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On a conjecture by Y. Last

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Abstract

We prove a conjecture due to Y. Last. The new determinantal representation for transmission coefficient of Jacobi matrix is obtained.

In this paper we consider discrete Schrödinger operator on the half-lattice with bounded real potential v

$$J = \begin{bmatrix} v_1 & 1 & 0 & 0 & \dots \\ 1 & v_2 & 1 & 0 & \dots \\ 0 & 1 & v_3 & 1 & \dots \\ 0 & 0 & 1 & v_4 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}. \tag{1}$$

In [1], Last posed the following problem

Conjecture 0.1. *Prove that the following conditions:* $v_n \to 0$ *and*

$$v_{n+q} - v_n \in \ell^2$$

($q \in \mathbb{Z}^+$*–fixed) guarantee that* $\sigma_{ac}(J) = [-2, 2]$.

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The symbol $\sigma_{ac}(J)$ conventionally denotes the absolutely continuous (a.c.) spectrum of self-adjoint operator J. In this paper, we give an affirmative answer to this question. The manuscript consists of two sections. The first one is mostly algebraic, it contains the determinantal formula for the so-called transmission coefficient that allows us to immediately treat the case $q = 1$. In the second part, we show how asymptotical methods for difference equations provide the solution for any q. The appendix contains elementary lemmas for harmonic functions which are used in the paper.

Recently, many results on the characterization of parameters in the Jacobi matrix through the spectral data were obtained and the ℓ^2-condition on coefficients was often involved in one form or another (see, e.g., [2–5]). This paper makes the next step in this direction by developing the technique suggested in [6].

We will use notations: $(\delta v)_n = v_{n+1} - v_n$, $(\delta^{(q)} v)_n = v_{n+q} - v_n$, $\chi_{j\in M}$ is the characteristic function of the set M. For the sequence $\alpha \in \ell^p$, the symbol $\|\alpha\|_p$ denotes its norm in ℓ^p. As usual, the symbol C denotes the positive constant which might take different values in different formulas. For any matrix $B \in \mathbb{C}^{k\times k}$, the symbol $\|B\|$ will denote its operator norm in $\mathbb{C}^k$. Consider a linear bounded operator A acting in the Hilbert space. Assume that it is Hilbert–Schmidt, i.e. $A \in S_2$. Then, we define the regularized determinant by the formula (see, e.g., [7])

$$\det{}_2(I + A) = \det(I + R_2(A))$$

where

$$R_2(A) = (I + A)\exp(-A) - I \in S_1.$$

The symbol S_p is reserved for the standard Schatten–von Neumann class.

1. Determinantal formula and $q = 1$

It will be convenient for us to start with Jacobi matrices on $\ell^2(\mathbb{Z})$. Let H be a discrete Schrödinger operator on $\ell^2(\mathbb{Z})$ with potential v. Later on, we will make the following choice for v. For positive indices, it will be taken from (1). For negative indices, it will be set to zero. By H_0 we will denote the "free" case, i.e. the case when the potential v is identically zero on all of $\mathbb{Z}$. Consider the right and the left shifts acting on $\ell^2(\mathbb{Z})$

$$(Rf)_n = f_{n-1}, \qquad L = R^*.$$

Obviously, $R = L^{-1}$. For any $z \in \mathbb{C}$, introduce the following diagonal operators $\Lambda(z) = \{\lambda_n(z)\}$ and $\Lambda^0(z) = \tilde{\lambda}(z) \cdot I$, where

$$\lambda_n(z) = \frac{z - v_n - \left[(z - v_n)^2 - 4\right]^{1/2}}{2}, \qquad \tilde{\lambda}(z) = \frac{z - \left[z^2 - 4\right]^{1/2}}{2} \tag{2}$$

and $\sqrt{\ }$ has a cut along the positive axis. Notice that both functions map the upper half-plane into the lower half of the unit disc, i.e. $\{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z < 0\}$. Let $\delta\Lambda(z) = \Lambda(z) - \Lambda_0(z)$. We also need the finite-dimensional versions of these operators. Take $m = 2n+1$-dimensional linear space $\operatorname{span}\{e_{-n}, \dots, e_n\}$ and let R_n and L_n be right and left cyclic shifts, respectively. They are unitary operators and $L_n = R_n^{-1}$. Let Λ_n, Λ_n^0, $\delta\Lambda_n = \Lambda_n - \Lambda_n^0$ be restrictions of Λ, Λ^0, and $\delta\Lambda$. Define $\omega_j = \lambda_{j+1} - \lambda_j$, $j = -n, \dots, n-1$, $\omega_n = \lambda_{-n} - \lambda_n$ and the corresponding diagonal operator $\Omega_n = \{\omega_j\}$, $j = -n, \dots, n$.

If $K_n = L_n + R_n - \Lambda_n - \Lambda_n^{-1}$, then we have an elementary

Lemma 1.1. *For any $z \notin \mathbb{R}$, we have*

$$(L_n - \Lambda_n)(R_n - \Lambda_n) = -\Lambda_n K_n - \Omega_n L_n = -K_n \Lambda_n - R_n \Omega_n$$
$$(R_n - \Lambda_n)(L_n - \Lambda_n) = -K_n \Lambda_n + \Omega_n L_n.$$

Proof. The proof is a straightforward calculation. □

Lemma 1.2. *For any $z \in \mathbb{C}^+$,*

$$\begin{aligned} \det K_n &= -\frac{\det(L_n - \Lambda_n)\det(R_n - \Lambda_n)}{\det \Lambda_n} \exp\left(\operatorname{tr}\left[\Omega_n L_n[(L_n - \Lambda_n)(R_n - \Lambda_n)]^{-1}\right]\right) \\ &\quad \times \det{}_2[I + \Omega_n L_n[(L_n - \Lambda_n)(R_n - \Lambda_n)]^{-1}] \\ &= -\frac{\det(L_n - \Lambda_n)\det(R_n - \Lambda_n)}{\det \Lambda_n} \\ &\quad \times \exp\left(\operatorname{tr}\left[\Omega_n L_n[(R_n - \Lambda_n)(L_n - \Lambda_n) - R_n\Omega_n - \Omega_n L_n]^{-1}\right]\right) \\ &\quad \times \det{}_2[I + \Omega_n L_n[(L_n - \Lambda_n)(R_n - \Lambda_n)]^{-1}] \end{aligned} \tag{3}$$

$$\begin{aligned} \det K_n &= -\frac{\det(L_n - \Lambda_n)\det(R_n - \Lambda_n)}{\det \Lambda_n} \exp\left(\operatorname{tr}\left[-\Omega_n L_n[(R_n - \Lambda_n)(L_n - \Lambda_n)]^{-1}\right]\right) \\ &\quad \times \det{}_2[I - \Omega_n L_n[(R_n - \Lambda_n)(L_n - \Lambda_n)]^{-1}] \end{aligned} \tag{4}$$

$$\begin{aligned} \det K_n &= -\frac{\det(L_n - \Lambda_n)\det(R_n - \Lambda_n)}{\det \Lambda_n} \\ &\quad \times \exp\left(\frac{1}{2}\operatorname{tr}\left[\Omega_n L_n (L_n - \Lambda_n)^{-1}(R_n - \Lambda_n)^{-1}(R_n\Omega_n + \Omega_n L_n)\right.\right. \\ &\quad \times \left.\left. (R_n - \Lambda_n)^{-1}(L_n - \Lambda_n)^{-1}\right]\right) \\ &\quad \times \left[\det{}_2[I + \Omega_n L_n (R_n - \Lambda_n)^{-1}(L_n - \Lambda_n)^{-1}]\right. \\ &\quad \times \left. \det{}_2[I - \Omega_n L_n (L_n - \Lambda_n)^{-1}(R_n - \Lambda_n)^{-1}]\right]^{1/2}. \end{aligned} \tag{5}$$

Proof. (3) and (4) follow immediately from the previous lemma. Multiplication of (3) and (4) yields (5) after application of the second resolvent identity:

$$(A + V)^{-1} - A^{-1} = -A^{-1}V(A + V)^{-1}$$

and taking the suitable square root. □

Writing down the formula (5) for the "free" case with

$$K_n^0 = L_n + R_n - \Lambda_n^0 - (\Lambda_n^0)^{-1}$$

and dividing, we have

$$\det\left[K_n/K_n^0\right] = \frac{\det(I-(L_n-\Lambda_n^0)^{-1}\delta\Lambda_n)\det(I-(R_n-\Lambda_n^0)^{-1}\delta\Lambda_n)}{\det\left[\Lambda_n/\Lambda_n^0\right]}$$
$$\times \exp\left(\frac{1}{2}\operatorname{tr}\left[\Omega_n L_n (L_n-\Lambda_n)^{-1}\right.\right.$$
$$\left.\left.\times\ (R_n-\Lambda_n)^{-1}(R_n\Omega_n+\Omega_n L_n)(R_n-\Lambda_n)^{-1}(L_n-\Lambda_n)^{-1}\right]\right)$$
$$\times \left[\det{}_2[I+\Omega_n L_n(R_n-\Lambda_n)^{-1}(L_n-\Lambda_n)^{-1}]\right.$$
$$\left.\times\ \det{}_2[I-\Omega_n L_n(L_n-\Lambda_n)^{-1}(R_n-\Lambda_n)^{-1}]\right]^{1/2}. \tag{6}$$

Later on we will need the following bound.

Lemma 1.3. *For any $z\in\mathbb{C}^+$ and v, we have the following estimates for the operator norms*

$$\|(L_n-\Lambda_n)^{-1}\| \le C(\operatorname{Im} z)^{-1}(1+\operatorname{Im} z), \qquad \|(L-\Lambda)^{-1}\| \le C(\operatorname{Im} z)^{-1}(1+\operatorname{Im} z) \tag{7}$$
$$\|(R_n-\Lambda_n)^{-1}\| \le C(\operatorname{Im} z)^{-1}(1+\operatorname{Im} z), \qquad \|(R-\Lambda)^{-1}\| \le C(\operatorname{Im} z)^{-1}(1+\operatorname{Im} z).$$

Proof. Writing $\tilde\lambda(z)$ in polar coordinates, one can prove that

$$|\tilde\lambda(z)| = \left|\frac{z-\sqrt{z^2-4}}{2}\right| \le \frac{\sqrt{4+(\operatorname{Im} z)^2}-\operatorname{Im} z}{2}, \quad z\in\mathbb{C}^+$$

and

$$\|\Lambda_n\| \le \frac{\sqrt{4+(\operatorname{Im} z)^2}-\operatorname{Im} z}{2}.$$

Since

$$\frac{C\operatorname{Im} z}{1+\operatorname{Im} z} < 1-\|\Lambda_n\| \le |((L_n-\Lambda_n)f, L_n f)| \le \|(L_n-\Lambda_n)f\|, \qquad \|f\|=1$$

we have the statement of the lemma. The statements for L, R_n, R have the same proofs. □

Taking $n\to\infty$ in (6), we get

Lemma 1.4. *Assume that v is supported on $|j|\le l$. For $z\in\mathbb{C}^+$,*

$$\det[(H-z)/(H_0-z)] = \frac{\det(I-(L-\Lambda^0)^{-1}\delta\Lambda)\det(I-(R-\Lambda^0)^{-1}\delta\Lambda)}{\det\left[\Lambda/\Lambda^0\right]}$$
$$\times \exp\left(\frac{1}{2}\operatorname{tr}\left[\Omega L(L-\Lambda)^{-1}(R-\Lambda)^{-1}(R\Omega+\Omega L)(R-\Lambda)^{-1}(L-\Lambda)^{-1}\right]\right)$$
$$\times \left[\det{}_2[I+\Omega L(R-\Lambda)^{-1}(L-\Lambda)^{-1}]\det{}_2[I-\Omega L(L-\Lambda)^{-1}(R-\Lambda)^{-1}]\right]^{1/2}. \tag{8}$$

Proof. Since v is compactly supported, all determinants and traces in (6) and (8) are taken of the finite matrices with size of order $\sim l$. Therefore, it is sufficient to check that the corresponding matrices converge componentwise, which follows from the bound (7). In the same way one can show that

$$\det[I + (K_n^0)^{-1}(K_n - K_n^0)] \to \det[I + (H_0 - z)^{-1}(H - H_0)].$$

Then, taking $n \to \infty$ in (6), one has (8). □

Lemma 1.5. *If $z \in \mathbb{C}^+$, then*

$$[(L - \Lambda^0)^{-1} f]_n = \sum_{k=-\infty}^{n} \tilde{\lambda}^{n-k}(z) f_{k-1}, \qquad [(R - \Lambda^0)^{-1} f]_n = \sum_{k=n}^{\infty} \tilde{\lambda}^{k-n}(z) f_{k+1}.$$

Proof. The sums are in $\ell^2(\mathbb{Z})$ by Young's inequality for convolutions since $|\tilde{\lambda}(z)| < 1$. The rest is a direct calculation. □

By inspection, we have

Lemma 1.6. *If $z \in \mathbb{C}^+$, then*

$$\det\left[I - (L - \Lambda^0)^{-1}\delta\Lambda\right] = 1, \qquad \det\left[I - (R - \Lambda^0)^{-1}\delta\Lambda\right] = 1$$

$$\det\left[\Lambda^0 \Lambda^{-1}\right] = \prod_{j=-l}^{l} \left[\tilde{\lambda}(z)/\lambda_j(z)\right].$$

Proof. It is a direct corollary from Lemma 1.5. □

Thus the formula (8) can be simplified and we have

Lemma 1.7. *For compactly supported v,*

$$\begin{aligned} \det[(H - z)/(H_0 - z)] &= \prod_{j=-\infty}^{\infty} \left[\tilde{\lambda}(z)/\lambda_j(z)\right] \\ &\times \exp\left(\frac{1}{2}\operatorname{tr}\left[\Omega L (L - \Lambda)^{-1}(R - \Lambda)^{-1}(R\Omega + \Omega L)(R - \Lambda)^{-1}(L - \Lambda)^{-1}\right]\right) \\ &\times \left[\det{}_2[I + \Omega L (R - \Lambda)^{-1}(L - \Lambda)^{-1}]\det{}_2[I - \Omega L (L - \Lambda)^{-1}(R - \Lambda)^{-1}\right]^{1/2}. \end{aligned} \qquad (9)$$

Consider the first factor. Take any sequence $v = \{v_j\}$, $j \in \mathbb{Z}$ of real numbers such that $v_j \to 0$ as $|j| \to \infty$ and let $v^N = v \cdot \chi_{|j|<N}$ be its truncation where N is large. For each N, introduce[1]

$$WKB_{v^N}(z) = \prod_{j=-N}^{N} \left[\tilde{\lambda}(z)/\lambda_j(z)\right]$$

where $\lambda_j(z)$ and $\tilde{\lambda}(z)$ are defined in (2). Obviously, if v is compactly supported, then $WKB_{v^N}(z)$ will coincide with the first factor in (9) as long as N is large.

[1] The use of symbol *WKB* is justified by analogous results in asymptotical theory of ordinary differential equation (see, e.g., [8]). This abbreviation is after Wentzel–Kramers–Brillouin.

Notice that, for any fixed δ, we have

$$|WKB_{v^N}(z)| = 1 \tag{10}$$

for any $z \in [-2+\delta, 2-\delta]$ and any N provided that $\|v\|_\infty$ is small.

In scattering theory, the inverse to the transmission coefficient is usually denoted by $a(z)$. We will introduce its modification (or, rather, regularization). It will be denoted by $a_m(z)$. For compactly supported v, consider the second and the third factors in (9). Let

$$\begin{aligned} a_m(z) &= \exp\left(\frac{1}{2}\operatorname{tr}\left[\Omega L(L-\Lambda)^{-1}(R-\Lambda)^{-1}(R\Omega+\Omega L)(R-\Lambda)^{-1}(L-\Lambda)^{-1}\right]\right) \\ &\quad\times \left[\det{}_2[I+\Omega L(R-\Lambda)^{-1}(L-\Lambda)^{-1}]\det{}_2[I-\Omega L(L-\Lambda)^{-1}(R-\Lambda)^{-1}]\right]^{1/2}. \end{aligned} \tag{11}$$

The relevance of $a_m(z)$ to the scattering will be clear from the proof of Theorem 1.1.

We want to control $a_m(z)$ for $\operatorname{Im} z > 0$ and $|z| < 4$. Specifically, we need estimates on the boundary behavior as z approaches $[-2, 2]$. For the Hilbert–Schmidt norm $\|\Omega\|_{S^2}$ of Ω, we have

$$\|\Omega\|_{S^2} \le C\|\delta v\|_2.$$

Combining this estimate with (7), we get

Lemma 1.8. *For* $\operatorname{Im} z > 0,\ |z| < 4$,

$$\ln|a_m(z)| \le C\frac{\|\delta v\|_2^2}{(\operatorname{Im} z)^4}. \tag{12}$$

Also, for any fixed $\epsilon > 0$ *and any* $z : \operatorname{Im} z > \epsilon,\ |z| < 4$, *we have*

$$|a_m(z)| > C(\epsilon, \|\delta v\|_2) > 0 \tag{13}$$

provided that $\|\delta v\|_2$ *is small.*

Proof. The estimates follow from Lemma 1.3 and from the properties of the trace and $\det_2$ (see [7], p. 107 (b)). □

Now, we are ready to prove the main statement of the first section.

Theorem 1.1. *Assume that* $v_n \to 0$ *and* $v_{n+1} - v_n \in \ell^2(\mathbb{Z}^+)$. *Then,* $\sigma_{ac}(J) = [-2, 2]$.

Proof. By Weyl's theorem, the essential spectrum of J is $[-2, 2]$. By the Kato–Rosenblum theorem, the support of a.c. spectrum does not change under the trace-class perturbations and $\sigma_{ac}(J) = \sigma_{ac}(J(L))$ where $J(L)$ has potential $v(L) = v \cdot \chi_{j>L}$.

$$\|v(L)\|_\infty + \|\delta v(L)\|_2 \to 0, \quad L \to \infty \tag{14}$$

and therefore we can assume $\|v\|_\infty + \|\delta v\|_2$ to be as small as we wish.

For large N, consider truncations v^N, i.e. $v^N = v \cdot \chi_{j<N}$. Also, take H^N on $\ell^2(\mathbb{Z})$ with potential $v_j = v_j^N$ for $j \ge 0$ and $v_j = 0$ for $j < 0$. Let J^N denotes analogous truncation for J. Then, the Jost function $\psi_n(k)$ is defined as the solution to $H^N\psi = (k+k^{-1})\psi$ that satisfies $\psi_n(k) = k^n$ for $n > N$. It is well known that such a solution exists for all $k \neq 0$. We will be interested in $k : \operatorname{Im} k \le 0, 0 < |k| \le 1$ which corresponds to $z = k + k^{-1} \in \overline{\mathbb{C}^+}$.

Since $v_j^N = 0$ for negative j, $\psi_n(k) = a^N(z)k^n + b^N(z)k^{-n}$ for $n < 0$. We will use the following well-known facts

$$a^N(z) = \det\left[\frac{H^N - z}{H_0^N - z}\right]$$

$$|a^N(z)| \geq 1 \quad \text{for } z \in (-2, 2) \tag{15}$$

$$\frac{1}{|a^N(z)|^2} = \frac{4|\sin\theta|}{|m^N(z) + \mathrm{e}^{\mathrm{i}\theta}|^2}\operatorname{Im} m^N(z), \qquad z = 2\cos\theta \in [-2, 2] \tag{16}$$

(see p. 346, [2]), here $m^N(z)$ is the Stieltjes transform of the spectral measure $\mathrm{d}\rho_N(\lambda)$ of J^N. From (15),

$$\int_{-2+\delta}^{2-\delta} \ln|a^N(z)|\mathrm{d}z > 0$$

for any small $\delta > 0$.

Consider the formula (9) for $a^N(z)$ and use (10), (11), and Lemma A.3 from Appendix with $f(z) = -\ln|a_m^N(z)|$, $a = -2+\delta$, $b = 2-\delta$. The estimates (12)–(14) guarantee its applicability because the function $a^N(z)$ is continuous up to the real line in the specified domain since v^N has a finite support. Thus, we have

$$-\int_{-2+\delta}^{2-\delta} \ln|a_m^N(z)|\mathrm{d}z > -C.$$

Therefore, due to (10) and (16),

$$\int_{-2+\delta}^{2-\delta} \ln\rho_N'(z)\mathrm{d}z \geq -C \tag{17}$$

uniformly in N. Since $\mathrm{d}\rho_N(\lambda) \to \mathrm{d}\rho(\lambda)$ in the weak sense [2], the semicontinuity argument from [3], Corollary 5.3 gives

$$\int_{-2+\delta}^{2-\delta} \ln\rho'(z)\mathrm{d}z \geq -C \tag{18}$$

for all $\delta > 0$. This implies that the a.c. part of the spectrum covers $[-2, 2]$. □

2. Last's conjecture for any q

In this section, we will apply the standard method of asymptotical analysis to study the Schrödinger difference relation, then the asymptotics obtained will be analyzed to conclude the presence of a.c. spectrum.

Consider a general solution

$$x_{n+1} + v_n x_n + x_{n-1} = z x_n, \qquad n = 1, 2, \ldots$$

$$X_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}, \qquad n = 0, 1, \ldots$$

$$X_n = (\Omega + V_n)X_{n-1}, \qquad \Omega = \begin{bmatrix} z & -1 \\ 1 & 0 \end{bmatrix}, V_n = v_n \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, n = 1, 2, \ldots.$$

If $Z_m = X_{mq}$, then

$$Z_{m+1} = T_m Z_m, \quad m = 0, 1, \dots \tag{19}$$

where

$$T_m = (\Omega + V_{mq+q}) \dots (\Omega + V_{mq+1}).$$

Let

$$k^{\pm 1}(z) = \frac{z \mp \sqrt{z^2 - 4}}{2}$$

so that $k(z)$ maps $\mathbb{C} \setminus [-2, 2]$ onto $\mathbb{D}$ conformally.

Notice that if $\tilde{P}, P$ and $\tilde{Q}, Q$ are $(q, q-1)$th and the $(q-1, q-2)$th polynomials corresponding to the first Jacobi coefficients $v_1, \dots, v_q$ and $(1, 0)^t$, $(0, 1)^t$ initial conditions, then

$$T_m = \begin{bmatrix} \tilde{P}_m & \tilde{Q}_m \\ P_m & Q_m \end{bmatrix}.$$

Notice that $\det T_m = 1$ and therefore $\tilde{P}_m Q_m - P_m \tilde{Q}_m = 1$. We also have

$$\Omega^q = \begin{bmatrix} \dfrac{k^{q+1} - k^{-q-1}}{k - k^{-1}} & -\dfrac{k^q - k^{-q}}{k - k^{-1}} \\ \dfrac{k^q - k^{-q}}{k - k^{-1}} & -\dfrac{k^{q-1} - k^{-q+1}}{k - k^{-1}} \end{bmatrix}$$

and therefore

$$\tilde{P}_m + Q_m = k^q + k^{-q} + d(k, v_{mq+1}, \dots, v_{mq+q}). \tag{20}$$

The function $d(\cdot)$ is a polynomial in $v_{mq+1}, \dots, v_{mq+q}$ and

$$d(k, v_{mq+1}, \dots, v_{mq+q}) \to 0, \quad m \to \infty. \tag{21}$$

Introduce $\lambda_{1(2)}^{(m)}$ by

$$\lambda_{1(2)}^{(m)} = \frac{\tilde{P}_m + Q_m \mp \sqrt{(\tilde{P}_m + Q_m)^2 - 4}}{2}. \tag{22}$$

These are the eigenvalues of T_m.

Let us take U_m

$$U_m = \begin{bmatrix} -Q_m + \lambda_1^{(m)} & -Q_m + \lambda_2^{(m)} \\ P_m & P_m \end{bmatrix}. \tag{23}$$

Then we have

$$U_{m+1}^{-1} U_m = \frac{1}{P_{m+1}(\lambda_1^{(m+1)} - \lambda_2^{(m+1)})} \times \begin{bmatrix} P_{m+1}(\lambda_1^{(m)} - Q_m) - P_m(\lambda_2^{(m+1)} - Q_{m+1}), & P_{m+1}(\lambda_2^{(m)} - Q_m) - P_m(\lambda_2^{(m+1)} - Q_{m+1}) \\ -P_{m+1}(\lambda_1^{(m)} - Q_m) + P_m(\lambda_1^{(m+1)} - Q_{m+1}), & -P_{m+1}(\lambda_2^{(m)} - Q_m) + P_m(\lambda_1^{(m+1)} - Q_{m+1}) \end{bmatrix}. \tag{24}$$

The matrix U_m can be used to diagonalize T_m as follows

$$T_m(z) = U_m(z)\Lambda_m(z)U_m^{-1}(z)$$

where

$$\Lambda_m(z) = \begin{bmatrix} \lambda_1^{(m)}(z) & 0 \\ 0 & \lambda_2^{(m)}(z) \end{bmatrix}.$$

Lemma 2.1. *The matrix T_m has eigenvalues $\lambda_{1(2)}^{(m)}(z)$ such that*

$$\begin{aligned} &\lambda_1^{(m)}(z)\cdot\lambda_2^{(m)}(z) = 1, \quad z\in\mathbb{C} \\ &|\lambda_{1(2)}^{(m)}(z)| = 1, \quad z\in[z_j+\delta(v), z_{j+1}-\delta(v)], \quad j=0,\ldots,q-1 \end{aligned} \tag{25}$$

and

$$\delta(v)\to 0$$

as

$$\zeta_m = \max\{|v_{mq+q}|,\ldots,|v_{mq+1}|\}\to 0.$$

Here $z_j = 2\cos(\pi-\pi j/q)$, $j=0,\ldots,q$.

Proof. The first identity follows from $\det T_m(z) = 1$. Since the function $d(\cdot)$ in (20) is real for real z, the second one is immediate from (20)–(22). □

Notice that Ω^q has eigenvalues

$$\omega_{1(2)} = k^{\mp q}(z) = \left(\frac{z\pm\sqrt{z^2-4}}{2}\right)^q.$$

We have the following elementary perturbation result

Lemma 2.2. *If $0\le \operatorname{Im} z\le 1$, $z_j+\delta < \operatorname{Re} z < z_{j+1}-\delta$, $j=0,\ldots,q-1$, then*

$$\begin{aligned} &\lambda_{1(2)}^{(m)}(z) = \omega_{1(2)}(z) + \underline{O}(\zeta_m) \\ &\sum_{m=0}^{\infty}\left|\lambda_{1(2)}^{(m+1)}(z) - \lambda_{1(2)}^{(m)}(z)\right|^2 < C(\delta) \end{aligned} \tag{26}$$

and $\delta > 0$ is fixed arbitrarily small number. Moreover, for all z in these domains we have the following estimate

$$\ln|\lambda_1^{(m)}(z)| = (C+\underline{O}(\zeta_m))\operatorname{Im} z \tag{27}$$

with some positive constant C. We also assume here that $m > m_0(\delta)$ and $m_0(\delta)$ is large depending on δ.

Proof. For $d(\cdot)$, we have $|d(k, v_{mq+1},\ldots,v_{mq+q})| < C|\zeta_m|$. It is also a polynomial in $v_{mq+1},\ldots,v_{mq+q}$. Then, (26) follows from the Mean Value Theorem and (20)–(22). To prove (27), we fix j and consider the following function

$$h(z) = \ln|\lambda_1^{(m)}(z)/\omega_1(z)|$$

harmonic in the domain of interest: $0 \le \operatorname{Im} z \le 1$, $z_j + \delta < \operatorname{Re} z < z_{j+1} - \delta$. On the real line, i.e., for $z_j + \delta < \operatorname{Re} z < z_{j+1} - \delta$, $\operatorname{Im} z = 0$, we have $h(z) = 0$ and at all other points we have

$$h(z) = \underline{O}(\zeta_m).$$

Therefore, the interpolation Lemma A.1 gives

$$h(z) = \underline{O}(\zeta_m) \operatorname{Im} z.$$

For $\omega_1(z)$ we have $|\omega_1(z)| > 1 + C \operatorname{Im} z$ with positive C, and one gets (27). □

Let us find Z_m in the form $Z_m = U_m S_m$ and

$$S_{m+1} = U_{m+1}^{-1} T_m U_m S_m = U_{m+1}^{-1} U_m \Lambda_m S_m = [U_{m+1}^{-1}(U_m - U_{m+1}) + I] \Lambda_m S_m. \tag{28}$$

Lemma 2.3. *If* $0 \le \operatorname{Im} z \le 1$, $z_j + \delta < \operatorname{Re} z < z_{j+1} - \delta$, $j = 0, \dots, q-1$, *then for the matrix norms we have*

$$\|U_{m+1}^{-1}(U_m - U_{m+1})\|_{\ell^2} < C.$$

Proof. Away from the points z_j, $\|U_{m+1}^{-1}\|$ is uniformly bounded and the proof follows immediately from (23) and (26). □

We need the following

Theorem 2.1. *Let*

$$\Psi_{n+1} = (I + W_n) \begin{bmatrix} \kappa_n & 0 \\ 0 & \kappa_n^{-1} \end{bmatrix} \Psi_n, \quad W_n = \begin{bmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{bmatrix}, \quad \Psi_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where $\kappa_n \in \mathbb{C}$, $C > |\kappa_n| > |\kappa| > 1$, *the sequence* $\zeta_n = \|W_n\| \in \ell^2(\mathbb{Z}^+)$ *and its* ℓ^2 *norm is small. Assume also that there is a constant* $0 \le \upsilon < 1$ *so that*

$$\left| \ln \prod_{n=k}^{l} |1 + \alpha_n| \right| \le C + \upsilon \sqrt{l-k}, \qquad \left| \ln \prod_{n=k}^{l} |1 + \delta_n| \right| \le C + \upsilon \sqrt{l-k}. \tag{29}$$

Then,

$$\Psi_n = p_n \begin{bmatrix} \phi_n \\ \nu_n \end{bmatrix},$$

where

$$p_n = \prod_{j=0}^{n-1} \kappa_j (1 + \alpha_j), \quad p_0 = 1$$

and

$$|\phi_n|, |\nu_n| \le C \exp\left(\frac{C}{|\kappa| - 1} \exp\left[\frac{C\upsilon^2}{|\kappa| - 1} \right] \right). \tag{30}$$

Moreover, for any fixed $\epsilon > 0$ *and any* $\kappa : |\kappa| > 1 + \epsilon$, *we have*

$$|\phi_n| > C > 0, \qquad |\nu_n| < C\|\zeta\|_2 \tag{31}$$

uniformly in n provided that $\|\zeta\|_2$ *is small enough.*

Proof. Let

$$\Psi_n = \begin{bmatrix} p_n & 0 \\ 0 & q_n \end{bmatrix} Y_n$$

where

$$q_n = \prod_{j=0}^{n-1} \kappa_j^{-1}(1+\delta_j), \qquad q_0 = 1.$$

Then,

$$Y_{n+1} = \begin{bmatrix} 1 & q_n p_{n+1}^{-1} \kappa_n^{-1} \beta_n \\ p_n q_{n+1}^{-1} \kappa_n \gamma_n & 1 \end{bmatrix} Y_n.$$

If

$$Y_n = \begin{bmatrix} \phi_n \\ \upsilon_n \end{bmatrix}$$

then

$$\nu_n = q_n p_n^{-1} \upsilon_n \tag{32}$$

and we have the following equations

$$\phi_n = 1 + \sum_{j=0}^{n-1} q_j p_{j+1}^{-1} \kappa_j^{-1} \beta_j \upsilon_j$$

$$\upsilon_n = \sum_{j=0}^{n-1} p_j q_{j+1}^{-1} \kappa_j \gamma_j \phi_j. \tag{33}$$

For ϕ_n:

$$\phi_n = 1 + \sum_{k=0}^{n-2} \phi_k \epsilon_{k,n}, \quad \epsilon_{k,n} = \gamma_k p_k q_{k+1}^{-1} \kappa_k \sum_{j=k+1}^{n-1} \kappa_j^{-1} q_j p_{j+1}^{-1} \beta_j.$$

For $\epsilon_{k,n}$,

$$\begin{aligned} |\epsilon_{k,n}| \le \epsilon_k &= C|\gamma_k| \sum_{j=k+1}^{\infty} |\beta_j| \cdot |\kappa|^{-2(j-k)} \cdot \prod_{l=k}^{j} \left| \frac{1+\delta_l}{1+\alpha_l} \right| \\ &< C|\gamma_k| \sum_{j=k+1}^{\infty} |\beta_j| |\kappa|^{-2(j-k)} \cdot \exp\left(C\upsilon\sqrt{j-k} \right). \end{aligned} \tag{34}$$

From the obvious inequality

$$\sum_{j=0}^{\infty} |\kappa|^{-j} \cdot \exp\left(C\upsilon\sqrt{j} \right) < \frac{C}{\ln|\kappa|} \exp\left[\frac{C\upsilon^2}{\ln|\kappa|} \right]$$

and Young's inequality for convolutions, we get an estimate for the ℓ^1 norm of the sequence ϵ introduced in (34):

$$\|\epsilon\|_1 \le \frac{C}{|\kappa|-1} \cdot \exp\left[\frac{C\upsilon^2}{|\kappa|-1} \right] \cdot \|\gamma\|_2 \cdot \|\beta\|_2. \tag{35}$$

This yields the same estimates for

$$\sum_{k=0}^{n} |\epsilon_{k,n}|$$

uniformly in n. Now, to prove (30), one can use the following lemma below.

Lemma 2.4. *If* $x_n, v_n \geq 0,\ x_0 = 1$, *and*

$$x_{n+1} \leq \sum_{j=0}^{n} v_j x_j$$

for all $n > 0$, *then*

$$x_n \leq v_0 \exp\left[\sum_{j=1}^{n-1} v_j\right], \quad n \geq 2; \qquad x_1 \leq v_0. \tag{36}$$

Proof. [2] Consider the functions

$$x(t) = \sum_{j=0}^{\infty} x_j \chi_{[j,j+1)}(t), \qquad v(t) = \sum_{j=0}^{\infty} v_j \chi_{[j,j+1)}(t).$$

We have

$$x(t) \leq v_0 + \int_1^{[t]-1} x(s)v(s)\mathrm{d}s \leq v_0 + \int_1^{t} x(s)v(s)\mathrm{d}s, \quad t > 1.$$

The application of Gronwall–Bellman inequality gives (36). □

The estimate for v_n and the line (31) are straightforward corollaries from the bound for $\|\phi\|_\infty$, (32) and (33), and the Cauchy–Schwarz inequality. □

Introduce

$$U_{m+1}^{-1}(U_m - U_{m+1}) = \begin{bmatrix} \alpha_m & \beta_m \\ \gamma_m & \delta_m \end{bmatrix}. \tag{37}$$

Then, (28) can be rewritten as

$$S_{m+1} = \left[I + \begin{bmatrix} \alpha_m & \beta_m \\ \gamma_m & \delta_m \end{bmatrix}\right] \times \begin{bmatrix} \lambda_1^{(m)} & 0 \\ 0 & \lambda_2^{(m)} \end{bmatrix} S_m. \tag{38}$$

Now, let us apply Theorem 2.1 to (38). For each $k = 0, \ldots, q-1$, consider z in the following domain: $0 \leq \operatorname{Im} z < 1$, $z_k + \delta < \operatorname{Re} z < z_{k+1} - \delta$ where δ is a small positive number. We have $\lambda_1^{(m)} \cdot \lambda_2^{(m)} = 1$ and $|\lambda_1^{(m)}| > (1 + C \operatorname{Im} z)$ by Lemma 2.1, (27). In our notations

$$W_m = \begin{bmatrix} \alpha_m & \beta_m \\ \gamma_m & \delta_m \end{bmatrix}$$

[2] This lemma can also be proved by induction.

and $\kappa_m = \lambda_1^{(m)}$. The estimate $\|W_m\| \in \ell^2(\mathbb{Z}^+)$ follows from Lemma 2.3. Now, let us control

$$\prod_{j=0}^{n}(1+\alpha_n), \qquad \prod_{j=0}^{n}(1+\delta_n)$$

and the constant $\upsilon(z)$ in (29).

Theorem 2.2. *For $z : 0 \le \operatorname{Im} z < 1,\ z_k + \delta < \operatorname{Re} z < z_{k+1} - \delta,\ k = 0, 1, \ldots, q-1$, we have*

$$\left| \ln \prod_{n=k}^{l} |1+\alpha_n| \right| \le C + (C \operatorname{Im} z)(l-k)^{1/2},$$

$$\left| \ln \prod_{n=k}^{l} |1+\delta_n| \right| \le C + (C \operatorname{Im} z)(l-k)^{1/2} \tag{39}$$

as long as $\|v\|_\infty$ is small.

Proof. Consider the product for α_n, the product for δ_n can be treated in the same way. In notations of (24) and (37),

$$1 + \alpha_m(z) = \frac{P_{m+1}(\lambda_1^{(m)} - Q_m) - P_m(\lambda_2^{(m+1)} - Q_{m+1})}{P_{m+1}(\lambda_1^{(m+1)} - \lambda_2^{(m+1)})} = 1 + t_m^1 + t_m^2 + t_m^3 + t_m^4 + t_m^5$$

where

$$t_m^1 = \frac{P_m Q_{m+1} - P_{m+1} Q_m}{P_{m+1}(\lambda_1^{(m+1)} - \lambda_2^{(m+1)})}$$

$$t_m^2 = -\frac{P_{m+1} - P_m}{2P_{m+1}}$$

$$t_m^3 = \frac{(P_{m+1} - P_m)(\tilde{P}_{m+1} + Q_{m+1})}{2P_{m+1}(\lambda_1^{(m+1)} - \lambda_2^{(m+1)})}$$

$$t_m^4 = -\frac{\tilde{P}_{m+1} - \tilde{P}_m + Q_{m+1} - Q_m}{2(\lambda_1^{(m+1)} - \lambda_2^{(m+1)})}$$

$$t_m^5 = -\frac{(\lambda_1^{(m+1)} - \lambda_2^{(m+1)}) - (\lambda_1^{(m)} - \lambda_2^{(m)})}{2(\lambda_1^{(m+1)} - \lambda_2^{(m+1)})}$$

since

$$\frac{P_{m+1}\lambda_1^{(m)} - P_m \lambda_2^{(m+1)}}{P_{m+1}(\lambda_1^{(m+1)} - \lambda_2^{(m+1)})} = \frac{P_{m+1}(\lambda_1^{(m)} - \lambda_2^{(m+1)})}{P_{m+1}(\lambda_1^{(m+1)} - \lambda_2^{(m+1)})} + \frac{(P_{m+1} - P_m)\lambda_2^{(m+1)}}{P_{m+1}(\lambda_1^{(m+1)} - \lambda_2^{(m+1)})}$$

$$\frac{\lambda_1^{(m)} - \lambda_2^{(m+1)}}{\lambda_1^{(m+1)} - \lambda_2^{(m+1)}} = 1 - \frac{\lambda_1^{(m+1)} - \lambda_1^{(m)}}{\lambda_1^{(m+1)} - \lambda_2^{(m+1)}}$$

$$\frac{\lambda_1^{(m+1)} - \lambda_1^{(m)}}{\lambda_1^{(m+1)} - \lambda_2^{(m+1)}} = \frac{(\lambda_1^{(m+1)} - \lambda_2^{(m+1)}) - (\lambda_1^{(m)} - \lambda_2^{(m)})}{2(\lambda_1^{(m+1)} - \lambda_2^{(m+1)})} + \frac{\tilde{P}_{m+1} - \tilde{P}_m + Q_{m+1} - Q_m}{2(\lambda_1^{(m+1)} - \lambda_2^{(m+1)})} \quad \text{as follows from (22)}$$

and

$$\frac{(P_{m+1}-P_m)\lambda_2^{(m+1)}}{P_{m+1}(\lambda_1^{(m+1)}-\lambda_2^{(m+1)})} = -\frac{P_{m+1}-P_m}{2P_{m+1}} + \frac{(P_{m+1}-P_m)(\tilde{P}_{m+1}+Q_{m+1})}{2P_{m+1}(\lambda_1^{(m+1)}-\lambda_2^{(m+1)})}.$$

Obviously, all $t_m^j \in \ell^2(\mathbb{Z}^+)$, $j = 1, \ldots, 5$ and are small if $\|v\|_\infty$ is small. Therefore, we have

$$\left|\mathrm{Re}\ln\prod_{n=k}^{l}(1+\alpha_n)\right| \le \left|\sum_{j=1}^{5}\mathrm{Re}\sum_{n=k}^{l} t_n^j\right| + C. \tag{40}$$

We need the following lemma.

Lemma 2.5. *Let $\epsilon_n \to \epsilon \in \mathbb{C}$, $|\epsilon_{n+1}-\epsilon_n| \in \ell^2$, and $f(z)$ is holomorphic around ϵ. Then,*

$$\left|\sum_{n=k}^{l}(\epsilon_{n+1}-\epsilon_n) f(\epsilon_n)\right| < C$$

for k and l large enough.

Proof. Consider $g(z)$ holomorphic around ϵ such that $g(\epsilon) = 0$ and $g'(z) = f(z)$. The Taylor formula around ϵ_n yields

$$g(\epsilon_{l+1}) - g(\epsilon_k) = \sum_{n=k}^{l} g(\epsilon_{n+1}) - g(\epsilon_n) = \sum_{n=k}^{l}\left[(\epsilon_{n+1}-\epsilon_n)f(\epsilon_n) + \underline{O}(|\epsilon_{n+1}-\epsilon_n|^2)\right]$$

which shows that the second term on the right-hand side is bounded. □

Taking $f(z) = z^{-1}$ in the lemma yields

$$\left|\sum_{n=k}^{l} t_n^j\right| < C$$

where $j = 2, 5$. Now, for the other j, consider the following functions

$$\mathrm{Re}\sum_{n=k}^{l} t_n^j(z)$$

which are harmonic near the intervals of interest. By Cauchy–Schwarz, we have

$$\left|\mathrm{Re}\sum_{n=k}^{l} t_n^j(z)\right| < C(l-k)^{1/2} \tag{41}$$

for any z in the specified domain. For real z within the intervals $[z_k+\delta, z_{k+1}-\delta]$,

$$\mathrm{Re}\sum_{n=k}^{l} t_n^j(z) = 0. \tag{42}$$

Indeed, all polynomials $P, \tilde{P}, Q, \tilde{Q}$ are real for real z. On the other hand, for real z within these intervals and small $\|v\|_\infty$, we have $\lambda_1^{(m)} = \overline{\lambda_2^{(m)}}$, and so $\lambda_1^{(m)} - \lambda_2^{(m)}$ is purely imaginary. Now, the theorem follows from (41) and (42) and interpolation Lemma A.1. □

As a corollary, we get

Theorem 2.3. *Consider* $z : 0 \le \operatorname{Im} z < 1,\ z_k + \delta < \operatorname{Re} z < z_{k+1} - \delta,\ k = 0, 1, \dots, q-1$ *and introduce* $\kappa_j(z)$ *and* $\alpha_j(z)$ *as before. If* $v_1 = v_2 = \cdots = v_q = 0$ *and* $X_0 = (k^{-1}(z), 1)^t$*, then we have the following estimates for the solution of the Schrödinger recursion:*

$$|x_{nq}(z)| \le \left| \prod_{j=0}^{n-1} \kappa_j(z)(1 + \alpha_j(z)) \right| \cdot \exp\left(\frac{C}{\operatorname{Im} z} \right)$$

where the first factor on the r.h.s. is uniformly bounded for $z \in (z_k + \delta, z_{k+1} - \delta)$*,* $k = 0, 1, \dots, q-1$*. Moreover, for any fixed* $\epsilon > 0$ *and any* $z : \operatorname{Im} z > \epsilon > 0$*, we have*

$$\left| x_{nq}(z) \left(\prod_{j=0}^{n-1} \kappa_j(z)(1 + \alpha_j(z)) \right)^{-1} \right| > C > 0$$

uniformly in n, provided that $\|v\|_\infty + \|\delta^{(q)} v\|_2$ *is small enough.*

Proof. We have

$$U_0^{-1} = \frac{1}{k^{-q} - k^q} \begin{bmatrix} 1 & -k \\ -1 & k^{-1} \end{bmatrix}.$$

If

$$X_{mq} = Z_m = U_m S_m \tag{43}$$

then we have recursion (28) for S_m and $S_0 = (k(z) - k^{-1}(z))(k^q(z) - k^{-q}(z))^{-1}(1, 0)^t$. From (43), an explicit form (23) for U_m, and Theorems 2.1 and 2.2, we get the statement of the theorem. □

Remark. One can easily see that the analysis suggested above proves the asymptotics for $x_n(z)$ in the corresponding domains of complex plane, not just bounds from above and below. It is also conceivable that the complicated product of $1 + \alpha_j$ in the asymptotics can be simplified and possibly eliminated (as for $q = 1$) by applying some analog of Lemma 2.5. We do not pursue it here and employ technique which is somewhat more powerful.

Now, we are ready to prove the main result of this paper.

Theorem 2.4. *If the Jacobi matrix* J *given by* (1) *has coefficients* $v_n \to 0$ *and* $v_{n+q} - v_n \in \ell^2(\mathbb{Z}^+)$ *for some* q*, then* $\sigma_{ac}(J) = [-2, 2]$*.*

Proof. By Weyl's theorem, the essential spectrum is $[-2, 2]$. Take any small $\delta > 0$. We will show that all intervals $[z_j + \delta, z_{j+1} - \delta]$, $j = 0, \dots, q-1$ are contained in the support the a.c. spectrum of J. Fix $\delta > 0$. Just like in Theorem 1.1, we can assume that $\|v\|_\infty + \|\delta^{(q)} v\|_2$ is as small as we wish.

Then, we will prove

$$\int_{z_j+\delta}^{z_{j+1}-\delta} \ln \rho'(z) \mathrm{d}z > -\infty, \quad j = 0, \dots, q-1 \tag{44}$$

where $\rho(z)$ is the spectral measure of the matrix J. Consider the truncated potential $v^N = v \cdot \chi_{j<N}$, the corresponding matrix J^N, and the spectral measure ρ^N, (we take $N = qm$). Since

the potential is finitely supported, there exists the Jost solution:

$$x^N_{n+1} + v^N_n x^N_n + x^N_{n-1} = zx^N_n$$

with the following asymptotics

$$x^N_n = k^n, \quad n > N.$$

Then, the factorization (see, [3], (1.32))

$$\pi \cdot (\rho^N)'(z) = \frac{\sqrt{4-z^2}}{|x^N_0(z)|^2}, \quad z \in [-2, 2]$$

holds.

Obviously, x^N_n is the solution of the Cauchy problem $x^N_N = k^N$, $x^N_{N+1} = k^{N+1}$ and x^N_0 can be obtained by solving the recursion "backward". Theorem 2.3 can be applied then. Introduce the function (modified Jost function)

$$f_N(z) = x^N_0(z) \left(k^N \prod_{j=0}^{m-1} (1 + \alpha_j(z))\kappa_j(z) \right)^{-1}$$

where α_j and κ_j are taken from Theorem 2.3 (with "backward" ordering for potential). From this theorem,

$$|f_N(z)| < \exp\left(\frac{C}{\operatorname{Im} z}\right)$$

uniformly in N as long as $z_j + \delta < \operatorname{Re} z < z_{j+1} - \delta, 0 < \operatorname{Im} z < 1$. On the real line,

$$\left| k^N \prod_{j=0}^{m-1} (1 + \alpha_j(z))\kappa_j(z) \right| \sim 1$$

and we obtain

$$\int_{z_j+\delta}^{z_{j+1}-\delta} \ln(\rho^N)'(z)\mathrm{d}z > -C_1 - C_2 \int_{z_j+\delta}^{z_{j+1}-\delta} \ln|f_N(z)|\mathrm{d}z \tag{45}$$

$$\int_{z_j+\delta}^{z_{j+1}-\delta} (-\ln|f_N(z)|)^+ \,\mathrm{d}z < C_1 \int_{z_j+\delta}^{z_{j+1}-\delta} \left(\ln(\rho^N)'(z)\right)^+ \mathrm{d}z + C_2 < C \tag{46}$$

uniformly in N. Moreover,

$$|f_N(z)| > C > 0$$

if $\operatorname{Im} z > \epsilon(v)$, also uniformly in N.

The function $-\ln|f_N(z)|$ is harmonic in $z_j + \delta < \operatorname{Re} z < z_{j+1} - \delta, 0 < \operatorname{Im} z < 1$ because $x_0(z)$ has no zeroes in $\mathbb{C}^+$ (otherwise the asymptotics at infinity would yield the complex eigenvalue for J^N which is impossible) and $1 + \alpha_j(z) \neq 0$ since $\|v\|_\infty$ is small. This function is also continuous up to the boundary since the potential is finitely supported. Due to (46), the Lemma A.3 of Appendix is then applicable. It yields

$$\int_{z_j+\delta}^{z_{j+1}-\delta} (-\ln|f_N(z)|)^- > -C \quad j = 0, \ldots, q-1$$

uniformly in N. By (45), we also have

$$\int_{z_j+\delta}^{z_{j+1}-\delta} \ln(\rho^N)'(z)\mathrm{d}z > C, \quad j = 0, \ldots, q-1.$$

Now, notice that $\mathrm{d}\rho^N \to \mathrm{d}\rho$ weakly as $N \to \infty$ (see, e.g., [2]) and the semicontinuity of the entropy argument from [3] (see Corollary 5.3) gives (44). Since $\delta > 0$ was arbitrary positive, that proves that the a.c. spectrum is supported on $[-2, 2]$. □

Remark. It would be nice to perform analogous analysis for the general case of Jacobi matrices. The determinantal representation of the Jost function for any q is also interesting to obtain. One can suggest more general conjecture: if $v_n \to 0$ and the Fourier transform $\hat{v}(x) \in L^2((-\pi, \pi), w(x))$, where $w(x)$ — some reasonable weight, then the a.c. spectrum contains $2K\pi^{-1}$, K being the support of this weight.

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Appendix

In this Appendix, we prove several auxiliary statements used in the main text.

Lemma A.1. *Assume that $f(z)$ is harmonic in $\Pi = a < \operatorname{Re} z < b$, $0 < \operatorname{Im} z < c$ and is continuous on the closure $\overline{\Pi}$. Then, two estimates*

$$|f(z)| < \gamma, \quad z \in \overline{\Pi}$$

and

$$f(z) = 0, \quad z \in [a, b]$$

imply

$$|f(z)| \le C\gamma \operatorname{Im} z, \qquad a + \delta < \operatorname{Re} z < b - \delta$$

and the constant C depends on the domain Π and δ only.

Proof. Map Π conformally onto the upper half-plane such that $[a, b]$ goes to, say, $[-1, 1]$. Let $g(\xi)$ be the transplantation of $f(z)$. Then, an application of the Poisson representation

$$g(x + \mathrm{i}y) = \frac{y}{\pi} \int_{\mathbb{R}} \frac{g(t)}{(t-x)^2 + y^2} \mathrm{d}t$$

provides the necessary estimate. □

Consider the domain $\Sigma = \{z = x + \mathrm{i}y : a < x < b, 0 < y < c\}$ in the complex plane. The next lemma is rather simple and is taken from [9]. Below, the constants C are all positive, f-independent, and can change from one formula to another; $f^{\pm}$ denotes the positive/negative parts of a real-valued function f (i.e., $f^+ = f$ if $f \ge 0$ and $f = 0$ otherwise, $f^- = f - f^+$).

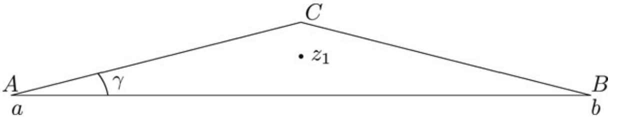

Fig. 1.

Lemma A.2. *Assume $f(z)$ is harmonic on Σ, continuous on $\overline{\Sigma}$, and*

$$\int_a^b f^+(x)\mathrm{d}x < C \tag{47}$$

$$f(z) < Cy^{-\alpha}, \quad z \in \Sigma, \alpha > 0 \tag{48}$$

and

$$f(z_1) > -C \tag{49}$$

for some $z_1 \in \Sigma$. Then,

$$\int_{a+\delta}^{b-\delta} f^-(x)\mathrm{d}x > -C(\delta).$$

Proof. Take isosceles triangle ABC as shown in Fig. 1 and write the mean value formula for f

$$\int_{AB} f(z)\omega(z)\mathrm{d}|z| + \int_{AC} f(z)\omega(z)\mathrm{d}|z| + \int_{BC} f(z)\omega(z)\mathrm{d}|z| = f(z_1)$$

where ω is the harmonic measure for triangle and z_1 assuming that z_1 is inside it. That can be rewritten as

$$\int_{AB} f^-(z)\omega(z)\mathrm{d}|z| + \int_{AC} f^-(z)\omega(z)\mathrm{d}|z| + \int_{BC} f^-(z)\omega(z)\mathrm{d}|z|$$
$$= f(z_1) - \int_{AB} f^+(z)\omega(z)\mathrm{d}|z| - \int_{AC} f^+(z)\omega(z)\mathrm{d}|z| - \int_{BC} f^+(z)\omega(z)\mathrm{d}|z|.$$

By taking the angle γ small enough we can always guarantee that ω will decay fast at A and B so that the integrals of f^+ over AC, BC are uniformly bounded due to (48). The integral of f^+ over AB is bounded due to (47). Therefore, since $f(z_1)$ is bounded from below by (49), we get the uniform estimate for the left-hand side. If z_1 is not inside the triangle, an analogous argument would work if one takes a slightly different shape (e.g., a curved triangle). □

In the main text, the condition that z_1 is inside a triangle can always be realized by assuming the norm of v to be small.

The slight modification of the argument provides the next lemma which is crucial for our considerations. It allows obtaining the entropy bounds from very rough estimates on modified Jost functions.

Lemma A.3. *Assume $f(z)$ is harmonic on Σ, continuous on $\overline{\Sigma}$, and*

$$\int_a^b f^+(x)\mathrm{d}x < C$$

$$f(z) > -Cy^{-\alpha}, \quad z \in \Sigma, \alpha > 0 \tag{50}$$

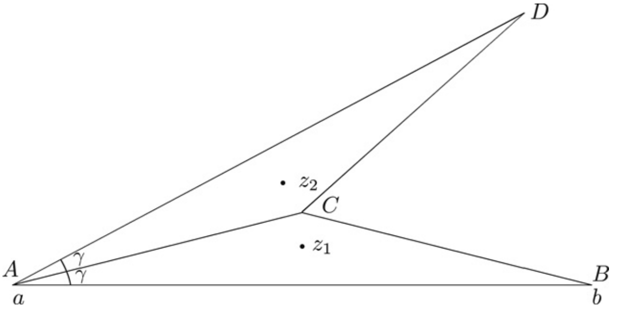

Fig. 2.

and

$$f(z) < C \tag{51}$$

for $\operatorname{Im} z > d(\alpha) = C(1+\alpha)^{-1} > 0$. *Then,*

$$\int_{a+\delta}^{b-\delta} f^-(x)\mathrm{d}x > -C(\delta).$$

Proof. Take two equal triangles ABC and ADC as shown in Fig. 2. As in the previous lemma,

$$\int_{AB} f^-(z)\omega(z)\mathrm{d}|z| + \int_{AC} f^-(z)\omega(z)\mathrm{d}|z| + \int_{CB} f^-(z)\omega(z)\mathrm{d}|z|$$
$$= f(z_1) - \int_{AB} f^+(z)\omega(z)\mathrm{d}|z| - \int_{AC} f^+(z)\omega(z)\mathrm{d}|z| - \int_{CB} f^+(z)\omega(z)\mathrm{d}|z|. \tag{52}$$

By the same arguments, we only need to provide a uniform estimate from above for the integrals over AC and BC. Consider the integral over AC, the other side can be treated in the same way. Consider ACD with z_2 being symmetric to z_1 with respect to the line AC. The harmonic measure for ACD and z_2 is equal to the same ω and we have

$$\int_{AD} f^+(z)\omega(z)\mathrm{d}|z| + \int_{AC} f^+(z)\omega(z)\mathrm{d}|z| + \int_{DC} f^+(z)\omega(z)\mathrm{d}|z|$$
$$= f(z_2) - \int_{AD} f^-(z)\omega(z)\mathrm{d}|z| - \int_{AC} f^-(z)\omega(z)\mathrm{d}|z| - \int_{DC} f^-(z)\omega(z)\mathrm{d}|z|.$$

The estimate (51) gives a bound for $f(z_2)$ from above and (50) controls the other terms on the right-hand side if the angle γ is small enough. Thus,

$$\int_{AC} f^+(z)\omega(z)\mathrm{d}|z| < C$$

and that finishes the proof due to (52). Choosing $d(\alpha) = C(1+\alpha)^{-1}$ with small C guarantees that (51) is applicable to z_2. □

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