Bimatroids and Gauss decomposition

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Abstract

In this paper we extend the notion of Gauss decomposition to a bimatroid. This is used to give an equivalent definition of a bimatroid. We show that representability of a bimatroid can be defined easily in terms of Gauss decomposition.

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1. Introduction

Let \( g \) be an \((m \times n)\)-matrix over a field \( k \), not necessarily square. There exists a (non-singular) lower-triangular square matrix \( b_L \) and an upper-triangular matrix \( b_R \) such that \( g = b_L u b_R \), where \( u \) is a uniquely determined \((0, 1)\)-matrix such that any row or column of \( u \) contains at most one 1. We say that the product \( b_L u b_R = g \) is a Gauss decomposition of \( g \). The set \( B_L^{-} u B_R \) of all matrices of this form with some fixed \( u \) is called a Gauss cell corresponding to \( u \); here, of course, \( B_L^{-} \) is the set of lower-triangular \((m \times m)\)- and \( B_R \) is the set of upper-triangular \((n \times n)\)-matrices over the field \( k \).

In [3], Kung introduced the notion of a bimatroid, which abstracts the nonsingularity properties of the minors of a matrix (at the same time, bimatroids were also defined by Schrijver [2], who called them “linking systems”). We consider an analogue of the Gauss decomposition for a bimatroid. Given a bimatroid \( X \), we define a pseudobijection \( u = u(X) \); this means that \( u(X) \) is a bijection between certain sets of rows and columns of \( X \). If \( X \) is represented by a matrix \( g \), then the middle term \( u \) in any Gauss decomposition of \( g \) is the matrix corresponding to \( u(X) \).

We extend the well-known Bruhat order on the set of permutations to the set of pseudobijections. Namely, we say that \( u \preceq v \) in this order if for any \( n_L \) and \( n_R \), the submatrix...
obtained by taking the first $n_L$ rows and the first $n_R$ columns of $u$ contains no less nonzero elements than the same submatrix of $v$. This definition coincides with a well-known description of the Bruhat order on the symmetric group; see, for example, [5, Proposition 7.3] (note that the order considered in [5] is dual to the Bruhat order).

Let $w_L$ be a permutation of rows of $X$ and $w_R$ a permutation of columns of $X$. Then we denote the pseudobijecton $u(w_L X w_R)$ by $f_X(w_L, w_R)$. We obtain the function $f_X : W_L \times W_R \rightarrow \tilde{W}_{LR}$, where $W_L$ is the group of row permutations, $W_R$ the group of column permutations, and $\tilde{W}_{LR}$ the set of pseudobijectons. We show that this function satisfies the following conditions:

$$f_X(w'_L w_L, w_R) \leq w'_L f_X(w_L, w_R),$$

and

$$f_X(w_L, w_R w'_R) \leq f_X(w_L, w_R) w'_R.$$

The function $f$, satisfying these conditions, is called a formal Gauss stratum.

In this paper we prove that the correspondence $X \mapsto f_X$ is a bijection. This means that the notion of the formal Gauss stratum is equivalent to the notion of the bimatroid and can be used for an alternative definition of a bimatroid. It seems that this definition can be extended to the case of arbitrary Coxeter groups instead of symmetric groups.

Notice further that a definition of a bimatroid as a formal Gauss stratum fits naturally into the theory of Coxeter matroids and matroid maps as developed by Borovik et al. [4]. Indeed, let $r$ be a natural number and $\phi$ a map from the symmetric group $W_L = \text{Sym}_m$ to $W_L / W^{(r)}$, where $W^{(r)}$ is the $r$th maximal parabolic subgroup of $W_L$. The set $W_L / W^{(r)}$ can be naturally identified with the set of all $r$-subsets in $L = [m]$. It was proved in [4] that the image of $\phi$ is a matroid on $L$ of rank $r$ if and only if it satisfies the inequality

$$w^{-1} \phi(w) \geq w^{-1} \phi(w'')$$

for any $w, w' \in W$. Here $\geq$ denotes the Bruhat order on $W_L / W^{(r)}$, or, equivalently, the Gale order on the set of all $r$-subsets in $L$ (see below). Let now the function $\psi$ be given by $\psi(w) = w \phi(w^{-1}) w_0$, where $w_0$ is the longest element in $W_L$. Since the multiplication by $w_0$ reverses the Bruhat order, one can easily check that the function $\psi$ satisfies the inequality

$$\psi(ww') \leq w \psi(w')$$

for all $w, w'' \in W$, which makes the definition of a bimatroid in terms of formal Gauss strata an analogue of the definition of matroids in terms of matroid maps.

However, the reader is not assumed to be acquainted with the theory of Coxeter matroids.

Now we return back to matrices. Suppose now that $f$ is a function from $W_L \times W_R$ to $\tilde{W}_{LR}$. In the space of all $(m \times n)$-matrices over a field $k$, consider the set

$$C_f = \bigcap_{w_L, w_R} w_L^{-1} B_L^{-1} f(w_L, w_R) B_R w_R^{-1}.$$

We call $C_f$ the Gauss stratum corresponding to $f$. We prove that if $C_f$ is nonempty, then $f$ is a formal Gauss stratum. Moreover, we show that $C_f$ is the set of all matrices $g$ such that the bimatroid represented by $g$ coincides with the bimatroid corresponding to $f$.

We assume that the reader has some basic knowledge of matroid theory. The first chapter of [1] is a good introduction to this theory.
2. Bimatroids

Let $L$ and $R$ be finite sets. Denote by $W_L$ (resp. $W_R$) the group of permutations of $L$ (resp. of $R$).

**Definition 1.** A subset $X \subseteq 2^L \times 2^R$ is called a bimatroid between $L$ and $R$ if it satisfies the following conditions:

1. If $(U, V) \in X$, then $\#U = \#V$, where $\#U$ is the number of elements of $U$.
2. $(\emptyset, \emptyset) \in X$.
3. If $(U_1, V_1) \in X$, $(U_2, V_2) \in X$, and $l \in U_2 \setminus U_1$, then at least one of the following holds:
   - (a) there exists an element $l' \in U_1 \setminus U_2$ such that $(l \cup U_1 \setminus l', V_1) \in X$ and $(l' \cup U_2 \setminus l, V_2) \in X$,
   - (b) there exists an element $r \in V_2 \setminus V_1$ such that $(U_1 \cup l, V_1 \cup r) \in X$ and $(U_2 \setminus l, V_2 \setminus r) \in X$.
4. If $(U_1, V_1) \in X$, $(U_2, V_2) \in X$, and $r \in V_2 \setminus V_1$, then at least one of the following holds:
   - (a) there exists an element $r' \in V_1 \setminus V_2$ such that $(U_1, r \cup V_1 \setminus r') \in X$ and $(U_2, r' \cup V_2 \setminus r) \in X$,
   - (b) there exists an element $l \in U_2 \setminus U_1$ such that $(U_1 \cup l, V_1 \cup r) \in X$ and $(U_2 \setminus l, V_2 \setminus r) \in X$.

**Proposition 2.1.** Suppose $X$ is a subset of $2^L \times 2^R$. Let $Z$ be the disjoint union $L \sqcup R$. Denote by $I$ the set

\[ \{U \cup (R \setminus V) \mid (U, V) \in X\}. \]

$X$ is a bimatroid if and only if $I$ is the set of bases of a matroid on $Z$.

**Proof.** The proof is found in [3]. □

Suppose $A$ and $B$ are arbitrary sets and $C \subseteq A \times B$; the set

\[ \{(b, a) \mid b \in B, a \in A, \text{ and } (a, b) \in C\} \subseteq B \times A \]

is called the transpose of $C$ and denoted by $^tC$.

It is readily seen that if $X$ is a bimatroid between $L$ and $R$, then the transpose of $X$, that is the set \{ $^tC \mid C \in X$ \} is also a bimatroid between $B$ and $A$.

Let $X$ be a bimatroid between $L$ and $R$, $w_L$ an element of $W_L$, and $w_R$ an element of $W_R$. By definition, put

\[ w_LXw_R = \{(U, V) \mid U \subseteq L, V \subseteq R, \text{ and } (w_L^{-1}U, w_RV) \in X\}. \]

**Lemma 2.2.** Let $X$ be a bimatroid, $w_L \in W_L$, and $w_R \in W_R$; then

1. $w_LXw_R$ is a bimatroid.
2. For any elements $w'_L \in W_L$ and $w'_R \in W_R$

   \[ (w'_Lw_L)X(w_RW'_R) = w'_L(w_LXw_R)w'_R. \]
3. $^t(w_LXw_R) = w_R^{-1}Xw_L^{-1}$. 
Proof. The proof is obvious. □

Lemma 2.3. Suppose $V \subseteq R$. Then the set

$$\{ U \subseteq L \mid \exists V' \subseteq V : (U, V') \in X \}$$

is the system of independent subsets of a matroid $X|V$, on $L$. In the same way, for any $U \subseteq L$, the set

$$\{ V \subseteq R \mid \exists U' \subseteq U : (U', V) \in X \}$$

is the system of independent subsets of a matroid $U|X$ on $R$.

Note that $U|X = ^t X|U$.

Proof. See [3]. □

Lemma 2.4. Suppose $(U, V) \in X$ and $V \subseteq V_0 \subseteq R$. If $U$ is a base of $X|V_0$, then $V$ is maximal independent subset of $V_0$ in $L|X$.

Proof. Let $V'$ be an independent set in $L|X$ such that $V \subseteq V' \subseteq V_0$. By Lemma 2.3, there exists a subset $U' \subseteq U$ such that $(U', V') \in X$. Therefore $U'$ is independent in $X|V_0$. Since $U$ is a basis of this matroid, we have

$$#V' = #U' \leq #U = #V.$$  

This implies that $V' = V$. □

Corollary 2.5. Suppose $V_0$ is a subset of $R$; then, the rank of the matroid $X|V_0$ equals the rank of $V_0$ in the matroid $L|X$.

3. Pseudobijections and Bruhat order

Assume that $L$ and $R$ are totally ordered. For any $l \in L$ denote by $L[l]$ the set

$$\{ l' \in L \mid l' \leq l \} \subseteq L.$$  

Suppose $U_1, U_2 \subseteq L$. If $(U_1 \cap L[l]) \supseteq (U_2 \cap L[l])$ for any $l \in L$, then we say that $U_1 \leq U_2$ in the Gale order. For $U_1 = U_2$ this order was introduced by Gale in [6]. In the following the word “smallest” is used as shorthand for “smallest with respect to the Gale order”; however, the word “maximal” means “maximal with respect to the inclusion”.

Lemma 3.1. If $U_1, U_2 \subseteq L$ and $U_1 \leq U_2$, then $#U_1 \geq #U_2$.

Proof. Let $l$ be the greatest element of $L$. Then $L[l] = L$ and

$$#U_1 = #(U_1 \cap L[l]) \geq #(U_2 \cap L[l]) = #U_2.$$  

□

Lemma 3.2. Suppose $U$ is a subset of $L$ and $l$ is an element of $L$ such that $U \subseteq L[l]$; then $L[l] \subseteq U$. If $#U = #L[l]$, then $U = L[l]$.

Proof. By definition, $#L[l] = #(L[l] \cap L[l]) \leq #(U \cap L[l])$. Since $U \cap L[l] \subseteq L[l]$, we obtain $L[l] = U \cap L[l]$. This completes the proof. □
Lemma 3.3. Suppose $U$ and $U'$ are subsets of $L$ and $l$ is an element of $L$ such that $U \subseteq U' \subseteq L[l]$. If $\#U \leq \#U'$, then $U \subseteq L[l]$.

Proof. Since $U \subseteq U'$, we have

$$\#(U \cap L[l]) \geq \#(U' \cap L[l]) = \#U' \geq \#U;$$

this means that $U \cap L[l] = U$ and $U \subseteq L[l]$. □

Lemma 3.4. If $U_1 \subseteq U_2 \subseteq L$, then $U_1 \geq U_2$.

Proof. Let $l$ be an element of $L$. Obviously, $U_1 \cap L[l] \subseteq U_2 \cap L[l]$. It follows that $\#(U_1 \cap L[l]) \leq \#(U_2 \cap L[l])$. □

It is well known that any matroid on $L$ has the smallest basis (see [6]).

Definition 2. A subset $u \subseteq L \times R$ is a pseudobijection between $L$ and $R$ if for any element $l \in L$ (resp. $r \in R$) there exists at most one $r \in R$ (resp. $l \in L$) such that $(l, r) \in u$. We denote the set of all pseudobijections between $L$ and $R$ by $\hat{W}_{LR}$.

In other words, $u$ is a graph of some bijection between subsets of $L$ and $R$. Let $V$ be a subset of $R$; denote by $uV$ the set

$$\{l \in L \mid \exists r \in V : (l, r) \in u\} \subseteq L.$$

Obviously, if $u$ is a pseudobijection between $L$ and $R$, then $^tu$ is a pseudobijection between $R$ and $L$.

Suppose $u$ is a pseudobijection between $L$ and $R$, $w_L$ an element of $W_L$, and $w_R$ an element of $W_R$. By definition, put

$$w_Luw_R = \{(l, r) \in L \times R \mid (w_L^{-1}l, w_Rr) \in u\}.$$

It is clear that $w_Luw_R$ is a pseudobijection.

Now, following [4, Theorem 5.17.3], we introduce an order on the set of pseudobijections. Suppose that $u$ and $u'$ are elements of $\hat{W}_{LR}$. We say that $u \leq u'$ in the Bruhat order if for any element $r$ of $R \ uR[r] \leq u'R[r]$.

4. Formal Gauss strata

Consider a bimatroid $X$ between $L$ and $R$.

Lemma 4.1. Suppose $r$ is an element of $R$, $U$ is a basis of $X \setminus R[r]$, and $V$ is a subset of $R[r]$ such that $(U, V) \in X$. Let $V_r$ be the closure of $R[r] \setminus r$ in the matroid $L \setminus X$. If $r \not\in V_r$, then $r \in V$.

Proof. Assume the converse. Then $V \subseteq R[r] \setminus r$ and the closure of $V$ is contained in $V_r$. Denote by $V'$ the union $V \cup r$; then $V'$ is independent in $L \setminus X$ and $V \subseteq V' \subseteq R[r]$. This contradicts Lemma 2.4. The lemma is proved. □

Proposition 4.2. Let $r$ be an element of $R$, $U$ the smallest basis of $X \setminus R[r]$, and $U'$ the smallest basis of $X \setminus (R[r] \setminus r)$. Suppose $r \not\in V_r$, where $V_r$ is the closure of $R[r] \setminus r$ in the matroid $L \setminus X$. Then there exists a unique element $l \in L$ such that $U = U' \cup l$. 

Proof. Since \( U \) (resp. \( U' \)) is independent in \( X|R[r] \) (resp. \( X|(R[r]\backslash r) \)), we see that there exists a subset \( V \subseteq R[r] \) (resp. \( V' \subseteq R[r]\backslash r \)) such that \( (U, V) \in X \) (resp. \( (U', V') \in X \)). Using Lemma 4.1, we get \( r \in V \backslash V' \). From Definition 1 it follows that at least one of the following holds:

Case 1. There exists an element \( r' \in V \backslash V' \) such that \( (U, r' \cup V \setminus r) \in X \). This contradicts Lemma 4.1.

Case 2. There exists an element \( l \in U \setminus U' \) such that \( (U \setminus l, V \setminus r) \in X \) and \( (U' \cup l, V' \cup r) \in X \). Since \( V \setminus r \subseteq R[r] \setminus r \), it follows that \( U \setminus l \) is an independent set in \( X|(R[r] \setminus r) \). Moreover, using Lemma 2.4, we get

\[
\#(U \setminus l) = \#(V \setminus r) = rk(R[r]) - 1 = rk(R[r] \setminus r) = \#V' = \#U',
\]

where \( rk \) means the rank function of the matroid \( L|X \). This means that \( U \setminus l \) is a basis of \( X|(R[r] \setminus r) \). In the same way, \( U' \cup l \) is a basis of \( X|R[r] \).

By assumption, \( U \subseteq U' \cup l \) and \( U' \subseteq U \setminus l \). Let \( l' \) be an element of \( L \); by definition, put

\[
\delta(l, l') = \begin{cases} 1 & \text{if } l \leq l' \\ 0 & \text{otherwise}. \end{cases}
\]

We have

\[
\#((U' \cup l) \cap L[l']) \leq \#(U \cap L[l']) = \#((U \setminus l) \cap L[l']) + \delta(l, l') \\
\leq \#(U' \cap L[l']) + \delta(l, l') = \#((U' \cup l) \cap L[l']).
\]

Therefore \( \#((U' \cup l) \cap L[l']) = \#(U \cap L[l']) \) for any \( l' \in L \). This means that \( U = U' \cup l \). This completes the proof. \( \Box \)

Corollary 4.3. There exists a unique pseudobijection \( u \) between \( L \) and \( R \), such that for any \( r \in R \) the set \( u R[r] \) is the smallest basis in the matroid \( X|R[r] \).

Proof. Suppose \( u \) satisfies the assumptions of the corollary. Let \( r \) be an element of \( R \). If \( r \) is the smallest element of \( R \), then \( R[r]\backslash r = \emptyset \) and \( u(R[r] \setminus r) = \emptyset \) is the smallest (and unique) basis of \( X|\emptyset \); otherwise denote by \( r' \) the maximal element of \( R \) that is less than \( r \). Then \( R[r]\backslash r = R[r'] \) and \( u(R[r] \setminus r) = u(R[r']) \) is the smallest basis of \( X|R[r'] \) = \( X|(R[r]\backslash r) \). Let us consider two cases.

Case 1. \( r \in V_r \). In this case, the closure of \( R[r]\backslash r \) in \( L|X \) coincides with the closure of \( R[r] \). In [3], Kung proved that the matroid \( X|V \) depends only on the closure of \( V \) in \( L|X \). Therefore the smallest basis of \( X|(R[r]\backslash r) \) is equal to the smallest basis of \( X|R[r] \). By the above, we have

\[
u(R[r]\backslash r) = u(R[r]).
\]

Since \( u \) is a pseudobijection, we obtain

\[
u r = (u R[r]) \setminus u(R[r]\setminus r) = \emptyset,
\]

and \( u \cap (L \times r) = \emptyset \).

Case 2. \( r \notin V_r \). From Proposition 4.2 it follows that \( u(R[r]) = u(R[r]\setminus r) \cup l \), where \( l \) is an element of \( L \). Arguing as above, we see that \( u \cap (L \times r) = (l, r) \).
We have proved that \( u \) is uniquely determined by \( X \). It follows easily that the set
\[
u = \{(l, r) \in L \times R \mid l = U_r \setminus U'_{r}\},
\]
where \( U_r \) (resp. \( U'_r \)) is the smallest basis of \( X|R[r] \) (resp. \( X|(R[r]\setminus r) \)), is the required pseudobijection. \( \square \)

We say that \( u \) is the pseudobijection associated with \( X \) and write \( u = u(X) \).

**Lemma 4.4.** Let \( u \) be a pseudobijection between \( L \) and \( R \), \( l \) an element of \( L \), and \( r \) an element of \( R \). Then
\[
\#(uR[r] \cap L[l]) = \#(l'uL[l] \cap R[r]).
\]

**Proof.** It is easily shown that both numbers are equal to \( \#(u \cap (L[l] \times R[r])). \)

**Remark.** We see that \( u \leq u' \) if and only if for any \( l \in L \) and \( r \in R \)
\[
\#(u \cap (L[l] \times R[r])) \geq \#(u' \cap (L[l] \times R[r]))
\]
where \( u \) and \( u' \) are pseudobijections. This description of the Bruhat order was stated in the introduction in a less formal way.

**Corollary 4.5.** Let \( u \) and \( u' \) be pseudobijections between \( L \) and \( R \). If \( u \leq u' \), then \( l'u \leq l'u' \).

**Proof.** Suppose \( u \leq u' \). Using Lemma 4.4 we get
\[
\#(l'uL[l] \cap R[r]) = \#(uR[r] \cap L[l]) \geq \#(u'R[r] \cap L[l]) = \#(l'u'L[l] \cap R[r]),
\]
where \( l \) is an element of \( L \) and \( r \) an element of \( R \). This means that for any \( l \in L \)
\[
l'uL[l] \leq l'u'L[l].
\]
It follows that \( l'u \leq l'u' \). \( \square \)

**Theorem 4.6.** \( u(X) = l'u(X) \).

**Proof.** Let \( l \) be an element of \( L \) and \( r \) an element of \( R \). Denote by \( U \) the set \( u(X)R[r] \cap L[l] \). Since \( u(X)R[r] \) is a basis of \( X|R[r] \), it follows that \( U \) is independent in \( X|R[r] \). By Lemma 2.3, there exists a subset \( V \subseteq R[r] \) such that \( (U, V) \in X \). Since \( U \subseteq L[l] \), we see that \( V \) is independent in \( L[l]|X \).

Let \( V' \) be a basis of \( L[l]|X = l'X|L[l] \) such that \( V \subseteq V' \). Since \( u(l'X)L[l] \) is the smallest basis of \( l'X|L[l] \), we see that \( \#(u(l'X)L[l] \cap R[r]) \geq \#(V' \cap R[r]) \). From Lemma 4.4 it follows that
\[
\#(l'u(X)L[l] \cap R[r]) = \#(u(X)R[r] \cap L[l]) = \#U = \#V \leq \#(V' \cap R[r]) \leq \#(u(l'X)L[l] \cap R[r]).
\]
This means that \( l'u(X)L[l] \geq u(l'X)L[l] \) and
\[
l'u(X) \geq u(l'X). \tag{1}
\]
If we replace \( X \) by \( l'X \) in (1), we obtain \( l'u(l'X) \geq u(X) \). Combining this with Corollary 4.5, we get \( u(l'X) \geq l'u(X) \). Hence \( u(l'X) = l'u(X) \). \( \square \)
By definition, we put
\[ f_X(w_L, w_R) = u(w_L X w_R). \]

**Proposition 4.7.** Let \( w_L, w'_L \) be elements of \( W_L \) and \( w_R, w'_R \) elements of \( W_R \). Then
\[ f_X(w'_L w_L, w_R) \leq w'_L f_X(w_L, w_R), \] (2)
and
\[ f_X(w_L, w_R w'_R) \leq f_X(w_L, w_R) w'_R. \] (3)

**Proof.** First let \( r \) be an element of \( R \). By definition, the set
\[ f_X(w_L, w_R) R[r] = u(w_L X w_R) R[r] \]
is a basis of the matroid \( w_L X w_R | R[r] \). It is clear that \( w'_L f_X(w_L, w_R) R[r] \) is a basis of the matroid \( w'_L w_L X w_R | R[r] \). But the set
\[ f_X(w'_L w_L, w_R) R[r] = u(w'_L w_L X w_R) R[r] \]
is the smallest basis of this matroid; hence
\[ w'_L f_X(w_L, w_R) R[r] \geq f_X(w'_L w_L, w_R) R[r]. \]
This proves (2).

Secondly, denote the bimatroid \( t X \) by \( Y \). From Theorem 4.6 it follows that
\[ t f_X(w_L, w_R) = t u(w_L X w_R) = u(t(w_L X w_R)) \]
\[ = u(w^{-1}_R t X w^{-1}_L) = f_Y(w^{-1}_R, w^{-1}_L). \]
Further, substituting \( Y \) for \( X \) in (2), we obtain
\[ t f_X(w_L, w_R w'_R) = f_Y(w^{-1}_R w^{-1}_L, w^{-1}_L) \]
\[ \leq w^{-1}_R f_Y(w^{-1}_R, w^{-1}_L) = w^{-1}_R f_X(w_L, w_R) = t(f_X(w_L, w_R) w'_R). \]
Now the application of Corollary 4.5 proves (3). \( \square \)

**Definition 3.** Let \( f \) be a function from \( W_L \times W_R \) to \( \tilde{W}_{LR} \). Suppose that for any elements \( w_L, w'_L \) in \( W_L \) and \( w_R, w'_R \) in \( W_R \)
\[ f(w'_L w_L, w_R) \leq w'_L f(w_L, w_R), \]
and
\[ f(w_L, w_R w'_R) \leq f(w_L, w_R) w'_R. \]
Then \( f \) is called a **formal Gauss stratum** between \( L \) and \( R \).

From Proposition 4.7 it follows that for any bimatroid \( X \) the function \( f_X \) is a formal Gauss stratum.

### 5. Reconstruction of a bimatroid—preliminaries

Suppose \( u \) is a pseudobijection; then any element \((l, r) \in u\) is called a **point** of \( u \).
Definition 4. Suppose $f$ is a formal Gauss stratum between $L$ and $R$. A pair $(U, V) \in 2^L \times 2^R$ is called $f$-admissible if there exists an element $w_L \in W_L$ and an element $w_R \in W_R$ such that the following conditions hold:

1. $w_L U = L[l]$, where $l$ is an element of $L$;
2. $w_R^{-1} V = R[r]$ where $r$ is an element of $R$;
3. $L[l] \times R[r]$ contains exactly $\#U = \#V$ points of $f(w_L, w_R)$.

Lemma 5.1. Let $U$ be a subset of $L$ and $V$ a subset of $R$. $(U, V) \in X$ for a given bimatroid $X$ if and only if $(U, V)$ is an $f_X$-admissible pair.

Proof. First suppose that $(U, V) \in X$. Let $w_L$ (resp. $w_R$) be an element of $W_L$ (resp. $W_R$) such that $w_L U = L[l]$ (resp. $w_R^{-1} V = R[r]$). By construction, $(L[l], R[r]) \in w_L X w_R$. This means that $L[l]$ is independent in $w_L X w_R$. From Corollary 2.5 it follows that $L[l]$ is a basis of this matroid. This means that $f_X(w_L, w_R)$ is a pseudobijection, we see that all conditions of Definition 4 are satisfied.

Conversely, suppose that $L[l] \times R[r]$ contains exactly $\#L[l] = \#R[r]$ points of $f_X(w_L, w_R)$. Since $f_X(w_L, w_R)$ is a pseudobijection, we have $f_X(w_L, w_R) R[r] = L[l]$. This means that $L[l]$ is a basis of $w_L X w_R$. From Corollary 2.5 it follows that $(L[l], R[r]) \in w_L X w_R$; in other words, $(U, V) \in X$. This completes the proof. □

Suppose $f$ is a formal Gauss stratum; denote the set of all $f$-admissible pairs by $X[f]$.

Lemma 5.2. Suppose $f$ is a formal Gauss stratum between $L$ and $R$. Let $\tilde{f}$ be the map of $W_R \times W_L$ to $\tilde{W}_{RL}$ such that $\tilde{f}(w_R, w_L) = f(w_L^{-1}, w_R^{-1})$ for all $w_L \in W_L$ and $w_R \in W_R$. Then $\tilde{f}$ is a formal Gauss stratum between $R$ and $L$.

Proof. The proof is by direct calculation. □

Lemma 5.3. Suppose that the pair $(U, V)$ is an element of $X[f]$, $w_L$ an element of $W_L$, and $w_R$ an element of $W_R$. If $w_L U \subseteq L[l]$ and $w_R^{-1} V = R[r]$, where $l$ is an element of $L$ and $r$ an element of $R$, then $L[l] \times R[r]$ contains exactly $\#R[r]$ points of $f(w_L, w_R)$.

Proof. By definition, there exists an element $w'_L \in W_L$ and an element $w'_R \in W_R$ such that $w'_L U = L[l']$, $w'_R^{-1} V = R[r']$, and $L[l'] \times \#R[r'] = \#V = \#R[r]$ contains $\#U = \#V$ points of $f(w'_L, w'_R)$. Since $f$ is a formal Gauss stratum, we get

$$f(w'_L, w'_R) R[r] \leq f(w'_L, w'_R) w_R^{-1} w_R R[r]$$

$$= f(w'_L, w'_R) w_R^{-1} V = f(w'_L, w'_R) R[r] = L[l'].$$

By Lemma 3.2, it follows that $f(w'_L, w'_R) R[r] = L[l']$ and

$$f(w_L, w_R) R[r] \leq w_L w_L^{-1} f(w'_L, w'_R) R[r] = w_L w_L^{-1} L[l'] = w_L U \subseteq L[l].$$

Using Lemma 3.3, we get

$$f(w_L, w_R) R[r] \subseteq L[l].$$

Moreover, by Lemma 3.1, it follows that

$$\#f(w_L, w_R) R[r] \geq \#w_L U = \#V = \#R[r],$$
This means that for any \( \tilde{\tau} \in R[r] \) there exists a (unique) element \( \tilde{\iota} \in L[l] \) such that \( (\tilde{\iota}, \tilde{\tau}) \in f(w_L, w_R) \). This completes the proof. \( \square \)

**Corollary 5.4.** Suppose that the pair \((U, V)\) is an element of \(X[f]\), \(w_L\) an element of \(W_L\), and \(w_R\) an element of \(W_R\). If \(w_L U = L[l] \) and \(w_R^{-1} V \subseteq R[r] \), where \(l\) is an element of \(L\) and \(r\) an element of \(R\), then \(L[l] \times R[r]\) contains \(\#L[l]\) points of \(f(w_L, w_R)\).

**Proof.** This follows easily from Lemmas 5.2 and 5.3. \( \square \)

**Lemma 5.5.** Suppose \(U\) is a subset of \(L\) and \(r\) an element of \(R\). If \(U \times R[r]\) contains \(\#U = \#R[r]\) points of \(f(w_L, w_R)\), then \((w_L^{-1} U, w_R R[r]) \in X[f]\).

**Proof.** Consider a permutation \(w'_L \in W_L\) such that \(w'_L U = L[l]\), where \(l\) is an element of \(L\). We have

\[
 f(w'_L w_L, w_R) R[r] \leq w'_L f(w_L, w_R) R[r] = w'_L U = L[l].
\]

Note also that

\[
\# f(w'_L w_L, w_R) R[r] \leq \# R[r] = \# U = \# L[l].
\]

By Lemma 3.3, it follows that \(f(w'_L w_L, w_R) R[r] \subseteq L[l]\). But using Lemma 3.1, we get \(\# f(w'_L w_L, w_R) R[r] \geq \# L[l]\). This means that

\[
f(w'_L w_L, w_R) R[r] = L[l].
\]

We see that for any element \(\tilde{\iota} \in L[l]\) there exists an element \(\tilde{\tau} \in R[r]\) such that \( (\tilde{\iota}, \tilde{\tau}) \in f(w'_L w_L, w_R)\). Therefore \(L[l] \times R[r]\) contains \(\# L[l] = \# U = \# R[r]\) points of \(f(w'_L w_L, w_R)\). It follows that \((w'_L w_L)^{-1} L[l], w_R R[r]) \in X[f]\). To conclude the proof, it remains to note that

\[
(w'_L w_L)^{-1} L[l] = w_L^{-1} w_R^{-1} L[l] = w_L^{-1} U. \quad \square
\]

**Corollary 5.6.** Suppose \(V\) is a subset of \(R\) and \(l\) an element of \(L\). If \(L[l] \times V\) contains \(\#V = \# L[l]\) points of \(f(w_L, w_R)\), then \((w_L^{-1} L[l], w_R V) \in X[f]\).

**Proof.** This follows easily from Lemmas 5.2 and 5.5. \( \square \)

**Lemma 5.7.** Suppose \(U\) is a subset of \(L\) and \(V\) a subset of \(R\) such that \((U, V) \in X[f]\). For any \(V' \subseteq V\) there exists a subset \(U' \subseteq U\) such that \((U', V') \in X[f]\).

**Proof.** Consider elements \(w_L \in W_L\) and \(w_R \in W_R\) such that \(w_L U = L[l]\), \(w_R^{-1} V = R[r]\), and \(w_R^{-1} V' = R[r']\), where \(r' \leq r\). By Lemma 5.3, it follows that \(L[l] \times R[r]\) contains exactly \(\# R[r] = \# V = \# U = \# L[l]\) points of \(f(w_L, w_R)\). This means that for any element \(\tilde{\tau} \in R[r]\) there exists an element \(\tilde{l} \in L[l]\) such that \((\tilde{l}, \tilde{\tau}) \in f(w_L, w_R)\).

Denote by \(\tilde{U}\) the set

\[
\{ \tilde{l} \in L[l] \mid \exists \tilde{\tau} \in R[r'] : (\tilde{l}, \tilde{\tau}) \in f(w_L, w_R) \}.
\]

By the above, \(\# \tilde{U} = \# R[r']\). It is clear that \(\tilde{U} \times R[r']\) contains \(\tilde{U} = \# R[r']\) points of \(f(w_L, w_R)\). From Lemma 5.5 it follows that \((w_L^{-1} \tilde{U}, w_R R[r']) \in X[f]\). To conclude the proof, it remains to note that \(w_R R[r'] = V'\) and \(w_L^{-1} \tilde{U} \subseteq w_L^{-1} L[l] = U\). \( \square \)

**Corollary 5.8.** Suppose \(U\) is a subset of \(L\) and \(V\) a subset of \(R\) such that \((U, V) \in X[f]\). For any \(U' \subseteq U\) there exists a subset \(V' \subseteq V\) such that \((U', V') \in X[f]\).
Proof. This follows easily from Lemmas 5.2 and 5.7. □

6. Reconstruction of a bimatroid

Proposition 6.1. If \((U_1, V_1) \in X[f], (U_2, V_2) \in X[f], \) and \(r \in V_2 \setminus V_1, \) then at least one of the following holds:

(1) there exists an element \(r' \in V_1 \setminus V_2 \) such that \((U_1, r \cup V_1 \setminus r') \in X[f], \) or

(2) there exists an element \(l \in U_2 \setminus U_1 \) such that \((U_1 \cup l, V_1 \cup r) \in X[f].\)

Proof. Denote by \(V_3 \) the set \((V_1 \cup r) \cap V_2.\) By Lemma 5.7, it follows that there exists a subset \(U_3 \subseteq U_2 \) such that \((U_3, V_3) \in X[f].\) Consider elements \(w_L \in W_L \) and \(w_R \in W_R \) such that \(w_L U_1 = L[l_1], w_L(U_1 \cup U_3) = L[l_2], w_R^{-1} V_3 = R[r_1],\) and \(w_R^{-1}(V_1 \cup r) = R[r_2],\) where \(l_1 \leq l_2\) and \(r_1 \leq r_2.\) Denote the pseudobijection \(f(w_L, w_R)\) by \(u.\)

Case 1. \(L[l_2] \times R[r_2]\) contains \#\(L[l_1] = #U_1 = #V_1 \cup r - 1 = #R[r_2] - 1\) points of \(u.\)

This means that there exists a unique element \(r'' \in R[r_2]\) such that \(L[l_1] \times r''\) does not contain a point of \(u.\) Let us consider two cases.

Case 1.a. \(L[l_2] \times r''\) contains a point \((l', r'')\) of \(u.\)

Clearly, \(l' \notin L[l_1].\) By Statement A, it follows that \((L[l_1] \cup l') \times R[r_2]\) contains \#\(L[l_1] + 1 = #L[l_1] + 1 = #R[r_2]\) points of \(u.\) Let \(l\) denote the element \(w_L^{-1}(l') \in w_L^{-1}(L[l_2] \setminus L[l_1]) = U_3 \setminus U_1 \subseteq U_2 \setminus U_1.\)

Using Lemma 5.5, we get
\((U_1 \cup l, V_1 \cup r) = (w_L^{-1}(L[l_1] \cup l'), w_R R[r_2]) \in X[f].\)

Case 1.b. \(L[l_2] \times r''\) does not contain a point of \(u.\)

From Lemma 5.3 it follows that \(L[l_2] \times R[r_1]\) contains \#\(R[r_1]\) points of \(u.\) This means that for any \(\tilde{r} \in R[r_1]\) \(L[l_2] \times \tilde{r}\) contains a point of \(u.\) It follows that \(r'' \notin R[r_1].\) Let \(r'\) denote the element \(w_R r'' \in w_R(R[r_2] \setminus R[r_1]) = (V_1 \cup r) \setminus V_3 = V_1 \setminus V_2.\)

By Statement A, it follows that \(L[l_1] \times (R[r_2] \setminus r'')\) contains \#\(L[l_1] = #R[r_2] - 1 = #R([r_2] \setminus r'')\) points of \(u.\) Using Corollary 5.6, we get
\((U_1, V_1 \setminus r' \cup r) = (w_L^{-1} L[l_1], w_R R[r_2] \setminus r'') \in X[f].\) □

Proposition 6.2. If \((U_1, V_1) \in X[f], (U_2, V_2) \in X[f], \) and \(l \in U_2 \setminus U_1, \) then at least one of the following holds:

(1) there exists an element \(l' \in U_1 \setminus U_2 \) such that \((l' \cup U_2 \setminus l, V_2) \in X[f],\) or

(2) there exists an element \(r \in V_2 \setminus V_1 \) such that \((U_2 \setminus l, V_2 \setminus r) \in X[f].\)

Proof. Denote by \(V_3 \) the set \(V_1 \cap V_2.\) By Lemma 5.7, it follows that there exists a subset \(U_3 \subseteq U_1 \) such that \((U_3, V_3) \in X[f].\) Consider elements \(w_L \in W_L \) and \(w_R \in W_R \) such that \(w_L(U_2 \setminus l) = L[l_1], w_L(U_2 \cup U_3 \setminus l) = L[l_2], w_R^{-1} V_3 = R[r_1],\) and \(w_R^{-1} V_2 = R[r_2],\) where \(l_1 \leq l_2\) and \(r_1 \leq r_2.\) Denote the pseudobijection \(f(w_L, w_R)\) by \(u.\)
By Corollary 5.8, it follows that there exists a subset $V' \subseteq V_2$ such that

$$(U_2 \setminus I, V') \in X[f].$$

We have

$$\#(V_2 \setminus V') = V_2 - \#V' = U_2 - \#(U_2 \setminus I) = 1.$$

Now note that $w_R^{-1}V' \subseteq w_R^{-1}V_2 = R[r_2]$. From Corollary 5.4 it follows that

**Statement B.** $L[I_1] \times R[r_2]$ contains $\#L[I_1] = \#(U_2 \setminus I) = V_2 - 1 = \#R[r_2] - 1$ points of $u$.

This means that there exists a unique element $r' \in R[r_2]$ such that $L[I_1] \times r'$ does not contain a point of $u$. Let us consider two cases.

**Case 1.** $r' \not\in R[r_1]$. Let $r$ denote the element

$$w_R r' \in w_R(R[r_2] \setminus R[r_1]) = V_2 \setminus V_3 = V_2 \setminus V_1.$$

By Statement B, it follows that $L[I_1] \times (R[r_2] \setminus r')$ contains $\#L[I_1]$ points of $u$. Using Corollary 5.6, we have

$$(U_2 \setminus I, V_2 \setminus r) = (w_L^{-1}L[I_1], w_R(R[r_2] \setminus r')) \in X[f].$$

**Case 2.** $r' \in R[r_1]$. From Lemma 5.3 it follows that $L[I_2] \times R[r_1]$ contains $\#R[r_1]$ points of $u$. Therefore there exists an element $l'' \in L[I_2]$ such that $(l'', r') \in u$. Clearly, $l'' \not\in L[I_1]$. Let $l'$ denote the element

$$w_L^{-1}l'' \in w_L^{-1}(L[I_2] \setminus L[I_1]) = (U_2 \cup U_3 \setminus I)(U_2 \setminus I) = U_3 \setminus U_2 \subseteq U_1 \setminus U_2.$$

By Statement B, it follows that $(L[I_1] \cup l'') \times R[r_2]$ contains

$$\#L[I_1] + 1 = \#(L[I_1] \cup l'') = R[r_2]$$

points of $u$. Using Lemma 5.5, we get

$$(l \cup U_2 \setminus I, V_2) = (w_2^{-1}(L[I_1] \cup l''), w_R R[r_2]) \in X[f].$$

**Theorem 6.3.** If $f$ is a formal Gauss stratum between $L$ and $R$, then $X[f]$ is a bimatroid.

**Proof.** Let $Z$ be the disjoint union $L \sqcup R$. Denote by $I$ the set

$$\{ U \cup (R \setminus V) \mid (U, V) \in X[f] \}. $$

Suppose $T_1$ and $T_2$ are elements of $I$ and $z$ is an element of $T_2 \setminus T_1$. Let us consider two cases.

**Case 1.** $z \in L$. This means that $z \in L \cap (T_2 \setminus T_1) = U_2 \setminus U_1$. By Proposition 6.2, it follows that one of the following holds:

1. There exists an element $l \in U_1 \setminus U_2 = L \cap (T_1 \setminus T_2)$ such that $(U_2 \cup l \setminus z, V_2)$ is an $f$-admissible pair. In other words,

$$l \cup T_2 \setminus z = (l \cup U_2 \setminus z) \cup (R \setminus V_2) \in I.$$

2. There exists an element $r \in V_2 \setminus V_1 = R \cap (T_1 \setminus T_2)$ such that $(U_2 \setminus z, V_2 \setminus r)$ is an $f$-admissible pair. This implies that

$$r \cup T_2 \setminus z = (U_2 \setminus z) \cup (R \setminus (V_2 \setminus r)) \in I.$$
Case 2. $z \in R$. This means that $z \in R \cap (T_2 \setminus T_1) = V_1 \setminus V_2$. By Proposition 6.1, it follows that one of the following holds:

1. There exists an element $r \in V_2 \setminus V_1 = R \cap (T_1 \setminus T_2)$ such that $(U_2, z \cup V_2 \setminus r)$ is an $f$-admissible pair. In other words,
   $$r \cup T_2 \setminus z = U_2 \cup (R \setminus (z \cup V_2 \setminus r)) \in I.$$

2. There exists an element $l \in U_1 \setminus U_2 = L \cap (T_1 \setminus T_2)$ such that $(U_2 \cup l, V_2 \cup z)$ is an $f$-admissible pair. This implies that
   $$l \cup T_2 \setminus z = (U_2 \cup l) \cup (R \setminus (V_2 \cup z)) \in I.$$

We see that for any element $z \in T_2 \setminus T_1$ there exists an element $z' \in T_1 \setminus T_2$ such that $z' \cup T_2 \setminus z \in I$. Equivalently, $I$ is the set of bases of a matroid on $Z$. By Proposition 2.1, it follows that $X[f]$ is a bimatroid.

**Theorem 6.4.** For any formal Gauss stratum $f$ between $L$ and $R$ there exists a unique bimatroid $X$ between $L$ and $R$ such that $f = f_X$.

**Proof.** First let us prove that $f_X[f] = f$. Let $w_L$ be an element of $W_L$, $w_R$ an element of $W_R$, and $r$ an element of $R$.

Denote by $U$ the set $f_X[f](w_L, w_R)R[R]$. By Theorem 6.3 and Corollary 4.3, $U$ is a basis of the matroid $w_LXw_R[R][R]$. It follows that there exists a subset $V \subseteq R[R]$ such that $(U, V) \in w_LX[f]w_R$. In other words, $(w_L^{-1}U, w_RV) \in X[f]$. Let $w'_R$ be an element of $W_L$ and $w'_R$ an element of $W_R$ such that $w'_LU = L[I]$ and $w'_RV = R[R']$. By Lemma 5.3, it follows that $L[I] \times R[R']$ contains $R[R'] = \#V = \#U = \#L[I]$ points of $f(w'_Lw_L, w_rw'_R)$. Therefore $f(w'_Lw_L, w_rw'_R)R[R'] = L[I]$. Since $f$ is a formal Gauss stratum, we obtain

$$f(w'_Lw_L, w_R)R[R] \leq f(w'_Lw_L, w_RW_R)w'_R^{-1}R[R] \supseteq f(w'_Lw_L, w_RW_R)w'_R^{-1}V = f(w'_Lw_L, w_RW_R)R[R'] = L[I].$$

From Lemma 3.4 it follows that $f(w'_Lw_L, w_R)R[R] \subseteq L[I]$. Using Lemma 3.2, we get $L[I] \subseteq f(w'_Lw_L, w_R)R[R]$. Hence

$$f(w_L, w_R)R[R] \leq w_L^{-1}f(w'_Lw_L, w_R)R[R] \supseteq w_R^{-1}L[I] = U = f_X[f](w_L, w_R)R[R].$$

By Lemma 3.4, it follows that $f(w_L, w_R)R[R] \leq f_X[f](w_L, w_R)R[R]$ and

$$f(w_L, w_R) \leq f_X[f](w_L, w_R). \quad (4)$$

Similarly, denote by $\tilde{U}$ the set $f(w_L, w_R)R[R]$. Let $\tilde{V}$ be the set

$$\{\tilde{r} \in R[R] \mid \tilde{w}_R^{-1}\tilde{r} \in \tilde{U} : \tilde{r}, \tilde{r} \in f(w_L, w_R)\}.$$ 

Clearly, $f(w_L, w_R)\tilde{V} = \tilde{U}$. Let $w_R$ be an element of $W_R$ such that $w_R^{-1}\tilde{V} = R[\tilde{r}]$. We have

$$f(w_L, w_Rw_R)R[\tilde{r}] \leq f(w_L, w_R)w_RR[\tilde{r}] = f(w_L, w_R)\tilde{V} = \tilde{U}.$$

Denote by $\tilde{U}$ the set $f(w_L, w_RW_R)R[\tilde{r}]$. From Lemma 3.1 it follows that

$$\#\tilde{U} \geq \#\tilde{U} = \#\tilde{V} = \#R[\tilde{r}].$$

Therefore $\tilde{U} \times R[\tilde{r}]$ contains $\#\tilde{U} = \#R[\tilde{r}]$ points of $f(w_L, w_RW_R)$. By Lemma 5.5, it follows that $(w_L^{-1}\tilde{U}, w_RW_R) = (w_L^{-1}\tilde{U}, w_RW_R)R[\tilde{r}] \in X[f]$. 

In other words, \( (\tilde{U}, \tilde{V}) \in w_LX[f]w_R \). This means that \( \tilde{U} \) is an independent set in \( w_LX[f]w_R|R[r] \). Let \( \hat{U} \) be a basis of \( w_LX[f]w_R|R[r] \) such that \( \hat{U} \subseteq \tilde{U} \). By Lemma 3.4, it follows that \( \hat{U} \geq \tilde{U} \). Since \( f_X(f(w_L, w_R)R[r] \) is the smallest basis of \( w_LXw_R|R[r] \), we have

\[
f(w_L, w_R)R[r] = \hat{U} \geq \tilde{U} \geq \tilde{U} \geq f_X(f(w_L, w_R)R[r].
\]

This means that \( f(w_L, w_R) \geq f_X(f(w_L, w_R) \). Combining this with (4), we get \( f(w_L, w_R) = f_X(f(w_L, w_R) \). Therefore \( f_X(f) = f \).

Finally, suppose \( X \) is a bimatroid such that \( f = f_X \). From Lemma 5.1 it follows that \( X = X[f] \). This completes the proof. \( \square \)

7. Gauss decomposition

Let \( k \) be a field. Suppose \( u \) is a pseudobijection; by definition, put

\[
ulr = \begin{cases} 1 & \text{if } (l, r) \in u \\ 0 & \text{otherwise}, \end{cases}
\]

where \( l \) and \( r \) are elements of \( L \) and \( R \) respectively. Denote the matrix \( (ulr) \) also by \( u \).

In the following, when referring to upper- or lower-triangular matrices, we assume they are non-singular. Denote the set of all lower-triangular \( L \times L \)-matrices by \( B_L^- \). It is clear that \( B_L^- \) is a group with respect to multiplication. Similarly, denote by \( B_R \) the group of upper-triangular \( R \times R \)-matrices. The following theorem belongs to the “folklore” of linear algebra; it can be proved by more-or-less standard argument.

**Theorem 7.1.** For any \( L \times R \)-matrix \( g \) there exist elements \( b_L \in B_L^-, b_R \in B_R \) and \( u \in \tilde{W}_{LR} \) such that \( g = b_Lub_R \).

We say that the product \( g = b_Lub_R \) is a Gauss decomposition of \( g \). Let \( u \) be a pseudobijection; the Gauss cell corresponding to \( u \) is the set

\[
B^-_LuB_R = \{ b_Lub_R | b_L \in B_L^-, b_R \in B_R \}.
\]

**Definition 5.** Let \( f \) be a map from \( W_L \times W_R \) to \( \tilde{W}_{LR} \). We say that the set

\[
C = \bigcap_{w_L \in W_L, w_R \in W_R} w_L^{-1}B^-_Lf(w_L, w_R)B_Rw_R^{-1}
\]

is the Gauss stratum corresponding to \( f \) and write \( C = C_f \).

In other words, \( C_f \) is the set of \( L \times R \)-matrices \( g \) such that

\[
w_Lg_wR = B^-_Lf(w_L, w_R)B_R
\]

for any \( w_L \in W_L \) and \( w_R \in W_R \). Suppose \( g \) is an \( L \times R \)-matrix. By definition, we put

\[
X(g) = \{ (U, V) \in 2^L \times 2^R | \#U = \#V \text{ and } \Delta_{UV}(g) \neq 0 \},
\]

where \( \Delta_{UV}(g) \) is the minor of \( g \) with columns indexed by \( U \) and rows indexed by \( V \). It was shown in [3], that \( X(g) \) is a bimatroid. A bimatroid \( X \) is called representable if there exists an \( L \times R \)-matrix \( g \) such that \( X = X(g) \).

Let \( g \) be an \( L \times R \)-matrix and \( r \) an element of \( R \). By \( g^{(r)} \) denote the \( L \times R|r| \)-matrix such that \( g_i^{(r)} = g_{ir} \) for any \( l \in L \) and \( r' \in R|r| \).
Lemma 7.2. Let \( r \) be an element of \( R \). The matroid \( X(g) \mid R[r] \) is the row matroid of \( g^{(r)} \).

Proof. By definition, we have
\[
\{ U \subseteq L \mid U \text{ is independent in } X(g) \mid R[r] \} = \{ U \subseteq L \mid \exists V \subseteq R[r] : (U, V) \in X(g) \} = \{ U \subseteq L \mid \exists V \subseteq R[r] : \Delta_{UV}(g) \neq 0 \} = \{ U \subseteq L \mid \text{rows of } g^{(r)} \text{ indexed by } U \text{ are linearly independent} \} = \{ U \subseteq L \mid U \text{ is independent in the row matroid of } g^{(r)} \}. \tag*{□}
\]

Proposition 7.3. Let \( g \) be a matrix, \( b_L \in B_L^{-} \) and \( b_R \in B_R \). Then \( u(X(b_Lgb_R)) = u(X(g)) \).

Proof. Let \( r \) be an element of \( R \). By \( b' \) denote the \( R[r] \times R[r] \)-matrix such that \( b'_{r',r''} = (b_R)_{r',r''} \) for any \( r', r'' \in R[r] \). Since \( b_R \) is upper-triangular, we have \( (b_R)_{r',r''} = 0 \) whenever \( r' \in R \setminus R[r] \) and \( r'' \in R[r] \). This implies that \( (gb_R)^{(r)} = g^{(r)}b' \).

Since \( b' \) is invertible, it follows that the row matroid of \( g^{(r)}b' \) is equal to the row matroid of \( g^{(r)} \). Using Lemma 7.2, we get \( X(g) \mid R[r] = X(gb_R) \mid R[r] \). From Corollary 4.3 it follows that
\[
u(X(g)) = u(X(gb_R)).
\]

Now note that \( \tilde{b}_L \) is an upper-triangular matrix. Using Theorem 4.6, we get
\[
u(X(g)) = u(X(gb_R)) = \tilde{t}u(\tilde{t}(gb_R)) = \tilde{t}u(\tilde{t}(gb_R)) = \tilde{t}u(\tilde{t}(gb_R)) = u(X(g)). \tag*{□}
\]

Corollary 7.4. If \( g \in B_L^{-}uB_R \) and \( u \in \tilde{W}_{LR} \), then \( u = u(X(g)) \).

Proof. Let \( b_L \) be an element of \( B_L^{-} \) and \( b_R \) be an element of \( B_R \) such that \( u = b_Lgb_R \). From Proposition 7.3 it follows that \( u(X(g)) = u(X(u)) \). It can easily be checked that \( u(X(u)) = u \) whenever \( u \) is a pseudobijection. This completes the proof. \( \Box \)

Corollary 7.5. For any \( L \times R \)-matrix \( g \) there exists a unique pseudobijection \( u \) between \( L \) and \( R \) such that \( g \in B_L^{-}uB_R \).

Proof. Suppose that \( g \in B_L^{-}uB_R \) and \( g \in B_L^{-}u'B_R \). By Corollary 7.4, we get \( u = u(X(g)) = u' \). \( \Box \)

Theorem 7.6. Let \( f \) be a function from \( W_L \times W_R \) to \( \tilde{W}_{LR} \). The Gauss stratum \( C_f \) is nonempty if and only if \( f \) is a formal Gauss stratum and the bimatroid \( X[f] \) is representable. Moreover, \( C_f \) is the set of matrices representing \( X[f] \).

Proof. We have
\[
C_f = \{ g \mid w_Lw_R \in B_L^{-}f(w_L, w_R)B_R \text{ for all } w_L \in W_L, w_R \in W_R \} = \{ g \mid u(X(w_Lw_R)) = f(w_L, w_R) \} = \{ g \mid u(w_LX(g)w_R) = f(w_L, w_R) \} = \{ g \mid f_X(g)(w_L, w_R) = f(w_L, w_R) \} = \{ g \mid f_X(g) = f \}.
\]
It follows that $C_f \neq \emptyset$ if and only if $f$ has the form $f_X(g)$, where $g$ is an $L \times R$-matrix. In other words, $C_f \neq \emptyset$ if and only if $f$ has the form $f_X$, where $X$ is a representable bimatroid between $L$ and $R$. To conclude the proof, it remains to note that $f = f_X$ if and only if $f$ is a formal Gauss stratum and $X = X[f]$. □

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References