t)(v) = \int_\Omega F(x,t) v(x)\, dx.$$

From the assumptions above, we have that $\mathscr{B}^\varepsilon \in \mathscr{L}(V,V')$. For each $\varepsilon > 0$, write $b^\varepsilon(x) = b(x, \frac{x}{\varepsilon})$ and assume that w_0 is a measurable function on Ω such that

$$\int_\Omega \sup_{y\in Y} \{b(x,y)\} |w_0(x)|^2 \, dx < \infty. \tag{3.6}$$

Note that the Y-periodicity of b and Lemma 1.3 of [1] then show that $\|b_\varepsilon^{1/2} w_0\|_{L^2(\Omega)}$ is bounded independently of ε.

PROPOSITION 3.1. *For each* $\varepsilon > 0$ *there is a unique solution* u^ε *in* $\mathscr{V}$ *of*

$$\frac{\partial}{\partial t}(\mathscr{B}^\varepsilon u^\varepsilon(t)) + \mathscr{A}^\varepsilon(t, u^\varepsilon(t)) = f(t) \quad \text{in } \mathscr{V}', \tag{3.7}$$

$$\lim_{t\to 0} b_\varepsilon^{1/2}(x) u^\varepsilon(t) = b_\varepsilon^{1/2}(x) w_0 \quad \text{in } L^2(\Omega). \tag{3.8}$$

Furthermore, for $1 \le i \le N$, $a_{ij}(\frac{x}{\varepsilon}, t, \partial u^\varepsilon/\partial x_j(x))$ *is bounded independently of* ε *in* $L^{p'}(\Omega \times (0,T))$ *and* $b_\varepsilon^{1/2} u^\varepsilon(t)$ *is bounded in* $L^\infty(0,T; L^2(\Omega))$.

Proof. Fix $\varepsilon > 0$ and $0 \le t \le T$. From (3.5) we have that there is a $C > 0$ (independent of t and ε) such that for any $v \in V$:

$$\mathscr{A}^\varepsilon(t,v)(v) \ge C\left[\|v\|_V^p - \|k(\cdot,t)\|_{L^{p'}(\Omega)}\right]. \tag{3.9}$$

Thus for each ε and t, $\mathscr{A}^\varepsilon(t,\cdot)$ is coercive and so its realization in $\mathscr{V}$ is coercive. From (3.4), $\mathscr{A}^\varepsilon(t,\cdot)\colon V \to V'$ is strictly monotone, and from (3.2) and (3.3) it is hemicontinuous and bounded. Furthermore, since, for each t with $0 \le t \le T$, $\mathscr{B}^\varepsilon$ is continuous, linear, symmetric, and monotone on $L^2(\Omega)$, we may define a seminorm on $L^2(\Omega)$ by $\langle \mathscr{B}^\varepsilon \cdot, \cdot \rangle$. Following Showalter [14], denote the completion of the resulting seminormed space by V_b^ε and the dual Hilbert space by $(V_b^\varepsilon)'$ with scalar product satisfying

$$(\mathscr{B}^\varepsilon u, \mathscr{B}^\varepsilon v)_{(V_b^\varepsilon)'} = \mathscr{B}^\varepsilon u(v).$$

From Corollary 3.6.3. of [14] we have that, given any ξ_0 in $(V_b^\varepsilon)'$, there is a solution of (3.7) satisfying

$$\lim_{t\to 0} \mathscr{B}^\varepsilon u^\varepsilon(t) = \xi_0. \tag{3.10}$$

Setting $\xi_0 = b_\varepsilon^{1/2} w_0$ with w_0 defined as above, (3.6) gives that ξ_0 lies in $(V_b^\varepsilon)'$. Thus (3.10) can be written as (3.8). Note that since the equation (3.7) holds in V' we have that

$$\left\langle \frac{\partial}{\partial t}(\mathscr{B}^\varepsilon u^\varepsilon(t)), u^\varepsilon(t) \right\rangle + \langle \mathscr{A}^\varepsilon(t, u^\varepsilon(t)), u^\varepsilon(t)\rangle = \langle f(t), u^\varepsilon(t)\rangle \tag{3.11}$$

for $0 \le t \le T$, where $\langle g, v \rangle = g(v)$ for $g \in V'$, $v \in V$. Since $\mathscr{B}^\varepsilon$ is symmetric and does not depend on t,

$$\frac{\partial}{\partial t}\left\langle (\mathscr{B}^\varepsilon u^\varepsilon), u^\varepsilon \right\rangle = \left\langle \mathscr{B}^\varepsilon \frac{\partial u^\varepsilon}{\partial t}, u^\varepsilon \right\rangle + \left\langle \mathscr{B}^\varepsilon u^\varepsilon, \frac{\partial u^\varepsilon}{\partial t} \right\rangle$$
$$= 2\left\langle \frac{\partial}{\partial t}(\mathscr{B}^\varepsilon u^\varepsilon), u^\varepsilon \right\rangle. \tag{3.12}$$

Suppose that u^ε and v^ε are both solutions of (3.7), (3.8). Then subtracting yields

$$\frac{\partial}{\partial t}(\mathscr{B}^\varepsilon(u^\varepsilon(t) - v^\varepsilon(t))) + \mathscr{A}^\varepsilon(t, u^\varepsilon(t)) - \mathscr{A}^\varepsilon(t, v^\varepsilon(t)) = 0 \quad \text{in } \mathscr{V}'.$$

Applying this to $u^\varepsilon - v^\varepsilon$, integrating in t and applying (3.12) gives

$$\tfrac{1}{2}\mathscr{B}^\varepsilon(u^\varepsilon(t) - v^\varepsilon(t))(u^\varepsilon(t) - v^\varepsilon(t))$$
$$+ \int_0^t (\mathscr{A}^\varepsilon(t, u^\varepsilon(t)) - \mathscr{A}^\varepsilon(t, v^\varepsilon(t)))(u^\varepsilon(t) - v^\varepsilon(t))\, dt = 0.$$

Since $\mathscr{B}^\varepsilon$ is monotone and $\mathscr{A}^\varepsilon$ is strictly monotone this gives $u^\varepsilon(t) - v^\varepsilon(t) = 0$, thus establishing uniqueness. Finally, we prove the boundedness results. Integrating (3.11) in t gives

$$\tfrac{1}{2}\langle (\mathscr{B}^\varepsilon u^\varepsilon(t)), u^\varepsilon(t) \rangle + \int_0^t \langle \mathscr{A}^\varepsilon(t, u^\varepsilon(t)), u^\varepsilon(t) \rangle\, dt$$
$$= \int_0^t \langle f(t), u^\varepsilon(t) \rangle\, dt + \tfrac{1}{2}\| b_\varepsilon^{1/2} w_0 \|_{L^2(\Omega)}^2. \tag{3.13}$$

Thus (3.9) and the monotonicity of $\mathscr{B}^\varepsilon$ imply that

$$\| u^\varepsilon \|_{\mathscr{V}}^p \le k_1 + k_2 \left(\| f \|_{\mathscr{V}'} \| u^\varepsilon \|_{\mathscr{V}} + \| b_\varepsilon^{1/2} w_0 \|_{L^2(\Omega)}^2 \right),$$

and since $p > 1$ and $b_\varepsilon^{1/2} w_0$ is uniformly bounded in $L^2(\Omega)$, this gives that u^ε is bounded in $\mathscr{V}$ (independently of ε). From (3.3):

$$\left\| a_{ij}\left(\frac{x}{\varepsilon}, t, \frac{\partial u^\varepsilon}{\partial x_j}(x) \right) \right\|_{L^{p'}(\Omega \times (0,T))}$$
$$\le c \sum_{j=1}^{N} \left\| \frac{\partial u^\varepsilon}{\partial x_j} \right\|_{L^p(\Omega \times (0,T))}^{p-1} + N \| k \|_{L^{p'}(\Omega \times (0,T))}. \tag{3.14}$$

∎

Remark 3.1. In the following we do not attempt to find optimal spaces for initial conditions since these change with each ε. The assumption (3.6) provides us with a rather large space from which to draw initial conditions for any $\varepsilon > 0$ and we see later that (3.6) is quite convenient.

With the bounds of Proposition 3.1 and Theorems 2.1 and 2.2, we can obtain functions $u \in L^p(0,T; W_0^{1,p}(\Omega))$, $u_1 \in L^p(\Omega \times (0,T); W_{\#}^{1,p}(Y)/\mathbf{R})$, $u^* \in L^2(\Omega \times Y)$, and g_i (for $1 \le i \le N$) in $L^{p'}(\Omega \times (0,T) \times Y)$ such that for some subsequence u^ε, $u^\varepsilon \overset{2}{\to} u$, and $u^\varepsilon \rightharpoonup u$ weakly in $\mathscr{V}$, $\nabla_x u^\varepsilon \overset{2}{\to} \nabla_x u(x,t) + \nabla_y u_1(x,t,y)$, $b_\varepsilon^{1/2} u^\varepsilon(T) \overset{2}{\to} u^*(x,y)$, and $a_{ij}(\frac{x}{\varepsilon}, t, \partial u^\varepsilon/\partial x_j(x,t)) \overset{2}{\to} g_i$ for $1 \le i \le N$.

For any smooth test function ψ we have

$$\frac{\partial}{\partial t}(\mathscr{B}^\varepsilon u^\varepsilon(t))\psi(t) + \mathscr{A}^\varepsilon(t, u^\varepsilon(t))\psi(t) = f(t)\psi(t).$$

Integrating this in t, we obtain

$$\begin{aligned} -\int_0^T\int_\Omega b\left(x, \frac{x}{\varepsilon}\right) u^\varepsilon(x,t)\psi'(x,t)\,dx\,dt + \int_0^T\int_\Omega a_{ij}\left(\frac{x}{\varepsilon}, t, \frac{\partial u^\varepsilon}{\partial x_j}\right)\frac{\partial \psi}{\partial x_i}\,dx\,dt \\ = \int_\Omega b\left(x, \frac{x}{\varepsilon}\right) u^\varepsilon(x,0)\psi(x,0)\,dx - \int_\Omega b\left(x, \frac{x}{\varepsilon}\right) u^\varepsilon(x,T)\psi(x,T)\,dx \\ + \int_0^T\int_\Omega F(x,t)\psi(x,t)\,dx\,dt. \end{aligned} \tag{3.15}$$

Choose $\varphi \in W^{1,p}(0,T; C_0^\infty(\Omega))$ and $\varphi_1 \in W^{1,p}(0,T; C_0^\infty(\Omega; C_{\#}^\infty(Y)))$. Note that φ_1 is Y-periodic in $y = \frac{x}{\varepsilon}$. Set $\psi(x,t) = \varphi(x,t) + \varepsilon\varphi_1(x,t,\frac{x}{\varepsilon})$ in (3.15) to obtain

$$\begin{aligned} -\int_0^T\int_\Omega b\left(x, \frac{x}{\varepsilon}\right) u^\varepsilon(x,t)\left(\varphi'(x,t) + \varepsilon\varphi_1'\left(x,t,\frac{x}{\varepsilon}\right)\right)dx\,dt \\ + \int_0^T\int_\Omega a_{ij}\left(\frac{x}{\varepsilon}, t, \frac{\partial u^\varepsilon}{\partial x_j}\right)\left(\frac{\partial\varphi}{\partial x_i} + \varepsilon\frac{\partial\varphi_1}{\partial x_i} + \frac{\partial\varphi_1}{\partial y_i}\right)dx\,dt \\ = \int_\Omega b^\varepsilon(x) w_0(x)\left(\varphi(x,0) + \varepsilon\varphi_1\left(x,0,\frac{x}{\varepsilon}\right)\right)dx \\ - \int_\Omega b^\varepsilon(x) u^\varepsilon(x,T)\left(\varphi(x,T) + \varepsilon\varphi_1\left(x,T,\frac{x}{\varepsilon}\right)\right)dx \\ + \int_0^T\int_\Omega F(x,t)\left(\varphi(x,t) + \varepsilon\varphi_1\left(x,t,\frac{x}{\varepsilon}\right)\right)dx\,dt, \end{aligned} \tag{3.16}$$

where the $'$ denotes $\frac{\partial}{\partial t}$. Note that $b\varphi$, $b\varphi'$, and $\partial\varphi/\partial x_i + \varepsilon(\partial\varphi_1/\partial x_i) + \partial\varphi_1/\partial y_i$ are admissible (in the sense of (2.1)). Using the two-scale convergence of u^ε, $\nabla_x u^\varepsilon$, $b_\varepsilon^{1/2} u^\varepsilon(T)$, and $a_{ij}(\frac{x}{\varepsilon}, t, \partial u/\partial x_j(x))$, and letting $\varepsilon \to 0$ in (3.16) we get

$$\begin{aligned}
-\int_0^T\int_\Omega\int_Y b(x,y)u\varphi'\,dy\,dx\,dt + \int_0^T\int_\Omega\int_Y g_i(x,y)\left(\frac{\partial\varphi}{\partial x_i} + \frac{\partial\varphi_1}{\partial y_i}\right)dy\,dx\,dt \\
= \int_0^T\int_\Omega F(x,t)\varphi(x,t)\,dx\,dt + \int_\Omega\int_Y b(x,y)w_0(x)\varphi(x,0)\,dy\,dx \\
- \int_\Omega\int_Y b^{1/2}(x,y)u^*(x,y)\varphi(x,T)\,dy\,dx. \qquad (3.17)
\end{aligned}$$

From (3.17) we note that u and $\vec{g}$ satisfy

$$\int_0^T\int_\Omega\int_Y g_i\frac{\partial\varphi_1}{\partial y_i}\,dy\,dx\,dt = 0,$$

and

$$\begin{aligned}
-\int_0^T\int_\Omega\left(\int_Y b\,dy\right)u\varphi'\,dx\,dt + \int_0^T\int_\Omega\left(\int_Y g_i\,dy\right)\left(\frac{\partial\varphi}{\partial x_i}\right)dx\,dt \\
= \int_0^T\int_\Omega F\varphi\,dx\,dt + \int_\Omega\left(\int_Y b\,dy\right)w_0(x)\varphi(x,0)\,dx \\
- \int_\Omega\int_Y b^{1/2}(x,y)u^*(x,y)\,dy\,\varphi(x,T)\,dx \qquad (3.18)
\end{aligned}$$

for any φ and φ_1 as chosen above. Thus

$$-\frac{\partial}{\partial y_i}g_i(x,y) = 0 \quad \text{in } [0,T]\times\Omega\times Y, \qquad (3.19)$$

and

$$\left(\int_Y b\,dy\right)\frac{\partial u}{\partial t} - \frac{\partial}{\partial x_i}\left(\int_Y g_i\,dy\right) = F \quad \text{in } [0,T]\times\Omega. \qquad (3.20)$$

Furthermore, applying (3.20) in (3.18) gives us

$$\left(\int_Y b\,dy\right)u(T) = \int_Y b^{1/2}u^*\,dy, \qquad (3.21)$$

and

$$\left(\int_Y b\,dy\right)u(0) = \left(\int_Y b\,dy\right)w_0. \tag{3.22}$$

Before proceeding we make two useful observations which we state as lemmas.

LEMMA 3.2.

$$\left(\int_Y b\,dy\right)^{1/2} u(0) = \left(\int_Y b(x,y)\,dy\right)^{1/2} w_0(x),$$

and $(\int_Y b(x,y)\,dy)^{1/2} w_0(x)$ *is in* $L^2(\Omega)$.

Proof. The equality comes directly from (3.22). From Lemma 1.3 of [1] we have that

$$\int_\Omega \int_Y b(x,y) w_0^2(x)\,dy\,dx = \lim_{\varepsilon\to 0} \int_\Omega \left(b_\varepsilon^{1/2}(x) w_0(x)\right)^2 dx$$

and thus (3.6) gives the stated result. ∎

LEMMA 3.3.

$$\lim_{\varepsilon\to 0} \int_\Omega b_\varepsilon(x)(u^\varepsilon(x,T))^2\,dx \ge \int_\Omega \int_Y b(x,y)\,dy |u(x,T)|^2\,dx$$

Proof. Applying Holder's inequality to (3.21) yields

$$\left(\int_Y b\,dy\right)^{1/2} u(T) \le \left(\int_Y (u^*)^2\,dy\right)^{1/2}$$

and Proposition 1.6 of [1] gives

$$\lim_{\varepsilon\to 0} \int_\Omega b_\varepsilon(x)(u^\varepsilon(x,T))^2\,dx \ge \|u^*\|^2_{L^2(\Omega\times Y)},$$

from which the result follows. ∎

We must now identify g_i in terms of a, u, and u_1. Following Allaire in [1], for $1 \le i \le N$ let φ_i and Φ be in $C_0^\infty(\Omega \times (0,T); C_\#^\infty(Y))$ and for $\varepsilon > 0$ and $\lambda > 0$ define test functions

$$\mu_i^\varepsilon(x) = \frac{\partial}{\partial x_i}\left[u(x) + \varepsilon\Phi\left(x,t,\frac{x}{\varepsilon}\right)\right] + \lambda\varphi_i\left(x,t,\frac{x}{\varepsilon}\right).$$

Note that $\mu_i^\varepsilon(x)$ and (because of the continuity assumption) $a_{ij}(\frac{x}{\varepsilon}, t, \mu_i^\varepsilon(x))$ are admissible test functions and

$$\mu_i^\varepsilon(x) \overset{2}{\to} \mu_i^0 = \frac{\partial u}{\partial x_i} + \frac{\partial \Phi}{\partial y_i}(x,t,y) + \lambda \varphi_i(x,t,y).$$

Using the monotonicity condition (3.4) yields

$$\int_0^T \int_\Omega \left(a_{ij}\left(\frac{x}{\varepsilon}, t, \frac{\partial u^\varepsilon}{\partial x_j}(x)\right) - a_{ij}\left(\frac{x}{\varepsilon}, t, \mu_j^\varepsilon(x)\right)\right) \times \left(\frac{\partial u^\varepsilon}{\partial x_i}(x,t) - \mu_i^\varepsilon(x)\right) dx\,dt \geq 0.$$

Expanding and employing (3.13) at $t = T$ yields

$$\int_0^T \int_\Omega F(x) u^\varepsilon(x,t)\, dx\,dt - \frac{1}{2}\left\| b_\varepsilon^{1/2} u^\varepsilon(T)\right\|_{L^2(\Omega)}^2 + \frac{1}{2}\|b_\varepsilon^{1/2} w_0\|_{L^2(\Omega)}^2 + \int_0^T \int_\Omega \left(-a_{ij}\left(\frac{x}{\varepsilon}, t, \mu_j^\varepsilon\right)\frac{\partial u^\varepsilon}{\partial x_i} - a_{ij}\left(\frac{x}{\varepsilon}, t, \frac{\partial u^\varepsilon}{\partial x_j}\right)\mu_i^\varepsilon + a_{ij}\left(\frac{x}{\varepsilon}, t, \mu_j^\varepsilon\right)\mu_i^\varepsilon\right) dx\,dt \geq 0.$$

Letting $\varepsilon \to 0$, the two-scale convergence of $u^\varepsilon(t)$ and $a_{ij}(\frac{x}{\varepsilon}, t, \partial u^\varepsilon/\partial x_j(x))$ and the continuity of a_{ij} give in the limit,

$$\int_0^T \int_\Omega F(x) u(x,t)\, dx\,dt - \lim_{\varepsilon\to 0} \frac{1}{2}\left\| b_\varepsilon^{1/2} u^\varepsilon(T)\right\|_{L^2(\Omega)}^2 + \frac{1}{2}\int_\Omega \int_Y b w_0^2\, dy\,dx + \int_0^T \int_\Omega \int_Y \left[-a_{ij}(y,t,\mu_j^0)\left(\frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i}\right) - g_i\,\mu_i^0\right] dy\,dx\,dt + \int_0^T \int_\Omega \int_y a_{ij}(y,t,\mu_j^0)\mu_i^0 \geq 0. \tag{3.23}$$

Since $a_{ij}(y,t,\xi)$ is continuous we may replace $\Phi(x,t,y)$ by a sequence converging strongly in $L^p(\Omega \times (0,T); W_\#^{1,p}(Y)/\mathbf{R})$ to $u_1(x,t,y)$, thus replacing μ_i^0 in (3.23) with $\partial u/\partial x_i + \partial u_1/\partial y_i + \lambda\varphi_i$. Integration by parts and (3.19) then give

$$\int_0^T\int_\Omega\left(F(x)+\frac{\partial}{\partial x_i}\left(\int_Y g_i\,dy\right)\right)u(x,t)\,dx\,dt$$
$$-\lim_{\varepsilon\to 0}\frac{1}{2}\left\|b_\varepsilon^{1/2}u^\varepsilon(T)\right\|^2_{L^2(\Omega)}+\frac{1}{2}\int_\Omega\int_Y b(x,y)w_0^2(x)\,dy\,dx$$
$$+\int_0^T\int_\Omega\int_Y\left[a_{ij}\left(y,t,\frac{\partial u}{\partial x_j}+\frac{\partial u_1}{\partial y_j}+\lambda\varphi_j\right)-g_i\right]\lambda\varphi_i\,dy\,dx\,dt\geq 0.$$

Finally, (3.20)–(3.22) show that the first two lines of the above sum to

$$\tfrac{1}{2}\int_\Omega\int_Y b(x,y)\,dy|u(x,T)|^2\,dx-\lim_{\varepsilon\to 0}\tfrac{1}{2}\left\|b_\varepsilon^{1/2}u^\varepsilon(T)\right\|^2_{L^2(\Omega)},$$

which by Lemma 3.3 is negative. Thus dividing by λ and letting $\lambda\to 0$ we see that for every φ_i,

$$\int_0^T\int_\Omega\int_Y\left[a_{ij}\left(y,t,\frac{\partial u}{\partial x_j}+\frac{\partial u_1}{\partial y_j}\right)-g_i\right]\varphi_i\,dy\,dx\,dt\geq 0.$$

We therefore have proved the desired result, namely, that

$$a_{ij}\left(y,t,\frac{\partial u}{\partial x_j}+\frac{\partial u_1}{\partial y_j}\right)=g_i,\quad\text{in }\Omega\times(0,T)\times Y.$$

We have shown that every subsequence of $u^\varepsilon(t)$ has a further subsequence for which the above convergence holds, and thus that the entire sequence converges. ∎

THEOREM 3.4. *With the spaces and operators defined as above, the solution* u^ε *of* (3.7) *satisfies the following*: $u^\varepsilon\rightharpoonup u$ *weakly in* $\mathscr{V}$ *and* $\nabla_x u^\varepsilon\xrightarrow{2}\nabla_x u(x,t)+\nabla_y u_1(x,t,y)$. *Furthermore, the unique two-scale limit* (u,u_1) *satisfies the two-scale homogenized system*

$$\frac{\partial}{\partial y_i}a_{ij}\left(y,t,\frac{\partial u}{\partial x_j}+\frac{\partial u_1}{\partial y_j}\right)=0\quad\text{in }\Omega\times(0,T)\times Y,\tag{3.24}$$

$$\left(\int_Y b\,dy\right)\frac{\partial u}{\partial t}-\frac{\partial}{\partial x_i}\left(\int_Y a_{ij}\left(y,t,\frac{\partial u}{\partial x_j}+\frac{\partial u_1}{\partial y_j}\right)dy\right)=F\quad\text{in }\Omega\times(0,T),\tag{3.25}$$

and u satisfies the homogenized initial condition

$$\left(\int_Y b\,dy\right)^{1/2} u(0) = \left(\int_Y b\,dy\right)^{1/2} w_0 \quad \text{in } L^2(\Omega). \tag{3.26}$$

Proof. Only uniqueness of solutions remains to be shown. If (u, u_1) and (v, v_1) are both solutions of (3.24)–(3.26) then subtracting yields

$$\left(\int_Y b\,dy\right)\frac{\partial(u-v)}{\partial t} - \frac{\partial}{\partial x_i}\left(\int_Y \left(a_{ij}\left(y, t, \frac{\partial u}{\partial x_j} + \frac{\partial u_1}{\partial y_j}\right)\right.\right.$$
$$\left.\left. - a_{ij}\left(y, t, \frac{\partial v}{\partial x_j} + \frac{\partial v_1}{\partial y_j}\right)\right) dy\right) = 0.$$

Multiplying by $u - v$ and integrating by parts in x gives

$$\int_0^t\int_\Omega \left(\int_Y b\,dy\right)\frac{\partial(u-v)}{\partial t}(u-v)\,dx\,dt$$
$$+ \int_0^t\int_\Omega\int_Y \left(a_{ij}\left(y, t, \frac{\partial u}{\partial x_j} + \frac{\partial u_1}{\partial y_j}\right) - a_{ij}\left(y, t, \frac{\partial v}{\partial x_j} + \frac{\partial v_1}{\partial y_j}\right)\right)$$
$$\times \frac{\partial}{\partial x_i}(u-v)\,dy\,dx\,dt = 0, \tag{3.27}$$

and from (3.24),

$$\int_Y \left(a_{ij}\left(y, t, \frac{\partial u}{\partial x_j} + \frac{\partial u_1}{\partial y_j}\right) - a_{ij}\left(y, t, \frac{\partial v}{\partial x_j} + \frac{\partial v_1}{\partial y_j}\right)\right)\frac{\partial}{\partial y_i}(u_1 - v_1)\,dy = 0. \tag{3.28}$$

The first term in (3.27) is non-negative (a statement similar to (3.12) holds here also) so adding the zero terms of (3.28) in (3.27) yields

$$\int_0^t\int_\Omega\int_Y \left(a_{ij}\left(y, t, \frac{\partial u}{\partial x_j} + \frac{\partial u_1}{\partial y_j}\right) - a_{ij}\left(y, t, \frac{\partial v}{\partial x_j} + \frac{\partial v_1}{\partial y_j}\right)\right)$$
$$\cdot\left(\frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i} - \frac{\partial v}{\partial x_i} - \frac{\partial v_1}{\partial y_i}\right) dy\,dx\,dt \le 0.$$

The strict monotonicity assumption (3.4) then implies that

$$\frac{\partial u}{\partial x_j} + \frac{\partial u_1}{\partial y_j} = \frac{\partial v}{\partial x_j} + \frac{\partial v_1}{\partial y_j} \tag{3.29}$$

for $j = 1, \ldots, N$. Integrating (3.29) over Y and the Y-periodicity of u_1 and v_1 give that $\partial u / \partial x_j = \partial v / \partial x_j$ and so that $u = v$ since both lie in $L^p(0, T; W_0^{1,p}(\Omega))$. Using this in (3.29) then gives $\nabla_y u_1 = \nabla_y v_1$. This is enough to give uniqueness of u_1 in $L^p(\Omega \times (0, T); W_\#^{1,p}(Y)/\mathbf{R})$. ∎

Remark 3.2. As is typical of this approach to homogenization, the two-scale homogenized problem (3.25) preserves the structure of the original problem. However, in contrast to the usual results, the form of the final problem is not a single partial differential equation, but a coupled system of equations involving both the macroscopic (x) and microscopic (y) variables, with the coupling specified by the non-linear relation (3.24). Finally, we note that the homogenized initial condition (3.26) holds at a.e. point $x \in \Omega$ for which $\int_Y b(x, y)\, dy > 0$.

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