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# Classification of discretely decomposable $A_{\mathfrak{q}}(\lambda)$ with respect to reductive symmetric pairs

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**Abstract**

We give a classification of the triples $(\mathfrak{g}, \mathfrak{g}', \mathfrak{q})$ such that Zuckerman's derived functor $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}}(\lambda)$ for a $\theta$-stable parabolic subalgebra $\mathfrak{q}$ is discretely decomposable with respect to a reductive symmetric pair $(\mathfrak{g}, \mathfrak{g}')$. The proof is based on the criterion for discretely decomposable restrictions by the first author and on Berger's classification of reductive symmetric pairs.


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## 1. Introduction

Branching problems in representation theory ask how an irreducible representation decomposes when restricted to a subgroup (or a subalgebra).

In the category of unitary representations of a locally compact group $G'$, one can describe irreducible decompositions by means of the direct integrals of Hilbert spaces. The object of our study is the restriction of an irreducible unitary representation $\pi$ of $G$ to its subgroup $G'$, in particular when $G$ and $G'$ are both reductive Lie groups. Then the irreducible decomposition is unique; however, it may contain a continuous spectrum in the direct integral of Hilbert spaces.

For a reductive Lie group $G$, we can consider branching problems also in the category of $(\mathfrak{g}, K)$-modules. If the underlying $(\mathfrak{g}, K)$-module $\pi_K$ is discretely decomposable as a $(\mathfrak{g}', K')$-module (see Definition 2.1), then the branching laws of the restrictions of the unitary representation $\pi$ to $G'$ and the $(\mathfrak{g}, K)$-module $\pi_K$ to $(\mathfrak{g}', K')$ are essentially the same in the following sense:

$$\pi|_{G'} \simeq \sum_{\tau \in \widehat{G'}}^{\oplus} m_\pi(\tau)\tau \quad \text{(Hilbert direct sum)},$$

$$\pi_K|_{(\mathfrak{g}', K')} \simeq \bigoplus_{\tau \in \widehat{G'}} m_\pi(\tau)\tau_{K'} \quad \text{(algebraic direct sum)},$$

where $\widehat{G'}$ is the set of equivalence classes of irreducible unitary representations of $G'$, and $\tau_{K'}$ is the underlying $(\mathfrak{g}', K')$-module of $\tau$. The key ingredient here is that the natural map

$$\mathrm{Hom}_{(\mathfrak{g}', K')}(\tau_{K'}, \pi_K) \to \mathrm{Hom}_{G', \mathrm{continuous}}(\tau, \pi) \tag{1.1}$$

is bijective, and therefore the dimensions of the spaces of homomorphisms coincide, giving the same multiplicity $m_\pi(\tau)$ in the branching laws.

It should be noted that (1.1) is not surjective in general and that the restriction of an irreducible and unitarizable $(\mathfrak{g}, K)$-module $\pi_K$ may not be decomposed into an algebraic direct sum of irreducible $(\mathfrak{g}', K')$-modules. Such a phenomenon happens whenever a continuous spectrum appears in the branching law of the restriction of the unitary representation $\pi$ to $G'$.

The aim of this article is to give a classification of the triples $(\mathfrak{g}, \mathfrak{g}', \pi_K)$ such that the $(\mathfrak{g}, K)$-module $\pi_K$ is discretely decomposable as a $(\mathfrak{g}', K')$-module in the setting where

$(\mathfrak{g}, \mathfrak{g}')$ is a reductive symmetric pair,

$\pi_K$ is Zuckerman's derived functor module $A_\mathfrak{q}(\lambda)$.

The condition for discrete decomposability does not change if we replace $K$ and $K'$ by their finite covering groups or subgroups of finite index. Thus, we may and do assume that $K$ is connected and $K' = K^\sigma$ (or equivalently $G' = G^\sigma$), where $\sigma$ is an involution of $G$ leaving

$K$ stable. Further, the condition for discrete decomposability of $A_{\mathfrak{q}}(\lambda)$ depends on a $\theta$-stable parabolic subalgebra $\mathfrak{q}$, but is independent of the parameter $\lambda$ in the good range.

Our main result is Theorem 4.1 with Tables C.1–C.4. They give a classification of the triples $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ for which $A_{\mathfrak{q}}(\lambda)$ is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module. The list is described up to the conjugacy of $K \times K$ as we explain at the beginning of Section 4. We find that quite a large part of such triples $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ appear as a 'family' containing $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{b})$ with $\mathfrak{b}$ a $\theta$-stable Borel subalgebra. We call them *discrete series type* (see Tables C.1–C.3), which include *holomorphic type* as a special case (see Proposition 2.15). Moreover, there are some other triples, which we refer to as *isolated type* (see Table C.4).

The tensor product of two representations is an example of the restriction with respect to symmetric pairs. Thus, a very special case of our theorem includes the classification of two discrete series representations $\pi_1$ and $\pi_2$ of $G'$ such that the tensor product representation $\pi_1 \otimes \pi_2$ decomposes discretely (see Corollary 3.2).

There exist irreducible symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$ for which any non-trivial $A_{\mathfrak{q}}(\lambda)$ is not discretely decomposable. We give a classification of all such pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$ in Theorem 4.12.

The proof is based on the criterion for the discretely decomposable restriction established in [4–6], see Theorem 2.8, and on the classification of reductive symmetric pairs $(\mathfrak{g}, \mathfrak{g}')$ by Berger [1] up to outer automorphisms of $\mathfrak{g}$.

## 2. Discretely decomposable $A_{\mathfrak{q}}(\lambda)$ for symmetric pairs

Let $G$ be a connected real reductive Lie group. We write $\mathfrak{g}$ for the Lie algebra of $G$ and $\mathfrak{g}_{\mathbb{C}}$ for its complexification. Analogous notation will be used for other Lie algebras.

Let $\sigma$ be an involutive automorphism of $G$, and we set $G^\sigma := \{g \in G : \sigma g = g\}$. Then $(G, G^\sigma)$ forms a reductive symmetric pair. Take a Cartan involution $\theta$ of $G$ which commutes with $\sigma$. Then $K := G^\theta$ and $K^\sigma = K \cap G^\sigma$ are maximal compact subgroups of $G$ and $G^\sigma$, respectively. We let $\theta$ and $\sigma$ also denote the induced involutions on $\mathfrak{g}$ and their complex linear extensions to $\mathfrak{g}_{\mathbb{C}}$. The Cartan decompositions are denoted by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $\mathfrak{g}^\sigma = \mathfrak{k}^\sigma + \mathfrak{p}^\sigma$, respectively.

We recall from [6] the following basic notion, which we shall apply to branching problems in the category of $(\mathfrak{g}, K)$-modules.

**Definition 2.1.** We say that a $(\mathfrak{g}, K)$-module $V$ is *discretely decomposable* if there exists an increasing filtration $\{V_n\}$ such that $V = \bigcup_{n=0}^{\infty} V_n$ and each $V_n$ is of finite length as a $(\mathfrak{g}, K)$-module.

**Remark 2.2** (*See [6, Lemma 1.3]*)**.** Suppose that $V$ is a unitarizable $(\mathfrak{g}, K)$-module. Then $V$ is discretely decomposable if and only if $V$ is isomorphic to the algebraic direct sum of irreducible $(\mathfrak{g}, K)$-modules.

Next, let us fix some notation concerning Zuckerman's derived functor modules $A_{\mathfrak{q}}(\lambda)$. Suppose $\mathfrak{q}$ is a $\theta$-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. The normalizer $L = N_G(\mathfrak{q})$ of $\mathfrak{q}$ is a connected reductive subgroup of $G$. Hence a unitary character $\mathbb{C}_\lambda$ of $L$ is determined by its differential $\lambda \in \sqrt{-1}\mathfrak{l}^*$. Associated to the data $(\mathfrak{q}, \lambda)$, one defines Zuckerman's derived functor module $A_{\mathfrak{q}}(\lambda)$ as in [3, (5.6)]. In our normalization, $A_{\mathfrak{q}}(0)$ is a unitarizable $(\mathfrak{g}, K)$-module with non-zero $(\mathfrak{g}, K)$-cohomologies, and in particular, has the same infinitesimal character as the trivial one-dimensional representation $\mathbb{C}$ of $\mathfrak{g}$. We note that if $\mathfrak{q} = \mathfrak{g}_{\mathbb{C}}$, then $L = G$ and $A_{\mathfrak{q}}(\lambda)$ is one-dimensional.

Take a fundamental Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$ and choose a positive root system $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ such that the set $\Delta(\mathfrak{q}, \mathfrak{h}_\mathbb{C})$ of roots for $\mathfrak{q}$ contains all the positive roots, and set $\Delta^+(\mathfrak{l}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) := \Delta(\mathfrak{l}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) \cap \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Let $\mathfrak{u}$ be the nilradical of $\mathfrak{q}$. Denote by $\rho$, $\rho_\mathfrak{l}$, and $\rho(\mathfrak{u}) \in \mathfrak{h}_\mathbb{C}^*$ half the sum of roots in $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$, $\Delta^+(\mathfrak{l}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$, and $\Delta(\mathfrak{u}, \mathfrak{h}_\mathbb{C})$, respectively. Let $\langle \cdot, \cdot \rangle$ be an invariant bilinear form on $\mathfrak{h}_\mathbb{C}^*$ that is positive definite on the real span of the roots. Following the terminology [3, Definitions 0.49 and 0.52], we say for a unitary character $\mathbb{C}_\lambda$ of $L$, $\lambda$ is in the *good range* if

$$\operatorname{Re}\langle \lambda + \rho, \alpha \rangle > 0 \quad \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}_\mathbb{C}),$$

and in the *weakly fair range* if

$$\operatorname{Re}\langle \lambda + \rho(\mathfrak{u}), \alpha \rangle \geq 0 \quad \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}_\mathbb{C}).$$

The $K$-finite Hermitian dual of the $(\mathfrak{g}, K)$-module $A_\mathfrak{q}(\lambda)$ in the normalization here is isomorphic to the cohomologically induced module $\mathcal{R}_\mathfrak{q}^S(\mathbb{C}_\nu)$ with $S = \dim_\mathbb{C}(\mathfrak{u} \cap \mathfrak{k}_\mathbb{C})$ and $\nu = \lambda + \rho(\mathfrak{u})$ in the normalization of [4]. Accordingly, the good range (resp. the weakly fair range) amounts to the condition on $\nu$ as

$$\operatorname{Re}\langle \nu + \rho_\mathfrak{l}, \alpha \rangle > 0 \quad \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}_\mathbb{C}) \quad (\text{resp. } \operatorname{Re}\langle \nu, \alpha \rangle \geq 0 \; \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}_\mathbb{C})).$$

We pin down some basic properties of the $(\mathfrak{g}, K)$-module $A_\mathfrak{q}(\lambda)$ ([3, Chapters VIII and IX]).

**Theorem 2.3.** *If $\lambda$ is in the weakly fair range, $A_\mathfrak{q}(\lambda)$ is unitarizable or zero. If $\lambda$ is in the good range, $A_\mathfrak{q}(\lambda)$ is non-zero and irreducible.*

**Theorem 2.4.** *Suppose that* $\operatorname{rank} \mathfrak{g}_\mathbb{C} = \operatorname{rank} \mathfrak{k}_\mathbb{C}$. *If $\mathfrak{q}$ is a $\theta$-stable Borel subalgebra of $\mathfrak{g}_\mathbb{C}$ and if $\lambda$ is in the good range, then $A_\mathfrak{q}(\lambda)$ is isomorphic to the underlying $(\mathfrak{g}, K)$-module of a discrete series representation of $G$. Conversely the underlying $(\mathfrak{g}, K)$-module of any discrete series representation of $G$ is isomorphic to $A_\mathfrak{q}(\lambda)$ for some $\theta$-stable Borel subalgebra $\mathfrak{q}$ and $\lambda$ in the good range.*

The goal of this article is to give a classification of the triples $(G, G^\sigma, \mathfrak{q})$ such that the $(\mathfrak{g}, K)$-module $A_\mathfrak{q}(\lambda)$ is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module. Since the discrete decomposability depends only on the triple $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ of Lie algebras and not on the Lie group $G$, our classification will be given in terms of the Lie algebras.

To pursue the classification, we prepare some further basic setups:

**Definition 2.5.** We say the pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is an *irreducible symmetric pair* if one of the following holds.

(1) $\mathfrak{g}$ is simple.
(2) $\mathfrak{g}'$ is simple and $\mathfrak{g} \simeq \mathfrak{g}' \oplus \mathfrak{g}'$; $\sigma$ acts by switching the factors.

Let $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}'$ and $\varphi$ a non-trivial automorphism of $\mathfrak{g}'$. Then there is also a symmetric pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ defined by the involution $\sigma(x, y) := (\varphi(y), \varphi^{-1}(x))$ for $x, y \in \mathfrak{g}'$. For the simplicity of the exposition, we exclude this case from the definition of irreducible pairs. This does not lose any generality for our purpose because we have an isomorphism $A_{\mathfrak{q}_1' \oplus \mathfrak{q}_2'}(\lambda_1, \lambda_2)|_{\mathfrak{g}^\sigma} \simeq A_{\mathfrak{q}_1' \oplus \varphi(\mathfrak{q}_2')}(\lambda_1, (\varphi^*)^{-1}\lambda_2)|_{\operatorname{diag}(\mathfrak{g}')}$ via the isomorphism $\mathfrak{g}^\sigma \simeq \operatorname{diag}(\mathfrak{g}'), (x, \varphi^{-1}(x)) \mapsto (x, x)$. Here, $\mathfrak{q}_1'$, $\mathfrak{q}_2'$ are parabolic subalgebras of $\mathfrak{g}_\mathbb{C}'$ and $\operatorname{diag}(\mathfrak{g}')$ is the diagonal in $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}'$. Therefore the discrete decomposability for the triple $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q}_1' \oplus \mathfrak{q}_2')$ is equivalent to that for the triple $(\mathfrak{g}, \mathrm{diag}(\mathfrak{g}'), \mathfrak{q}_1' \oplus \varphi(\mathfrak{q}_2'))$. We shall treat the latter case in Section 3.

We should remark that our definition differs from the one in [1], where the pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ was called irreducible if $\mathfrak{g}^{-\sigma}$ is an irreducible $\mathfrak{g}^\sigma$-module. For example, $(\mathfrak{sl}(n, \mathbb{R}), \mathfrak{sl}(m, \mathbb{R}) \oplus \mathfrak{sl}(n-m, \mathbb{R}) \oplus \mathbb{R})$ is an irreducible pair for Definition 2.5, while it is not for the definition of [1]. Both definitions are the same for Riemannian symmetric pairs.

Any semisimple symmetric pair is isomorphic to the direct sum of irreducible symmetric pairs. In particular, branching problems of $A_\mathfrak{q}(\lambda)$ with respect to reductive symmetric pairs reduce to the case of irreducible symmetric pairs because any $\theta$-stable parabolic subalgebra $\mathfrak{q}$ is obviously written as the direct sum of $\theta$-stable parabolic subalgebras of each factor.

To describe $\theta$-stable parabolic subalgebras of $\mathfrak{g}_\mathbb{C}$, it is convenient to use the following convention:

**Definition 2.6.** We say that a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}_\mathbb{C}$ is given by a vector $a \in \sqrt{-1}\mathfrak{k}$ if $\mathfrak{q}$ is the sum of non-negative eigenspaces of $\mathrm{ad}(a)$.

Then $\mathfrak{q}$ is a $\theta$-stable parabolic subalgebra with a Levi decomposition $\mathfrak{q} = \mathfrak{l}_\mathbb{C} + \mathfrak{u}$, where $\mathfrak{l}_\mathbb{C}$ and $\mathfrak{u}$ are the sums of zero and positive eigenspaces of $\mathrm{ad}(a)$, respectively. Note that any $\theta$-stable parabolic subalgebras are obtained in this way.

Needless to say, the defining element $a$ of a $\theta$-stable parabolic subalgebra $\mathfrak{q}$ is not unique. However, we adopt this convention in our classification (Tables C.1, C.3 and C.4) because it is not hard to compute $\mathfrak{q}$ and $L = N_G(\mathfrak{q})$ from the defining element $a$ by using the root system.

Replacing $\mathfrak{q}$ by $\mathrm{Ad}(k)\mathfrak{q}$ for $k \in K$ if necessary, we restrict ourselves to consider the following setting.

**Setting 2.7.** (1) Suppose that $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is an irreducible symmetric pair and the involution $\sigma$ commutes with a Cartan involution $\theta$. Fix a $\sigma$-stable Cartan subalgebra $\mathfrak{t} = \mathfrak{t}^\sigma + \mathfrak{t}^{-\sigma}$ of $\mathfrak{k}$ such that $\mathfrak{t}^{-\sigma}$ is maximal abelian in $\mathfrak{k}^{-\sigma}$. Choose a positive system $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ that is compatible with some positive system of the restricted root system $\Sigma^+(\mathfrak{k}_\mathbb{C}, \sqrt{-1}\mathfrak{t}^{-\sigma})$.

(2) Let $\mathfrak{q}$ be a $\theta$-stable parabolic subalgebra of $\mathfrak{g}_\mathbb{C}$. We assume that $\mathfrak{q}$ is given by a $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$-dominant vector $a \in \sqrt{-1}\mathfrak{t}$.

Since $\mathfrak{t}$ is $\sigma$-stable, $\sigma$ acts on the complexification $\mathfrak{t}_\mathbb{C}$ and also on the dual space $\mathfrak{t}_\mathbb{C}^*$, which is denoted by the same letter $\sigma$. We note that $\mathfrak{p}_\mathbb{C}$ and the nilradical $\mathfrak{u}$ of $\mathfrak{q}$ are $\mathfrak{t}_\mathbb{C}$-invariant subspaces. We write $\Delta(\mathfrak{p}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$, $\Delta(\mathfrak{u} \cap \mathfrak{p}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ for the sets of the weights of $\mathfrak{t}_\mathbb{C}$ in $\mathfrak{p}_\mathbb{C}$, $\mathfrak{u} \cap \mathfrak{p}_\mathbb{C}$, respectively. Here is a summary on equivalent conditions for discretely decomposable restrictions of $A_\mathfrak{q}(\lambda)$ with respect to reductive symmetric pairs. We shall use condition (iii) for our classification of the triples $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$.

**Theorem 2.8.** *In Setting 2.7, the following eight conditions on the triple $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ are equivalent:*

- (i) $A_\mathfrak{q}(\lambda)$ *is non-zero and discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module for some $\lambda$ in the weakly fair range.*
- (i′) $A_\mathfrak{q}(\lambda)$ *is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module for any $\lambda$ in the weakly fair range.*
- (ii) $\mathbb{R}_+\langle \mathfrak{u} \cap \mathfrak{p}_\mathbb{C} \rangle \cap \sqrt{-1}(\mathfrak{t}^{-\sigma})^* = \{0\}$. *Here, we define*

$$\mathbb{R}_+\langle \mathfrak{u} \cap \mathfrak{p}_\mathbb{C} \rangle := \left\{ \sum_{\alpha \in \Delta(\mathfrak{u} \cap \mathfrak{p}_\mathbb{C}, \mathfrak{t}_\mathbb{C})} n_\alpha \alpha : n_\alpha \in \mathbb{R}_{\geq 0} \right\}.$$

(ii′) *There exists* $b \in \sqrt{-1}\mathfrak{t}^\sigma$ *such that* $\langle \mathrm{pr}_+(\mathbb{R}_+\langle \mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}\rangle), b\rangle > 0$, *where* $\mathrm{pr}_+ : \sqrt{-1}\mathfrak{t}^* \to \sqrt{-1}(\mathfrak{t}^\sigma)^*$ *is the restriction map.*

(iii) $\sigma\alpha(a) \geq 0$ *whenever* $\alpha \in \Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ *satisfies* $\alpha(a) > 0$.

(iii′) $\sigma(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}) \subset \mathfrak{q}$.

(iv) *Let* $\mathcal{V}(A_{\mathfrak{q}}(\lambda))$ *be the associated variety of* $A_{\mathfrak{q}}(\lambda)$ *and* $\mathrm{pr}_+ : \mathfrak{g}_{\mathbb{C}}^* \to (\mathfrak{g}_{\mathbb{C}}^\sigma)^*$ *the restriction map. Then* $\mathrm{pr}_+\mathcal{V}(A_{\mathfrak{q}}(\lambda))$ *is contained in the nilpotent cone of* $(\mathfrak{g}_{\mathbb{C}}^\sigma)^*$ *for any* $\lambda$ *in the weakly fair range.*

(v) *Each* $K^\sigma$*-type occurs in* $A_{\mathfrak{q}}(\lambda)$ *with finite multiplicity for any* $\lambda$ *in the weakly fair range.*

If one of, and hence any of, these equivalent conditions holds, we say that the triple $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ satisfies the *discrete decomposability condition*.

**Proof.** The equivalence of (i), (i′), (ii), (iii′), (iv), and (v) was established in [4–6]. To be more precise, the implication (ii) ⇒ (v) was proved in [4], and an alternative proof based on micro-local analysis was given in [5]. The opposite direction (v) ⇒ (i′) ⇒ (i) ⇒ (iv) ⇒ (iii′) ⇒ (ii) was proved in [6]. The conditions (ii′) and (iii) are just reformulations of (ii) and (iii′), respectively. □

We end this section with a number of direct consequences of Theorem 2.8, namely, one equivalent condition (Proposition 2.9), two sufficient conditions (Propositions 2.10 and 2.15) and two necessary conditions (Propositions 2.16 and 2.17) for the discrete decomposability of $A_{\mathfrak{q}}(\lambda)$ as a $(\mathfrak{g}^\sigma, K^\sigma)$-module.

Suppose that an involution $\sigma$ of $G$ commutes with a Cartan involution $\theta$. Then the composition $\theta\sigma$ becomes another involution of $G$. The symmetric pair $(\mathfrak{g}, \mathfrak{g}^{\theta\sigma})$ is called the *associated pair* of $(\mathfrak{g}, \mathfrak{g}^\sigma)$.

Since $\sigma = \theta\sigma$ on $\mathfrak{t}$, we get from condition (iii) in Theorem 2.8 the following proposition:

**Proposition 2.9.** *For* $\lambda$ *in the weakly fair range,* $A_{\mathfrak{q}}(\lambda)$ *is discretely decomposable as a* $(\mathfrak{g}^\sigma, K^\sigma)$*-module if and only if it is discretely decomposable as a* $(\mathfrak{g}^{\theta\sigma}, K^{\theta\sigma})$*-module.*

The following proposition is a direct consequence of condition (ii) in Theorem 2.8:

**Proposition 2.10.** *Let* $\mathfrak{q}_1$ *and* $\mathfrak{q}_2$ *be* $\theta$*-stable parabolic subalgebras of* $\mathfrak{g}_{\mathbb{C}}$ *such that* $\mathfrak{q}_1 \subset \mathfrak{q}_2$. *If* $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q}_1)$ *satisfies the discrete decomposability condition, then so does* $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q}_2)$.

Yet another easy consequence of Theorem 2.8 concerns the triples $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ for *holomorphic* $\mathfrak{q}$ (Definition 2.12) defined for a Hermitian Lie algebra $\mathfrak{g}$ below:

**Definition 2.11.** Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a real non-compact simple Lie algebra. We say $\mathfrak{g}$ is of *Hermitian type* and the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ is a *Hermitian symmetric pair* if the center $\mathfrak{z}_K$ of $\mathfrak{k}$ is one-dimensional.

If $\mathfrak{g}$ is of Hermitian type, then $\mathfrak{p}_{\mathbb{C}}$, regarded as a $K$-module by the adjoint action, decomposes into the direct sum of two irreducible submodules, say, $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ + \mathfrak{p}_-$. Then the Riemannian symmetric space $G/K$ becomes a Hermitian symmetric space by choosing $\mathfrak{p}_-$ as a holomorphic tangent space at the base point.

**Definition 2.12.** Suppose that $\mathfrak{g}$ is a simple Lie algebra of Hermitian type. A $\theta$-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}_{\mathbb{C}}$ is said to be *holomorphic* (resp. *anti-holomorphic*) if $\mathfrak{q} \supset \mathfrak{p}_+$ (resp. $\mathfrak{q} \supset \mathfrak{p}_-$).

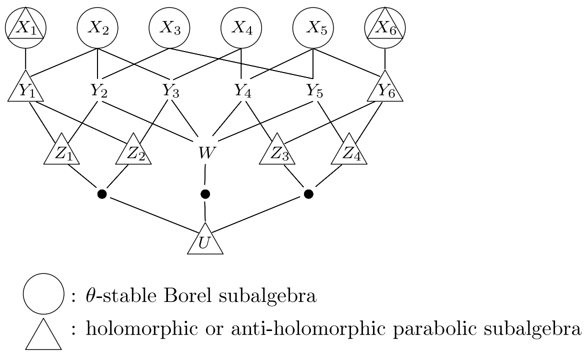


Fig. 1. $K$-conjugacy classes of $\theta$-stable parabolic subalgebras for $\mathfrak{su}(2,2)$.

See Table C.1 for the conditions on a defining element $a$ for parabolic subalgebra $\mathfrak{q}$ to be holomorphic or anti-holomorphic.

If a $\theta$-stable parabolic subalgebra $\mathfrak{q}$ is holomorphic and if $A_{\mathfrak{q}}(\lambda)$ is non-zero and irreducible (in particular, if $\lambda$ is in the good range), then $A_{\mathfrak{q}}(\lambda)$ is a lowest weight module with respect to a Borel subalgebra containing $\mathfrak{p}_+$. Similarly, if $\mathfrak{q}$ is anti-holomorphic, $A_{\mathfrak{q}}(\lambda)$ is a highest weight module. If $\mathfrak{q} \cap \mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+$ (resp. $\mathfrak{q} \cap \mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_-$) and $\lambda$ is in the good range, then $A_{\mathfrak{q}}(\lambda)$ is the underlying $(\mathfrak{g}, K)$-module of a holomorphic (resp. anti-holomorphic) discrete series representation of $G$.

**Definition 2.13.** Suppose that $\mathfrak{g}$ is a simple Lie algebra of Hermitian type, so the center $\mathfrak{z}_K$ of $\mathfrak{k}$ is one-dimensional. We say a symmetric pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is of *holomorphic type* if $\mathfrak{z}_K \subset \mathfrak{g}^\sigma$, or equivalently if $\sigma$ induces a holomorphic involution on the Hermitian symmetric space $G/K$.

It follows immediately from $\mathfrak{k}^\sigma = \mathfrak{k}^{\theta\sigma}$ that the pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is of holomorphic type if and only if the associated pair $(\mathfrak{g}, \mathfrak{g}^{\theta\sigma})$ is of holomorphic type. See Table C.2 for the classification of symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$ of holomorphic type.

**Example 2.14.** Let $\mathfrak{g} = \mathfrak{su}(2,2) \simeq \mathfrak{so}(4,2)$. Suppose we are in Setting 2.7, and use the notation of Setting A.1 for $\mathfrak{t}$ and $e_i$. In particular, $\mathfrak{q}$ is given by $a = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4$ with $a_1 \geq a_2$ and $a_3 \geq a_4$. Fig. 1 follows the notation in [4]. It shows 18 $\theta$-stable parabolic subalgebras of $\mathfrak{g}_{\mathbb{C}}$, which form a complete set of representatives of $\mathfrak{q}$ up to $K$-conjugacy and the equivalence relation among the $\theta$-stable parabolic subalgebras $\mathfrak{q}_1 \sim \mathfrak{q}_2$ defined by a $(\mathfrak{g}, K)$-isomorphism $A_{\mathfrak{q}_1}(0) \simeq A_{\mathfrak{q}_2}(0)$. The correspondence is: $X_1 \leftrightarrow a_1 > a_2 > a_3 > a_4$, $X_2 \leftrightarrow a_1 > a_3 > a_2 > a_4$, $X_3 \leftrightarrow a_1 > a_3 > a_4 > a_2$, $X_4 \leftrightarrow a_3 > a_1 > a_2 > a_4$, $X_5 \leftrightarrow a_3 > a_1 > a_4 > a_2$, $X_6 \leftrightarrow a_3 > a_4 > a_1 > a_2$, $Y_1 \leftrightarrow a_1 > a_2 = a_3 > a_4$, $Y_2 \leftrightarrow a_1 > a_3 > a_2 = a_4$, $Y_3 \leftrightarrow a_1 = a_3 > a_2 > a_4$, $Y_4 \leftrightarrow a_3 > a_1 > a_2 = a_4$, $Y_5 \leftrightarrow a_1 = a_3 > a_4 > a_2$, $Y_6 \leftrightarrow a_3 > a_1 = a_4 > a_2$, $Z_1 \leftrightarrow a_1 > a_2 = a_3 = a_4$, $Z_2 \leftrightarrow a_1 = a_2 = a_3 > a_4$, $Z_3 \leftrightarrow a_3 > a_1 = a_2 = a_4$, $Z_4 \leftrightarrow a_1 = a_3 = a_4 > a_2$, $W \leftrightarrow a_1 = a_3 > a_2 = a_4$, $U \leftrightarrow a_1 = a_2 = a_3 = a_4$. We see that $X_1, \ldots, X_6$ yield $\theta$-stable Borel subalgebras and $X_1, X_6, Y_1, Y_6, Z_1, Z_2, Z_3, Z_4, U$ yield holomorphic or anti-holomorphic parabolic subalgebras.

The following theorem can be deduced from [7, Theorem 7.4]. For the convenience of the reader, we give an alternative proof by using the criterion Theorem 2.8(iii).

**Proposition 2.15.** *Suppose that a symmetric pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is of holomorphic type and a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}_{\mathbb{C}}$ is holomorphic or anti-holomorphic. Then $A_{\mathfrak{q}}(\lambda)$ is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module for any $\lambda$ in the weakly fair range.*

**Proof.** Choose $z \in \sqrt{-1}\mathfrak{z}_K$ such that $\Delta(\mathfrak{p}_+, \mathfrak{t}_{\mathbb{C}}) = \{\alpha \in \Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) : \alpha(z) > 0\}$. We observe $\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}} \subset \mathfrak{p}_+$ if the $\theta$-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$ is holomorphic. Thus, if $\alpha(a) > 0$ for $\alpha \in \Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, then $\alpha \in \Delta(\mathfrak{p}_+, \mathfrak{t}_{\mathbb{C}})$. Since $\sigma(z) = z$, the $\sigma$-action on $\mathfrak{t}_{\mathbb{C}}^*$ stabilizes $\Delta(\mathfrak{p}_+, \mathfrak{t}_{\mathbb{C}})$. Then $\sigma\alpha \in \Delta(\mathfrak{p}_+, \mathfrak{t}_{\mathbb{C}})$ and hence $\sigma\alpha(a) \geq 0$. Thus, Theorem 2.8(iii) is satisfied. □

Conversely, Theorem 2.8 gives a simple, necessary condition on a pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ such that at least one infinite dimensional $A_{\mathfrak{q}}(\lambda)$ is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module.

**Proposition 2.16.** *Let $\mathfrak{g}$ be a simple non-compact Lie algebra and $\sigma$ an involution of $\mathfrak{g}$ commuting with $\theta$. Suppose that $\lambda$ is in the weakly fair range, $\mathfrak{q} \neq \mathfrak{g}_{\mathbb{C}}$, and $A_{\mathfrak{q}}(\lambda)$ is non-zero. If $A_{\mathfrak{q}}(\lambda)$ is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module, then $\mathfrak{t}^\sigma \neq 0$, or equivalently $\mathfrak{k}^\sigma + \sqrt{-1}\mathfrak{k}^{-\sigma}$ is not a split real form of $\mathfrak{k}_{\mathbb{C}}$.*

**Proof.** Suppose $\mathfrak{t}^\sigma = 0$. Then $\sigma$ acts by $-1$ on $\mathfrak{t}$ and hence on $\Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Therefore if $\alpha(a) > 0$ for some $\alpha \in \Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, then $\sigma\alpha(a) < 0$. By Theorem 2.8, $A_{\mathfrak{q}}(\lambda)$ is not discretely decomposable. If $\alpha(a) = 0$ for all $\alpha \in \Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, then $\mathfrak{p}_{\mathbb{C}} \subset \mathfrak{q}$. Therefore $\mathfrak{q}$ must coincide with $\mathfrak{g}_{\mathbb{C}}$, which is not the case.

Finally, we note that $\mathfrak{k}^\sigma + \sqrt{-1}\mathfrak{k}^{-\sigma}$ is a real form of $\mathfrak{k}_{\mathbb{C}}$, where $\sigma$ acts as a Cartan involution. Thus, we see from Setting 2.7(1) that $\mathfrak{t}^\sigma = 0$ if and only if $\mathfrak{k}^\sigma + \sqrt{-1}\mathfrak{k}^{-\sigma}$ is a split real form of $\mathfrak{k}_{\mathbb{C}}$. □

The following proposition also presents a necessary condition for $A_{\mathfrak{q}}(\lambda)$ to be discretely decomposable, which is stronger than the one in Proposition 2.16. Let $\alpha_0$ be the highest weight of the irreducible representation of $\mathfrak{k}_{\mathbb{C}}$ on $\mathfrak{p}_{\mathbb{C}}$ (if $\mathfrak{g}$ is not of Hermitian) or on $\mathfrak{p}_+$ (if $\mathfrak{g}$ is of Hermitian).

**Proposition 2.17.** *Let $\mathfrak{g}$ be a simple non-compact Lie algebra and $\sigma$ an involution of $\mathfrak{g}$ commuting with $\theta$. Suppose that $\mathfrak{q} \neq \mathfrak{g}_{\mathbb{C}}$, $\lambda$ is in the weakly fair range, and $A_{\mathfrak{q}}(\lambda)$ is non-zero. If one of the following three assumptions hold, then $A_{\mathfrak{q}}(\lambda)$ is not discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module.*

(1) *$\mathfrak{g}$ is not of Hermitian type and $-\sigma\alpha_0$ is $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$-dominant.*
(2) *$\mathfrak{g}$ is of Hermitian type, $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is not of holomorphic type, and $-\sigma\alpha_0$ is $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$-dominant.*
(3) *$\mathfrak{g}$ is of Hermitian type, $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is of holomorphic type, $-\sigma\alpha_0$ is $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$-dominant, and $\mathfrak{q}$ is neither holomorphic nor anti-holomorphic.*

**Proof.** We assume that the parabolic subalgebra $\mathfrak{q}$ is given by a $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$-dominant vector $a$.

(1) Since $-\sigma\alpha_0$ is an extremal weight of $\Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and $-\sigma\alpha_0$ is dominant, $-\sigma\alpha_0$ is the highest weight of $\mathfrak{p}_{\mathbb{C}}$ and hence $-\sigma\alpha_0 = \alpha_0$. If $\alpha_0(a) \leq 0$, then $\alpha(a) \leq 0$ for all $\alpha \in \Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Hence $\alpha(a) = 0$ for all $\alpha \in \Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and then $\mathfrak{q} = \mathfrak{g}_{\mathbb{C}}$, contradicting our assumption. We therefore have $\alpha_0(a) > 0$ and $\sigma\alpha_0(a) = -\alpha_0(a) < 0$ so condition (iii) in Theorem 2.8 fails.

(2) Similarly to the proof of case (1), $-\sigma\alpha_0$ must be the highest weight of $\mathfrak{p}_+$ and hence $-\sigma\alpha_0 = \alpha_0$. Let $\alpha_0'$ be the highest weight of $\mathfrak{p}_-$. Then $-\sigma\alpha_0'$ is dominant and hence $-\sigma\alpha_0' = \alpha_0'$. If $\alpha_0(a) \leq 0$ and $\alpha_0'(a) \leq 0$, then $\alpha(a) \leq 0$ for all $\alpha \in \Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. This implies $\mathfrak{q} = \mathfrak{g}_{\mathbb{C}}$,

contradicting our assumption. Hence we must have $\alpha_0(a) > 0$ or $\alpha_0'(a) > 0$. If $\alpha_0(a) > 0$, then $\sigma\alpha_0(a) < 0$ so condition (iii) in Theorem 2.8 fails. Similarly for the case $\alpha_0'(a) > 0$.

(3) Similarly to the proof of case (1), $\sigma\alpha_0$ is the lowest weight of $\mathfrak{p}_+$. If $\alpha_0(a) \le 0$, then $\alpha(a) \le 0$ for all $\alpha \in \Delta(\mathfrak{p}_+, \mathfrak{t}_{\mathbb{C}})$. This implies that $\mathfrak{q}$ is anti-holomorphic, contradicting our assumption. In the same way, $\sigma\alpha_0(a) \ge 0$ implies that $\mathfrak{q}$ is holomorphic, a contradiction. Therefore $\alpha_0(a) > 0$ and $\sigma\alpha_0(a) < 0$ so condition (iii) in Theorem 2.8 fails. $\square$

The key assumption of Proposition 2.17 is that $-\sigma\alpha_0$ is dominant. In order to give a simple criterion to verify this, we consider the Satake diagram of the reductive Lie algebra $\mathfrak{k}^\sigma + \sqrt{-1}\mathfrak{k}^{-\sigma}$, which is a real form of $\mathfrak{k}_{\mathbb{C}}$ (see [2, Chapter X] for the Satake diagram). Each vertex is associated to a simple root of $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Then we add a vertex $\star$, indicating the highest weight $\alpha_0 (\in \mathfrak{t}_{\mathbb{C}}^*)$ of $\mathfrak{p}_{\mathbb{C}}$ or $\mathfrak{p}_+$. We connect this new vertex to the vertex associated to $\alpha_i$ if $\langle \alpha_0, \alpha_i \rangle > 0$. We can immediately tell whether $-\sigma\alpha_0$ is dominant from this diagram:

**Proposition 2.18.** *$-\sigma\alpha_0$ is $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$-dominant if and only if no black circle is connected to the new vertex $\star$.*

**Proof.** Write $\Delta^+ = \Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ for simplicity. Suppose that the vertex $\star$ is connected to the black circle associated to a simple root $\alpha_i$. Then $\langle -\sigma\alpha_0, \alpha_i \rangle = -\langle \alpha_0, \sigma\alpha_i \rangle = -\langle \alpha_0, \alpha_i \rangle < 0$ and hence $-\sigma\alpha_0$ is not $\Delta^+$-dominant.

Conversely, assume that there is no black circle connected to the vertex $\star$. Suppose that $\alpha \in -\sigma\Delta^+$. Then $-\sigma\alpha \in \Delta^+$ and hence $\langle -\sigma\alpha_0, \alpha \rangle = \langle \alpha_0, -\sigma\alpha \rangle \ge 0$. Suppose that $\alpha \in \Delta^+ \setminus -\sigma\Delta^+$. Since $\Delta^+$ is compatible with a positive restricted root system $\Sigma^+(\mathfrak{k}_{\mathbb{C}}, \sqrt{-1}\mathfrak{t}^{-\sigma})$, it follows that $\sigma\alpha = \alpha$ and $\alpha$ can be written as a linear sum of roots associated to black circles. Our assumption implies that $\alpha_0$ is orthogonal to any roots associated to black circles and hence orthogonal to $\alpha$. Thus, $\langle -\sigma\alpha_0, \alpha \rangle \ge 0$ for all $\alpha \in \Delta^+$. $\square$

Owing to Proposition 2.18, we can classify all the pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$ such that $-\sigma\alpha_0$ is $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$-dominant and $\mathfrak{t}^\sigma \ne 0$. See Appendix B for the list of the diagrams for all such pairs.

## 3. Discretely decomposable tensor product

The tensor product of two representations is a special case of the restriction with respect to a symmetric pair, namely, it is regarded as the restriction of an outer tensor product representation of the direct sum $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}'$, when restricted to the subalgebra $\mathfrak{g}^\sigma := \mathrm{diag}(\mathfrak{g}')$. In this section we discuss when the tensor product of $(\mathfrak{g}', K')$-modules $A_{\mathfrak{q}_1'}(\lambda_1) \otimes A_{\mathfrak{q}_2'}(\lambda_2)$ decomposes discretely. This is a branching problem of the $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}}(\lambda)$ with respect to $(\mathfrak{g}^\sigma, K^\sigma) := (\mathrm{diag}(\mathfrak{g}'), \mathrm{diag}(K'))$, where $K = K' \times K'$, $\mathfrak{q} = \mathfrak{q}_1' \oplus \mathfrak{q}_2'$ and $\lambda = (\lambda_1, \lambda_2)$.

**Theorem 3.1.** *Let $\mathfrak{g}'$ be a non-compact simple Lie algebra. Let $\mathfrak{q}_1'$ and $\mathfrak{q}_2'$ be $\theta$-stable parabolic subalgebras of $\mathfrak{g}_{\mathbb{C}}'$, not equal to $\mathfrak{g}_{\mathbb{C}}'$. Then the following three conditions on $\mathfrak{q}_1'$ and $\mathfrak{q}_2'$ are equivalent.*

(i) *The tensor product $A_{\mathfrak{q}_1'}(\lambda_1) \otimes A_{\mathfrak{q}_2'}(\lambda_2)$ is non-zero and discretely decomposable as a $(\mathfrak{g}', K')$-module for some $\lambda_1$ and $\lambda_2$ in the weakly fair range.*

(i′) *The tensor product $A_{\mathfrak{q}_1'}(\lambda_1) \otimes A_{\mathfrak{q}_2'}(\lambda_2)$ is discretely decomposable as a $(\mathfrak{g}', K')$-module for any $\lambda_1$ and $\lambda_2$ in the weakly fair range.*

(ii) *$\mathfrak{g}'$ is of Hermitian type and both $\mathfrak{q}'_1$ and $\mathfrak{q}'_2$ are simultaneously holomorphic or anti-holomorphic.*

**Proof.** Let $\mathfrak{t}'$ be a Cartan subalgebra of $\mathfrak{k}'$. Fix a positive system $\Delta^+(\mathfrak{k}'_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}})$. Suppose that $\mathfrak{q}'_1$ and $\mathfrak{q}'_2$ are given by $a_1 \in \sqrt{-1}\mathfrak{t}'$ and $a_2 \in \sqrt{-1}\mathfrak{t}'$, respectively. We set $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}'$, $\mathfrak{k} = \mathfrak{k}' \oplus \mathfrak{k}'$, $\mathfrak{t} = \mathfrak{t}' \oplus \mathfrak{t}'$, and $\mathfrak{q} = \mathfrak{q}'_1 \oplus \mathfrak{q}'_2$. Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$ and $\mathfrak{q}$ is a $\theta$-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. We define the involution $\sigma$ of $\mathfrak{g}$ as $\sigma(x, y) := (y, x)$ for $x, y \in \mathfrak{g}'$. Let $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ be the union of $\Delta^+(\mathfrak{k}'_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}})$ in the first factor and $-\Delta^+(\mathfrak{k}'_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}})$ in the second factor, so the condition of Setting 2.7(1) is satisfied. We assume that the defining element $a = (a_1, a_2)$ of $\mathfrak{q}$ is $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$-dominant. This means that $a_1$ and $-a_2$ are $\Delta^+(\mathfrak{k}'_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}})$-dominant. Then condition (iii) in Theorem 2.8 amounts to that

(i$''$) $\alpha(a_2) \geq 0$ whenever $\alpha \in \Delta(\mathfrak{p}'_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}})$ satisfies $\alpha(a_1) > 0$.

(i$''$) implies that $\alpha(a_1) \geq 0$ whenever $\alpha \in \Delta(\mathfrak{p}'_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}})$ satisfies $\alpha(a_2) > 0$.

By Theorem 2.8, it suffices to prove that (i$''$) is equivalent to (ii).

(ii) $\Rightarrow$ (i$''$): This is similar to Proposition 2.15. Suppose that $(\mathfrak{g}', \mathfrak{k}')$ is a Hermitian symmetric pair. We assume $\mathfrak{q}'_1$ and $\mathfrak{q}'_2$ are holomorphic with respect to $\mathfrak{p}'_{\mathbb{C}} = \mathfrak{p}'_+ + \mathfrak{p}'_-$. If $\alpha \in \Delta(\mathfrak{p}'_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}})$ satisfies $\alpha(a_1) > 0$, then $\alpha \in \Delta(\mathfrak{p}'_+, \mathfrak{t}'_{\mathbb{C}})$. Since $\Delta(\mathfrak{p}'_+, \mathfrak{t}'_{\mathbb{C}}) \subset \Delta(\mathfrak{q}'_2, \mathfrak{t}_{\mathbb{C}})$, it follows that $\alpha(a_2) \geq 0$ and hence (i$''$) holds. The same argument works when $\mathfrak{q}'_1$ and $\mathfrak{q}'_2$ are anti-holomorphic.

(i$''$) $\Rightarrow$ (ii): Suppose that $(\mathfrak{g}', \mathfrak{k}')$ is not a Hermitian symmetric pair. This means that $\mathfrak{k}'_{\mathbb{C}}$ is semisimple and acts irreducibly on $\mathfrak{p}'_{\mathbb{C}}$ by the adjoint action. Let us show $\alpha_0(a_1) > 0$ and $\alpha_0(a_2) < 0$ if $\alpha_0 \in \Delta(\mathfrak{p}'_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}})$ is the highest weight of $\mathfrak{p}'_{\mathbb{C}}$ with respect to $\Delta^+(\mathfrak{k}'_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}})$. First we observe that $\alpha_0(a_1) \geq 0$ because $a_1$ is dominant and $\mathfrak{k}'_{\mathbb{C}}$ is semisimple. If $\alpha_0(a_1) = 0$, then we would have $\mathfrak{p}'_{\mathbb{C}} \subset \mathfrak{q}'_1$, which would result in $\mathfrak{q}'_1 = \mathfrak{g}'_{\mathbb{C}}$, contradicting our assumption. Therefore $\alpha_0(a_1) > 0$. In the same way, we have $\alpha_0(a_2) < 0$. Hence (i$''$) fails.

Suppose now that $(\mathfrak{g}', \mathfrak{k}')$ is a Hermitian symmetric pair and fix a decomposition $\mathfrak{p}'_{\mathbb{C}} = \mathfrak{p}'_+ + \mathfrak{p}'_-$. We assume that (i$''$) holds. Let $\alpha_0 \in \Delta(\mathfrak{p}'_+, \mathfrak{t}'_{\mathbb{C}})$ be the highest weight of $\mathfrak{p}'_+$ with respect to $\Delta^+(\mathfrak{k}'_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}})$. Since $\Delta(\mathfrak{p}'_+, \mathfrak{t}'_{\mathbb{C}}) = -\Delta(\mathfrak{p}'_-, \mathfrak{t}'_{\mathbb{C}})$, we see that $-\alpha_0$ is the lowest weight of $\mathfrak{p}'_-$.

Now we assume that $\mathfrak{q}'_1$ is not anti-holomorphic, namely $\mathfrak{p}'_- \not\subset \mathfrak{q}'_1$. Then $\alpha_0(a_1) > 0$ because $\mathfrak{p}'_- \not\subset \mathfrak{q}'_1$ and $\alpha_0$ is the highest weight. Then (i$''$) implies that $\alpha_0(a_2) \geq 0$. Since $-a_2$ is dominant, $\alpha(-a_2) \leq \alpha_0(-a_2) \leq 0$ for every $\alpha \in \Delta(\mathfrak{p}'_+, \mathfrak{t}'_{\mathbb{C}})$. Therefore $\mathfrak{q}'_2$ is holomorphic. In particular, $\mathfrak{q}'_2$ is not anti-holomorphic, which in turn implies that $\mathfrak{q}'_1$ is holomorphic by the same argument.

Likewise, if we assume that $\mathfrak{q}'_1$ is not holomorphic, we see that $\mathfrak{q}'_1$ and $\mathfrak{q}'_2$ are anti-holomorphic. $\square$

In view of Theorems 2.4 and 3.1, we can tell when the tensor product of two discrete series representations decomposes discretely.

**Corollary 3.2.** *Suppose that $V_1$ and $V_2$ are the underlying $(\mathfrak{g}', K')$-modules of discrete series representations. Then $V_1 \otimes V_2$ is discretely decomposable as a $(\mathfrak{g}', K')$-module if and only if they are simultaneously holomorphic (or anti-holomorphic) discrete series representations.*

## 4. Classification of discretely decomposable $A_{\mathfrak{q}}(\lambda)$

The classification of the triples $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ goes as follows. The tensor product case was treated in Section 3. Consider the case where $\mathfrak{g}$ is simple. We fix a simple Lie algebra $\mathfrak{g}$ with a Cartan involution $\theta$. Suppose that $(\mathfrak{g}, \mathfrak{g}^{\sigma_1}, \mathfrak{q}_1)$ and $(\mathfrak{g}, \mathfrak{g}^{\sigma_2}, \mathfrak{q}_2)$ are triples such that $\mathrm{Ad}(k)\sigma_1\mathrm{Ad}(k^{-1}) = \sigma_2$ and $\mathrm{Ad}(k')\mathfrak{q}_1 = \mathfrak{q}_2$ for $k, k' \in K$. Then there is an isomorphism $A_{\mathfrak{q}_1}(\lambda_1)|_{\mathfrak{g}^{\sigma_1}} \simeq A_{\mathfrak{q}_2}(\lambda_2)|_{\mathfrak{g}^{\sigma_2}}$ via the isomorphism $\mathrm{Ad}(k) : \mathfrak{g}^{\sigma_1} \to \mathfrak{g}^{\sigma_2}$ if $\mathrm{Ad}^*(k')\lambda_1 = \lambda_2$. In this sense the branching problems with respect to $(\mathfrak{g}, \mathfrak{g}^{\sigma_1}, \mathfrak{q}_1)$ and $(\mathfrak{g}, \mathfrak{g}^{\sigma_2}, \mathfrak{q}_2)$ are equivalent. Thus, we will classify the triples $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ with the discrete decomposability condition up to the adjoint action $\mathrm{Ad}(K) \times \mathrm{Ad}(K)$. Here, $\sigma$ is an involution commuting with $\theta$ and $\mathfrak{q}$ is a $\theta$-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

Retain the notation and the assumption in Setting 2.7. In particular, the parabolic subalgebra $\mathfrak{q}$ is given by a $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$-dominant vector $a \in \sqrt{-1}\mathfrak{t}$. The classification of $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ with the discrete decomposability condition is given as conditions on the coordinates $a_i$ of $a$.

**Theorem 4.1.** *Let $(\mathfrak{g}, \mathfrak{g}^\sigma)$ be an irreducible symmetric pair such that $\sigma$ commutes with $\theta$ and let $\mathfrak{q}$ be a $\theta$-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$, not equal to $\mathfrak{g}_{\mathbb{C}}$. Suppose that $\lambda$ is in the weakly fair range and that $A_{\mathfrak{q}}(\lambda)$ is non-zero. Then $A_{\mathfrak{q}}(\lambda)$ is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module if and only if one of the following conditions on the triple $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ holds.*

(1) $\mathfrak{g}$ *is compact.*
(2) $\sigma = \theta$.
(3) $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}'$ *and* $\mathfrak{q} = \mathfrak{q}'_1 \oplus \mathfrak{q}'_2$. *Further,* $\mathfrak{g}'$ *is of Hermitian type and both of the parabolic subalgebras* $\mathfrak{q}'_1$ *and* $\mathfrak{q}'_2$ *of* $\mathfrak{g}'_{\mathbb{C}}$ *are holomorphic, or they are anti-holomorphic (see Table C.1 for holomorphic and anti-holomorphic parabolic subalgebras).*
(4) *The symmetric pair* $(\mathfrak{g}, \mathfrak{g}^\sigma)$ *is of holomorphic type (see Table C.2 for the classification) and the parabolic subalgebra* $\mathfrak{q}$ *is either holomorphic or anti-holomorphic.*
(5) *The triple* $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ *is isomorphic to one of those listed in Table C.3 or in Table C.4, where the parabolic subalgebra* $\mathfrak{q}$ *is given by the conditions on* $a$.

*In Tables C.1, C.3 and C.4, we have assumed that the defining element $a$ of $\mathfrak{q}$ is dominant with respect to $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ (see Appendix A for concrete conditions on the coordinates of $a$) and list only additional conditions for the discrete decomposability.*

**Proof.** If $\mathfrak{g}$ is compact, namely, if $\mathfrak{g}$ is isomorphic to the Lie algebra of a compact Lie group, then the discrete decomposability follows obviously.

We divide irreducible symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$ into the following four cases.

Case 1. In the tensor product case, a necessary and sufficient condition for the discrete decomposability was obtained in Theorem 3.1.

In the rest of the proof, we assume that $\mathfrak{g}$ is non-compact and simple.

Case 2. Suppose that $\mathfrak{t}^\sigma = 0$ or the assumption (1) or (2) in Proposition 2.17 is satisfied for a symmetric pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$. Then it follows from Propositions 2.16 and 2.17 that the triple $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ does not satisfy the discrete decomposability condition for any $\theta$-stable parabolic subalgebra $\mathfrak{q}$ other than $\mathfrak{g}_{\mathbb{C}}$.

Case 3. Suppose that assumption (3) in Proposition 2.17 is satisfied for a symmetric pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$. Then the triple $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ satisfies the discrete decomposability condition if and only if $\mathfrak{q}$ is holomorphic or anti-holomorphic.

We can verify which irreducible pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$ belong to Case 2 or Case 3. The condition $\mathfrak{t}^\sigma = 0$ holds if and only if $\mathfrak{k}^\sigma + \sqrt{-1}\mathfrak{k}^{-\sigma}$ is a split real form of $\mathfrak{k}_\mathbb{C}$, so it is easily verified. The case $\mathfrak{t}^\sigma \neq 0$ is less easy. We give a list of all the pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$ such that $-\sigma\alpha_0$ is $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$-dominant and $\mathfrak{t}^\sigma \neq 0$ in Appendix B. The verification of the dominancy of $-\sigma\alpha_0$ is reduced to a simple combinatorial problem by using the Satake diagram as we noted at the end of Section 2.

Case 4. The classification of the triples $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ with the discrete decomposability condition for the remaining symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is more delicate. For this, we apply the criterion Theorem 2.8(iii). This criterion reduces to simple computations for only the pair $(\mathfrak{k}, \mathfrak{k}^\sigma)$ and the set of weights $\Delta(\mathfrak{p}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. We then carry out the computation in a case-by-case way.

To be more precise, we classify the $K$-conjugacy classes of symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$, building on Berger's classification of symmetric pairs [1]. We postpone this until Section 5.

In Setting 2.7, we gave a symmetric pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$, followed by the choice of a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ and a positive system $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ that satisfy the compatibility condition with respect to $\sigma$ and finally we set a $\theta$-stable parabolic subalgebra $\mathfrak{q}$ given by a $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$-dominant vector $a \in \sqrt{-1}\mathfrak{t}$. In the following, however, we do this in a different order. We fix $\mathfrak{t}$ and $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ before $\sigma$ is given. This does not lose the generality because all the pairs $(\mathfrak{t}, \Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C}))$ are $K$-conjugate (recall that we treat $\sigma$ and $\mathfrak{q}$ up to $K \times K$-conjugacy). Then choose $\sigma$ that satisfies the conditions in Setting 2.7(1) with respect to $(\mathfrak{t}, \Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C}))$. Each $K$-conjugacy class of $\theta$-stable parabolic subalgebras of $\mathfrak{g}_\mathbb{C}$ has a unique representative $\mathfrak{q}$ which is given by a dominant vector $a \in \sqrt{-1}\mathfrak{t}$.

Let $\mathfrak{g}$ be a non-compact simple Lie algebra. Choose coordinates $e_i$ of $\mathfrak{t}_\mathbb{C}$ and write the defining element $a$ of $\mathfrak{q}$ as $a = \sum a_i e_i$ (see Appendix A). Fix a positive system $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. We assume that $a$ is $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$-dominant. For a given $K$-conjugacy class of symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$, we choose a representative $\sigma$ that satisfies the conditions in Setting 2.7(1). We describe the restriction of $\sigma$ to $\mathfrak{t}$ and then the $\sigma$-action on the set of weights $\Delta(\mathfrak{p}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. Now the condition Theorem 2.8(iii) amounts to conditions on the coordinates $a_i$.

We illustrate computations in the following two examples. Other cases are verified similarly. □

**Example 4.2.** Let $(\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{su}(m, n),\ \mathfrak{su}(m, k) \oplus \mathfrak{su}(n-k) \oplus \mathfrak{u}(1))$ for $k, n-k \geq 1$. We fix $\mathfrak{t}$, $\{\epsilon_i\}$, and $\{e_i\}$ as in Setting A.1. Choose $\sigma$ that satisfies the conditions in Setting 2.7(1), so the restriction of $\sigma$ to $\mathfrak{t}_\mathbb{C}$ can be written as $\sigma(e_i) = e_{\sigma(i)}$ for $1 \leq i \leq m+n$, where

$$\begin{aligned}
&\sigma(i) = i && \text{for } 1 \leq i \leq m,\\
&\sigma(m+j) = m+n-j+1 && \text{for } 1 \leq j \leq \min\{k, n-k\} \text{ or } \max\{k, n-k\} < j \leq n,\\
&\sigma(m+j) = m+j && \text{for } \min\{k, n-k\} < j \leq \max\{k, n-k\}.
\end{aligned}$$

Suppose that $\mathfrak{q}$ is given by a dominant vector $a = a_1 e_1 + \cdots + a_{m+n} e_{m+n} \in \sqrt{-1}\mathfrak{t}$, namely $a_1 \geq \cdots \geq a_m$ and $a_{m+1} \geq \cdots \geq a_{m+n}$ as in Setting A.1. If the condition (iii) in Theorem 2.8 is satisfied, then $a_i - a_{m+n} > 0$ implies $a_i - a_{m+1} \geq 0$ for $1 \leq i \leq m$. As a consequence, we see that the triple $(\mathfrak{su}(m, n),\ \mathfrak{su}(m, k) \oplus \mathfrak{su}(n-k) \oplus \mathfrak{u}(1),\ \mathfrak{q})$ satisfies the discrete decomposability condition if and only if

(1) $a_{m+n} \geq a_1$,
(2) there exists an integer $1 \leq l \leq m-1$ such that $a_1 \geq \cdots \geq a_l \geq a_{m+1} \geq \cdots \geq a_{m+n} \geq a_{l+1} \geq \cdots \geq a_m$, or

(3) $a_m \geq a_{m+1}$.

These triples are listed in Table C.3.

**Example 4.3.** Let $(\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{f}_{4(-20)}, \mathfrak{so}(8,1))$. Here, the exceptional Lie algebra $\mathfrak{f}_{4(-20)}$ is a real form of $\mathfrak{f}_4^{\mathbb{C}}$ with real rank one. We fix $\mathfrak{t}$, $\{\epsilon_i\}$, and $\{e_i\}$ as in Setting A.12. Choose $\sigma$ that satisfies the conditions in Setting 2.7(1), so the restriction of $\sigma$ to $\mathfrak{t}_{\mathbb{C}}$ can be written as

$$\sigma e_1 = -e_1,$$
$$\sigma e_i = e_i \quad \text{for } 2 \leq i \leq 4.$$

Suppose that $\mathfrak{q}$ is given by $a = a_1 e_1 + \cdots + a_4 e_4 \in \sqrt{-1}\mathfrak{t}$, namely $a_1 \geq \cdots \geq a_4 \geq 0$ as in Setting A.12. If the condition (iii) in Theorem 2.8 is satisfied, then $\frac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4)(a) \leq 0$ or $\frac{1}{2}(-\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4)(a) \geq 0$. The former implies $a_1 = a_2 = a_3 = a_4$ and the latter implies $a_1 = a_2 \geq a_3 = a_4 = 0$. Hence the triple $(\mathfrak{f}_{4(-20)}, \mathfrak{so}(8,1), \mathfrak{q})$ satisfies the discrete decomposability condition if and only if $(a_1, a_2, a_3, a_4) = (s, s, s, s)$ or $(s, s, 0, 0)$ for $s \geq 0$. These triples are listed in Table C.4.

From our classification result, we see that:

**Corollary 4.4.** *In the setting of Theorem 4.1, suppose that $\mathfrak{q}$ is a Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$. If $A_{\mathfrak{q}}(\lambda)$ is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module, then $\sigma = \theta$ or* $\operatorname{rank} \mathfrak{g}_{\mathbb{C}} = \operatorname{rank} \mathfrak{k}_{\mathbb{C}}$. *In particular, $A_{\mathfrak{q}}(\lambda)$ is isomorphic to the underlying $(\mathfrak{g}, K)$-module of a discrete series representation in the latter case as far as $\lambda$ is in the good range.*

**Remark 4.5.** The triples $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ in Table C.3 have the following property: there exists a $\theta$-stable Borel subalgebra $\mathfrak{b}$ contained in $\mathfrak{q}$ such that $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{b})$ also satisfies the discrete decomposability condition. This is also the case for (1)–(4) in Theorem 4.1. Then Proposition 2.10 implies that every $\theta$-stable parabolic subalgebra containing $\mathfrak{b}$ satisfies the discrete decomposability condition. We call such triples $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ *discrete series type*. The triples in Table C.3 together with (1)–(4) in Theorem 4.1 give all triples of discrete series type.

**Remark 4.6.** The remaining case is (5) in Theorem 4.1 for Table C.4. We call triples $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ in Table C.4 *isolated type*. For generic $m, n$ and $k$, discrete series type and isolated type are exclusive. However, for particular $m, n$ or $k$ there may be overlaps (see Remark 4.7(6), (7)).

**Remark 4.7.** We did not intend to write cases (1)–(5) in Theorem 4.1 in an exclusive way. Also there are some overlaps among the tables. What follows from (2)–(5) below shows overlaps between cases (4) and (5) in Theorem 4.1. (6) and (7) discuss some overlaps between Tables C.3 and C.4.

(1) Table C.2 includes the case $\sigma = \theta$ with $\mathfrak{g}$ Hermitian.
(2) The symmetric pair $(\mathfrak{su}(m,n), \mathfrak{su}(m,k) \oplus \mathfrak{su}(n-k) \oplus \mathfrak{u}(1))$ in Table C.3 is of holomorphic type and the parabolic subalgebra $\mathfrak{q}$ is holomorphic (anti-holomorphic) if $a_m \geq a_{m+1}$ $(a_{m+n} \geq a_1)$.
(3) The symmetric pairs $(\mathfrak{so}(2m, 2n), \mathfrak{so}(2m,k) \oplus \mathfrak{so}(2n-k))$ and $(\mathfrak{so}(2m, 2n+1), \mathfrak{so}(2m,k) \oplus \mathfrak{so}(2n-k+1))$ in Table C.3 are of holomorphic type if $m = 1$ and $\mathfrak{q}$ is holomorphic or anti-holomorphic if $m = 1$, $|a_1| \geq |a_2|$.

(4) The symmetric pair $(\mathfrak{so}(2m, 2n),\ \mathfrak{u}(m, n))$ in Table C.3 is of holomorphic type if $m = 1$ or $n = 1$ and the parabolic subalgebra $\mathfrak{q}$ is holomorphic or anti-holomorphic if $m = 1,\ a = a_1 e_1$ or $n = 1,\ a = a_{m+1} e_{m+1}$.

(5) The symmetric pairs $(\mathfrak{so}^*(2n),\ \mathfrak{so}^*(2n-2) \oplus \mathfrak{so}(2))$ and $(\mathfrak{so}^*(2n),\ \mathfrak{u}(n-1, 1))$ in Table C.4 are of holomorphic type and the parabolic subalgebra $\mathfrak{q}$ is holomorphic or anti-holomorphic if $k = 1$ or $n - 1$.

(6) The triple $(\mathfrak{so}(2m, 2n),\ \mathfrak{u}(m, n), \mathfrak{q})$ for $m = 2$ and $a = a_1 e_1$ in Table C.4 is also listed in Table C.3.

(7) The triple $(\mathfrak{su}(2m, 2n),\ \mathfrak{sp}(m, n), \mathfrak{q})$ for $m = 1$ and $(a_1, a_2;\ a_3, \ldots, a_{2n+2}) = (s, 0;\ t, 0, \ldots, 0)(s \geq t)$, $(0, -s;\ 0, \ldots, 0, -t)(s \geq t)$, $(s, -t;\ 0, \ldots, 0)(s, t \geq 0)$ mod $\mathbb{I}_{2n+2}$ in Table C.4 are also listed in Table C.3.

(8) There are also coincidences of Lie algebras with small rank such as $\mathfrak{sp}(2, \mathbb{R}) \simeq \mathfrak{so}(2, 3)$, $\mathfrak{so}(2, 4) \simeq \mathfrak{su}(2, 2)$, $\mathfrak{so}(3, 3) \simeq \mathfrak{sl}(4, \mathbb{R})$, and $\mathfrak{so}^*(6) \simeq \mathfrak{su}(1, 3)$.

**Remark 4.8.** Our classification of the triples $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ is carried out up to $K \times K$-conjugacy as we noted in the beginning of this section. In some cases, there exist more than one $K$-conjugacy classes of $(\mathfrak{g}, \mathfrak{g}^\sigma)$ for a given Lie algebra isomorphism class of $\mathfrak{g}^\sigma$. To save space, we did not distinguish between some different $K$-conjugacy classes in the tables if the discrete decomposability conditions with respect to them are the same. For given Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, we define $\mathcal{S}$, $\mathcal{T}$ and $\phi$ as we shall explain in (5.1). The elements of $\mathcal{S}$ correspond to the $K$-conjugacy classes of involutions $\sigma$ of $\mathfrak{g}$ such that $\theta\sigma = \sigma\theta$ and $\mathfrak{g}^\sigma$ is isomorphic to $\mathfrak{g}'$. The discrete decomposability condition only depends on their images in $\mathcal{T}$ by $\phi$.

(1) Let $\mathfrak{g} = \mathfrak{so}^*(8)$ and $\mathfrak{g}' = \mathfrak{so}^*(2) \oplus \mathfrak{so}^*(6) \simeq \mathfrak{u}(1, 3)$. There are two $K$-conjugacy classes of involutions $\sigma$ such that $\theta\sigma = \sigma\theta$ and $\mathfrak{g}^\sigma$ is isomorphic to $\mathfrak{g}'$, and one is the associated pair of the other. We list the two pairs $(\mathfrak{so}^*(8), \mathfrak{so}^*(2) \oplus \mathfrak{so}^*(6))$ and $(\mathfrak{so}^*(8), \mathfrak{u}(1, 3))$ in Table C.2, which are not $K$-conjugate to each other. This is case (1) in Proposition 5.1.

(2) Let $\mathfrak{g} = \mathfrak{su}(m, n)$. Among two types of symmetric pairs $(\mathfrak{su}(m, n), \mathfrak{su}(m, k) \oplus \mathfrak{su}(n - k) \oplus \mathfrak{u}(1))$ and $(\mathfrak{su}(m, n), \mathfrak{su}(m - k) \oplus \mathfrak{su}(k, n) \oplus \mathfrak{u}(1))$, we list only the former type in Tables C.3 and C.4 because the latter type can be treated by interchanging $m$ and $n$. Similarly for $\mathfrak{g} = \mathfrak{so}(m, n)$ or $\mathfrak{g} = \mathfrak{sp}(m, n)$.

(3) Let $\mathfrak{g} = \mathfrak{so}(2m, 2n)$ and $\mathfrak{g}' = \mathfrak{u}(m, n)$. Consider $K$-conjugacy classes of involutions $\sigma$ such that $\theta\sigma = \sigma\theta$ and $\mathfrak{g}^\sigma$ is isomorphic to $\mathfrak{g}'$. Then $\mathfrak{g}^{\theta\sigma}$ is also isomorphic to $\mathfrak{g}'$.

If both $m$ and $n$ are odd, then $|\mathrm{Im}\ \phi| = 1$ and $|\mathcal{S}| = 2$. This is case (1) in Proposition 5.1.

If $m$ is even and $n$ is odd, then $|\mathrm{Im}\ \phi| = 2$ and $|\mathcal{S}| = 2$. For every $\sigma$, case (3) in Proposition 5.1 occurs. The same holds if $m$ is odd and $n$ is even.

If both $m$ and $n$ are even, then $|\mathrm{Im}\ \phi| = 4$ and $|\mathcal{S}| = 4$. For every $\sigma$, case (3) in Proposition 5.1 occurs.

It turns out that the discrete decomposability condition depends on the $K$-conjugacy classes of $\sigma$ only if $m = 2$ (or $n = 2$). For $m = 2$ and $n \neq 2$, we write in Table C.3 as

$$\mathfrak{g}^\sigma = \mathfrak{u}(2, n)_1 \quad \text{if } \sigma(e_1) = -e_2,$$
$$\mathfrak{g}^\sigma = \mathfrak{u}(2, n)_2 \quad \text{if } \sigma(e_1) = e_2,$$

(if $m = 2$ and $n = 2k$ ($k > 1$), there are four $K$-conjugacy classes, so we group them two and two). For $m = n = 2$, we write in Table C.3 as

$$\begin{aligned}
\mathfrak{g}^\sigma &= \mathfrak{u}(2,2)_{11} \quad \text{if } \sigma(e_1) = -e_2 \text{ and } \sigma(e_3) = -e_4,\\
\mathfrak{g}^\sigma &= \mathfrak{u}(2,2)_{12} \quad \text{if } \sigma(e_1) = -e_2 \text{ and } \sigma(e_3) = e_4,\\
\mathfrak{g}^\sigma &= \mathfrak{u}(2,2)_{21} \quad \text{if } \sigma(e_1) = e_2 \text{ and } \sigma(e_3) = -e_4,\\
\mathfrak{g}^\sigma &= \mathfrak{u}(2,2)_{22} \quad \text{if } \sigma(e_1) = e_2 \text{ and } \sigma(e_3) = e_4.
\end{aligned}$$

Let $\mathfrak{g}$ be a simple non-compact Lie algebra and $\sigma(\neq \theta)$ an involution commuting with $\theta$. We illustrate by examples how to obtain all $\theta$-stable parabolic subalgebras $\mathfrak{q}$ of $\mathfrak{g}_\mathbb{C}$ such that $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ satisfy the discrete decomposability condition.

**Example 4.9.** Let $(\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{so}(4,2), \mathfrak{u}(2,1))$. Fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k} = \mathfrak{so}(4) \oplus \mathfrak{so}(2)$, a positive system $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$, and a basis of $\mathfrak{t}_\mathbb{C}$ as in Setting A.2. We use the letters $a_i'$, $e_i'$ instead of $a_i$, $e_i$ in Setting A.2. Suppose that $\mathfrak{g}^\sigma = \mathfrak{u}(2,1)_1$ in the notation of Remark 4.8. We assume $\mathfrak{q}$ is given by a $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$-dominant vector $a = a_1'e_1' + a_2'e_2' + a_3'e_3'$ as in Setting 2.7(2). According to Table C.2, the pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is of holomorphic type. Hence all holomorphic or anti-holomorphic parabolic subalgebras $\mathfrak{q}$ satisfy the discrete decomposability condition. According to Table C.1, $\mathfrak{q}$ is holomorphic or anti-holomorphic if and only if $|a_3'| \geq a_1'$. The pair $(\mathfrak{so}(4,2), \mathfrak{u}(2,1)_1)$ is listed in Table C.3. This says that $\mathfrak{q}$ satisfies the discrete decomposability condition if $-a_2' \geq |a_3'|$. The pair $(\mathfrak{so}(4,2), \mathfrak{u}(2,1))$ is also listed in Table C.4. This says that $\mathfrak{q}$ satisfies the discrete decomposability condition if $a = a_1'e_1'$ or $a = a_3'e_3'$. We also have an isomorphism

$$(\mathfrak{so}(4,2), \mathfrak{u}(2,1)) \simeq (\mathfrak{su}(2,2), \mathfrak{su}(2,1) \oplus \mathfrak{su}(1) \oplus \mathfrak{u}(1)).$$

Regard $\mathfrak{t}$ as a Cartan subalgebra of $\mathfrak{su}(2,2)$ and define $a_i$ and $e_i$ as in Setting A.1. Then we have

$$\begin{aligned}
a_1' &= \frac{1}{2}(a_1 - a_2 + a_3 - a_4), \qquad a_2' = \frac{1}{2}(-a_1 + a_2 + a_3 - a_4),\\
a_3' &= \frac{1}{2}(a_1 + a_2 - a_3 - a_4).
\end{aligned}$$

The pair $(\mathfrak{su}(2,2), \mathfrak{su}(2,1) \oplus \mathfrak{su}(1) \oplus \mathfrak{u}(1))$ is listed in Table C.3 and Table C.4. However, it turns out that no parabolic subalgebra other than that obtained in the previous argument satisfies the discrete decomposability condition. As a consequence, a parabolic subalgebra $\mathfrak{q}$ satisfies the discrete decomposability condition if and only if $\mathfrak{q}$ is given by $a$ for $|a_3'| \geq a_1'$ or $-a_2' \geq |a_3'|$ under the assumptions in Settings 2.7 and A.2. They correspond to $X_1, X_3, X_6, Y_1, Y_2, Y_5, Y_6, Z_1, Z_2, Z_3, Z_4, W$, or $U$. In all cases, the triples $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ are of discrete series type (Fig. 2).

**Example 4.10.** Let $(\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{su}(2,2), \mathfrak{sp}(1,1))$. Fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$, a positive system $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$, and $e_i$ as in Setting A.1. We assume $\mathfrak{q}$ is given by $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$-dominant vector $a = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4$ as in Setting 2.7(2). The pair $(\mathfrak{su}(2,2), \mathfrak{sp}(1,1))$ is listed in Table C.3. This says that $\mathfrak{q}$ satisfies the discrete decomposability condition if $a_1 \geq a_3 \geq a_4 \geq a_2$ or $a_3 \geq a_1 \geq a_2 \geq a_4$. The pair $(\mathfrak{su}(2,2), \mathfrak{sp}(1,1))$ is also listed in Table C.4 and we have an isomorphism

$$(\mathfrak{su}(2,2), \mathfrak{sp}(1,1)) \simeq (\mathfrak{so}(4,2), \mathfrak{so}(4,1)).$$

It turns out that $\mathfrak{q}$ satisfies the discrete decomposability condition if and only if $a_1 \geq a_3 \geq a_4 \geq a_2$ or $a_3 \geq a_1 \geq a_2 \geq a_4$ under the assumptions in Settings 2.7 and A.1. They correspond to

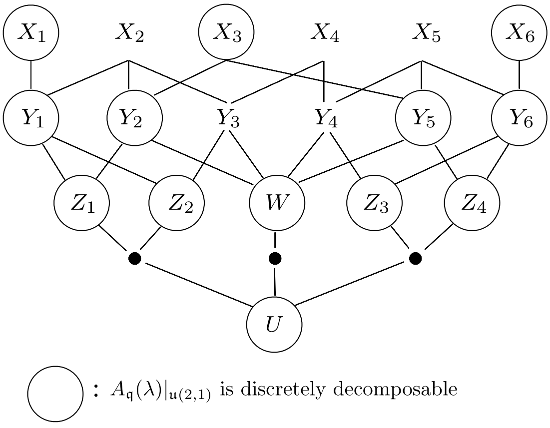


Fig. 2. $\mathfrak{so}(4, 2) \downarrow \mathfrak{u}(2, 1)$.

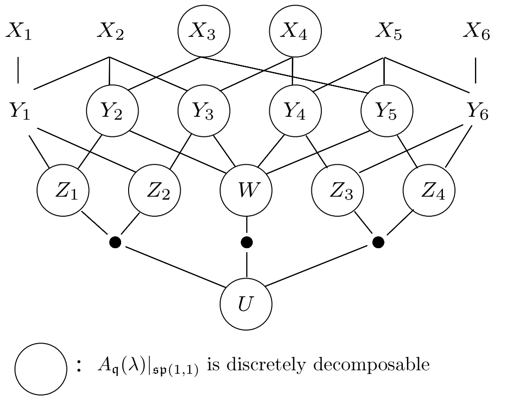


Fig. 3. $\mathfrak{su}(2, 2) \downarrow \mathfrak{sp}(1, 1)$.

$X_3$, $X_4$, $Y_2$, $Y_3$, $Y_4$, $Y_5$, $Z_1$, $Z_2$, $Z_3$, $Z_4$, $W$, or $U$. In all cases, the triples $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ are of discrete series type (Fig. 3).

We see from our classification that, in most cases, the center of $L = N_G(\mathfrak{q})$ is contained in $K$ if $A_{\mathfrak{q}}(\lambda)$ is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module. We classify the cases where $L$ has a split center, or equivalently $\lambda$ can be non-zero on $\mathfrak{l} \cap \mathfrak{p}$.

**Corollary 4.11.** *Let $(\mathfrak{g}, \mathfrak{g}^\sigma)$ be an irreducible symmetric pair such that $\sigma \neq \theta$. Suppose that $A_{\mathfrak{q}}(\lambda)$ is non-zero and discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module with $\lambda$ in the weakly fair range. Then $L$ has a split center if and only if*

$$\begin{aligned}(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{l}) = {} & (\mathfrak{sl}(2n, \mathbb{C}),\ \mathfrak{sp}(n, \mathbb{C}),\ \mathfrak{sl}(2n-1, \mathbb{C}) \oplus \mathbb{C}),\\ & (\mathfrak{sl}(2n, \mathbb{C}),\ \mathfrak{su}^*(2n),\ \mathfrak{sl}(2n-1, \mathbb{C}) \oplus \mathbb{C}),\\ & (\mathfrak{so}(2n, \mathbb{C}),\ \mathfrak{so}(2n-1, \mathbb{C}),\ \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{C}),\ \textit{or}\\ & (\mathfrak{so}(2n, \mathbb{C}),\ \mathfrak{so}(2n-1, 1),\ \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{C}).\end{aligned}$$

**Proof.** If $\operatorname{rank}\mathfrak{g}_{\mathbb{C}} = \operatorname{rank}\mathfrak{k}_{\mathbb{C}}$, then a fundamental Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$ is contained in $\mathfrak{k}$. In this case, the center of $L$ is contained in $K$.

Suppose that $\operatorname{rank}\mathfrak{g}_{\mathbb{C}} > \operatorname{rank}\mathfrak{k}_{\mathbb{C}}$. Then Theorem 4.1 implies that the pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is isomorphic to $(\mathfrak{so}(2m+1, 2n+1), \mathfrak{so}(2m+1, k) \oplus \mathfrak{so}(2n-k+1))$, $(\mathfrak{sl}(2n, \mathbb{C}), \mathfrak{sp}(n, \mathbb{C}))$, $(\mathfrak{sl}(2n, \mathbb{C}), \mathfrak{su}^*(2n))$, $(\mathfrak{so}(2n, \mathbb{C}), \mathfrak{so}(2n-1, \mathbb{C}))$, or $(\mathfrak{so}(2n, \mathbb{C}), \mathfrak{so}(2n-1, 1))$.

Let $(\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{so}(2m+1, 2n+1), \mathfrak{so}(2m+1, k) \oplus \mathfrak{so}(2n-k+1))$. By Theorem 4.1, we may assume that $\mathfrak{q}$ is given by $a$ with $a_{m+1} = \cdots = a_{m+n} = 0$ (see Table C.4). Then $\mathfrak{l}$ is a direct sum of $\mathfrak{so}(2l-1, 2n+1) (l \geq 1)$ and compact factors. Hence the center of $L$ is contained in $K$.

For the remaining four pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$, we have $\mathfrak{l} = \mathfrak{sl}(2n-1, \mathbb{C}) \oplus \mathbb{C}$ if $\mathfrak{g} = \mathfrak{sl}(2n, \mathbb{C})$ and $\mathfrak{l} = \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{C}$ if $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ (see Table C.4). Therefore $L$ has a split center in these cases.

As another consequence of Theorem 4.1, we get all the pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$ which do not have discretely decomposable restrictions $A_{\mathfrak{q}}(\lambda)|_{\mathfrak{g}^\sigma}$. We use the notation of [2, Chapter X] for exceptional Lie algebras.

**Theorem 4.12.** *Let $(\mathfrak{g}, \mathfrak{g}^\sigma)$ be an irreducible symmetric pair such that $\mathfrak{g}$ is non-compact and that $\sigma$ $(\neq \theta)$ commutes with $\theta$. The following two conditions on the pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ are equivalent.*

(i) *There is no $\theta$-stable parabolic subalgebra $\mathfrak{q}$ $(\neq \mathfrak{g}_{\mathbb{C}})$ such that the triple $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ satisfies the discrete decomposability condition (see Theorem 2.8).*
(ii) *One of the following cases occurs.*
   (1) $\mathfrak{g} \simeq \mathfrak{g}' \oplus \mathfrak{g}'$ *with $\mathfrak{g}'$ not of Hermitian type.*
   (2) *The simple Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(n, \mathbb{R}) (n \geq 5)$, $\mathfrak{so}(1, n)$, $\mathfrak{su}^*(2n)$, $\mathfrak{sl}(2n+1, \mathbb{C})$, $\mathfrak{so}(2n+1, \mathbb{C})$, $\mathfrak{sp}(n, \mathbb{C})$, $\mathfrak{g}_{2(2)}$, $\mathfrak{e}_{6(6)}$, $\mathfrak{e}_{6(-26)}$, $\mathfrak{e}_{7(7)}$, $\mathfrak{e}_{8(8)}$, $\mathfrak{g}_2^{\mathbb{C}}$, $\mathfrak{f}_4^{\mathbb{C}}$, $\mathfrak{e}_6^{\mathbb{C}}$, $\mathfrak{e}_7^{\mathbb{C}}$, or $\mathfrak{e}_8^{\mathbb{C}}$.*
   (3) $\mathfrak{k}^\sigma + \sqrt{-1}\mathfrak{k}^{-\sigma}$ *is a split real form of $\mathfrak{k}_{\mathbb{C}}$.*
   (4) *The pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is isomorphic to one of those listed in Table C.5.*

## 5. $K$-conjugacy classes of reductive symmetric pairs

In [1], irreducible symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$ are classified up to outer automorphisms of $\mathfrak{g}$. For our purpose, we need its refinement. To classify the $K$-conjugacy classes of $(\mathfrak{g}, \mathfrak{g}^\sigma)$, we have to tell whether or not two symmetric pairs $(\mathfrak{g}, \mathfrak{g}^{\sigma_1})$ and $(\mathfrak{g}, \mathfrak{g}^{\sigma_2})$ are $K$-conjugate to each other when $\mathfrak{g}^{\sigma_1}$ is isomorphic to $\mathfrak{g}^{\sigma_2}$ by an outer automorphism of $\mathfrak{g}$.

For this, we fix a reductive Lie algebra $\mathfrak{g}'$ with a Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{p}'$. Denote by $\mathcal{S} \equiv \mathcal{S}(\mathfrak{g}, \mathfrak{g}')$ the set of $K$-conjugacy classes of involutions $\sigma$ of $\mathfrak{g}$ such that $\sigma$ commutes with $\theta$ and that there is an isomorphism $\varphi : \mathfrak{g}^\sigma \xrightarrow{\sim} \mathfrak{g}'$ of Lie algebras with $\varphi(\mathfrak{k}^\sigma) = \mathfrak{k}'$. Similarly, denote by $\mathcal{T} \equiv \mathcal{T}(\mathfrak{k}, \mathfrak{k}')$ the set of $K$-conjugacy classes of involutions $\sigma$ of $\mathfrak{k}$ such that $\mathfrak{k}^\sigma$ is isomorphic to $\mathfrak{k}'$. We allow the case where $\sigma$ is the identity in the definition of $\mathcal{T}$. Then the restriction $\sigma \mapsto \sigma|_{\mathfrak{k}}$ induces a map:

$$\phi : \mathcal{S} \to \mathcal{T}. \tag{5.1}$$

The aim of this section is to classify the set $\mathcal{S}$. This is carried out by studying $\mathcal{T}$ and $\phi$.

First let us study $\mathcal{T} \equiv \mathcal{T}(\mathfrak{k}, \mathfrak{k}')$. There is a one-to-one correspondence between $\mathcal{T}$ and the set of $K$-conjugacy classes of real forms $\mathfrak{k}^\sigma + \sqrt{-1}\mathfrak{k}^{-\sigma}$ of $\mathfrak{k}_{\mathbb{C}}$ such that $\mathfrak{k}^\sigma \simeq \mathfrak{k}'$. Therefore the elements of $\mathcal{T}$ correspond to the Satake diagrams of real forms $\mathfrak{k}_0$ of $\mathfrak{k}_{\mathbb{C}}$ such that a maximal compact subalgebra of $\mathfrak{k}_0$ is isomorphic to $\mathfrak{k}'$. For a simple compact Lie algebra $\mathfrak{k}$, we see from the list of Satake diagrams ([2, Chapter X]) that

$$\begin{aligned}|\mathcal{T}| = 3 \quad &\text{if } (\mathfrak{k}, \mathfrak{k}') \simeq (\mathfrak{so}(8), \mathfrak{u}(4)) \simeq (\mathfrak{so}(8), \mathfrak{so}(2) \oplus \mathfrak{so}(6)),\\ &\qquad\qquad\quad (\mathfrak{so}(8), \mathfrak{so}(7)),\\ &\qquad\qquad\quad (\mathfrak{so}(8), \mathfrak{so}(3) \oplus \mathfrak{so}(5)),\\ |\mathcal{T}| = 2 \quad &\text{if } (\mathfrak{k}, \mathfrak{k}') \simeq (\mathfrak{so}(4n), \mathfrak{u}(2n)) \quad (n \geq 3),\end{aligned}$$

and $|\mathcal{T}| \leq 1$ if otherwise. For $\mathfrak{k}$ not simple, there may exist outer automorphisms which interchange simple factors. In such a case, the cardinality of $\mathcal{T}$ may also be greater than one.

Second we study the map $\phi$.

**Proposition 5.1.** *Let $x \in \mathcal{T}$. Suppose that the fiber $\phi^{-1}(x)$ is non-empty. Choose an involution $\sigma$ of $\mathfrak{g}$ which represents an element of $\phi^{-1}(x)$. Then one of the following three cases occurs.*

(1) $|\phi^{-1}(x)| = 2$ *and $\{\sigma, \theta\sigma\}$ is a complete set of representatives of $\phi^{-1}(x)$. In particular, $\mathfrak{g}^{\sigma}$ and $\mathfrak{g}^{\theta\sigma}$ are isomorphic as Lie algebras, but they are not $K$-conjugate to each other.*
(2) $|\phi^{-1}(x)| = 1$ *and $\mathfrak{g}^{\sigma}$ is not isomorphic to $\mathfrak{g}^{\theta\sigma}$ as a Lie algebra.*
(3) $|\phi^{-1}(x)| = 1$ *and $\mathfrak{g}^{\sigma}$ is $K$-conjugate to $\mathfrak{g}^{\theta\sigma}$.*

Let $y_1, y_2 \in \phi^{-1}(x)$. We can choose two involutions $\sigma_1$ and $\sigma_2$ of $\mathfrak{g}$ which represent $y_1$ and $y_2$, respectively, such that $\mathfrak{k}^{\sigma_1} = \mathfrak{k}^{\sigma_2}$. Therefore, the proof of Proposition 5.1 reduces to the following lemma.

**Lemma 5.2.** *Let $\sigma_1$ and $\sigma_2$ be involutions of a simple Lie algebra $\mathfrak{g}$ that commute with a Cartan involution $\theta$. If $\mathfrak{k}^{\sigma_1} = \mathfrak{k}^{\sigma_2}$, then $\sigma_1$ is $K$-conjugate to $\sigma_2$ or $\sigma_1 = \theta\sigma_2$.*

**Proof of the Lemma 5.2.** Since $\sigma_1 = \sigma_2$ on $\mathfrak{k}$, the composition $\tau = \sigma_1\sigma_2$ is an automorphism of $\mathfrak{g}$ that is the identity map on $\mathfrak{k}$. Then the restriction $\tau|_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{p}$ is an isomorphism of ad($\mathfrak{k}$)-modules.

Suppose that $\mathfrak{g}$ is not of Hermitian type. Then $\mathfrak{p}_{\mathbb{C}}$ is a simple $\mathfrak{k}$-module. Hence $\tau$ acts on $\mathfrak{p}$ as a scalar. Because $\tau = 1$ on $\mathfrak{k}$ and $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$, we have $\tau = 1$ or $-1$ on $\mathfrak{p}$. Therefore $\sigma_1 = \sigma_2$ or $\sigma_1 = \theta\sigma_2$.

Suppose that $\mathfrak{g}$ is of Hermitian type. Then $\mathfrak{p}_{\mathbb{C}}$ decomposes as a $\mathfrak{k}$-module: $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ + \mathfrak{p}_-$. We extend $\tau$ to a complex linear automorphism of $\mathfrak{g}_{\mathbb{C}}$ and use the same letter. Since $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are non-isomorphic simple $\mathfrak{k}$-modules, there are constants $c_+, c_- \in \mathbb{C}$ such that $\tau = c_+$ on $\mathfrak{p}_+$ and $\tau = c_-$ on $\mathfrak{p}_-$. In light that $[\mathfrak{p}_+, \mathfrak{p}_-] = \mathfrak{k}$ and $\tau = 1$ on $\mathfrak{k}$, we have $c_+c_- = 1$. We write $\overline{\mathfrak{p}_+}$ for the complex conjugate of $\mathfrak{p}_+$ with respect to the real form $\mathfrak{g}$. Since $\overline{\mathfrak{p}_+} = \mathfrak{p}_-$ and $\tau$ commutes with the complex conjugates, we have $\overline{c_+} = c_-$. Let $z \in \mathfrak{z}_K$ be a non-zero element of the center of $\mathfrak{k}$. Then we can write $\tau = \mathrm{Ad}(\exp(tz))$ for $t \in \mathbb{R}$. Since $\sigma_1\tau = \sigma_2$ is an involution, it follows that $\tau^{-1} = \sigma_1\tau\sigma_1 = \mathrm{Ad}(\exp(t\sigma_1 z))$. If the symmetric pair $(\mathfrak{g}, \mathfrak{g}^{\sigma_1})$ is of holomorphic type, then $\sigma_1 z = z$ and hence $\tau = \tau^{-1}$. Therefore, $c_+ = 1$ or $-1$ and it follows that $\sigma_1 = \sigma_2$ or $\sigma_1 = \sigma_2\theta$. If the symmetric pair $(\mathfrak{g}, \mathfrak{g}^{\sigma_1})$ is not of holomorphic type, then $\sigma_1 z = -z$. In this case, $\mathrm{Ad}(\exp(-tz/2))\sigma_1\mathrm{Ad}(\exp(tz/2)) = \sigma_2$, so $\sigma_1$ is $K$-conjugate to $\sigma_2$.

When $\mathfrak{g}^{\sigma}$ is isomorphic to $\mathfrak{g}^{\theta\sigma}$ as a Lie algebra, we use a case-by-case analysis to tell whether $\sigma$ and $\theta\sigma$ are $K$-conjugate and we conclude that:

**Proposition 5.3.** *For a symmetric pair* $(\mathfrak{g}, \mathfrak{g}')$ *with* $\mathfrak{g}$ *simple, Proposition* 5.1(1) *occurs if and only if* $(\mathfrak{g}, \mathfrak{g}')$ *is isomorphic to* $(\mathfrak{so}(4m+2, 4n+2), \mathfrak{u}(2m+1, 2n+1))$.

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## Appendix A. Setup for $\theta$-stable parabolic subalgebras

In this appendix we fix a positive system $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ with respect to a Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ of $\mathfrak{k}_{\mathbb{C}}$ and present the set of weights $\Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ for each simple Lie algebra $\mathfrak{g}$. We also write down the conditions for $a \in \sqrt{-1}\mathfrak{t}$ to be $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$-dominant in terms of the coordinates $a_i$, which are used in Tables C.1, C.3 and C.4.

In what follows, we do not include $\mathfrak{g}$ that has no non-trivial triple $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ satisfying the discrete decomposability condition (see Theorem 4.12(2)). We will define $\epsilon_i \in \mathfrak{t}_{\mathbb{C}}^*$ and $e_i \in \mathfrak{t}_{\mathbb{C}}$. If $\mathfrak{g}$ is not equal to $\mathfrak{su}(m,n)$, $\mathfrak{sl}(2n, \mathbb{C})$, $\mathfrak{e}_{6(2)}$, or $\mathfrak{e}_{7(-25)}$, then $\{\epsilon_i\}$ is a basis of $\mathfrak{t}_{\mathbb{C}}^*$ and $\{e_i\}$ is a dual basis of $\{\epsilon_i\}$.

**Setting A.1.** Let $\mathfrak{g} = \mathfrak{su}(m,n)$. Choose $\epsilon_1, \dots, \epsilon_{m+n} \in \mathfrak{t}_{\mathbb{C}}^*$ such that

$$\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\epsilon_i - \epsilon_j\}_{1\le i<j\le m} \cup \{\epsilon_{m+i} - \epsilon_{m+j}\}_{1\le i<j\le n},$$
$$\Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\pm(\epsilon_i - \epsilon_{m+j})\}_{1\le i\le m,\ 1\le j\le n}.$$

Define $e_1, \dots, e_{m+n} \in \mathfrak{t}_{\mathbb{C}}$ such that $(\epsilon_i - \epsilon_j)(e_k) = \delta_{ik} - \delta_{jk}$ and then $e_1 + \cdots + e_{m+n} = 0$. The dominant condition on $a = a_1e_1 + \cdots + a_{m+n}e_{m+n} \in \sqrt{-1}\mathfrak{t}$ amounts to that $a_1 \ge a_2 \ge \cdots \ge a_m$ and $a_{m+1} \ge a_{m+2} \ge \cdots \ge a_{m+n}$.

**Setting A.2.** Let $\mathfrak{g} = \mathfrak{so}(2m, 2n)$. Choose $\epsilon_1, \dots, \epsilon_{m+n} \in \mathfrak{t}_{\mathbb{C}}^*$ such that

$$\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\epsilon_i \pm \epsilon_j\}_{1\le i<j\le m} \cup \{\epsilon_{m+i} \pm \epsilon_{m+j}\}_{1\le i<j\le n},$$

$$\Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\pm\epsilon_i \pm \epsilon_{m+j}\}_{1\le i\le m,\ 1\le j\le n}.$$

Denote by $e_1, \dots, e_{m+n} \in \mathfrak{t}_{\mathbb{C}}$ the dual basis of $\epsilon_1, \dots, \epsilon_{m+n}$. The dominant condition on $a = a_1e_1 + \cdots + a_{m+n}e_{m+n} \in \sqrt{-1}\mathfrak{t}$ amounts to that $a_1 \ge \cdots \ge a_{m-1} \ge |a_m|$ and $a_{m+1} \ge \cdots \ge a_{m+n-1} \ge |a_{m+n}|$.

**Setting A.3.** Let $\mathfrak{g} = \mathfrak{so}(2m, 2n+1)$. Choose $\epsilon_1, \dots, \epsilon_{m+n} \in \mathfrak{t}_{\mathbb{C}}^*$ such that

$$\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\epsilon_i \pm \epsilon_j\}_{1\le i<j\le m} \cup \{\epsilon_{m+i} \pm \epsilon_{m+j}\}_{1\le i<j\le n} \cup \{\epsilon_{m+i}\}_{1\le i\le n},$$

$$\Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\pm\epsilon_i \pm \epsilon_{m+j}\}_{1\le i\le m,\ 1\le j\le n} \cup \{\pm\epsilon_i\}_{1\le i\le m}.$$

Denote by $e_1, \dots, e_{m+n} \in \mathfrak{t}_{\mathbb{C}}$ the dual basis of $\epsilon_1, \dots, \epsilon_{m+n}$. The dominant condition on $a = a_1e_1 + \cdots + a_{m+n}e_{m+n} \in \sqrt{-1}\mathfrak{t}$ amounts to that $a_1 \ge \cdots \ge a_{m-1} \ge |a_m|$ and $a_{m+1} \ge \cdots \ge a_{m+n} \ge 0$.

**Setting A.4.** Let $\mathfrak{g} = \mathfrak{so}(2m+1, 2n)$. Choose $\epsilon_1, \ldots, \epsilon_{m+n} \in \mathfrak{t}_{\mathbb{C}}^*$ such that

$$\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\epsilon_i \pm \epsilon_j\}_{1\le i<j\le m} \cup \{\epsilon_{m+i} \pm \epsilon_{m+j}\}_{1\le i<j\le n} \cup \{\epsilon_i\}_{1\le i\le m},$$

$$\Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\pm\epsilon_i \pm \epsilon_{m+j}\}_{1\le i\le m,\ 1\le j\le n} \cup \{\pm\epsilon_{m+i}\}_{1\le i\le n}.$$

Denote by $e_1, \ldots, e_{m+n} \in \mathfrak{t}_{\mathbb{C}}$ the dual basis of $\epsilon_1, \ldots, \epsilon_{m+n}$. The dominant condition on $a = a_1 e_1 + \cdots + a_{m+n} e_{m+n} \in \sqrt{-1}\mathfrak{t}$ amounts to that $a_1 \ge \cdots \ge a_m \ge 0$ and $a_{m+1} \ge \cdots \ge a_{m+n-1} \ge |a_{m+n}|$.

**Setting A.5.** Let $\mathfrak{g} = \mathfrak{so}(2m+1, 2n+1)$. Choose $\epsilon_1, \ldots, \epsilon_{m+n} \in \mathfrak{t}_{\mathbb{C}}^*$ such that

$$\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\epsilon_i \pm \epsilon_j\}_{1\le i<j\le m} \cup \{\epsilon_{m+i} \pm \epsilon_{m+j}\}_{1\le i<j\le n} \cup \{\epsilon_i\}_{1\le i\le m} \cup \{\epsilon_{m+i}\}_{1\le i\le n},$$
$$\Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\pm\epsilon_i \pm \epsilon_{m+j}\}_{1\le i\le m,\ 1\le j\le n} \cup \{\pm\epsilon_i\}_{1\le i\le m} \cup \{\pm\epsilon_{m+i}\}_{1\le i\le n} \cup \{0\}.$$

Denote by $e_1, \ldots, e_{m+n} \in \mathfrak{t}_{\mathbb{C}}$ the dual basis of $\epsilon_1, \ldots, \epsilon_{m+n}$. The dominant condition on $a = a_1 e_1 + \cdots + a_{m+n} e_{m+n} \in \sqrt{-1}\mathfrak{t}$ amounts to that $a_1 \ge \cdots \ge a_m \ge 0$ and $a_{m+1} \ge \cdots \ge a_{m+n} \ge 0$.

**Setting A.6.** Let $\mathfrak{g} = \mathfrak{sp}(m, n)$. Choose $\epsilon_1, \ldots, \epsilon_{m+n} \in \mathfrak{t}_{\mathbb{C}}^*$ such that

$$\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\epsilon_i \pm \epsilon_j\}_{1\le i<j\le m} \cup \{\epsilon_{m+i} \pm \epsilon_{m+j}\}_{1\le i<j\le n} \cup \{2\epsilon_i\}_{1\le i\le m} \cup \{2\epsilon_{m+i}\}_{1\le i\le n},$$
$$\Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\pm\epsilon_i \pm \epsilon_{m+j}\}_{1\le i\le m,\ 1\le j\le n}.$$

Denote by $e_1, \ldots, e_{m+n} \in \mathfrak{t}_{\mathbb{C}}$ the dual basis of $\epsilon_1, \ldots, \epsilon_{m+n}$. The dominant condition on $a = a_1 e_1 + \cdots + a_{m+n} e_{m+n} \in \sqrt{-1}\mathfrak{t}$ amounts to that $a_1 \ge \cdots \ge a_m \ge 0$ and $a_{m+1} \ge \cdots \ge a_{m+n} \ge 0$.

**Setting A.7.** Let $\mathfrak{g} = \mathfrak{so}^*(2n)$. Choose $\epsilon_1, \ldots, \epsilon_n \in \mathfrak{t}_{\mathbb{C}}^*$ such that

$$\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\epsilon_i - \epsilon_j\}_{1\le i<j\le n},$$
$$\Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\pm(\epsilon_i + \epsilon_j)\}_{1\le i<j\le n}.$$

Denote by $e_1, \ldots, e_n \in \mathfrak{t}_{\mathbb{C}}$ the dual basis of $\epsilon_1, \ldots, \epsilon_n$. The dominant condition on $a = a_1 e_1 + \cdots + a_n e_n \in \sqrt{-1}\mathfrak{t}$ amounts to that $a_1 \ge \cdots \ge a_n$.

**Setting A.8.** Let $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$. Choose $\epsilon_1, \ldots, \epsilon_n \in \mathfrak{t}_{\mathbb{C}}^*$ such that

$$\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\epsilon_i - \epsilon_j\}_{1\le i<j\le n},$$
$$\Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\pm 2\epsilon_i\}_{1\le i\le n} \cup \{\pm(\epsilon_i + \epsilon_j)\}_{1\le i<j\le n}.$$

Denote by $e_1, \ldots, e_n \in \mathfrak{t}_{\mathbb{C}}$ the dual basis of $\epsilon_1, \ldots, \epsilon_n$. The dominant condition on $a = a_1 e_1 + \cdots + a_n e_n \in \sqrt{-1}\mathfrak{t}$ amounts to that $a_1 \geq \cdots \geq a_n$.

**Setting A.9.** Let $\mathfrak{g} = \mathfrak{sl}(2n, \mathbb{C})$. Choose $\epsilon_1, \ldots, \epsilon_{2n} \in \mathfrak{t}_{\mathbb{C}}^*$ such that

$$\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\epsilon_i - \epsilon_j\}_{1 \leq i < j \leq 2n},$$
$$\Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\pm(\epsilon_i - \epsilon_j)\}_{1 \leq i < j \leq 2n} \cup \{0\}.$$

Define $e_1, \ldots, e_{2n} \in \mathfrak{t}_{\mathbb{C}}$ such that $(\epsilon_i - \epsilon_j)(e_k) = \delta_{ik} - \delta_{jk}$ and then $e_1 + \cdots + e_{2n} = 0$. The dominant condition on $a = a_1 e_1 + \cdots + a_{2n} e_{2n} \in \sqrt{-1}\mathfrak{t}$ amounts to that $a_1 \geq \cdots \geq a_{2n}$.

**Setting A.10.** Let $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$. Choose $\epsilon_1, \ldots, \epsilon_n \in \mathfrak{t}_{\mathbb{C}}^*$ such that

$$\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq n},$$
$$\Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\pm\epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq n} \cup \{0\}.$$

Denote by $e_1, \ldots, e_n \in \mathfrak{t}_{\mathbb{C}}$ the dual basis of $\epsilon_1, \ldots, \epsilon_n$. The dominant condition on $a = a_1 e_1 + \cdots + a_n e_n \in \sqrt{-1}\mathfrak{t}$ amounts to that $a_1 \geq \cdots \geq a_{n-1} \geq |a_n|$.

For real exceptional Lie algebras, we follow the notation of [2, Chapter X].

**Setting A.11.** Let $\mathfrak{g} = \mathfrak{f}_{4(4)} (\equiv \mathfrak{f}_4^1)$ so that $\mathfrak{k}_{\mathbb{C}} = \mathfrak{sp}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$. Choose $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \mathfrak{t}_{\mathbb{C}}^*$ such that

$$\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq 3} \cup \{2\epsilon_i\}_{1 \leq i \leq 3} \cup \{2\epsilon_4\},$$
$$\Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4\} \cup \{\pm\epsilon_i \pm \epsilon_4\}_{1 \leq i \leq 3}.$$

Denote by $e_1, e_2, e_3, e_4 \in \mathfrak{t}_{\mathbb{C}}$ the dual basis of $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$. The dominant condition on $a = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 \in \sqrt{-1}\mathfrak{t}$ amounts to that $a_1 \geq a_2 \geq a_3 \geq 0$ and $a_4 \geq 0$.

**Setting A.12.** Let $\mathfrak{g} = \mathfrak{f}_{4(-20)} (\equiv \mathfrak{f}_4^2)$ so that $\mathfrak{k}_{\mathbb{C}} = \mathfrak{so}(9, \mathbb{C})$. Choose $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \mathfrak{t}_{\mathbb{C}}^*$ such that

$$\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq 4} \cup \{\epsilon_i\}_{1 \leq i \leq 4},$$
$$\Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \left\{\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\right\}.$$

Denote by $e_1, e_2, e_3, e_4 \in \mathfrak{t}_{\mathbb{C}}$ the dual basis of $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$. The dominant condition on $a = a_1 e_1 + \cdots + a_4 e_4 \in \sqrt{-1}\mathfrak{t}$ amounts to that $a_1 \geq a_2 \geq a_3 \geq a_4 \geq 0$.

**Setting A.13.** Let $\mathfrak{g} = \mathfrak{e}_{6(2)} (\equiv \mathfrak{e}_6^2)$ so that $\mathfrak{k}_{\mathbb{C}} = \mathfrak{sl}(6, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$. Choose $\epsilon_1, \ldots, \epsilon_7 \in \mathfrak{t}_{\mathbb{C}}^*$ such that

$$\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\epsilon_i - \epsilon_j\}_{1 \leq i < j \leq 6} \cup \{2\epsilon_7\},$$
$$\Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \left\{\frac{1}{2}\left(\sum_{i=1}^{6}(-1)^{k(i)}\epsilon_i\right) \pm \epsilon_7 :\ k(i) \in \{0, 1\},\ k(1) + \cdots + k(6) = 3\right\}.$$

Define $e_1, \ldots, e_7 \in \mathfrak{t}_{\mathbb{C}}$ such that $(\epsilon_i - \epsilon_j)(e_k) = \delta_{ik} - \delta_{jk}$, $\epsilon_7(e_7) = 1$, and $(\epsilon_i - \epsilon_j)(e_7) = \epsilon_7(e_k) = 0$ for $1 \leq i, j, k \leq 6$. Then $e_1 + \cdots + e_6 = 0$. The dominant condition on $a = a_1 e_1 + \cdots + a_7 e_7 \in \sqrt{-1}\mathfrak{t}$ amounts to that $a_1 \geq \cdots \geq a_6$ and $a_7 \geq 0$.

**Setting A.14.** Let $\mathfrak{g} = \mathfrak{e}_{6(-14)} (\equiv \mathfrak{e}_6^3)$ so that $\mathfrak{k}_{\mathbb{C}} = \mathfrak{so}(10, \mathbb{C}) \oplus \mathbb{C}$. Choose $\epsilon_1, \dots, \epsilon_6 \in \mathfrak{t}_{\mathbb{C}}^*$ such that

$$\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\epsilon_i \pm \epsilon_j\}_{1 \le i < j \le 5},$$
$$\Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \left\{ \frac{1}{2} \left( \sum_{i=1}^{6} (-1)^{k(i)} \epsilon_i \right) : \ k(1) + \dots + k(6) \text{ odd} \right\}.$$

Denote by $e_1, \dots, e_6 \in \mathfrak{t}_{\mathbb{C}}$ the dual basis of $\epsilon_1, \dots, \epsilon_6$. The dominant condition on $a = a_1 e_1 + \dots + a_6 e_6 \in \sqrt{-1}\mathfrak{t}$ amounts to that $a_1 \ge \dots \ge a_4 \ge |a_5|$.

**Setting A.15.** Let $\mathfrak{g} = \mathfrak{e}_{7(-5)} (\equiv \mathfrak{e}_7^2)$ so that $\mathfrak{k}_{\mathbb{C}} = \mathfrak{so}(12, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$. Choose $\epsilon_1, \dots, \epsilon_7 \in \mathfrak{t}_{\mathbb{C}}^*$ such that

$$\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\epsilon_i \pm \epsilon_j\}_{1 \le i < j \le 6} \cup \{2\epsilon_7\},$$
$$\Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \left\{ \frac{1}{2} \left( \sum_{i=1}^{6} (-1)^{k(i)} \epsilon_i \right) \pm \epsilon_7 : \ k(1) + \dots + k(6) \text{ odd} \right\}.$$

Denote by $e_1, \dots, e_7 \in \mathfrak{t}_{\mathbb{C}}$ the dual basis of $\epsilon_1, \dots, \epsilon_7$. The dominant condition on $a = a_1 e_1 + \dots + a_7 e_7 \in \sqrt{-1}\mathfrak{t}$ amounts to that $a_1 \ge \dots \ge a_5 \ge |a_6|$ and $a_7 \ge 0$.

**Setting A.16.** Let $\mathfrak{g} = \mathfrak{e}_{7(-25)} (\equiv \mathfrak{e}_7^3)$ so that $\mathfrak{k}_{\mathbb{C}} = \mathfrak{e}_6^{\mathbb{C}} \oplus \mathbb{C}$. Choose $\epsilon_1, \dots, \epsilon_8 \in \mathfrak{t}_{\mathbb{C}}^*$ such that

$$\begin{aligned} \Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) &= \{\epsilon_i \pm \epsilon_j\}_{1 \le j < i \le 5} \\ &\quad \cup \left\{ \frac{1}{2} \left( \epsilon_8 - \epsilon_7 - \epsilon_6 + \sum_{i=1}^{5} (-1)^{k(i)} \epsilon_i \right) : \ k(1) + \dots + k(5) \text{ even} \right\}, \\ \Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) &= \{\pm\epsilon_6 \pm \epsilon_i\}_{1 \le i \le 5} \cup \{\pm(\epsilon_8 - \epsilon_7)\} \\ &\quad \cup \left\{ \pm\frac{1}{2} \left( \epsilon_8 - \epsilon_7 + \epsilon_6 + \sum_{i=1}^{5} (-1)^{k(i)} \epsilon_i \right) : \ k(1) + \dots + k(5) \text{ odd} \right\}. \end{aligned}$$

Define $e_1, \dots, e_8 \in \mathfrak{t}_{\mathbb{C}}$ such that $\epsilon_i(e_j) = \delta_{ij}$ for $1 \le i \le 6, \ 1 \le j \le 8$ and that $(\epsilon_8 - \epsilon_7)(e_i) = \delta_{i8} - \delta_{i7}$ for $1 \le i \le 8$. Then $e_8 + e_7 = 0$. The dominant condition on $a = a_1 e_1 + \dots + a_8 e_8 \in \sqrt{-1}\mathfrak{t}$ amounts to that $a_5 \ge \dots \ge a_2 \ge |a_1|$ and $a_8 - a_7 - a_6 - a_5 - a_4 - a_3 - a_2 + a_1 \ge 0$.

**Setting A.17.** Let $\mathfrak{g} = \mathfrak{e}_{8(-24)} (\equiv \mathfrak{e}_8^2)$ so that $\mathfrak{k}_{\mathbb{C}} = \mathfrak{e}_7^{\mathbb{C}} \oplus \mathfrak{sl}(2, \mathbb{C})$. Choose $\epsilon_1, \dots, \epsilon_8 \in \mathfrak{t}_{\mathbb{C}}^*$ such that

$$\begin{aligned} \Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) &= \{\epsilon_i \pm \epsilon_j\}_{1 \le j < i \le 6} \cup \{\epsilon_8 \pm \epsilon_7\} \\ &\quad \cup \left\{ \frac{1}{2} \left( \epsilon_8 - \epsilon_7 + \sum_{i=1}^{6} (-1)^{k(i)} \epsilon_i \right) : \ k(1) + \dots + k(6) \text{ odd} \right\}, \\ \Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) &= \{\pm\epsilon_7 \pm \epsilon_i\}_{1 \le i \le 6} \cup \{\pm\epsilon_8 \pm \epsilon_i\}_{1 \le i \le 6} \\ &\quad \cup \left\{ \pm\frac{1}{2} \left( \epsilon_8 + \epsilon_7 + \sum_{i=1}^{6} (-1)^{k(i)} \epsilon_i \right) : \ k(1) + \dots + k(6) \text{ even} \right\}. \end{aligned}$$

Denote by $e_1, \dots, e_8 \in \mathfrak{t}_{\mathbb{C}}$ the dual basis of $\epsilon_1, \dots, \epsilon_8$. The dominant condition on $a = a_1 e_1 + \dots + a_8 e_8 \in \sqrt{-1}\mathfrak{t}$ amounts to that $a_6 \ge \dots \ge a_2 \ge |a_1|$ and $a_8 - a_7 - a_6 - a_5 - a_4 - a_3 - a_2 + a_1 \ge 0$.

## Appendix B. List of symmetric pairs satisfying the assumption of Proposition 2.17

In this appendix we assume that $\mathfrak{g}$ is a non-compact simple Lie algebra and classify all the irreducible symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$ satisfying the following two conditions:

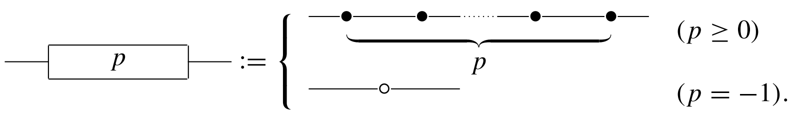
(1) $-\sigma\alpha_0$ is dominant with respect to $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$,
(2) $\mathfrak{t}^\sigma \neq 0$,

where $\alpha_0$ is the highest weight of $\mathfrak{p}_\mathbb{C}$ with respect to $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ (if $\mathfrak{g}$ is not of Hermitian type) or that of $\mathfrak{p}_+$ (if $\mathfrak{g}$ is of Hermitian type). Condition (1) is the key assumption in Proposition 2.17. Recall we have assumed that $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ and $\sigma$ satisfy the compatibility condition of Setting 2.7(1). If $\mathfrak{t}^\sigma = 0$, then we can apply Proposition 2.16 and we see there is no parabolic subalgebra other than $\mathfrak{g}_\mathbb{C}$ satisfying the discrete decomposability condition. To save space, we do not list the pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$ with $\mathfrak{t}^\sigma = 0$ because we can easily verify the condition $\mathfrak{t}^\sigma = 0$, which is equivalent to that $\mathfrak{k}^\sigma + \sqrt{-1}\mathfrak{k}^{-\sigma}$ is a split real form of $\mathfrak{k}_\mathbb{C}$.

In view of Proposition 2.18, the classification of such pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is carried out by using diagrams. We thus write the Satake diagram of $\mathfrak{k}^\sigma + \sqrt{-1}\mathfrak{k}^{-\sigma}$ and add a vertex $\star$, which is associated to the weight $\alpha_0$ as explained in the paragraph before Proposition 2.18.

In Appendix B.1, we list all the pairs satisfying Assumption (1) or (2) in Proposition 2.17 and $\mathfrak{t}^\sigma \neq 0$. For these pairs, no parabolic subalgebra other than $\mathfrak{g}_\mathbb{C}$ satisfies the discrete decomposability condition. In Appendix B.2, we list all the pairs satisfying Assumption (3) of Proposition 2.17 and $\mathfrak{t}^\sigma \neq 0$. By Propositions 2.15 and 2.17, the triple $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ satisfies the discrete decomposability condition if and only if $\mathfrak{q}$ is holomorphic or anti-holomorphic (Definition 2.12).

In what follows, it is convenient to use the following symbol:



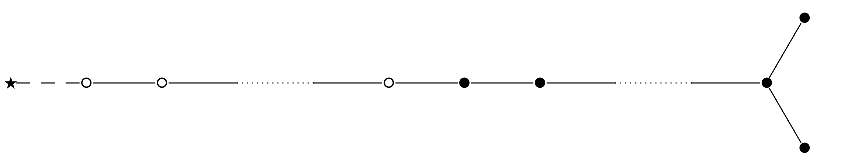

We note that the diagram for $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is the same as that for the associated pair $(\mathfrak{g}, \mathfrak{g}^{\theta\sigma})$.

### *B.1. Case of non-holomorphic symmetric pairs*

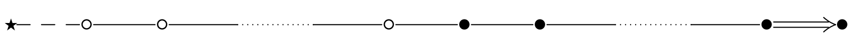
#### *B.1.1*

$$m, n-m \geq 1, \quad |n-2m| \geq 2$$
$$(\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{sl}(n, \mathbb{R}), \mathfrak{sl}(m, \mathbb{R}) \oplus \mathfrak{sl}(n-m, \mathbb{R}) \oplus \mathbb{R})$$
$$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{sl}(n, \mathbb{R}), \mathfrak{so}(m, n-m)).$$

Case: $n$ even



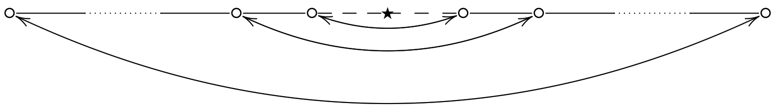
Case: $n$ odd



*B.1.2*

$n \geq 2$

$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{su}(n, n), \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R})$

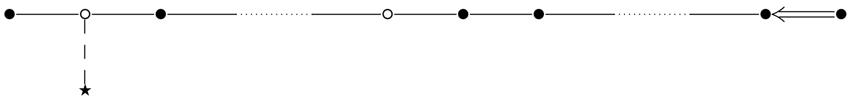


*B.1.3*

$m, n - m \geq 1$

$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{su}^*(2n), \mathfrak{su}^*(2m) \oplus \mathfrak{su}^*(2n - 2m) \oplus \mathbb{R})$

$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{su}^*(2n), \mathfrak{sp}(m, n - m))$

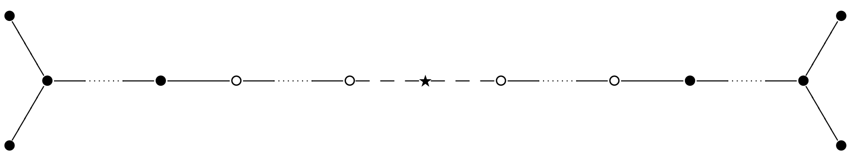


*B.1.4*

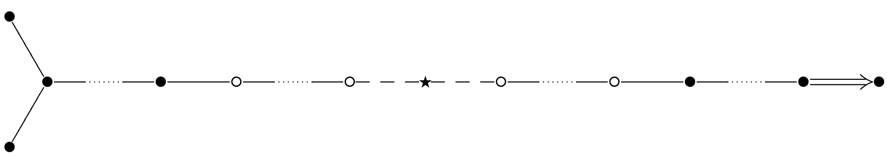
$k, l, m - k, n - l \geq 1, \quad \max\{|m - 2k|, |n - 2l|\} \geq 2$

$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{so}(m, n), \mathfrak{so}(k, l) \oplus \mathfrak{so}(m - k, n - l)).$

Case: $m, n$ even



Case: $m + n$ odd



Case: $m, n$ odd

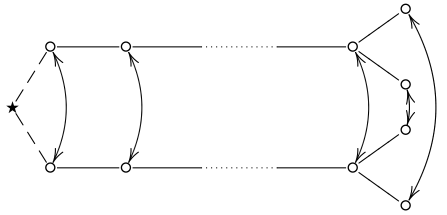
*B.1.5*

$n \geq 3$

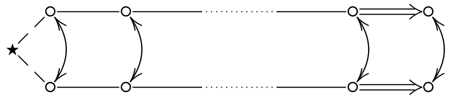
$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{so}(n, n), \mathfrak{so}(n, \mathbb{C}))$

$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{so}(n, n), \mathfrak{gl}(n, \mathbb{R})).$

Case: $n$ even



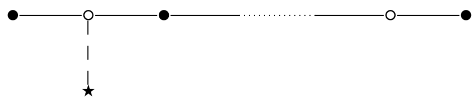
Case: $n$ odd



*B.1.6*

$n \geq 2$

$(\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{so}^*(4n), \mathfrak{su}^*(2n) \oplus \mathbb{R})$

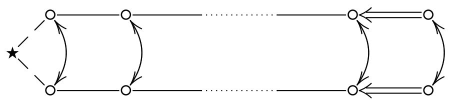


*B.1.7*

$n \geq 1$

$(\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{sp}(n, n), \mathfrak{sp}(n, \mathbb{C}))$

$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{sp}(n, n), \mathfrak{su}^*(2n) \oplus \mathbb{R})$

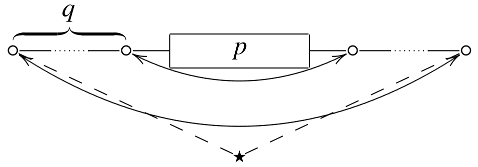


*B.1.8*

$m, n - m \geq 1$

$(\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{sl}(n, \mathbb{C}), \mathfrak{sl}(m, \mathbb{C}) \oplus \mathfrak{sl}(n - m, \mathbb{C}) \oplus \mathbb{C})$

$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{sl}(n, \mathbb{C}), \mathfrak{su}(m, n - m))$



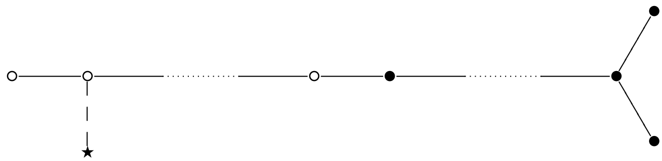

$$p = \begin{cases} n - 2m - 1 & \text{if } m < n - m \\ 2m - n - 1 & \text{if } m > n - m \\ -1 & \text{if } m = n - m \end{cases} \qquad q = \begin{cases} m & \text{if } m < n - m \\ n - m & \text{if } m > n - m \\ m - 1 & \text{if } m = n - m. \end{cases}$$

*B.1.9*

$m, 2n - m \geq 2, \quad |2n - 2m| \geq 2$

$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{so}(2n, \mathbb{C}), \mathfrak{so}(m, \mathbb{C}) \oplus \mathfrak{so}(2n - m, \mathbb{C}))$

$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{so}(2n, \mathbb{C}), \mathfrak{so}(m, 2n - m))$

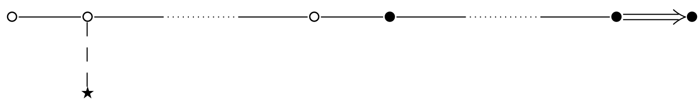


*B.1.10*

$m, 2n - m + 1 \geq 2, \qquad |2n - 2m + 1| \geq 2$

$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{so}(2n + 1, \mathbb{C}), \mathfrak{so}(m, \mathbb{C}) \oplus \mathfrak{so}(2n - m + 1, \mathbb{C}))$

$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{so}(2n + 1, \mathbb{C}), \mathfrak{so}(m, 2n - m + 1))$



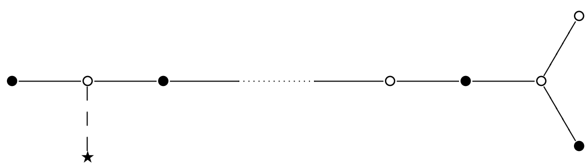
*B.1.11*

$n \geq 4$

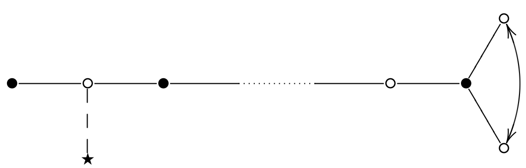
$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{so}(2n, \mathbb{C}), \mathfrak{gl}(n, \mathbb{C}))$

$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{so}(2n, \mathbb{C}), \mathfrak{so}^*(2n))$.

Case: $n$ even



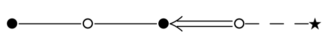
Case: $n$ odd



*B.1.12*

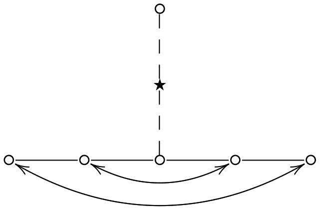
$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{e}_{6(6)}, \mathfrak{so}(5, 5) \oplus \mathbb{R})$

$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{e}_{6(6)}, \mathfrak{sp}(2, 2))$



*B.1.13*

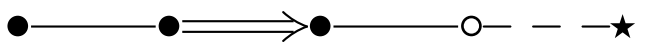
$(\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{e}_{6(2)}, \mathfrak{su}(3,3) \oplus \mathfrak{sl}(2,\mathbb{R}))$



*B.1.14*

$(\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{e}_{6(-26)}, \mathfrak{so}(1,9) \oplus \mathbb{R})$

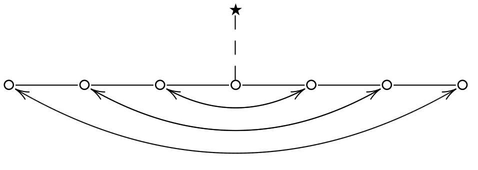
$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{e}_{6(-26)}, \mathfrak{f}_{4(-20)})$



*B.1.15*

$(\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{e}_{7(7)}, \mathfrak{so}(6,6) \oplus \mathfrak{sl}(2,\mathbb{R}))$

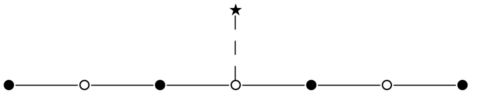
$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{e}_{7(7)}, \mathfrak{su}(4,4))$



*B.1.16*

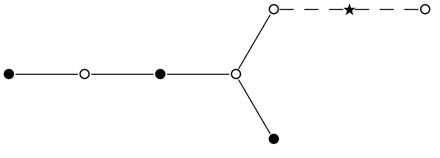
$(\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{e}_{7(7)}, \mathfrak{su}^*(8))$

$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(6)} \oplus \mathbb{R})$



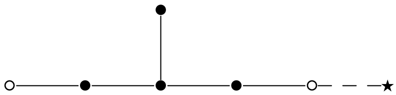
*B.1.17*

$(\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{e}_{7(-5)}, \mathfrak{so}^*(12) \oplus \mathfrak{sl}(2,\mathbb{R}))$



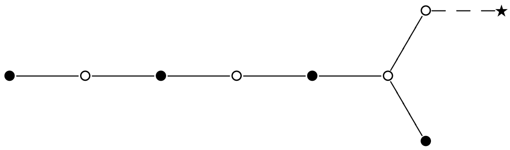
*B.1.18*

$$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-26)} \oplus \mathbb{R})$$



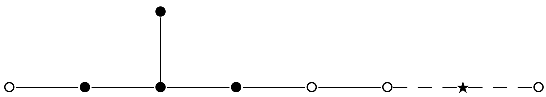
*B.1.19*

$$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{e}_{8(8)}, \mathfrak{so}^*(16))$$
$$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{e}_{8(8)}, \mathfrak{e}_{7(7)} \oplus \mathfrak{sl}(2, \mathbb{R}))$$



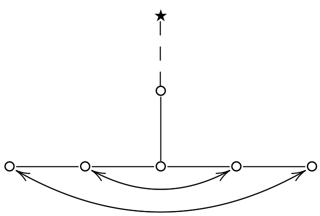
*B.1.20*

$$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{e}_{8(-24)}, \mathfrak{e}_{7(-25)} \oplus \mathfrak{sl}(2, \mathbb{R}))$$



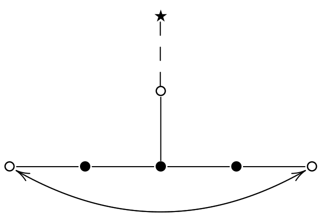
*B.1.21*

$$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{e}_6^{\mathbb{C}}, \mathfrak{sl}(6, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}))$$
$$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{e}_6^{\mathbb{C}}, \mathfrak{e}_{6(2)})$$



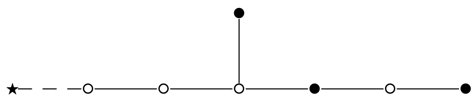
*B.1.22*

$$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{e}_6^{\mathbb{C}}, \mathfrak{so}(10, \mathbb{C}) \oplus \mathbb{C})$$
$$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{e}_6^{\mathbb{C}}, \mathfrak{e}_{6(-14)})$$



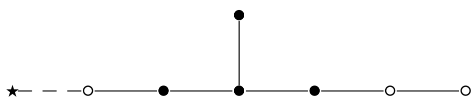
*B.1.23*

$$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{e}_7^{\mathbb{C}}, \mathfrak{so}(12, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}))$$
$$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{e}_7^{\mathbb{C}}, \mathfrak{e}_{7(-5)})$$



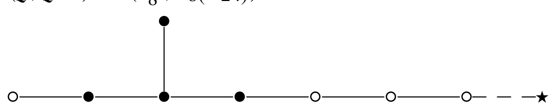
*B.1.24*

$$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{e}_7^{\mathbb{C}}, \mathfrak{e}_6^{\mathbb{C}} \oplus \mathbb{C})$$
$$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{e}_7^{\mathbb{C}}, \mathfrak{e}_{7(-25)})$$



*B.1.25*

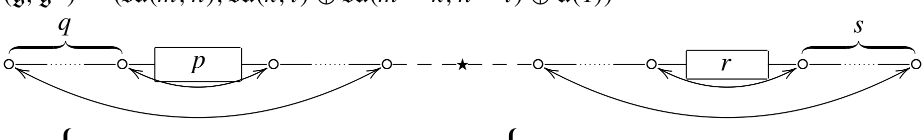
$$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{e}_8^{\mathbb{C}}, \mathfrak{e}_7^{\mathbb{C}} \oplus \mathfrak{sl}(2, \mathbb{C}))$$
$$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{e}_8^{\mathbb{C}}, \mathfrak{e}_{8(-24)})$$



## B.2. *Case of holomorphic symmetric pairs*

*B.2.1*

$$k, l, m-k, n-l \geq 1$$
$$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{su}(m, n), \mathfrak{su}(k, l) \oplus \mathfrak{su}(m-k, n-l) \oplus \mathfrak{u}(1))$$

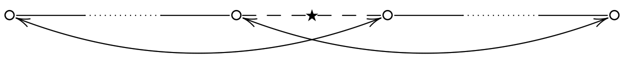



$$p = \begin{cases} m-2k-1 & \text{if } k < m-k \\ 2k-m-1 & \text{if } k > m-k \\ -1 & \text{if } k = m-k \end{cases} \qquad q = \begin{cases} k & \text{if } k < m-k \\ m-k & \text{if } k > m-k \\ k-1 & \text{if } k = m-k \end{cases}$$

$$r = \begin{cases} n-2l-1 & \text{if } l < n-l \\ 2l-n-1 & \text{if } l > n-l \\ -1 & \text{if } l = n-l \end{cases} \qquad s = \begin{cases} l & \text{if } l < n-l \\ n-l & \text{if } l > n-l \\ l-1 & \text{if } l = n-l. \end{cases}$$

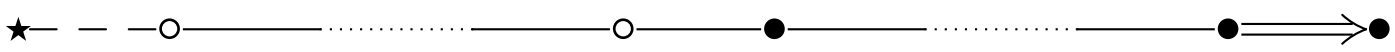
*B.2.2*

$n \geq 2$
$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{su}(n, n), \mathfrak{so}^*(2n))$
$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{su}(n, n), \mathfrak{sp}(n, \mathbb{R}))$



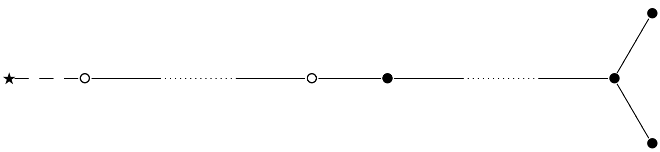
*B.2.3*

$m, 2n - m + 1 \geq 1$
$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{so}(2, 2n + 1), \mathfrak{so}(2, m) \oplus \mathfrak{so}(2n - m + 1))$



*B.2.4*

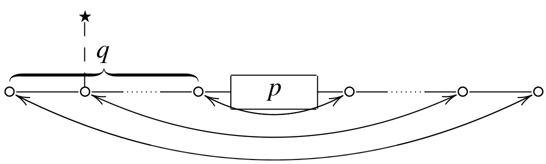
$m, 2n - m \geq 1$
$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{so}(2, 2n), \mathfrak{so}(2, m) \oplus \mathfrak{so}(2n - m))$



*B.2.5*

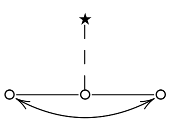
$m, n - m \geq 2$
$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{so}^*(2n), \mathfrak{u}(m, n - m))$
$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{so}^*(2n), \mathfrak{so}^*(2m) \oplus \mathfrak{so}^*(2n - 2m))$.

Case: $n \geq 5$




$$p = \begin{cases} n - 2m - 1 & \text{if } m < n - m \\ 2m - n - 1 & \text{if } m > n - m \\ -1 & \text{if } m = n - m \end{cases} \qquad q = \begin{cases} m & \text{if } m < n - m \\ n - m & \text{if } m > n - m \\ m - 1 & \text{if } m = n - m. \end{cases}$$

Case: $m = 2, n = 4$

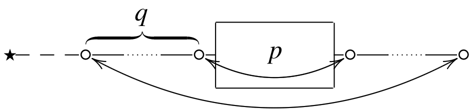


*B.2.6*

$$m, n-m \geq 1$$

$$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{sp}(n, \mathbb{R}), \mathfrak{u}(m, n-m))$$

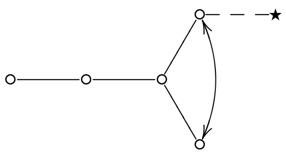
$$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{sp}(n, \mathbb{R}), \mathfrak{sp}(m, \mathbb{R}) \oplus \mathfrak{sp}(n-m, \mathbb{R}))$$




$$p = \begin{cases} n-2m-1 & \text{if } m < n-m \\ 2m-n-1 & \text{if } m > n-m \\ -1 & \text{if } m = n-m \end{cases} \qquad q = \begin{cases} m & \text{if } m < n-m \\ n-m & \text{if } m > n-m \\ m-1 & \text{if } m = n-m. \end{cases}$$

*B.2.7*

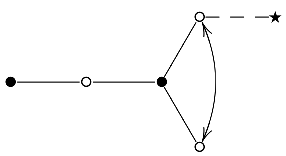
$$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{e}_{6(-14)}, \mathfrak{su}(4, 2) \oplus \mathfrak{su}(2))$$



*B.2.8*

$$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{e}_{6(-14)}, \mathfrak{so}^*(10) \oplus \mathfrak{so}(2))$$

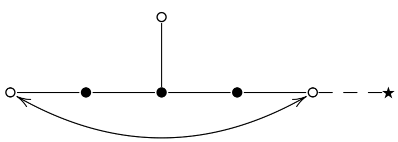
$$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{e}_{6(-14)}, \mathfrak{su}(5, 1) \oplus \mathfrak{sl}(2, \mathbb{R}))$$



*B.2.9*

$$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2))$$

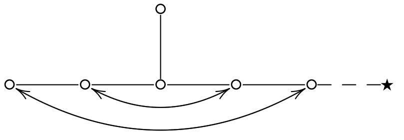
$$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{e}_{7(-25)}, \mathfrak{so}(2, 10) \oplus \mathfrak{sl}(2, \mathbb{R}))$$



*B.2.10*

$$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{e}_{7(-25)}, \mathfrak{su}(6,2))$$

$$(\mathfrak{g}, \mathfrak{g}^{\theta\sigma}) = (\mathfrak{e}_{7(-25)}, \mathfrak{so}^*(12) \oplus \mathfrak{su}(2))$$



## Appendix C. Tables

See Tables C.1–C.5.

Table C.1
Holomorphic parabolic subalgebras.

| $\mathfrak{g}$ | $a = a_1e_1 + a_2e_2 + \cdots$ | |
|---|---|---|
| | Holomorphic | Anti-holomorphic |
| $\mathfrak{su}(m,n)$ | $a_m \geq a_{m+1}$ | $a_{m+n} \geq a_1$ |
| $\mathfrak{so}(2,2n)$ | $a_1 \geq a_2$ | $-a_1 \geq a_2$ |
| $\mathfrak{so}(2,2n+1)$ | $a_1 \geq a_2$ | $-a_1 \geq a_2$ |
| $\mathfrak{so}^*(2n)$ | $a_{n-1} + a_n \geq 0$ | $a_1 + a_2 \leq 0$ |
| $\mathfrak{sp}(n,\mathbb{R})$ | $a_n \geq 0$ | $a_1 \leq 0$ |
| $\mathfrak{e}_{6(-14)}$ | $a_6 \geq a_1 + a_2 + a_3 + a_4 + a_5$ | $-a_6 \geq a_1 + a_2 + a_3 + a_4 - a_5$ |
| $\mathfrak{e}_{7(-25)}$ | $a_6 \geq a_5$ | $a_8 \leq a_7$ |

See Appendix A for the notation of $a$ and the dominant condition.

Table C.2
Symmetric pairs of holomorphic type.

| $\mathfrak{g}$ | $\mathfrak{g}^{\sigma}$ |
|---|---|
| $\mathfrak{su}(m,n)$ $m \neq n$ | $\mathfrak{su}(k,l) \oplus \mathfrak{su}(m-k,n-l) \oplus \mathfrak{u}(1)$ |
| $\mathfrak{su}(n,n)$ | $\mathfrak{su}(k,l) \oplus \mathfrak{su}(n-k,n-l) \oplus \mathfrak{u}(1)$<br>$\mathfrak{so}^*(2n)$<br>$\mathfrak{sp}(n,\mathbb{R})$ |
| $\mathfrak{so}(2,2n)$ | $\mathfrak{so}(2,k) \oplus \mathfrak{so}(2n-k)$<br>$\mathfrak{u}(1,n)$ |
| $\mathfrak{so}(2,2n+1)$ | $\mathfrak{so}(2,k) \oplus \mathfrak{so}(2n-k+1)$ |
| $\mathfrak{so}^*(2n)$ | $\mathfrak{u}(m,n-m)$<br>$\mathfrak{so}^*(2m) \oplus \mathfrak{so}^*(2n-2m)$ |
| $\mathfrak{sp}(n,\mathbb{R})$ | $\mathfrak{u}(m,n-m)$<br>$\mathfrak{sp}(m,\mathbb{R}) \oplus \mathfrak{sp}(n-m,\mathbb{R})$ |

(*continued on next page*)

Table C.2 (*continued*)

| $\mathfrak{g}$ | $\mathfrak{g}^\sigma$ |
|---|---|
| $\mathfrak{e}_{6(-14)}$ | $\mathfrak{so}(10) \oplus \mathfrak{so}(2)$<br>$\mathfrak{so}(2, 8) \oplus \mathfrak{so}(2)$<br>$\mathfrak{su}(4, 2) \oplus \mathfrak{su}(2)$<br>$\mathfrak{so}^*(10) \oplus \mathfrak{so}(2)$<br>$\mathfrak{su}(5, 1) \oplus \mathfrak{sl}(2, \mathbb{R})$ |
| $\mathfrak{e}_{7(-25)}$ | $\mathfrak{e}_{6(-78)} \oplus \mathfrak{so}(2)$<br>$\mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2)$<br>$\mathfrak{so}(2, 10) \oplus \mathfrak{sl}(2, \mathbb{R})$<br>$\mathfrak{su}(6, 2)$<br>$\mathfrak{so}^*(12) \oplus \mathfrak{su}(2)$ |

Table C.3
$(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ of discrete series type.

| $\mathfrak{g}$ | $\mathfrak{g}^\sigma$ | $a = a_1e_1 + a_2e_2 + \cdots$ |
|---|---|---|
| $\mathfrak{su}(m, n)$ | $\mathfrak{su}(m, k) \oplus \mathfrak{su}(n-k) \oplus \mathfrak{u}(1)$ | $a_{m+n} \geq a_1$,<br>$a_l \geq a_{m+1}$ and $a_{m+n} \geq a_{l+1} (1 \leq \exists l \leq m-1)$,<br>or $a_m \geq a_{m+1}$ |
| $\mathfrak{su}(2, 2n)$<br>$n \neq 1$ | $\mathfrak{sp}(1, n)$ | $a_1 \geq a_3$ and $a_{2n+2} \geq a_2$ |
| $\mathfrak{su}(2, 2)$ | $\mathfrak{sp}(1, 1)$ | $a_1 \geq a_3 \geq a_4 \geq a_2$<br>or $a_3 \geq a_1 \geq a_2 \geq a_4$ |
| $\mathfrak{so}(2m, 2n)$ | $\mathfrak{so}(2m, k) \oplus \mathfrak{so}(2n-k)$ | $\lvert a_m\rvert \geq \lvert a_{m+1}\rvert$ |
| $\mathfrak{so}(2m, 2n+1)$ | $\mathfrak{so}(2m, k) \oplus \mathfrak{so}(2n-k+1)$ | $\lvert a_m\rvert \geq a_{m+1}$ |
| $\mathfrak{so}(4, 2n)$<br>$n \neq 2$ | $\mathfrak{u}(2, n)_1$<br>$\mathfrak{u}(2, n)_2$ | $-a_2 \geq \lvert a_3\rvert$<br>$a_2 \geq \lvert a_3\rvert$ |
| $\mathfrak{so}(4, 4)$ | $\mathfrak{u}(2, 2)_{11}$<br>$\mathfrak{u}(2, 2)_{12}$<br>$\mathfrak{u}(2, 2)_{21}$<br>$\mathfrak{u}(2, 2)_{22}$ | $-a_2 \geq a_3$ or $-a_4 \geq a_1$<br>$-a_2 \geq a_3$ or $a_4 \geq a_1$<br>$a_2 \geq a_3$ or $-a_4 \geq a_1$<br>$a_2 \geq a_3$ or $a_4 \geq a_1$ |
| $\mathfrak{sp}(m, n)$ | $\mathfrak{sp}(m, k) \oplus \mathfrak{sp}(n-k)$ | $a_m \geq a_{m+1}$ |
| $\mathfrak{f}_{4(4)}$ | $\mathfrak{sp}(2, 1) \oplus \mathfrak{su}(2)$<br>$\mathfrak{so}(5, 4)$ | $a_1 + a_2 + a_3 \leq a_4$<br>$a_1 + a_2 + a_3 \leq a_4$ |
| $\mathfrak{e}_{6(2)}$ | $\mathfrak{so}(6, 4) \oplus \mathfrak{so}(2)$<br>$\mathfrak{su}(4, 2) \oplus \mathfrak{su}(2)$<br>$\mathfrak{sp}(3, 1)$<br>$\mathfrak{f}_{4(4)}$ | $a_1 + a_2 + a_3 - a_4 - a_5 - a_6 \leq 2a_7$<br>$a_1 + a_2 + a_3 - a_4 - a_5 - a_6 \leq 2a_7$<br>$a_1 + a_2 + a_3 - a_4 - a_5 - a_6 \leq 2a_7$<br>$a_1 + a_2 + a_3 - a_4 - a_5 - a_6 \leq 2a_7$ |
| $\mathfrak{e}_{7(-5)}$ | $\mathfrak{so}(8, 4) \oplus \mathfrak{su}(2)$<br>$\mathfrak{su}(6, 2)$<br>$\mathfrak{e}_{6(2)} \oplus \mathfrak{so}(2)$ | $a_1 + a_2 + a_3 + a_4 + a_5 - a_6 \leq 2a_7$<br>$a_1 + a_2 + a_3 + a_4 + a_5 - a_6 \leq 2a_7$<br>$a_1 + a_2 + a_3 + a_4 + a_5 - a_6 \leq 2a_7$ |
| $\mathfrak{e}_{8(-24)}$ | $\mathfrak{so}(12, 4)$<br>$\mathfrak{e}_{7(-5)} \oplus \mathfrak{su}(2)$ | $a_7 \geq a_6$<br>$a_7 \geq a_6$ |

See Appendix A for the notation of $a$ and the dominant condition.

Table C.4
$(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ of isolated type.

| $\mathfrak{g}$ | $\mathfrak{g}^\sigma$ | $a = a_1e_1 + a_2e_2 + \cdots$ |
|---|---|---|
| $\mathfrak{su}(2m, 2n)$ | $\mathfrak{sp}(m, n)$ | $(a_1, \dots, a_{2m};\ a_{2m+1}, \dots, a_{2m+2n})$<br>$= (s, 0, \dots, 0;\ t, 0, \dots, 0)$, mod $\mathbb{I}_{2m+2n} (s, t \geq 0)$<br>$(0, \dots, 0, -s;\ 0, \dots, 0, -t)$, mod $\mathbb{I}_{2m+2n} (s, t \geq 0)$<br>$(s, 0, \dots, 0, -t;\ 0, \dots, 0)$ mod $\mathbb{I}_{2m+2n} (s, t \geq 0)$<br>or $(0, \dots, 0;\ s, 0, \dots, 0, -t)$mod $\mathbb{I}_{2m+2n} (s, t \geq 0)$ |
| $\mathfrak{so}(2m+1, 2n)$ | $\mathfrak{so}(2m+1, k) \oplus \mathfrak{so}(2n-k)$ | $a_{m+1} = \cdots = a_{m+n} = 0$ |
| $\mathfrak{so}(2m+1, 2n+1)$ | $\mathfrak{so}(2m+1, k) \oplus \mathfrak{so}(2n-k+1)$ | $a_{m+1} = \cdots = a_{m+n} = 0$ |
| $\mathfrak{so}(2m, 2n)$ | $\mathfrak{u}(m, n)$ | $(a_1, \dots, a_m;\ a_{m+1}, \dots, a_{m+n})$<br>$= (s, 0, \dots, 0;\ 0, \dots, 0)$<br>or $(0, \dots, 0;\ s, 0, \dots, 0)$ |
| $\mathfrak{so}^*(2n)$ | $\mathfrak{so}^*(2n-2) \oplus \mathfrak{so}(2)$<br><br><br>$\mathfrak{u}(n-1, 1)$ | $(a_1, \dots, a_n) = (\underbrace{s, \dots, s}_{k}, -s, \dots, -s)$<br>$(1 \leq \exists k \leq n-1)$<br>$(a_1, \dots, a_n) = (\underbrace{s, \dots, s}_{k}, -s, \dots, -s)$<br>$(1 \leq \exists k \leq n-1)$ |
| $\mathfrak{sp}(m, n)$ | $\mathfrak{sp}(k, l) \oplus \mathfrak{sp}(m-k, n-l)$<br>$k, l, m-k, n-l \geq 1$<br><br>$\mathfrak{sp}(m, k) \oplus \mathfrak{sp}(n-k)$ | $(a_1, \dots, a_m;\ a_{m+1}, \dots, a_{m+n})$<br>$= (s, 0, \dots, 0;\ 0, \dots, 0)$<br>or $(0, \dots, 0;\ s, 0, \dots, 0)$<br>$(a_1, \dots, a_m;\ a_{m+1}, \dots, a_{m+n})$<br>$= (0, \dots, 0;\ s, 0, \dots, 0)$,<br>$a_{l-1} \geq a_{m+1}$ and $a_l = a_{m+2} = 0$ $(2 \leq \exists l \leq m)$ |
| $\mathfrak{sl}(2n, \mathbb{C})$ | $\mathfrak{sp}(n, \mathbb{C})$<br><br>$\mathfrak{su}^*(2n)$ | $(a_1, \dots, a_{2n}) = (s, 0, \dots, 0)$ mod $\mathbb{I}_{2n}$<br>or $(0, \dots, 0, s)$ mod $\mathbb{I}_{2n}$<br>$(a_1, \dots, a_{2n}) = (s, 0, \dots, 0)$ mod $\mathbb{I}_{2n}$<br>or $(0, \dots, 0, s)$ mod $\mathbb{I}_{2n}$ |
| $\mathfrak{so}(2n, \mathbb{C})$ | $\mathfrak{so}(2n-1, \mathbb{C})$<br>$\mathfrak{so}(2n-1, 1)$ | $(a_1, \dots, a_n) = (s, \dots, s)$<br>$(a_1, \dots, a_n) = (s, \dots, s)$ |
| $\mathfrak{f}_{4(-20)}$ | $\mathfrak{so}(8, 1)$ | $(a_1, a_2, a_3, a_4) = (s, s, s, s)$ or $(s, s, 0, 0)$ |
| $\mathfrak{e}_{6(2)}$ | $\mathfrak{so}^*(10) \oplus \mathfrak{so}(2)$ | $(a_1, \dots, a_7) = (s, s, s, s, t, t, 0)$,<br>or $(s, s, t, t, t, t, 0)$ |
| $\mathfrak{e}_{6(-14)}$ | $\mathfrak{so}(2, 8) \oplus \mathfrak{so}(2)$ | $(a_1, \dots, a_6) = (s, s, s, s, s, s)$,<br>or $(s, s, s, s, -s, -s)$ |
| $\mathfrak{e}_{6(-14)}$ | $\mathfrak{f}_{4(-20)}$ | $(a_1, \dots, a_6) = (s, s, 0, 0, 0, 0)$,<br>or $(s, s, s, s, t, t)$ |
| $\mathfrak{e}_{7(-5)}$ | $\mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2)$ | $(a_1, \dots, a_7) = (s, s, s, s, s, s, 0)$ |

$s, t \in \mathbb{R}$, $\mathbb{I}_n = (\underbrace{1, \dots, 1}_{n})$

See Appendix A for the notation of $a$ and the dominant condition.

Table C.5
no $A_{\mathfrak{q}}(\lambda)$ is discretely decomposable, case (4) of Theorem 4.12.

| $\mathfrak{g}$ | $\mathfrak{g}^\sigma$ | |
|---|---|---|
| $\mathfrak{su}(n,n)$ | $\mathfrak{sl}(n,\mathbb{C})\oplus\mathbb{R}$ | $n\ge 1$ |
| $\mathfrak{so}(m,n)$ | $\mathfrak{so}(k,l)\oplus\mathfrak{so}(m-k,n-l)$ | $k,l,m-k,n-l\ge 1$ |
| $\mathfrak{so}(n,n)$ | $\mathfrak{so}(n,\mathbb{C})$<br>$\mathfrak{gl}(n,\mathbb{R})$ | $n\ge 3$<br>$n\ge 3$ |
| $\mathfrak{so}^*(4n)$ | $\mathfrak{su}^*(2n)\oplus\mathbb{R}$ | $n\ge 2$ |
| $\mathfrak{sp}(n,n)$ | $\mathfrak{sp}(n,\mathbb{C})$<br>$\mathfrak{su}^*(2n)\oplus\mathbb{R}$ | $n\ge 1$<br>$n\ge 1$ |
| $\mathfrak{sp}(2n,\mathbb{R})$ | $\mathfrak{sp}(n,\mathbb{C})$ | $n\ge 2$ |
| $\mathfrak{sl}(2n,\mathbb{C})$ | $\mathfrak{sl}(m,\mathbb{C})\oplus\mathfrak{sl}(2n-m,\mathbb{C})\oplus\mathbb{C}$<br>$\mathfrak{su}(m,2n-m)$ | $m,2n-m\ge 1$<br>$m,2n-m\ge 1$ |
| $\mathfrak{so}(2n,\mathbb{C})$ | $\mathfrak{so}(m,\mathbb{C})\oplus\mathfrak{so}(2n-m,\mathbb{C})$<br>$\mathfrak{so}(m,2n-m)$<br>$\mathfrak{gl}(n,\mathbb{C})$<br>$\mathfrak{so}^*(2n)$ | $m,2n-m\ge 2$<br>$m,2n-m\ge 2$<br>$n\ge 3$<br>$n\ge 3$ |
| $\mathfrak{e}_{6(2)}$ | $\mathfrak{su}(3,3)\oplus\mathfrak{sl}(2,\mathbb{R})$ | |
| $\mathfrak{e}_{7(-5)}$ | $\mathfrak{so}^*(12)\oplus\mathfrak{sl}(2,\mathbb{R})$ | |
| $\mathfrak{e}_{7(-25)}$ | $\mathfrak{e}_{6(-26)}\oplus\mathbb{R}$ | |
| $\mathfrak{e}_{8(-24)}$ | $\mathfrak{e}_{7(-25)}\oplus\mathfrak{sl}(2,\mathbb{R})$ | |

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