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Nonlinear progressive internal gravity wave on fluid of trapezoidal bottom

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Abstract

In the present paper we discuss a theoretical model for the interfacial profiles of progressive nonlinear waves which result from introducing a trapezoidal obstacle, of finite height, attached to the bottom below the flow of a stratified, ideal, two layer fluid, bounded from above by a rigid boundary. Assuming a very large horizontal length compared with the vertical height allows us to apply the shallow-water approximation theory, and, consequently, a nonlinear perturbation method is presented to construct per power series the analytical solution for the interfacial profiles of the progressive waves. The dependence of the interfacial profile on the trapezoidal obstacle size, as well as its dependence on some flow parameters, such as the ratios of depths and densities of the two fluids, have been studied and illustrated.

Keywords: Shallow-water approximation theory; Nonlinear perturbation method; Froude number

AMS classification: 76; 76B

1. Introduction

1.1. The problem

Problem of gravity nonlinear waves has been initiated by Boussinesq [1] in 1871. Then the problem started to attract the attention of many well known contributors like Rayleigh [15] in 1883 and Korteweg and de Vries [8] in 1895. The work on internal waves has been started in the middle of the twentieth century by Keulegan [6] in 1953 followed by Long [11] in 1956, where they applied Boussinesq's and Rayleigh's techniques, respectively, on free surface waves to the problems of internal waves. The problem of flow pattern over obstacles appeared to be of great importance. Lamb [9] in 1945 was the first to give the essential features of the flow of an ideal fluid in an open

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channel in the presence of an obstruction in the channel. In 1955, Long [10] and later McIntyre [12] in 1972, considered the case of a steady and uniform stratification over obstacles of finite height. The effect of the irregularities of the bottom on surface gravity waves, has been studied by Kakutani [5] in 1971 who used a reductive perturbation method. In 1989, Kevorkian and Yu [7] studied the shallow water waves excited by a small amplitude bottom disturbance in the presence of a uniform incoming flow.

In the present work we study the interfacial profiles of nonlinear progressive waves which propagate at the interface of a two-layer immiscible, incompressible perfect fluid, in a two-dimensional open channel with a trapezoidal obstacle, attached to the bottom below the flow which is bounded above by a horizontal rigid boundary. The effect of the irregularities of the bottom surface is not only interesting academically but also very important practically. Every seabed or riverbed is, in fact, obviously irregular, see [5].

Our main objective is to calculate the shape of the interfacial waves due to the effect of both geometrical and flow parameters.

In Section 4, we extend the mathematical technique applied by Helal and Molines [4] in 1981. The technique of nonlinear perturbation, presented in Sections 5 and 6, has been applied to find an analytical expression of the interfacial waves. Boutros et al. [2] in 1991 applied the same technique to study the internal waves over a ramp.

Finally, in Section 6, the effect of the density ratio, R, the thickness ratio, H, and the trapezoidal height, L, has been studied and the results have been plotted.

1.2. Notation

The geometrical configuration of gravity waves over a trapezoidal obstacle is shown in Fig. 1.

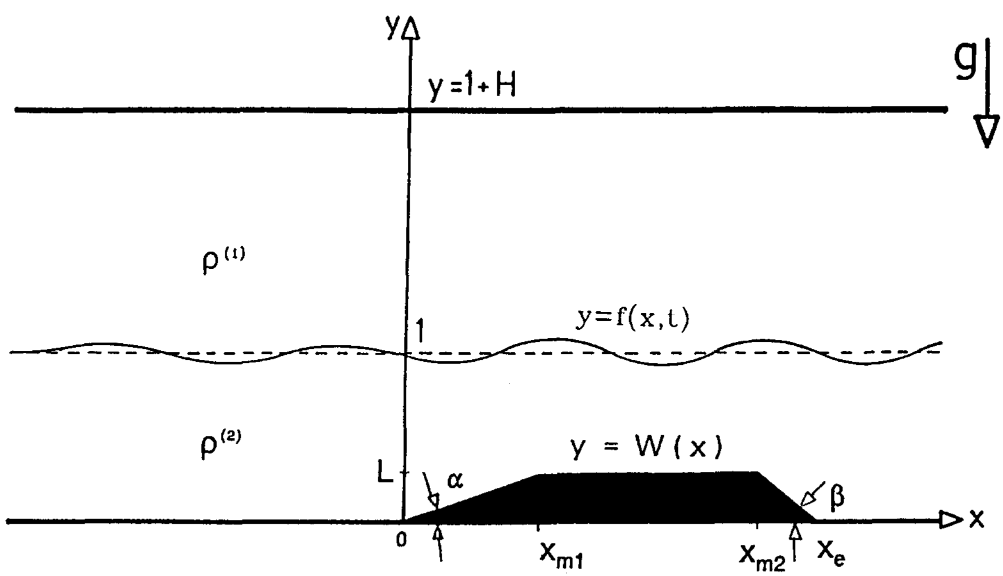

Fig. 1. Geometrical configuration of gravity waves over a trapezoidal obstacle in nondimensionalized variables.

The following notations are used in the text:

$y=W(x)$	form of the trapezoidal obstacle
α, β	inclination angles of the trapezoidal bottom
x_{m1}, x_{m2}, x_e	horizontal coordinates of the trapezoidal bottom
L	height of the obstacle
H	thickness of the upper layer
$\rho^{(1)}, \rho^{(2)}$	constant density of the upper and lower layer, respectively
F	Froude number
$\phi^{(1)}, \phi^{(2)}$	velocity potential of the upper and lower layer, respectively
$y=f(x,t)$	form of the interfacial wave
U	velocity of the flow far up/down stream

2. Mathematical formulation of the problem

The motion of an inviscid, incompressible and irrotational fluid may be described by the harmonic equation for the velocity potential $\phi^{*(i)}(X^*, Y^*, \tau)$, $(i=1,2)$,

$$\phi^{(1)}_{X^*X^*} + \phi^{(1)}_{Y^*Y^*} = 0, \quad f^* < Y^* < H_1^* + H_2^*, \quad -\infty < X^* < \infty, \tag{2.1.1}$$

$$\phi^{(2)}_{X^*X^*} + \phi^{(2)}_{Y^*Y^*} = 0, \quad W^* < Y^* < f^*, \quad -\infty < X^* < \infty. \tag{2.1.2}$$

For an inviscid fluid, the fluid must flow along the bottom surface, hence we have the following boundary condition at the bottom:

$$\phi^{*(2)}_{Y^*} = \phi^{*(2)}_{X^*} W^*_{X^*} \quad \text{at } Y^* = W^*(X^*) \tag{2.2.1}$$

and on the solid upper layer

$$\phi^{*(1)}_{Y^*} = 0 \quad \text{at } Y^* = H_1^* + H_2^*. \tag{2.2.2}$$

On the other hand, the matching conditions at the interface are given as follows:
At $Y^* = f^*$

$$\begin{aligned} &\phi^{*(i)}_{Y^*} = f^*_\tau + \phi^{*(i)}_{X^*} f^*_{X^*} \quad i=1,2, \\ &\rho^{(1)}\left\{\phi^{*(1)}_\tau + \frac{1}{2}[(\phi^{*(1)}_{X^*})^2 + (\phi^{*(1)}_{Y^*})^2] + \frac{U^{*2}}{F^2}\left(\frac{f^*}{H_2^*} - 1\right)\right\} \\ &\quad = \rho^{(2)}\left\{\phi^{*(2)}_\tau + \frac{1}{2}[(\phi^{*(2)}_{X^*})^2 + (\phi^{*(2)}_{Y^*})^2] + \frac{U^{*2}}{F^2}\left(\frac{f^*}{H_2^*} - 1\right)\right\}, \end{aligned} \tag{2.3}$$

where $F = U^*/\sqrt{gH_2^*}$ is the Froude number.

Moreover, assuming that the fluids are in an undisturbed uniform state up/down stream at infinity, we impose the following boundary conditions with respect to X^*

$$\phi^{*(i)}_{X^*} = U^*, \quad i=1,2 \quad \text{as } X^* \to \pm\infty. \tag{2.4}$$

3. Normalized basic equations

All variables are non-dimensionalized by using the characteristic length of the lower layer, H_2^*, and time $(g/H_2^*)^{-1/2}$, and accordingly

$$H=\frac{H_1^*}{H_2^*},\qquad U=\frac{U^*}{[gH_2^*]^{1/2}}\quad\text{and}\quad \phi^{(i)}=\frac{\phi^{*(i)}}{(H_2^*[gH_2^*]^{1/2})}. \tag{3.1}$$

Hence the boundary condition (2.4) takes the form

$$\phi_X^{(i)}=U,\quad i=1,2\quad \text{as } X\to\pm\infty. \tag{3.2}$$

Since the fluid is perfect we ignore surface tension for $y=f(x,t)$.

An essential step which makes our problem easier to handle is to define an appropriate *stretching* of the horizontal coordinate while leaving the vertical coordinate unchanged due to the fact that the horizontal dimensions are much greater than the vertical dimensions. Thus we define

$$x=\varepsilon X,\quad y=Y,\quad t=\varepsilon\tau, \tag{3.3}$$

where ε is a small parameter. Considering the case of critical flow ($F=1$ and $U=1$), the basic equations for this system can be written as follows:

$$\varepsilon^2\phi_{xx}^{(1)}+\phi_{yy}^{(1)}=0,\quad f<y<1+H,\quad -\infty<x<\infty; \tag{3.4.1}$$

$$\varepsilon^2\phi_{xx}^{(2)}+\phi_{yy}^{(2)}=0,\quad W<y<f,\quad -\infty<x<\infty, \tag{3.4.2}$$

subject to the following conditions:
(i) *Boundary conditions*:
At $y=f$

$$\phi_y^{(i)}=\varepsilon f_t+\varepsilon^2\phi_x^{(i)}f_x\quad (i=1,2),$$
$$R\left\{\varepsilon\phi_t^{(1)}+\frac{1}{2}[\varepsilon^2(\phi_x^{(1)})^2+(\phi_y^{(1)})^2]+f-1\right\}$$
$$=\left\{\varepsilon\phi_t^{(2)}+\frac{1}{2}[\varepsilon^2(\phi_x^{(2)})^2+(\phi_y^{(2)})^2]+f-1\right\}; \tag{3.4.3}$$

At $y=W(x)$

$$\phi_y^{(2)}=\varepsilon^2\phi_x^{(2)}W_x; \tag{3.4.4}$$

At $y=1+H$

$$\phi_y^{(1)}=0; \tag{3.4.5}$$

$$\varepsilon\phi_x^{(i)}=1,\quad i=1,2\quad \text{as } x\to\pm\infty. \tag{3.4.6}$$

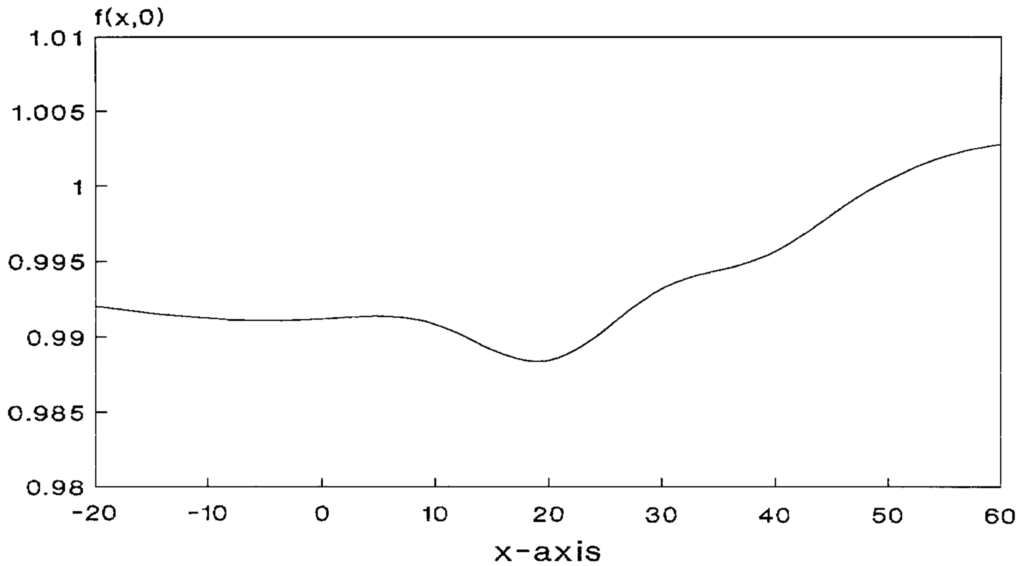

Fig. 2. The initial waveform over a trapezoidal bottom with $x_{m1}=13$, $x_{m2}=23$, $x_e=40$, $L=0.25$, $\alpha=0.01561$ and $\beta=-0.0192$.

(ii) *Initial condition*:

The initial profile of the interfacial wave, $f(x,0)$, is shown in Fig. 2. The density ratio $R=\rho^{(1)}/\rho^{(2)}$ (less than unity) and the thickness ratio H are two characteristic parameters of the system, and $W(x)$ has the form

$$W(x)=ax+b, \tag{3.5}$$

where

$$(a,b)=\begin{cases}(0,0) & x\leqslant 0,\\ (\alpha,0) & 0\leqslant x\leqslant x_{m1},\\ (0,L) & x_{m1}\leqslant x\leqslant x_{m2},\\ (-\beta,\beta x_e) & x_{m2}\leqslant x\leqslant x_e,\\ (0,0) & x>x_e,\end{cases} \tag{3.6}$$

where $\alpha\approx\tan\alpha$ and $\beta\approx\tan\beta$ for small α and β, respectively.

4. The shallow-water theory

Since we consider weakly nonlinear waves, we expand the dependent variables as a power series in the same parameter ε around the undisturbed uniform state. Following Helal and Molines [4], we get

$$\phi^{(i)}=\sum_{n=1}^{\infty}\varepsilon^{2n-1}G_{2n-1}^{(i)}(x,y,t), \quad i=1,2,$$
$$f=\sum_{n=1}^{\infty}\varepsilon^{2n}f_{2n}(x,y,t) \tag{4.1}$$

with $f_0=1$. The scale parameter ε, which is assumed to be small, provides a measure of weakness of dispersion.

The boundary conditions on the interface, given by Eqs. (3.4.3), are expanded as a Taylor expansion of the type

$$[V]_{y=y_0+\varepsilon^2 A}=\sum_{n=0}^{\infty}\frac{(\varepsilon^2 A)^n}{n!}\left[\frac{\partial^n V}{\partial y^n}\right]_{y_0}. \tag{4.2}$$

When (3.3), (3.5), using the expansion (4.2), are inserted into Eqs. (3.4) and powers of ε are sorted out, we get an ordered set of equations to be solved.

5. Orders of approximations

5.1. The first-order approximation

Equations of the first-order approximation, finally give, for $i=1,2$

$$G_1^{(i)}=B^{(i)}(x,t), \tag{5.1.1}$$

where $B^{(i)}(x,t)$ are unknown functions to be determined.

5.2. The second-order approximation

From the equations obtained from the second-order approximation, we conclude that

$$B_x^{(i)}=0, \quad (i=1,2) \quad \text{as } x\to\pm\infty \tag{5.2.1}$$

and

$$f_2(x,t)=\frac{1}{1-R}[RB_t^{(1)}-B_t^{(2)}]. \tag{5.2.2}$$

5.3. The third- and fourth-order approximations

Equations of the third- and fourth-order approximation finally give, for $i=1,2$

$$G_3^{(i)}=-\tfrac{1}{2}y^2B_{xx}^{(i)}+yC^{(i)}(x,t)+D^{(i)}(x,t), \tag{5.3.1}$$

where $C^{(i)}(x,t)$ and $D^{(i)}(x,t)$ are arbitrary functions which satisfy the following boundary conditions:

$$C_x^{(i)} = 0 \quad i=1,2 \quad \text{as } x \to \pm\infty, \tag{5.3.2}$$

$$C^{(2)}(x,t) = (WB_x^{(2)})_x \quad \text{at } y = W(x), \tag{5.3.3}$$

$$D_x^{(i)} = 0 \quad i=1,2 \quad \text{as } x \to \pm\infty. \tag{5.3.4}$$

Substituting Eq. (5.3.1) in the equations that are obtained from the third- and fourth-order approximation, we obtain

$$(H+1)B_{xx}^{(1)} - C^{(1)} = 0, \tag{5.3.5}$$

and for $i=1,2$

$$B_{xx}^{(i)} - C^{(i)} + \frac{1}{1-R}(RB_{tt}^{(1)} - B_{tt}^{(2)}) = 0. \tag{5.3.6}$$

From Eqs. (5.3.3), (5.3.5), and (5.3.6) we get

$$\Box_1 B^{(1)} = B_{tt}^{(2)} \tag{5.3.7}$$

$$\Box_2 B^{(2)} = RB_{tt}^{(1)} \tag{5.3.8}$$

where $\Box_1, \Box_2$ are the differential operators defined by

$$\Box_1 \equiv -H(1-R)\frac{\partial^2}{\partial x^2} + R\frac{\partial^2}{\partial t^2}, \tag{5.3.9}$$

$$\Box_2 \equiv -(1-R)(1-W)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2} + (1-R)\frac{\partial W}{\partial x}\frac{\partial}{\partial x}. \tag{5.3.10}$$

From equations (5.3.7)–(5.3.10) we get, after eliminating $B^{(1)}$ and substituting for $W(x)$, the following differential equation for the unknown function $B^2(x,t)$:

$$\begin{aligned} &-H(1-R)(1-b-ax)B_{xxxx}^{(2)} + [H + R(1-b-ax)]B_{xxtt}^{(2)} \\ &-RaB_{xtt}^{(2)} + 3Ha(1-R)B_{xxx}^{(2)} = 0, \end{aligned} \tag{5.3.11}$$

and we get the following relation for $f_4(x,t)$:

$$\begin{aligned} f_4(x,t) = \frac{1}{1-R}\Big\{ & R\left[-\frac{1}{2}B_{xxt}^{(1)} + C_t^{(1)} + D_t^{(t)} + \frac{1}{2}(B_x^{(1)})^2\right] \\ & + \frac{1}{2}B_{xxt}^{(2)} - C_t^{(2)} - D_t^{(2)} - \frac{1}{2}(B_x^{(2)})^2\Big\}. \end{aligned} \tag{5.3.12}$$

5.4. The fifth- and sixth-order approximations

Equations of the fifth and sixth-order approximation yield, for $i=1,2$

$$G_5^{(i)} = \frac{y^4}{24}B_{xxxx}^{(i)} - \frac{y^3}{6}C_{xx}^{(i)}(x,t) - \frac{y^2}{2}D_{xx}^{(i)}(x,t) + yE^{(i)}(x,t) + F^{(i)}(x,t), \tag{5.4.1}$$

where $E^{(i)}(x,t)$ and $F^{(i)}(x,t)$ are arbitrary functions which satisfy the following conditions:

$$E_x^{(i)}=0 \quad (i=1,2), \quad \text{as } x\to \pm\infty, \tag{5.4.2}$$

$$E^{(2)}(x,t)=\left(-\frac{W^3}{3!}B_{xxx}^{(2)}+\frac{W^2}{2!}C_x^{(2)}(x,t)+WD_x^{(2)}\right)_x \quad \text{at } y=W(x), \tag{5.4.3}$$

$$F_x^{(i)}=0 \quad (i=1,2) \quad \text{as } x\to \pm\infty. \tag{5.4.4}$$

Introducing Eqs. (5.3.1)–(5.4.1) in the boundary conditions, we obtain the following relations:

$$\frac{(H+1)^3}{3!}B_{xxxx}^{(1)}-\frac{(H+1)^2}{2!}C_{xx}^{(1)}-(H+1)D_{xx}^{(1)}+E^{(1)}=0 \tag{5.4.5}$$

and for $i=1,2$

$$\begin{aligned}&\frac{1}{3!}B_{xxxx}^{(i)}-\frac{1}{2!}C_{xx}^{(i)}-D_{xx}^{(i)}+E^{(i)}+\frac{1}{1-R}[(B_t^{(2)}-RB_t^{(1)})B_{xx}^{(i)}\\ &\quad+(B_{xt}^{(2)}-RB_{xt}^{(1)})B_x^{(i)}-\frac{1}{2}B_{xxtt}^{(2)}+C_{tt}^{(2)}+D_{tt}^{(2)}\\ &\quad-R\left(-\frac{1}{2}B_{xxtt}^{(1)}+C_{tt}^{(1)}+D_{tt}^{(1)}\right)+B_x^{(2)}B_{xt}^{(2)}-RB_x^{(1)}B_{xt}^{(1)}]=0.\end{aligned} \tag{5.4.6}$$

Thus the problem is now reduced to solving Eqs. (5.3.5) and (5.3.6) for $B^{(i)}$ and $C^{(i)}$ and next Eqs. (5.4.2), (5.4.3) and (5.4.6) for $D^{(i)}$ and $E^{(i)}$, $i=1,2$.

6. Case of the progressive wave

It must be remarked that our procedure is valid as long as $a\gg\varepsilon^2$; otherwise a two-parameter analysis has to be carried out. Moreover, we shall invoke the smallness of "a" and write perturbation expansions for $B^{(i)}$, $(i=1,2)$, in the form

$$B^{(i)}=B_0^{(i)}+aB_1^{(i)}+a^2B_2^{(i)}+\cdots. \tag{6.1}$$

Substituting (6.1) in (5.3.11) and equating coefficients of $a^{(j)}$, $j=0,1,2,\ldots$ we get the following system of differential equations:

$$\Box B_j^{(2)}=\Lambda B_{j-1}^{(2)}, \quad j=0,1,2,\ldots, \quad B_{-1}^{(2)}=0, \tag{6.2}$$

where $\Box,\Lambda$ are two differential operators defined as

$$\Box\equiv -H(1-R)(1-b)\frac{\partial^4}{\partial x^4}+[H+R(1-b)]\frac{\partial^4}{\partial x^2\partial t^2} \tag{6.3}$$

and

$$\Lambda\equiv -xH(1-R)\frac{\partial^4}{\partial x^4}+xR\frac{\partial^4}{\partial x^2\partial t^2}-3H(1-R)\frac{\partial^3}{\partial x^3}+R\frac{\partial^3}{\partial x\partial t^2}. \tag{6.4}$$

Eq. (6.2), for $j=0$, has the following general solution, for the case of pure progressive waves:

$$B_0^{(i)}=B_0^{(i)}(\xi)$$

with

$$\xi=x-\gamma t, \qquad \gamma^2=\frac{H(1-b)(1-R)}{H+(1-b)R}. \tag{6.5}$$

From Eqs. (3.5), (5.3.3), and (6.1) we get

$$C^{(2)}=\sum_{n=0}^{\infty} a^n[bB_{n,xx}^{(2)}+(xB_{n-1,x}^{(2)})_x], \quad B_{-1}^{(2)}=0. \tag{6.6}$$

Again substituting Eqs. (6.1), (6.6) in Eq. (5.3.6) we get, after equating coefficients of $a^n, n=0,1,2,\ldots$

$$B_{0,x}^{(2)}=\lambda B_{0,x}^{(1)}, \tag{6.7}$$

$$B_{1,x}^{(2)}=\frac{x}{1-b}B_{0,x}^{(2)}+\lambda B_{1,x}^{(1)}, \tag{6.8}$$

where

$$\lambda=\frac{H}{1-b}. \tag{6.9}$$

The elimination of $E^{(1)}$ in Eqs. (5.4.5) and (5.4.6) gives for "a", the following system of differential equations:

$$\left(H-\frac{\gamma^2R}{1-R}\right)D_{\xi\xi}^{(1)}+\left(\frac{\gamma^2}{1-R}\right)D_{\xi\xi}^{(2)}=P_1B_{0,\xi\xi\xi\xi}^{(1)}+Q_1B_{0,\xi}^{(1)}B_{0,\xi\xi}^{(1)}, \tag{6.10}$$

$$\left(\frac{\gamma^2R}{1-R}\right)D_{\xi\xi}^{(1)}+\left(1-b-\frac{\gamma^2}{1-R}\right)D_{\xi\xi}^{(2)}=P_2B_{0,\xi\xi\xi\xi}^{(1)}+Q_2B_{0,\xi}^{(1)}B_{0,\xi\xi}^{(1)}, \tag{6.11}$$

where

$$P_1=\frac{-H(2H^2+6H+3)}{6}+\frac{\gamma^2[\lambda(1-2b)+R(2H+1)]}{2(1-R)}, \tag{6.12}$$

$$Q_1=\frac{\gamma}{1-R}(\lambda^2+2\lambda-3R), \tag{6.13}$$

$$P_2=\frac{\lambda}{6}(1-3b+2b^3)+\frac{\gamma^2}{2(1-R)}[(2b-1)\lambda-R(2H+1)], \tag{6.14}$$

and

$$Q_2=\frac{\gamma}{1-R}[R(2\lambda+1)-3\lambda^2]. \tag{6.15}$$

For the non-trivial solution of $D_{\xi\xi}^{(1)}$ and $D_{\xi\xi}^{(2)}$, the following differential equation for $B_0^{(1)}$ should be satisfied:

$$M_1B_{0,\xi\xi\xi\xi}^{(1)}+M_2B_{0,\xi}^{(1)}B_{0,\xi\xi}^{(1)}=0, \tag{6.16}$$

where

$$M_1 = \left(1 - b - \frac{\gamma^2}{1-R}\right)P_1 - \left(\frac{\gamma^2}{1-R}\right)P_2, \tag{6.17}$$

and

$$M_2 = \left(1 - b - \frac{\gamma^2}{1-R}\right)Q_1 - \left(\frac{\gamma^2}{1-R}\right)Q_2. \tag{6.18}$$

Define

$$\Gamma = B^{(1)}_{0,\xi}. \tag{6.19}$$

Thus Eq. (6.16), by virtue of Eq. (6.19), will be transformed to the Boussinesq equation

$$M_1\Gamma_{\xi\xi\xi} + M_2\Gamma\Gamma_\xi = 0. \tag{6.20}$$

Helal and Molines [4] mention that the general solution of Eq. (6.20) was found by Byrd and Friedmann [3] to be, in terms of the Jacobi elliptic functions $\mathrm{sn}(u,k)$,

$$B^{(1)}_{0,\xi} = Y_1\left[1 - \frac{3k^2}{k^2+1}\mathrm{sn}^2(\delta\xi,k^2)\right], \tag{6.21}$$

where Y_1 is the greatest of the roots of the polynomial resulting from integrating Eq. (6.20) twice and k is the modulus of the Jacobean elliptic function, and

$$\delta = \frac{1}{2}\left(-\frac{3AY_1}{k^2+1}\right)^{1/2}. \tag{6.22}$$

For small values of k the above elliptic function could be calculated in terms of trigonometric functions, see Milne–Thomson [14], thus we have

$$\begin{aligned} B^{(1)}_{0,\xi} = Y_1\Bigg\{1 - \frac{3k^2}{k^2+1}\Bigg[&\left(\frac{1}{2}+\frac{k^2}{8}+\frac{k^4}{16}\right) + \left(\frac{k^4-64}{128}\right)\cos 2\delta\xi \\ &- \left(\frac{8k^2+k^4}{64}\right)\cos 4\delta\xi - \frac{k^4}{128}\cos 6\delta\xi - \delta\xi\left\{\left(\frac{k^2}{2}+\frac{k^4}{8}\right)\sin 2\delta\xi\right. \\ &\left.+\frac{k^4}{16}\sin 4\delta\xi\right\} + \delta^2\xi^2\left\{\frac{k^4}{8}+\frac{k^4}{8}\cos 2\delta\xi\right\}\Bigg]\Bigg\}. \end{aligned} \tag{6.23}$$

Substituting in Eq. (6.2), for $B^{(2)}_{0,x}$ and $B^{(2)}_{0,t}$, we get the following fourth-order linear partial differential equation

$$\begin{aligned} \Box B^{(2)}_1 = &\sum_{n=1}^{3}(A_n x\sin 2n\delta\xi + A_{n+6}\cos 2n\delta\xi) + \delta\xi\sum_{n=1}^{2}(A_{n+3}x\cos 2n\delta\xi + A_{n+10}\sin 2n\delta\xi) \\ &+ \delta^2\xi^2(A_6 x\sin 2\delta\xi + A_{13}\cos 2\delta\xi) + A_{10}, \end{aligned} \tag{6.24}$$

where the coefficients $A_1, A_2, \ldots, A_{13}$ are given at the end of the paper, as Appendix 1.

Solving Eq. (6.24) for the unknown $B_1^{(2)}$, following Miller [13], and calculating $B_{1,t}^{(2)}$ we get

$$B_{1,t}^{(2)} = B_{0,t}^{(2)} + r_1 t^3 + r_2 x^2 t + (r_3 + r_4 x^2 + r_5 xt + r_6 t^2)\sin 2\delta\xi + (r_7 + r_8 x^2 + r_9 xt + r_{10} t^2)\sin 4\delta\xi + r_{11}\sin 6\delta\xi + (r_{12} + r_{13}x + r_{14}t + r_{15}x^3 + r_{16}x^2 t + r_{17}xt^2 + r_{18}t^3)\cos 2\delta\xi + (r_{19}x + r_{20}t)\cos 4\delta\xi + (r_{21}x + r_{22}t)\cos 6\delta\xi, \tag{6.25}$$

where the coefficients $r_1, r_2, \ldots, r_{22}$ are also given at the end of the paper, as Appendix 2.

Taking into consideration the value of $B_{0,x}^{(1)}$ from Eq. (6.23), we can get $B_{0,x}^{(2)}$ and thus, using (6.25) for $B_{1,t}^{(2)}$ we can get $B_{1,t}^{(1)}$

$$B_{1,t}^{(1)} = \frac{1}{\lambda}\left(B_{1,t}^{(2)} - \frac{x}{1-b}B_{0,t}^{(2)}\right). \tag{6.26}$$

In order to account for the nonlinear effects the $O(\varepsilon^4)$ equations have to be considered as well. Thus bearing in mind the linear system of Eqs. (6.10) and (6.11), the principal and secondary determinants of this system, we come to the result that

$$D_t^{(i)} = 0, \quad i = 1, 2. \tag{6.27}$$

Hence $f_4(x,t)$ may be rewritten in the simplified form

$$f_4(x,t) = \frac{1}{2(1-R)}\left\{\left((\lambda - R) + 2(1+H)R - 2\lambda(ax+b) + \frac{ax\lambda(1-2b)}{(1-b)}\right)B_{0,xxt}^{(1)} + (R(2H+1) + \lambda(1-2b))B_{1,xxt}^{(1)} + \left(2(R-\lambda^2) - \frac{2ax\lambda^2(ax+2)}{1-b}\right)(B_{0,x}^{(1)})^2 + a^2(R-a)(B_{1,x}^{(1)})^2 + \frac{2a\lambda b}{(b-1)}B_{0,xt}^{(1)} + 2a\left(R - \lambda^2 - \frac{ax\lambda^2}{1-b}\right)B_{0,x}^{(1)}B_{1,x}^{(1)}\right\}. \tag{6.28}$$

Hence $f(x,t)$ will take the form

$$f(x,t) = 1 + \varepsilon^2\left\{\frac{(R-\lambda)(b-1) + \lambda bx}{(1-R)(b-1)}B_{0,t}^{(1)} + \frac{a(R-\lambda)}{1-R}B_{1,t}^{(1)}\right\} + \varepsilon^4 f_4(x,t) + O(\varepsilon^6), \tag{6.29}$$

where $f_4(x,t)$ is given by (6.28) and $B_{0,t}^{(1)}$ and $B_{1,t}^{(1)}$ are given by (6.23) and (6.25), respectively.

7. Presentation of results and discussion

The number of terms which has been obtained seems to be a good measure for the purpose of illustrating the effect of the parameters: the density ratio R, the thickness ratio H, and the obstacle height L. The error, difference between the fourth and second order approximations, in the interfacial profile for the two approximations is of order 10^{-6}. Thus we limit our calculations up to the second-order approximation, as well as we considered the following values for the description of the trapezoidal obstacle: $x_{m1}=13$, $x_{m2}=23$, $x_e=40$ and $L=0.25$.

In Fig. 3, we illustrate the effect of the density ratio R, on the wave profiles at the interfacial surface. Different values of R have been considered, for fixed values of H, L, and t. It is clear that as R decreases, there is a kind of violent oscillations in the obstacle region. This phenomena vanishes gradually as R increases. The presence of the upper fluid has the effect of decreasing the velocity of propagation of the wave which consequently causes the decrease of the potential energy

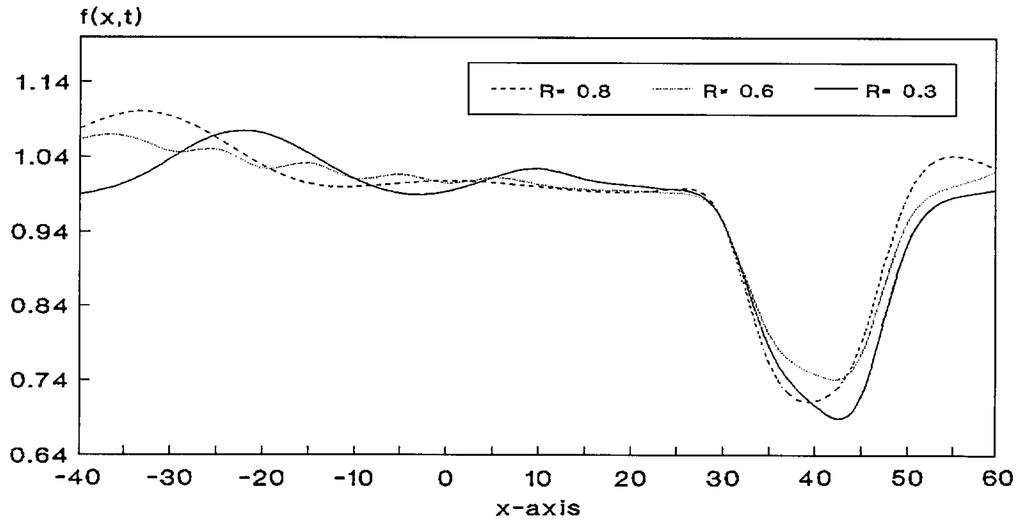

Fig. 3. Effect of the density ratio R on the interfacial wave.

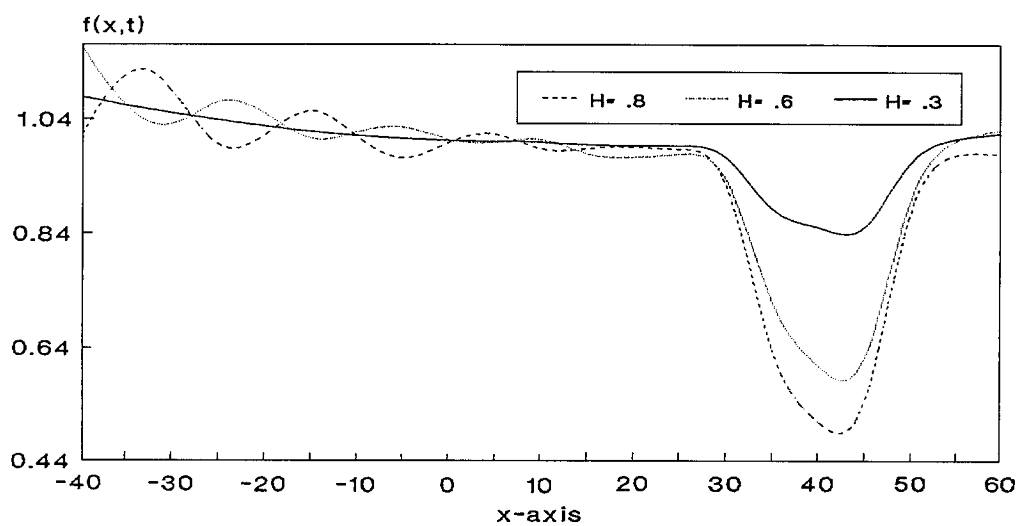

Fig. 4. Effect of the thickness ratio H on the interfacial wave.

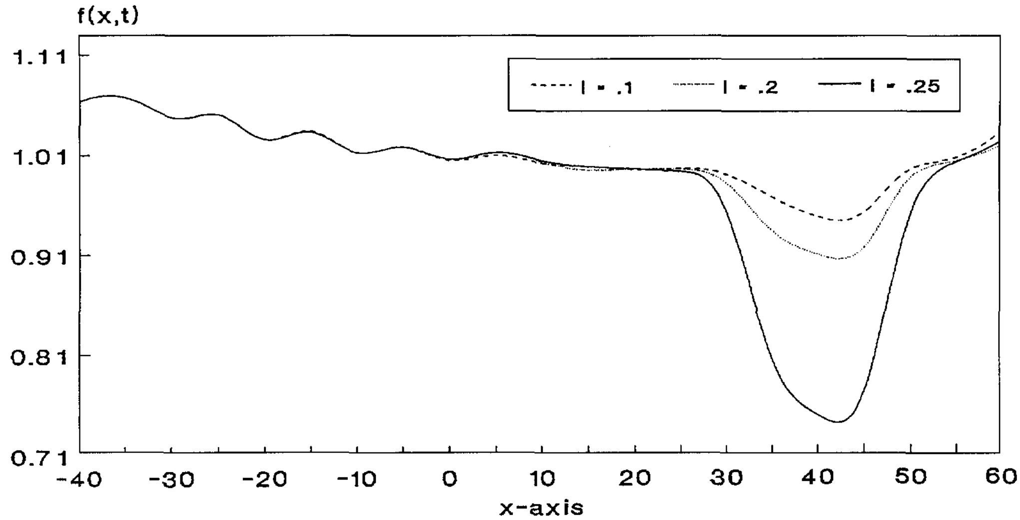

Fig. 5. Effect of the obstacle height L on the interfacial wave.

of a given deformation of the interface as well as the increase of the inertia, from which the period of oscillation starts to increase and the waves vanish far downstream.

Fig. 4 shows different interfacial wave profiles, $f(x,t)$, for different values of the thickness ratio, H, while the other parameters R, L, and t are fixed. It is clear that as H increases, there is an increase in the amplitude of the wave along the obstacle interval, as well as a noticeable drop in the wave at the beginning of the downstream interval. As the thickness of the upper layer decreases, the irregularities of the interfacial waveform vanishes.

In Fig. 5, we study the effect of changing the triangle height, L. Different values of L have been considered, for fixed values of R, H, and t. For the interfacial wave, as L increases a kind of violent disturbance in the wave profile appears, starting by a sudden decrease in the profile. In the upstream interval, there is no effect of the obstacle height. The behaviour of that solution can be interpreted, following Kakutani [5], as follows: a given smooth waveform will propagate along the characteristic curves, gradually steepen its shape due to nonlinear interactions, and then the dispersive term will begin to play its role to balance this steeping.

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Appendix 1

$$A_1 = W_1(-4 + 6k^2 + \tfrac{1}{16}k^4), \qquad A_2 = W_1(2k^4 - 8k^2), \qquad A_3 = W_1(-\tfrac{27}{16}k^4),$$
$$A_4 = W_1(4k^2 - 2k^4), \qquad A_5 = 4W_1k^4, \qquad A_6 = W_1k^4,$$

where

$$W_1 = (\gamma^2 - H(1-R))\left(\frac{-3Y_1k^2\delta^3}{k^2+1}\right)$$

and

$$A_7 = W_2(2 - 2k^2 - \tfrac{9}{32}k^4), \qquad A_8 = W_2(2k^2 - \tfrac{1}{4}k^4), \qquad A_9 = \tfrac{9}{32}W_2k^4,$$
$$A_{10} = \tfrac{1}{4}W_2k^4, \qquad A_{11} = W_2(2k^2 - \tfrac{1}{2}k^4), \qquad A_{12} = W_2k^4, \qquad A_{13} = -\tfrac{1}{2}W_2k^4,$$

where

$$W_2 = (\gamma^2 - 3H(1-R))\left(\frac{-3Y_1k^2\delta^2}{k^2+1}\right)$$

and

$$A_{14} = H(1-R)(b-1), \qquad A_{15} = H + 1 - b, \qquad A_{16} = \frac{1}{4\gamma\delta}(2A_1 - A_4 - A_6),$$
$$A_{17} = \frac{1}{2\gamma}\delta A_{12}, \qquad A_{18} = \frac{1}{2\gamma}(A_{11} - A_{13}), \qquad A_{19} = \frac{1}{16\gamma\delta}(4A_2 - A_5),$$
$$A_{20} = \frac{1}{4\gamma}A_{12}, \qquad A_{21} = \frac{1}{6\gamma\delta}A_3, \qquad A_{22} = \frac{1}{4\gamma\delta}(A_{13} - 2A_7 - A_{11}),$$
$$A_{23} = -\frac{1}{2\gamma}(A_4 + A_6), \qquad A_{24} = -\frac{1}{2\gamma}\delta A_{13}, \qquad A_{25} = -\frac{1}{16\gamma\delta}(4A_8 + A_{12}),$$
$$A_{26} = -\frac{1}{4\gamma}A_5, \qquad A_{27} = -\frac{1}{6\gamma\delta}A_9, \qquad A_{28} = \frac{1}{A_{15} - 2\gamma^2},$$
$$A_{29} = -\frac{2}{\gamma}A_{14}A_{28}, \qquad A_{30} = -2A_{28}, \qquad A_{31} = A_{15}A_{28},$$
$$A_{32} = -\frac{1}{\gamma}A_{15}A_{28}, \qquad A_{33} = \gamma A_{15}A_{28}, \qquad A_{34} = \frac{1}{2\gamma}A_{14}A_{28},$$
$$A_{35} = 3\gamma A_{28}, \qquad A_{36} = \frac{1}{2\gamma}A_{15}A_{28}, \qquad A_{37} = \frac{1}{2\gamma}A_{28},$$
$$A_{38} = (\gamma^2A_{15} + 2A_{14})A_{28}, \qquad A_{39} = \frac{1}{2\gamma}(6A_{14} + \gamma^2A_{15})A_{28}, \qquad A_{40} = 2A_{15}A_{28},$$
$$A_{41} = \frac{1}{4\gamma\delta}(2A_1 - A_4 - A_6 + 2\delta[A_{11} - A_{13}]), \qquad A_{42} = -\delta A_6, \qquad A_{43} = \tfrac{1}{2}\gamma\delta A_6,$$

$$A_{44} = \tfrac{1}{2}(A_{13} - A_{11}), \qquad A_{45} = \frac{1}{16\gamma\delta}(4A_2 - A_5 + 4\delta A_{12}), \qquad A_{46} = -\tfrac{1}{4}A_{12},$$

$$A_{47} = -\frac{1}{2\gamma}(A_4 + A_6 + \delta A_{13}), \qquad A_{48} = \tfrac{1}{2}(A_4 + A_6 + 2\delta A_{13}),$$

$$A_{49} = -\tfrac{1}{2}\gamma\delta A_{13}, \qquad A_{50} = \tfrac{1}{4}A_5, \qquad A_{51} = 3A_{38}^2 A_{35}, \qquad A_{52} = 3A_{38}^2 A_{40},$$

$$A_{53} = 6A_{38}(A_{30}A_{39} + A_{32}A_{40}), \qquad A_{54} = 6A_{38}A_{30}A_{35}, \qquad A_{55} = 6A_{38}(A_{30}A_{40} + A_{32}A_{35}),$$

$$A_{56} = -3A_{35}^2 A_{38}, \qquad A_{57} = -3A_{38}(A_{40}^2 + 2A_{39}A_{35}), \qquad A_{58} = 2A_{35}A_{40}A_{38},$$

$$A_{59} = A_{29} + 2A_{31}A_{38} - 2A_{39}A_{40} + 3A_{38}^2 A_{32} + A_{57}, \qquad A_{60} = A_{30} - A_{35}^2,$$

$$A_{61} = A_{31} + 2A_{38}A_{32} - A_{40}^2 - 2A_{39}A_{35} + 3A_{38}^2 A_{30} + A_{58},$$

$$A_{62} = A_{32} + 2A_{38}A_{30} - 2A_{35}A_{40} + A_{56},$$

$$A_{63} = 2(A_{38}A_{36} + A_{32}A_{40} + A_{32}A_{39}) - A_{40} - 6A_{39}A_{35}A_{40} + A_{53},$$

$$A_{64} = A_{36} + 2(A_{31}A_{35} + A_{32}A_{40}) - 3A_{35}(A_{35}A_{39} + A_{40}^2) + A_{55},$$

$$A_{65} = 2(A_{30}A_{40} + A_{32}A_{35}) + A_{54} - 3A_{35}^2 A_{40},$$

$$A_{66} = A_{39} + 2A_{38}A_{40} + A_{51}, \qquad A_{67} = A_{40} + 2A_{38}A_{35}, \qquad A_{68} = 2A_{38}A_{39} + A_{52},$$

$$A_{69} = A_{25} + \frac{1}{16\delta^2}(2A_{61}A_{26} + A_{62}A_{50}),$$

$$A_{70} = A_{50} + 2A_{38}A_{26}, \qquad A_{71} = \tfrac{1}{2}A_{38}(A_{50} + 2A_{38}A_{26}),$$

$$A_{72} = \frac{1}{4\delta}(A_{35}A_{50} + 2A_{26}A_{67}), \qquad A_{73} = \frac{1}{4\delta}(2A_{66}A_{26} + A_{67}A_{50}),$$

$$A_{74} = A_{22} + \frac{1}{2\delta^2}(A_{60}A_{49} + A_{61}A_{47} + 2A_{62}A_{48}),$$

$$A_{75} = A_{48} + 2A_{38}A_{47},$$

$$A_{76} = A_{49} + \tfrac{1}{2}A_{38}(A_{48} + 2A_{38}A_{47}),$$

$$A_{77} = \frac{1}{2\delta}(A_{35}A_{48} + 2A_{67}A_{47}),$$

$$A_{78} = \frac{1}{\delta}(A_{35}A_{49} + 2A_{67}A_{48} + A_{66}A_{47}),$$

$$A_{79} = \frac{1}{4\delta}(A_{35}A_{46} + A_{67}A_{45}),$$

$$A_{80} = A_{41} + \frac{1}{2\delta^2}(A_{60}A_{43} + 3A_{61}A_{17} + A_{62}A_{42}),$$

$$A_{81} = A_{42} + 3A_{38}A_{17},$$

$$A_{82} = A_{43} + A_{38}A_{42} + 3A_{38}^2 A_{17},$$

$$A_{83} = A_{44} + A_{38}A_{41} + \frac{1}{2\delta^2}(A_{61}A_{42} + A_{62}A_{43}),$$

$$A_{84} = \tfrac{1}{3}(A_{38}A_{43} + A_{38}^2 A_{42}) + A_{38}^3 A_{17},$$

$$A_{85} = \frac{1}{4\delta^3}(3A_{63}A_{17} + A_{64}A_{42} + A_{65}A_{43}) + \frac{1}{2\delta}(A_{35}A_{44} + A_{67}A_{41}),$$

$$A_{86} = \frac{1}{\delta}(3A_{66}A_{17} + A_{35}A_{43} + A_{67}A_{42}),$$

$$A_{87} = \frac{1}{2\delta}(A_{66}A_{42} + A_{67}A_{43} + 3A_{68}A_{17}),$$

$$A_{88} = \frac{1}{2\delta}(A_{35}A_{42} + 3A_{67}A_{17}).$$

Appendix 2

$$r_1 = -\tfrac{1}{6}A_{10},$$

$$r_2 = \frac{1}{8\delta^3}A_{37}(2\gamma\delta(A_{74} - A_{85}) - A_{78} - A_{83}),$$

$$r_3 = \frac{1}{8\delta^3}A_{37}(2\gamma\delta(A_{47} - A_{88}) - A_{81}),$$

$$r_4 = \frac{1}{4\delta^3}A_{37}(\gamma\delta(A_{75} - A_{86}) - A_{82}),$$

$$r_5 = \frac{1}{8\delta^3}A_{37}(2\gamma\delta(A_{76} - A_{87}) - 3A_{84}),$$

$$r_6 = \frac{1}{64\delta^3}A_{37}(4\gamma\delta(A_{69} - A_{79}) - A_{73} - A_{46} - A_{38}A_{45}),$$

$$r_7 = \frac{1}{16\delta^2}\gamma A_{37}A_{26}, \qquad r_8 = \frac{1}{16\delta^2}\gamma A_{37}A_{70}, \qquad r_9 = \frac{1}{16\delta^2}\gamma A_{37}A_{71},$$

$$r_{10} = \frac{1}{216\delta^3}A_{37}(\gamma(6\delta A_{27} - A_{67}) - A_{38}A_{21}),$$

$$r_{11} = \frac{1}{4\delta^2}A_{37}A_{80},$$

$$r_{12} = \frac{1}{8\delta^3}A_{37}(2\gamma\delta A_{77} + A_{75} - A_{86}),$$

$$r_{13} = \frac{1}{4\delta^3}A_{37}(\gamma\delta(A_{78} + A_{83}) + A_{76} - A_{87}),$$

$$r_{14} = \frac{1}{4\delta^2}\gamma A_{37}A_{88}, \qquad r_{15} = \frac{1}{4\delta^2}\gamma A_{37}A_{81}, \qquad r_{16} = \frac{1}{4\delta^2}\gamma A_{37}A_{82}, \qquad r_{17} = \frac{1}{4\delta^2}\gamma A_{37}A_{84},$$

$$r_{18} = \frac{1}{64\delta^3}A_{37}(A_{70} + 4\gamma\delta(A_{72} + A_{45})),$$

$$r_{19} = \frac{1}{32\delta^3}A_{37}(A_{71} + 2\gamma\delta(A_{73} + A_{46} + A_{38}A_{45})),$$

$$r_{20}=\frac{1}{36\delta^2}\gamma A_{37}A_{21},\qquad r_{21}=\frac{1}{36\delta^2}\gamma A_{37}A_{38}A_{21},$$
$$r_{22}=\frac{1}{36\delta^2}\gamma A_{38}A_{21}A_{43}.$$

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