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Regularity theory for the fractional harmonic oscillator ☆

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Abstract

In this paper we develop the theory of Schauder estimates for the fractional harmonic oscillator $H^\sigma = (-\Delta + |x|^2)^\sigma$, $0 < \sigma < 1$. More precisely, a new class of smooth functions $C_H^{k,\alpha}$ is defined, in which we study the action of H^σ. In fact these spaces are those adapted to the operator H, hence the suited ones for this type of regularity estimates. In order to prove our results, an analysis of the interaction of the Hermite–Riesz transforms with the Hölder spaces $C_H^{k,\alpha}$ is needed, that we believe of independent interest.

Keywords: Fractional harmonic oscillator; Schauder estimate; Fractional integral; Riesz transform

1. Introduction

For a given partial differential operator L, the analysis of its regularity properties with respect to Hölder classes is one of the tools employed in the theory to prove important facts about partial differential equations. Indeed, being a bit imprecise, it is well known that if f is a Hölder continuous function with exponent α, then the equation $-\Delta u = f$ has a unique solution u, whose second order derivatives belong to C^α, and $\|u\|_{C^{2,\alpha}}$ is controlled by $\|f\|_{C^\alpha}$. This result was first applied to obtain classical solutions of second order elliptic equations of the form $Lu = f$ (see for instance [7, Chapter 6]). Recently, and motivated by the obstacle problem for the fractional

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Laplacian, L. Silvestre proved in [11] (see also his thesis [10]) the regularity properties for the operator $(-\Delta)^\sigma$, $0<\sigma<1$, when acting on Hölder spaces. For more applications see [5] and [6].

Let H be the most basic Schrödinger operator in $\mathbb{R}^n$, $n \geqslant 1$, the harmonic oscillator:

$$H = -\Delta + |x|^2.$$

The fractional powers H^σ, $0<\sigma<1$, were introduced in [13].

The aim of this paper is to prove regularity estimates in Hölder classes for the fractional harmonic oscillator H^σ. For this purpose, we define new Hölder spaces $C_H^{k,\alpha}$, different than the classical Hölder spaces $C^{k,\alpha}$, in which the smoothness properties of H^σ are analyzed, see Definition 1.1 and Theorem A.

The classes $C_H^{k,\alpha}$ are the natural spaces associated to H. This becomes evident, for instance, in the fact that the Hermite–Riesz transforms have the expected behavior: they preserve them, see Theorem 4.1. Also the fractional integrals produce a kind of "inverse fractional derivative" process when acting in $C_H^{k,\alpha}$, see Theorem B.

Our estimates, together with Harnack's inequality for H^σ proved in [13], are the basic regularity estimates one expects to get for the fractional powers of a second order operator. Moreover, we found the right spaces for which Schauder estimates are appropriated. We expect to obtain the correct regularity estimates for nonlinear problems related to the fractional harmonic oscillator in these spaces. Applications will appear elsewhere.

Let us introduce the definition of H^σ. For a function f in Schwartz's class $\mathcal{S}$ and $0<\sigma<1$, the fractional harmonic oscillator H^σ is given by the classical formula

$$H^\sigma f(x) = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \big(e^{-tH} f(x) - f(x)\big) \frac{dt}{t^{1+\sigma}}, \tag{1.1}$$

where $v(x,t) = e^{-tH} f(x)$ is the solution of the heat-diffusion equation $\partial_t v + Hv = 0$ in $\mathbb{R}^n \times (0,\infty)$, with initial datum $v(x,0) = f(x)$ on $\mathbb{R}^n$. In [13] it is shown that

$$H^\sigma f(x) = \int_{\mathbb{R}^n} \big(f(x) - f(z)\big) F_\sigma(x,z)\,dz + f(x) B_\sigma(x), \quad x \in \mathbb{R}^n,\ f \in \mathcal{S}, \tag{1.2}$$

where

$$F_\sigma(x,z) = \frac{1}{-\Gamma(-\sigma)} \int_0^\infty G_t(x,z) \frac{dt}{t^{1+\sigma}},$$

$$B_\sigma(x) = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left[\int_{\mathbb{R}^n} G_t(x,z)\,dz - 1 \right] \frac{dt}{t^{1+\sigma}}, \tag{1.3}$$

and $G_t(x,z)$ is the kernel of the heat-diffusion semigroup generated by H, see (3.1).

Next we define the Hölder spaces in which the regularity properties of the operators will be considered.

Definition 1.1. Let $0 < \alpha \leqslant 1$. A continuous function $u : \mathbb{R}^n \to \mathbb{R}$ belongs to the Hermite–Hölder space $C_H^{0,\alpha}$ associated to H, if there exists a constant C, depending only on u and α, such that

$$\left|u(x_1) - u(x_2)\right| \leqslant C|x_1 - x_2|^\alpha, \quad \text{and} \quad \left|u(x)\right| \leqslant \frac{C}{(1+|x|)^\alpha},$$

for all $x_1, x_2, x \in \mathbb{R}^n$. With the notation

$$[u]_{C^{0,\alpha}} = \sup_{\substack{x_1,x_2\in\mathbb{R}^n \\ x_1\neq x_2}} \frac{|u(x_1)-u(x_2)|}{|x_1-x_2|^\alpha}, \quad \text{and} \quad [u]_{M^\alpha} = \sup_{x\in\mathbb{R}^n} \left|\left(1+|x|\right)^\alpha u(x)\right|,$$

we define the norm in the spaces $C_H^{0,\alpha}$ to be

$$\|u\|_{C_H^{0,\alpha}} = [u]_{C^{0,\alpha}} + [u]_{M^\alpha}.$$

When working with the harmonic oscillator H some special first order partial differential operators are considered, see (3.3), which are the natural *derivatives*. Then the classes $C_H^{k,\alpha}$ can be defined in a usual way, see Definition 3.1. We present now our first main result.

Theorem A. *Let $\alpha \in (0,1]$ and $\sigma \in (0,1)$.*

(A1) *Let $u \in C_H^{0,\alpha}$ and $2\sigma < \alpha$. Then $H^\sigma u \in C_H^{0,\alpha-2\sigma}$ and*

$$\left\|H^\sigma u\right\|_{C_H^{0,\alpha-2\sigma}} \leqslant C\|u\|_{C_H^{0,\alpha}}.$$

(A2) *Let $u \in C_H^{1,\alpha}$ and $2\sigma < \alpha$. Then $H^\sigma u \in C_H^{1,\alpha-2\sigma}$ and*

$$\left\|H^\sigma u\right\|_{C_H^{1,\alpha-2\sigma}} \leqslant C\|u\|_{C_H^{1,\alpha}}.$$

(A3) *Let $u \in C_H^{1,\alpha}$ and $2\sigma \geqslant \alpha$, with $\alpha - 2\sigma + 1 \neq 0$. Then $H^\sigma u \in C_H^{0,\alpha-2\sigma+1}$ and*

$$\left\|H^\sigma u\right\|_{C_H^{0,\alpha-2\sigma+1}} \leqslant C\|u\|_{C_H^{1,\alpha}}.$$

(A4) *Let $u \in C_H^{k,\alpha}$ and assume that $k+\alpha-2\sigma$ is not an integer. Then $H^\sigma u \in C_H^{l,\beta}$ where l is the integer part of $k+\alpha-2\sigma$ and $\beta = k+\alpha-2\sigma-l$.*

The last theorem can be interpreted as saying that the Hölder spaces $C_H^{k,\alpha}$ are the reasonable classes in order to obtain Schauder type estimates for H^σ. Indeed, if we define the negative powers of H, i.e. the fractional integral operators

$$H^{-\sigma} f(x) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tH} f(x) \frac{dt}{t^{1-\sigma}}, \quad 0 < \sigma \leqslant 1,$$

then we are able to prove our second main result:

Theorem B. *Let* $u \in C_H^{0,\alpha}$*, for some* $0 < \alpha \leqslant 1$*, and* $0 < \sigma \leqslant 1$*.*

(B1) *If* $\alpha + 2\sigma \leqslant 1$*, then* $H^{-\sigma}u \in C_H^{0,\alpha+2\sigma}$ *and*

$$\|H^{-\sigma}u\|_{C_H^{0,\alpha+2\sigma}} \leqslant C\|u\|_{C_H^{0,\alpha}}.$$

(B2) *If* $1 < \alpha + 2\sigma \leqslant 2$*, then* $H^{-\sigma}u \in C_H^{1,\alpha+2\sigma-1}$ *and*

$$\|H^{-\sigma}u\|_{C_H^{1,\alpha+2\sigma-1}} \leqslant C\|u\|_{C_H^{0,\alpha}}.$$

(B3) *If* $2 < \alpha + 2\sigma \leqslant 3$*, then* $H^{-\sigma}u \in C_H^{2,\alpha+2\sigma-2}$ *and*

$$\|H^{-\sigma}u\|_{C_H^{2,\alpha+2\sigma-2}} \leqslant C\|u\|_{C_H^{0,\alpha}}.$$

It is worth noting that, if in Theorem B(B3) we take $\sigma = 1$, we get the Schauder estimate for the solution to $Hu = f$, in $\mathbb{R}^n$, with $f \in C_H^{0,\alpha}$.

To prove the two main theorems of this paper we need a study of the first and second order Hermite–Riesz transforms, $\mathcal{R}_i$ and $\mathcal{R}_{ij}$, acting on the spaces $C_H^{0,\alpha}$, that we believe of independent interest. See the final part of Section 3 and Theorem 4.1.

The first main task in our paper is to obtain explicit pointwise expressions for all the operators involved, when they are applied to functions belonging to the spaces $C_H^{k,\alpha}$, and the second one is to actually prove the regularity estimates. Section 2 contains two abstract propositions dealing with these two aspects: Proposition 2.1 takes care of the pointwise formulas and Proposition 2.3 contains a regularity result. We will apply, in a systematic way, both propositions in order to reach our objectives: see Section 3 for all the pointwise formulas, and Section 4 for the proofs of Theorems A and B. In Section 5 we collect all the computational lemmas used in the previous sections.

In some recent papers, B. Bongioanni, E. Harboure and O. Salinas studied the boundedness of fractional integrals (see [2]) and Riesz transforms (see [3]), associated to a certain class of Schrödinger operators $\mathcal{L} = -\Delta + V$, in spaces of $BMO_{\mathcal{L}}^{\beta}$ type, $0 \leqslant \beta < 1$, using Harmonic Analysis techniques. In [2, Proposition 4], they showed that the spaces $BMO_{\mathcal{L}}^{\beta}$ coincide with a Hölder type space $\Lambda_{\mathcal{L}}^{\beta}$, $0 < \beta < 1$, with equivalent norms. In the case $V = |x|^2$, our space $C_H^{0,\beta}$ coincides with their space Λ_H^{β}, for $0 < \beta < 1$.

A natural question to think about is the possibility of getting (at least the local part of) our results by modifying, in an appropriate way, the kernel of the classical fractional Laplace operator. In our opinion our procedure is the natural one and we haven't found a smooth bridge to pass from one case to the other. Even more, some recent (local) results by R.F. Bass in [1] about stable-like operators cannot be applied in our case because, clearly, his assumption on the kernel $A(x,h)$ [1, Assumption 1.1] is not fulfilled by our kernel $F_\sigma(x,z)$, see Lemma 5.4 below. Moreover, he does not allow for $\alpha + \beta$ to be an integer [3, Assumption 1.1], but we do. We want to complete the thought of this paragraph by establishing the parallelism with possible definitions of Sobolev spaces for the harmonic oscillator that were considered by R. Radha and S. Thangavelu in [9], by S. Thangavelu in [15] and also by B. Bongioanni and J.L. Torrea in [4].

In that case natural definitions of Sobolev spaces were given in order to get the results for the harmonic oscillator.

Throughout this paper, the letter C denotes a positive constant that may change in each occurrence and it will depend on the parameters involved (whenever it is necessary, we point out this dependence with subscripts), and Γ stands for the Gamma function. Recall that $\Gamma(-\sigma) = -\Gamma(1-\sigma)/\sigma$, $0 < \sigma < 1$. Without mentioning it, we will repeatedly apply the inequality $r^\eta e^{-r} \leqslant C_\eta e^{-r/2}$, $\eta \geqslant 0$, $r > 0$.

2. Two abstract results

Proposition 2.1. *Let T be a bounded operator on $\mathcal{S}$ such that $\langle Tf, g\rangle = \langle f, Tg\rangle$, for all $f, g \in \mathcal{S}$. Assume that*

$$Tf(x) = \int_{\mathbb{R}^n} \big(f(x) - f(z)\big) K(x,z)\,dz + f(x)B(x), \quad x \in \mathbb{R}^n,$$

where the kernel K verifies

$$\big|K(x,z)\big| \leqslant \frac{C}{|x-z|^{n+\gamma}} e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}, \quad x, z \in \mathbb{R}^n, \tag{2.1}$$

for some $-n \leqslant \gamma < 1$, and B is a continuous function with polynomial growth at infinity. Let $u \in C_H^{0,\gamma+\varepsilon}$, with $0 < \gamma + \varepsilon \leqslant 1$, $\varepsilon > 0$. Then Tu is well defined as a tempered distribution and it coincides with the continuous function

$$Tu(x) = \int_{\mathbb{R}^n} \big(u(x) - u(z)\big) K(x,z)\,dz + u(x)B(x), \quad x \in \mathbb{R}^n. \tag{2.2}$$

Proof. By (2.1) and the smoothness of u, the integral in (2.2) is absolutely convergent. Since B has polynomial growth at infinity, the right-hand side of (2.2) defines a tempered distribution. Let us take $\frac{n}{\gamma+\varepsilon} < p < \infty$. Then, the finiteness of $[u]_{M^{\gamma+\varepsilon}}$ implies that $u \in L^p(\mathbb{R}^n)$, and Tu is well defined as a tempered distribution. Fix an arbitrary positive number η and suppose that $R > 0$. Let $f_j(x) := \zeta(x/j)(u * W_{1/j})(x)$, $j \in \mathbb{N}$, where $W_t(z) = (4\pi t)^{-n/2} e^{-|z|^2/(4t)}$ is the Gauss–Weierstrass kernel and ζ is a nonnegative smooth cutoff function (that is, $\zeta \in C_c^\infty(\mathbb{R}^n)$, $0 \leqslant \zeta \leqslant 1$, $\zeta \equiv 1$ in $B_1(0)$, $\zeta \equiv 0$ in $B_2^c(0)$, and $|\nabla\zeta| < C$ in $\mathbb{R}^n$). Note that each f_j belongs to $\mathcal{S}$. It is easy to check that the sequence $\{f_j\}_{j\in\mathbb{N}}$ converges to u in $L^p(\mathbb{R}^n)$ and uniformly in $B_R(x)$ for each $x \in \mathbb{R}^n$, and $[f_j]_{C^{0,\gamma+\varepsilon}} \leqslant C\|u\|_{C_H^{0,\gamma+\varepsilon}} =: M$. As $j \to \infty$, $Tf_j \to Tu$ in $\mathcal{S}'$. Since B is a continuous function, $f_j B$ converges uniformly to uB in $B_R(x_0)$, $x_0 \in \mathbb{R}^n$. There exists $0 < \delta < R/2$ such that

$$M \int_{B_\delta(0)} |z|^{\varepsilon-n}\,dz \leqslant \frac{\eta}{3}.$$

For $x \in B_{R/2}(x_0)$, we write

$$\int_{\mathbb{R}^n} \big(f_j(x) - f_j(z)\big) K(x,z)\,dz = \int_{B_\delta(x)} + \int_{B_\delta^c(x)} = I + II.$$

Then, by the choice of δ,

$$|I| + \left| \int_{B_\delta(x)} \big(u(x) - u(z)\big) K(x,z)\,dz \right| \leqslant \frac{2}{3}\eta.$$

We also have

$$\left| II - \int_{B_\delta^c(x)} \big(u(x) - u(z)\big) K(x,z)\,dz \right| \leqslant C\big|f_j(x) - u(x)\big| + C\Big(\int_{B_\delta^c(x)} \big|f_j(z) - u(z)\big|^p\,dz \Big)^{1/p}$$
$$\leqslant \frac{\eta}{3},$$

for sufficiently large j, uniformly in $x \in B_{R/2}(x_0)$. Therefore,

$$\int_{\mathbb{R}^n} \big(f_j(x) - f_j(z)\big) K(x,z)\,dz \rightrightarrows \int_{\mathbb{R}^n} \big(u(x) - u(z)\big) K(x,z)\,dz, \quad j \to \infty,$$

in $B_{R/2}(x_0)$. Hence, by uniqueness of the limits, Tu is a function that coincides with (2.2), and it is a continuous function because it is the uniform limit of continuous functions. □

Remark 2.2. In the context of Proposition 2.1, assume that, instead of having estimate (2.1) on the kernel, we just know that $|K(x,z)| \leqslant \Phi(x-z)$, where $\Phi \in L^{p'}(\mathbb{R}^n)$, and p' is the conjugate exponent of some p such that $\frac{n}{\gamma+\varepsilon} < p < \infty$. Then, it is enough to take $u \in C_H^{0,\alpha}$, for some $0 < \alpha \leqslant 1$, to get the same conclusion, since the approximation procedure given in the proof above can also be applied in this situation.

Proposition 2.3. *Let T be an operator satisfying the hypotheses of Proposition* 2.1*, with $0 \leqslant \gamma < 1$ and $0 < \gamma + \varepsilon \leqslant 1$, for some $0 < \varepsilon < 1$. Assume that the kernel K and the function B also satisfy*:

(a) $|K(x_1,z) - K(x_2,z)| \leqslant C\frac{|x_1-x_2|}{|x_2-z|^{n+1+\gamma}} e^{-\frac{|z||x_2-z|^2}{C}} e^{-\frac{|x_2-z|^2}{C}}$*, when* $|x_1 - z| > 2|x_1 - x_2|$.
(b) *There exists a constant $C > 0$ such that* $|\int_{|x-z|>r} K(x,z)\,dz| \leqslant Cr^{-\gamma}$*, for all* $x \in \mathbb{R}^n$.
(c) *For all $x \in \mathbb{R}^n$,* $|B(x)| \leqslant C(1+|x|)^\gamma$*, and* $\nabla B \in L^\infty(\mathbb{R}^n)$.

Then T maps $C_H^{0,\gamma+\varepsilon}$ into $C_H^{0,\varepsilon}$ continuously.

Proof. Given $x_1, x_2 \in \mathbb{R}^n$, let $B = B(x_1, 2|x_1 - x_2|)$, $\widetilde{B} = B(x_2, 4|x_1 - x_2|)$ and $B' = B(x_2, |x_1 - x_2|)$. We write

$$\begin{aligned} Tu(x) &= \int_B \big(u(x) - u(z)\big) K(x,z)\,dz + \int_{B^c} \big(u(x) - u(z)\big) K(x,z)\,dz + u(x)B(x) \\ &= I(x) + II(x) + III(x). \end{aligned}$$

By (2.1) we have

$$\begin{aligned} &\big|I(x_1) - I(x_2)\big| \\ &\quad\leqslant \int_B \big|\big(u(x_1) - u(z)\big)K(x_1,z)\big|\,dz + \int_{\widetilde{B}} \big|\big(u(x_2) - u(z)\big)K(x_2,z)\big|\,dz \\ &\quad\leqslant C[u]_{C^{0,\gamma+\varepsilon}}\bigg[\int_B \frac{|x_1 - z|^{\gamma+\varepsilon}}{|x_1 - z|^{n+\gamma}}\,dz + \int_{\widetilde{B}} \frac{|x_2 - z|^{\gamma+\varepsilon}}{|x_2 - z|^{n+\gamma}}\,dz\bigg] = C[u]_{C^{0,\gamma+\varepsilon}}|x_1 - x_2|^{\varepsilon}. \end{aligned}$$

For the difference $II(x_1) - II(x_2)$, we add the term $\pm u(x_2)K(x_1, z)$ and we use the smoothness and cancellation properties of the kernel $K(x,z)$ (hypotheses (a) and (b)) to get:

$$\begin{aligned} &\big|II(x_1) - II(x_2)\big| \\ &\quad\leqslant \int_{B^c} \big|\big(u(x_2) - u(z)\big)\big(K(x_1,z) - K(x_2,z)\big)\big|\,dz + \bigg|\int_{B^c} \big(u(x_1) - u(x_2)\big)K(x_1,z)\,dz\bigg| \\ &\quad\leqslant C[u]_{C^{0,\gamma+\varepsilon}}\bigg[\int_{B^c} |x_2 - z|^{\gamma+\varepsilon} \frac{|x_1 - x_2|}{|x_2 - z|^{n+1+\gamma}}\,dz + |x_1 - x_2|^{\gamma+\varepsilon}\bigg|\int_{B^c} K(x_1,z)\,dz\bigg|\bigg] \\ &\quad\leqslant C[u]_{C^{0,\gamma+\varepsilon}}\bigg[\int_{(B')^c} \frac{|x_1 - x_2|}{|x_2 - z|^{n+1-\varepsilon}}\,dz + |x_1 - x_2|^{\varepsilon}\bigg] = C[u]_{C^{0,\gamma+\varepsilon}}|x_1 - x_2|^{\varepsilon}. \end{aligned}$$

If $|x_1 - x_2| < \frac{1}{1+|x_1|}$, by (c),

$$\begin{aligned} \frac{|III(x_1) - III(x_2)|}{|x_1 - x_2|^{\varepsilon}} &\leqslant \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^{\gamma+\varepsilon}}\big|B(x_1)\big||x_1 - x_2|^{\gamma} + \big|u(x_2)\big|\frac{|B(x_1) - B(x_2)|}{|x_1 - x_2|^{\varepsilon}} \\ &\leqslant C[u]_{C^{0,\gamma+\varepsilon}} + [u]_{M^{\gamma+\varepsilon}}\|\nabla B\|_{L^\infty(\mathbb{R}^n)}|x_1 - x_2|^{1-\varepsilon} \leqslant C\|u\|_{C_H^{0,\gamma+\varepsilon}}. \end{aligned}$$

Assume that $|x_1 - x_2| \geqslant \frac{1}{1+|x_1|}$. Then $1 + |x_1| \leqslant 1 + |x_2| + |x_1 - x_2|$, which implies

$$\frac{1+|x_1|}{1+|x_2|} \leqslant 1 + \frac{|x_1 - x_2|}{1+|x_2|},$$

and then,

$$\frac{1}{1+|x_2|} \leqslant \frac{1}{1+|x_1|} + \frac{|x_1 - x_2|}{(1+|x_1|)(1+|x_2|)} \leqslant 2|x_1 - x_2|.$$

With this and hypothesis (c) we have

$$\frac{|III(x_1)-III(x_2)|}{|x_1-x_2|^{\varepsilon}} \leqslant \left|u(x_1)B(x_1)\right|\left(1+|x_1|\right)^{\varepsilon} + \left|u(x_2)B_{\sigma}(x_2)\right|2^{\varepsilon}\left(1+|x_2|\right)^{\varepsilon} \leqslant C[u]_{M^{\gamma+\varepsilon}}.$$

Let us finally study the growth of $Tu(x)$. For the multiplicative term uB we clearly have $|u(x)B(x)| \leqslant C[u]_{M^{\gamma+\varepsilon}}(1+|x|)^{-\varepsilon}$. Consider next the integral part in the formula for $Tu(x)$, (2.2). Since Tu and B are continuous functions, it is enough to consider $|x|>2$. We write

$$\left|\int_{\mathbb{R}^n}\big(u(x)-u(z)\big)K(x,z)\,dz\right| = \left|\left(\int_{|x-z|<\frac{1}{1+|x|}} + \int_{|x-z|\geqslant\frac{1}{1+|x|}}\right)dz\right|.$$

On the one hand,

$$\begin{aligned}\int_{|x-z|<\frac{1}{1+|x|}} \left|u(x)-u(z)\right|\left|K(x,z)\right|dz &\leqslant C[u]_{C^{0,\gamma+\varepsilon}} \int_{|x-z|<\frac{1}{1+|x|}} \frac{|x-z|^{\gamma+\varepsilon}}{|x-z|^{n+\gamma}}\,dz \\ &= [u]_{C^{0,\gamma+\varepsilon}}\frac{C}{(1+|x|)^{\varepsilon}}.\end{aligned}$$

On the other hand, by (b),

$$\left|u(x)\right|\left|\int_{|x-z|\geqslant\frac{1}{1+|x|}} K(x,z)\,dz\right| \leqslant \frac{[u]_{M^{\gamma+\varepsilon}}}{(1+|x|)^{\gamma+\varepsilon}}C\left(1+|x|\right)^{\gamma} = [u]_{M^{\gamma+\varepsilon}}\frac{C}{(1+|x|)^{\varepsilon}}.$$

Since $|x-z| \geqslant \frac{1}{1+|x|}$ implies that $\frac{1}{1+|z|} \leqslant 2|x-z|$, applying (2.1) we get

$$\begin{aligned}\int_{|x-z|\geqslant\frac{1}{1+|x|}} \left|u(z)\right|\left|K(x,z)\right|dz &\leqslant C[u]_{M^{\gamma+\varepsilon}} \int_{|x-z|\geqslant\frac{1}{1+|x|}} \frac{1}{(1+|z|)^{\gamma+\varepsilon}}\frac{e^{-\frac{|x||x-z|}{C}}e^{-\frac{|x-z|^2}{C}}}{|x-z|^{n+\gamma}}\,dz \\ &\leqslant C[u]_{M^{\gamma+\varepsilon}} \int_{|x-z|\geqslant\frac{1}{1+|x|}} |x-z|^{\gamma+\varepsilon}\frac{e^{-\frac{|x||x-z|}{C}}e^{-\frac{|x-z|^2}{C}}}{|x-z|^{n+\gamma}}\,dz \\ &= C[u]_{M^{\gamma+\varepsilon}} \sum_{j=0}^{\infty}\int_{|x-z|\sim\frac{2^j}{1+|x|}} \frac{e^{-\frac{|x||x-z|}{C}}e^{-\frac{|x-z|^2}{C}}}{|x-z|^{n-\varepsilon}}\,dz \\ &\leqslant [u]_{M^{\gamma+\varepsilon}}\frac{C}{(1+|x|)^{\varepsilon}}\sum_{j=0}^{\infty}2^{j\varepsilon}e^{-\frac{2^j}{C'}} = [u]_{M^{\gamma+\varepsilon}}\frac{C}{(1+|x|)^{\varepsilon}},\end{aligned}$$

where in the last line the constant C' appearing in the exponential is independent of x because $|x||x-z| \sim 2^j$. By pasting the estimates above the result is proved. □

3. The operators

In this section we give the pointwise definitions, in the class $C_H^{k,\alpha}$, of all the operators involved.

3.1. The heat-diffusion semigroup: e^{-tH}

In our paper, the kernel of the heat-diffusion semigroup generated by H will play an essential role. We shall need the pointwise formula for it.

Recall (see [14]) that the eigenfunctions of H are the multi-dimensional Hermite functions $h_\nu(x) = e^{-|x|^2/2}\Psi_\nu(x)$, $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}_0^n$, where Ψ_ν are the multi-dimensional Hermite polynomials, with positive eigenvalues: $Hh_\nu = (2|\nu| + n)h_\nu$, for all $\nu \in \mathbb{N}_0^n$, $|\nu| = \nu_1 + \cdots + \nu_n$. Moreover, $\overline{\operatorname{span}\{h_\nu\colon \nu \in \mathbb{N}_0^n\}} = L^2(\mathbb{R}^n)$. The heat-diffusion semigroup generated by H is given as an integral operator: for $u \in \bigcup_{1 \leqslant p \leqslant \infty} L^p(\mathbb{R}^n)$,

$$\begin{aligned} e^{-tH}u(x) &= \int_{\mathbb{R}^n} G_t(x,z)u(z)\,dz \\ &= \int_{\mathbb{R}^n} \Bigg[\sum_{j=0}^{\infty} e^{-t(2j+n)} \sum_{|\nu|=j} h_\nu(x)h_\nu(z)\Bigg] u(z)\,dz \\ &= \int_{\mathbb{R}^n} \frac{e^{-[\frac{1}{2}|x-z|^2 \coth 2t + x\cdot z \tanh t]}}{(2\pi \sinh 2t)^{n/2}} u(z)\,dz. \end{aligned} \tag{3.1}$$

Note that, for all $\nu \in \mathbb{N}_0^n$, $e^{-tH}h_\nu(x) = e^{-t(2|\nu|+n)}h_\nu(x)$, $t \geqslant 0$. With the following change of parameters (due to S. Meda)

$$t = \frac{1}{2}\log\frac{1+s}{1-s}, \quad t \in (0,\infty),\ s \in (0,1), \tag{3.2}$$

the heat-diffusion kernel can be expressed as

$$\begin{aligned} G_{t(s)}(x,z) &= \sum_{j=0}^{\infty} \left(\frac{1-s}{1+s}\right)^{j+n/2} \sum_{|\nu|=j} h_\nu(x)h_\nu(z) \\ &= \left(\frac{1-s^2}{4\pi s}\right)^{n/2} e^{-\frac{1}{4}[s|x+z|^2 + \frac{1}{s}|x-z|^2]}, \quad s \in (0,1). \end{aligned}$$

3.2. The fractional operators: H^σ *and* $(H \pm 2k)^\sigma$, $k \in \mathbb{N}$

Let us first analyze the fractional harmonic oscillator H^σ. Let $\langle f, h_\nu\rangle = \int_{\mathbb{R}^n} f(z)h_\nu(z)\,dz$. If $f \in \mathcal{S}$, the Hermite series expansion $\sum_\nu \langle f, h_\nu\rangle h_\nu = \sum_{k=0}^\infty \sum_{|\nu|=k} \langle f, h_\nu\rangle h_\nu$ converges to f uniformly in $\mathbb{R}^n$ (and also in $L^2(\mathbb{R}^n)$), since $\|h_\nu\|_{L^\infty(\mathbb{R}^n)} \leqslant C$, for all $\nu \in \mathbb{N}_0^n$, and, for each $m \in \mathbb{N}$, we have $|\langle f, h_\nu\rangle| \leqslant \|H^m f\|_{L^2(\mathbb{R}^n)}(2|\nu| + n)^{-m}$. As $e^{-tH} f(x) = \sum_\nu e^{-t(2|\nu|+n)}\langle f, h_\nu\rangle h_\nu$, from (1.1) we get $H^\sigma f = \sum_\nu (2|\nu| + n)^\sigma \langle f, h_\nu\rangle h_\nu$, and the series converges uniformly in $\mathbb{R}^n$. As a consequence of the last reasonings, H^σ is a bounded operator in $\mathcal{S}$. Note that, by using Hermite series expansions, we can check that $\langle H^\sigma f, g\rangle = \langle f, H^\sigma g\rangle$, for all $f, g \in \mathcal{S}$, and $H^1 f = Hf$, $H^0 f = f$.

The proof of the identity (1.2) can be found in [13]. We sketch it here for completeness. Since $e^{-tH}1(x)$ is not a constant function, we have

$$\begin{aligned}
&\frac{1}{\Gamma(-\sigma)}\int_0^\infty \big(e^{-tH}f(x) - f(x)\big)\frac{dt}{t^{1+\sigma}} \\
&= \frac{1}{\Gamma(-\sigma)}\int_0^\infty \bigg(\int_{\mathbb{R}^n} G_t(x,z)f(z)\,dz - f(x)\bigg)\frac{dt}{t^{1+\sigma}} \\
&= \frac{1}{\Gamma(-\sigma)}\int_0^\infty \bigg[\int_{\mathbb{R}^n} G_t(x,z)\big(f(z)-f(x)\big)\,dz + f(x)\bigg(\int_{\mathbb{R}^n} G_t(x,z)\,dz - 1\bigg)\bigg]\frac{dt}{t^{1+\sigma}} \\
&= \frac{1}{\Gamma(-\sigma)}\int_0^\infty \int_{\mathbb{R}^n} G_t(x,z)\big(f(z)-f(x)\big)\,dz\,\frac{dt}{t^{1+\sigma}} + f(x)\frac{1}{\Gamma(-\sigma)}\int_0^\infty \big(e^{-tH}1(x)-1\big)\frac{dt}{t^{1+\sigma}} \\
&= \int_{\mathbb{R}^n} \big(f(x)-f(z)\big)F_\sigma(x,z)\,dz + f(x)B_\sigma(x).
\end{aligned}$$

The subtle point in the calculations above is to justify the last equality. If $0 < \sigma < 1/2$, the last integral is absolutely convergent. In the case $1/2 \leqslant \sigma < 1$, a cancellation is involved (which is also exploited in the proof of Theorem 3.2 below), that allows to show that the integral converges as a principal value.

As we said in the Introduction, some type of *derivatives* (first order partial differential operators) are usually considered when working with the operator H. Recall the factorization

$$H = \frac{1}{2}\sum_{i=1}^n (A_i A_{-i} + A_{-i}A_i),$$

where

$$A_i = \partial_{x_i} + x_i, \qquad A_{-i} = A_i^* = -\partial_{x_i} + x_i, \quad i = 1,\dots,n. \tag{3.3}$$

In the Harmonic Analysis associated to H, the operators A_i, $1 \leqslant |i| \leqslant n$, play the role of the classical partial derivatives ∂_{x_i} in the Euclidean Harmonic Analysis (see [14,4,8,12]). Now, it is natural to consider the classes of functions whose k-th *derivatives* are in $C_H^{0,\alpha}$.

Definition 3.1. For each $k \in \mathbb{N}$, we define the Hermite–Hölder space $C_H^{k,\alpha}$, $0 < \alpha \leqslant 1$, as the set of all functions $u \in C^k(\mathbb{R}^n)$ such that the following norm is finite:

$$\|u\|_{C_H^{k,\alpha}} = [u]_{M^\alpha} + \sum_{\substack{1\leqslant |i_1|,\dots,|i_m|\leqslant n\\ 1\leqslant m\leqslant k}} [A_{i_1}\cdots A_{i_m}u]_{M^\alpha} + \sum_{1\leqslant |i_1|,\dots,|i_k|\leqslant n} [A_{i_1}\cdots A_{i_k}u]_{C^{0,\alpha}}.$$

We are ready to show that the pointwise formula for $H^\sigma u$, when u belongs to the Hölder class $C_H^{k,\alpha}$, is the same as (1.2).

Theorem 3.2. *Let* $0 < \alpha \leqslant 1$ *and* $0 < \sigma < 1$.

(1) *If* $0 < \alpha - 2\sigma < 1$ *and* $u \in C_H^{0,\alpha}$, *then*

$$H^\sigma u(x) = \int_{\mathbb{R}^n} \big(u(x) - u(z)\big) F_\sigma(x,z)\,dz + u(x) B_\sigma(x), \quad x \in \mathbb{R}^n, \tag{3.4}$$

and the integral converges absolutely.

(2) *If* $-1 < \alpha - 2\sigma \leqslant 0$ *and* $u \in C_H^{1,\alpha}$, *then* $H^\sigma u(x)$ *is given by* (3.4), *where the integral converges as a principal value.*

(3) *When* $-2 < \alpha - 2\sigma \leqslant -1$, *it is enough to take* $u \in C_H^{1,1}$ *to have the conclusion of* (2).

In the three cases above, $H^\sigma u \in C(\mathbb{R}^n)$.

Proof. If $0 < \alpha - 2\sigma < 1$, then $\sigma < 1/2$. The properties of F_σ and B_σ established in Lemmas 5.4 and 5.5 (see Section 5), allow us to apply Proposition 2.1, with $K(x,z) = F_\sigma(x,z)$, $B = B_\sigma$ and $\gamma = 2\sigma < 1$, to get (1).

Under the hypotheses of (2), we will take advantage of a cancellation to show that the integral in (3.4) is well defined. Suppose that $\delta > 0$. By Lemma 5.4,

$$\int_{|x-z|\geqslant\delta} \big|u(x) - u(z)\big| F_\sigma(x,z)\,dz \leqslant C_\delta \|u\|_{L^\infty(\mathbb{R}^n)} \int_{|x-z|\geqslant\delta} e^{-\frac{|x-z|^2}{C}}\,dz < \infty.$$

For $\rho \in \mathbb{R}$, the change of parameters (3.2) produces

$$\frac{dt}{t^{1+\rho}} = d\mu_\rho(s) := \frac{ds}{(1-s^2)(\frac{1}{2}\log\frac{1+s}{1-s})^{1+\rho}}, \quad t \in (0,\infty),\ s \in (0,1),$$

so that,

$$F_\sigma(x,z) = \frac{1}{-\Gamma(-\sigma)} \int_0^1 \left(\frac{1-s^2}{4\pi s}\right)^{n/2} e^{-\frac{1}{4}[s|x+z|^2 + \frac{1}{s}|x-z|^2]}\,d\mu_\sigma(s), \tag{3.5}$$

which, up to the multiplicative constant $1/(-\Gamma(-\sigma))$, gives

$$\begin{aligned} I &= \int_{|z|<\delta} \big(u(x) - u(x-z)\big) F_\sigma(x, x-z)\,dz \\ &= \int_0^\delta r^{n-1} \int_{|z'|=1} \big(u(x) - u(x-rz')\big) \int_0^1 \left(\frac{1-s^2}{4\pi s}\right)^{n/2} e^{-\frac{1}{4}[s|2x-rz'|^2 + \frac{r^2}{s}]}\,d\mu_\sigma(s)\,dS(z')\,dr. \end{aligned}$$

By the smoothness of u, $u(x)-u(x-rz')=\nabla u(x)(rz')+R_1u(x,rz')$, with $|R_1u(x,rz')|\leqslant [\nabla u]_{C^{0,\alpha}}r^{1+\alpha}$. We apply the Mean Value Theorem to the function $\psi(x)=\psi_{s,r}(x)=e^{-\frac{1}{4}[s|x|^2+\frac{r^2}{s}]}$, to see that $e^{-\frac{1}{4}[s|2x-rz'|^2+\frac{r^2}{s}]}=e^{-\frac{1}{4}[s|2x|^2+\frac{r^2}{s}]}+R_0\psi(x,rz')$, with $|R_0\psi(x,rz')|\leqslant Cs^{1/2}re^{-r^2/(8s)}$. Therefore,

$$\begin{aligned} I &= \int_0^\delta r^{n-1}\int_{|z'|=1}\nabla u(x)\big(rz'\big)\int_0^1\left(\frac{1-s^2}{4\pi s}\right)^{n/2}R_0\psi\big(x,rz'\big)\,d\mu_\sigma(s)\,dS\big(z'\big)\,dr \\ &\quad+\int_0^\delta r^{n-1}\int_{|z'|=1}R_1u\big(x,rz'\big)\int_0^1\left(\frac{1-s^2}{4\pi s}\right)^{n/2} \\ &\quad\times\big(e^{-\frac{1}{4}[s|2x|^2+\frac{r^2}{s}]}+R_0\psi\big(x,rz'\big)\big)\,d\mu_\sigma(s)\,dS\big(z'\big)\,dr \\ &=: I_1+I_2. \end{aligned}$$

Note that

$$d\mu_\rho(s)\sim\frac{ds}{s^{1+\rho}},\quad s\sim 0,\qquad d\mu_\rho(s)\sim\frac{ds}{(1-s)(-\log(1-s))^{1+\rho}},\quad s\sim 1. \tag{3.6}$$

With the estimates on R_1u and $R_0\psi$ given above and (3.6), we obtain

$$\begin{aligned} |I_1| &\leqslant C\big|\nabla u(x)\big|\int_0^\delta r^{n+1}\int_0^1\left(\frac{1-s}{s}\right)^{n/2}s^{1/2}e^{-\frac{r^2}{8s}}\,d\mu_\sigma(s)\,dr \\ &\leqslant C\big|\nabla u(x)\big|\int_0^\delta\frac{r^{n+1}}{r^{n-1+2\sigma}}\,dr=C\delta^{3-2\sigma}, \end{aligned}$$

and

$$\begin{aligned} |I_2| &\leqslant C[\nabla u]_{C^{0,\alpha}}\int_0^\delta r^{n+\alpha}\int_0^1\left(\frac{1-s}{s}\right)^{n/2}e^{-\frac{r^2}{4s}}\,d\mu_\sigma(s)\,dr \\ &\leqslant C[\nabla u]_{C^{0,\alpha}}\int_0^\delta\frac{r^{n+\alpha}}{r^{n+2\sigma}}\,dr=C\delta^{\alpha-2\sigma+1}. \end{aligned}$$

Thus, the integral in (3.4) converges as a principal value. The same happens if we take $u\in C_H^{1,1}$: we repeat the argument above, but applying in I_2 the estimate $|R_1u(x,rz')|\leqslant[\nabla u]_{C^{0,1}}\cdot r^2$.

To obtain the conclusions of (2) and (3), we note that the approximation procedure used in the proof of Proposition 2.1 can be applied here (with the estimate $[\nabla f_j]_{C^{0,\alpha}}\leqslant C\|u\|_{C_H^{1,\alpha}}=M$). □

Remark 3.3. As in [13] (and [10,11] for $(-\Delta)^\sigma$), some easy maximum and comparison principles can be derived from Theorem 3.2. For instance, take $0<\alpha\leqslant 1$, $0<\sigma<1$, and u,v in the class $C_H^{0,\alpha}$ (or $C_H^{1,\alpha}$, depending on the value of $\alpha-2\sigma$), such that $u\geqslant v$ in $\mathbb{R}^n$, with $u(x_0)=v(x_0)$ for some $x_0\in\mathbb{R}^n$. Then $H^\sigma u(x_0)\leqslant H^\sigma v(x_0)$. Moreover, $H^\sigma u(x_0)=H^\sigma v(x_0)$ only when $u\equiv v$.

In order to prove the regularity estimates for H^σ, we will have to work with the derivatives of H^σ, that is, with operators of the type A_iH^σ, $1\leqslant |i|\leqslant n$. We recall that, for all $\nu\in\mathbb{N}_0^n$, we have

$$A_ih_\nu=(2\nu_i)^{1/2}h_{\nu-e_i},\qquad A_{-i}h_\nu=(2\nu_i+2)^{1/2}h_{\nu+e_i},\quad 1\leqslant i\leqslant n,$$

where e_i is the i-th coordinate vector in $\mathbb{N}_0^n$. Then, for all $f\in\mathcal{S}$ and $1\leqslant i\leqslant n$,

$$A_if=\sum_\nu(2\nu_i)^{1/2}\langle f,h_\nu\rangle h_{\nu-e_i},\qquad A_{-i}f=\sum_\nu(2\nu_i+2)^{1/2}\langle f,h_\nu\rangle h_{\nu+e_i},$$

and both series converge uniformly in $\mathbb{R}^n$.

Remark 3.4. Let $b\in\mathbb{R}$. Then, by using Hermite series expansions, it is easy to check that for all $f\in\mathcal{S}$ and $1\leqslant i\leqslant n$, we have

$$\begin{aligned}A_iH^bf&=(H+2)^bA_if, &\qquad H^bA_if&=A_i(H-2)^bf,\\ A_{-i}H^bf&=(H-2)^bA_{-i}f, &\qquad H^bA_{-i}f&=A_{-i}(H+2)^bf,\end{aligned}$$

where we defined $(H\pm 2)^bh_\nu:=(2|\nu|+n\pm 2)^bh_\nu$.

Consequently, we need to study the operators $(H\pm 2k)^\sigma$, $k\in\mathbb{N}$.

Let us start with $(H+2k)^\sigma$, k being a positive integer. For $f\in\mathcal{S}$ and $k\in\mathbb{N}$ we define

$$(H+2k)^\sigma f(x)=\sum_\nu\big(2|\nu|+n+2k\big)^\sigma\langle f,h_\nu\rangle h_\nu(x),\quad x\in\mathbb{R}^n.$$

The series above converges in $L^2(\mathbb{R}^n)$ and uniformly in $\mathbb{R}^n$, it defines a Schwartz's class function, and

$$(H+2k)^\sigma f(x)=\frac{1}{\Gamma(-\sigma)}\int_0^\infty\big(e^{-2kt}e^{-tH}f(x)-f(x)\big)\frac{dt}{t^{1+\sigma}},\quad x\in\mathbb{R}^n.$$

By using Lemmas 5.4 and 5.5 stated in Section 5, the following theorem can be proved in a parallel way to Theorem 3.2. We leave the details to the interested reader.

Theorem 3.5. *Let u be as in Theorem 3.2. Then $(H+2k)^\sigma u\in\mathcal{S}'\cap C(\mathbb{R}^n)$, and*

$$(H+2k)^\sigma u(x)=\int_{\mathbb{R}^n}\big(u(x)-u(z)\big)F_{2k,\sigma}(x,z)\,dz+u(x)B_{2k,\sigma}(x),\quad x\in\mathbb{R}^n,$$

where

$$F_{2k,\sigma}(x,z)=\frac{1}{-\Gamma(-\sigma)}\int_0^\infty e^{-2kt}G_t(x,z)\,\frac{dt}{t^{1+\sigma}}=\frac{1}{-\Gamma(-\sigma)}\int_0^1\left(\frac{1-s}{1+s}\right)^k G_{t(s)}(x,z)\,d\mu_\sigma(s),$$

and

$$B_{2k,\sigma}(x)=\frac{1}{\Gamma(-\sigma)}\int_0^1\left[\left(\frac{1-s}{1+s}\right)^k\left(\frac{1-s^2}{2\pi(1+s^2)}\right)^{n/2}e^{-\frac{s}{1+s^2}|x|^2}-1\right]d\mu_\sigma(s).$$

Consider next the operators $(H-2k)^\sigma$, $k\in\mathbb{N}$. We say that a function $f\in\mathcal{S}$ belongs to the space $\mathcal{S}_k$ if

$$\int_{\mathbb{R}^n} f(z)h_\nu(z)\,dz=0,\quad \text{for all } \nu\in\mathbb{N}_0^n \text{ such that } |\nu|<k.$$

For $f\in\mathcal{S}_k$ define

$$(H-2k)^\sigma f(x)=\sum_{|\nu|\geqslant k}\big(2|\nu|+n-2k\big)^\sigma\langle f,h_\nu\rangle h_\nu(x).$$

Note that, on $\mathcal{S}_k$, the operator $(H-2k)^\sigma$ is positive. Let

$$\phi_{2k}(x)=\phi_{2k}(x,z,s)=\left[\sum_{j=0}^{k-1}\left(\frac{1-s}{1+s}\right)^{j+n/2}\sum_{|\nu|=j}h_\nu(x)h_\nu(z)\right]\chi_{(1/2,1)}(s),$$

the sum of the first $(k-1)$-terms of the series defining $G_{t(s)}(x,z)$, for $s\in(1/2,1)$. Then, the heat-diffusion semigroup generated by $H-2k$:

$$\begin{aligned}e^{-t(H-2k)}f(x)&=\int_{\mathbb{R}^n}e^{2kt}G_t(x,z)f(z)\,dz=\int_{\mathbb{R}^n}\left(\frac{1+s}{1-s}\right)^k G_{t(s)}(x,z)f(z)\,dz\\&=e^{-t(s)(H-2k)}f(x),\end{aligned}$$

can be written as

$$e^{-t(s)(H-2k)}f(x)=\int_{\mathbb{R}^n}\left(\frac{1+s}{1-s}\right)^k\big[G_{t(s)}(x,z)-\phi_{2k}(x,z,s)\big]f(z)\,dz,\quad f\in\mathcal{S}_k.$$

Moreover,

$$(H-2k)^\sigma f(x) = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-t(H-2k)} f(x) - f(x)\right) \frac{dt}{t^{1+\sigma}}$$
$$= \frac{1}{\Gamma(-\sigma)} \int_0^1 \left(e^{-t(s)(H-2k)} f(x) - f(x)\right) d\mu_\sigma(s).$$

The following idea is taken from [8]. By the n-dimensional Mehler's formula (see [14, p. 6]),

$$M_r(x,z) := \sum_{j=0}^\infty r^j \sum_{|\nu|=j} h_\nu(x) h_\nu(z)$$
$$= \frac{1}{\pi^{n/2}(1-r^2)^{n/2}} e^{-\frac{1}{4}[\frac{1-r}{1+r}|x+z|^2 + \frac{1+r}{1-r}|x-z|^2]}, \quad r \in (0,1). \tag{3.7}$$

Then, for all $r \in (0, 1/3)$,

$$\left|\frac{d^k}{dr^k} M_r(x,z)\right| \leqslant C\left(1+|x+z|^2+|x-z|^2\right)^k e^{-\frac{1}{4}[\frac{1-r}{1+r}|x+z|^2+\frac{1+r}{1-r}|x-z|^2]} \leqslant Ce^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}},$$

where in the second inequality we applied Lemma 5.1 of Section 5, with $s = \frac{1-r}{1+r}$. Thus, by Taylor's formula,

$$\left|\sum_{j=k}^\infty r^j \sum_{|\nu|=j} h_\nu(x) h_\nu(z)\right| \leqslant Cr^k e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}, \quad r \in (0, 1/3).$$

Therefore, letting $r = \frac{1-s}{1+s}$ above, we obtain

$$\left|G_{t(s)}(x,z) - \phi_{2k}(x,z,s)\right| = \left|\sum_{j=k}^\infty \left(\frac{1-s}{1+s}\right)^{j+n/2} \sum_{|\nu|=j} h_\nu(x) h_\nu(z)\right| \tag{3.8}$$
$$\leqslant C\left(\frac{1-s}{1+s}\right)^{k+n/2} e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}, \quad \text{for all } s \in (1/2, 1). \tag{3.9}$$

If $u \in C_H^{k,\alpha}$, then we have

$$\int_{\mathbb{R}^n} A_{-i_1} \cdots A_{-i_k} u(x) h_\nu(x)\, dx = 0, \quad 1 \leqslant i_1, \ldots, i_k \leqslant n, \ |\nu| < k.$$

Theorem 3.6. *Let $0 < \alpha \leqslant 1$ and $0 < \sigma < 1$. Assume that $0 < \alpha - 2\sigma < 1$ and take $u \in C_H^{k,\alpha}$. If $v(x) = (A_{-i_1} \cdots A_{-i_k} u)(x)$, $1 \leqslant i_1, \ldots, i_k \leqslant n$, then $A_{-i_1} \cdots A_{-i_k} H^\sigma u \in \mathcal{S}' \cap C(\mathbb{R}^n)$, and, for all $x \in \mathbb{R}^n$,*

$$A_{-i_1}\cdots A_{-i_k}H^\sigma u(x)=(H-2k)^\sigma v(x)$$
$$=\int_{\mathbb{R}^n}\big(v(x)-v(z)\big)F_{-2k,\sigma}(x,z)\,dz+v(x)B_{-2k,\sigma}(x),\qquad(3.10)$$

where

$$F_{-2k,\sigma}(x,z)=\frac{1}{-\Gamma(-\sigma)}\int_0^1\left(\frac{1+s}{1-s}\right)^k\big[G_{t(s)}(x,z)-\phi_{2k}(x,z,s)\big]\,d\mu_\sigma(s),$$

and

$$B_{-2k,\sigma}(x)=\frac{1}{\Gamma(-\sigma)}\int_0^1\left[\left(\frac{1+s}{1-s}\right)^k\int_{\mathbb{R}^n}\big[G_{t(s)}(x,z)-\phi_{2k}(x,z,s)\big]\,dz-1\right]d\mu_\sigma(s).$$

The integral in (3.10) *is absolutely convergent.*

Proof. Even if we have good estimates for $F_{-2k,\sigma}$ and $B_{-2k,\sigma}$ (see Lemmas 5.4 and 5.5), we cannot apply directly Proposition 2.1 here because the test space for $(H-2k)^\sigma$ is not $\mathcal{S}$, but $\mathcal{S}_k$. Nevertheless, the same ideas will work. Indeed, using Lemmas 5.4 and 5.5, it can be checked that the conclusion is valid when u is a Schwartz's class function (and then $v\in\mathcal{S}_k$), and, for the general result, we can apply the approximation procedure given in the proof of Proposition 2.1, noting that $(A_{-i_1}\cdots A_{-i_k}f_j)(x)$ can be used to approximate $v(x)$. □

3.3. *The fractional integral*: $H^{-\sigma}$

For $f\in\mathcal{S}$, the fractional integral $H^{-\sigma}f$, $0<\sigma\leqslant 1$, is given by

$$H^{-\sigma}f(x)=\frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-tH}f(x)\,\frac{dt}{t^{1-\sigma}}=\sum_\nu\frac{1}{(2|\nu|+n)^\sigma}\langle f,h_\nu\rangle h_\nu(x),$$

and $H^{-\sigma}$ is a continuous and symmetric operator in $\mathcal{S}$. Moreover, $H^{-\sigma}f=(H^\sigma)^{-1}f$, $f\in\mathcal{S}$. By writing down the expression of the heat-diffusion semigroup and applying Fubini's Theorem,

$$H^{-\sigma}f(x)=\int_{\mathbb{R}^n}\left[\frac{1}{\Gamma(\sigma)}\int_0^\infty G_t(x,z)\,\frac{dt}{t^{1-\sigma}}\right]f(z)\,dz=\int_{\mathbb{R}^n}F_{-\sigma}(x,z)f(z)\,dz.$$

In [4] it is shown that the definition of $H^{-\sigma}$ extends to $f\in L^p(\mathbb{R}^n)$, $1\leqslant p\leqslant\infty$, via the previous integral formula.

Observe that, for $f \in \mathcal{S}$, we have

$$H^{-\sigma} f(x) = \int_{\mathbb{R}^n} \big(f(z) - f(x)\big) F_{-\sigma}(x,z)\,dz + f(x) H^{-\sigma} 1(x), \quad \text{for all } x \in \mathbb{R}^n,$$

where

$$H^{-\sigma} 1(x) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tH} 1(x) \frac{dt}{t^{1-\sigma}} = \int_{\mathbb{R}^n} F_{-\sigma}(x,z)\,dz. \tag{3.11}$$

The next theorem shows that the operator $H^{-\sigma}$ can be defined in $C_H^{0,\alpha}$ precisely by this formula.

Theorem 3.7. *For $u \in C_H^{0,\alpha}$, $0 < \alpha \leqslant 1$, and $0 < \sigma \leqslant 1$, $H^{-\sigma} u \in \mathcal{S}' \cap C(\mathbb{R}^n)$, and*

$$H^{-\sigma} u(x) = \int_{\mathbb{R}^n} \big(u(z) - u(x)\big) F_{-\sigma}(x,z)\,dz + u(x) H^{-\sigma} 1(x), \quad x \in \mathbb{R}^n.$$

Proof. In Lemmas 5.6 and 5.7 we collect the properties of the kernel $F_{-\sigma}(x,z)$ and the function $H^{-\sigma}1(x)$. When $n > 2\sigma$, an application of Proposition 2.1 with $\gamma = -2\sigma$ and $\varepsilon = \alpha + 2\sigma$ gives the result. For the case $n \leqslant 2\sigma$, we use Remark 2.2. □

We shall also need to work with the derivatives of $H^{-\sigma} u$. The following theorem gives the pointwise formula that will be used along the paper.

Theorem 3.8. *Take $0 < \alpha \leqslant 1$ and $0 < \sigma \leqslant 1$, such that $\alpha + 2\sigma > 1$. If $u \in C_H^{0,\alpha}$ then, for each $1 \leqslant |i| \leqslant n$, we have $A_i H^{-\sigma} u \in \mathcal{S}' \cap C(\mathbb{R}^n)$, and*

$$A_i H^{-\sigma} u(x) = \int_{\mathbb{R}^n} \big(u(z) - u(x)\big) A_i F_{-\sigma}(x,z)\,dz + u(x) A_i H^{-\sigma} 1(x), \quad x \in \mathbb{R}^n.$$

Proof. Let us first prove the result when $u = f \in \mathcal{S}$. It is enough to consider $1 \leqslant i \leqslant n$. We have

$$A_i H^{-\sigma} f(x) = A_i \int_{\mathbb{R}^n} \big(f(z) - f(x)\big) F_{-\sigma}(x,z)\,dz + \partial_{x_i} f(x) H^{-\sigma} 1(x) + f(x) A_i H^{-\sigma} 1(x).$$

We want to put the A_i inside the integral. In order to do that, we apply a classical approximation argument given in the proof of Lemma 4.1 of [7], that we sketch here. By estimate (5.10), Lemma 5.7 (see Section 5), and the fact that $\alpha + 2\sigma > 1$, the function

$$\begin{aligned} g(x) &= \int_{\mathbb{R}^n} \partial_{x_i} \big[\big(f(z) - f(x)\big) F_{-\sigma}(x,z)\big]\,dz \\ &= \int_{\mathbb{R}^n} \big(f(z) - f(x)\big) \partial_{x_i} F_{-\sigma}(x,z)\,dz - \partial_{x_i} f(x) H^{-\sigma} 1(x) \end{aligned}$$

is well defined. Fix a function $\phi \in C^1(\mathbb{R})$ satisfying $0 \leqslant \phi \leqslant 1$, $\phi(t) = 0$ for $t \leqslant 1$, $\phi(t) = 1$ for $t \geqslant 2$, and $0 \leqslant \phi' \leqslant 2$. Define, for $0 < \varepsilon < 1/2$,

$$h_\varepsilon(x) = \int_{\mathbb{R}^n} \big(f(z) - f(x)\big) F_{-\sigma}(x,z) \phi\big(\varepsilon^{-1}|x-z|\big)\, dz.$$

Then estimate (5.9) implies that, as $\varepsilon \to 0$, $h_\varepsilon(x)$ converges uniformly in $\mathbb{R}^n$ to

$$\int_{\mathbb{R}^n} \big(f(z) - f(x)\big) F_{-\sigma}(x,z)\, dz.$$

Moreover, $h_\varepsilon \in C^1(\mathbb{R}^n)$, and, again by (5.9) and (5.10),

$$\begin{aligned}
\big|g(x) - \partial_{x_i} h_\varepsilon(x)\big| &\leqslant \int_{\mathbb{R}^n} \big|\partial_{x_i}\big[\big(f(z) - f(x)\big) F_{-\sigma}(x,z)\big(1 - \phi\big(\varepsilon^{-1}|x-z|\big)\big)\big]\big|\, dz \\
&\leqslant C_f \int_{|x-z|<2\varepsilon} \left[F_{-\sigma}(x,z) + |x-z|\big|\nabla_x F_{-\sigma}(x,z)\big| + |x-z| F_{-\sigma}(x,z)\frac{1}{\varepsilon}\right] dz \\
&\leqslant C_f \Phi_{n,\sigma}(\varepsilon),
\end{aligned}$$

where $\Phi_{n,\sigma}(\varepsilon) \to 0$, as $\varepsilon \to 0$, uniformly in $x \in \mathbb{R}^n$. Thus,

$$\partial_{x_i} \int_{\mathbb{R}^n} \big(f(z) - f(x)\big) F_{-\sigma}(x,z)\, dz = \int_{\mathbb{R}^n} \big(f(z) - f(x)\big) \partial_{x_i} F_{-\sigma}(x,z)\, dz - \partial_{x_i} f(x) H^{-\sigma} 1(x),$$

and the theorem is valid when u is a Schwartz function. For the general case, $u \in C_H^{0,\alpha}$, we argue as follows. If $n > 2\sigma - 1$, then, by (5.10), Proposition 2.1 can be applied with $\gamma = 1 - 2\sigma$ and $\varepsilon = \alpha + 2\sigma - 1$, and, if $n = 2\sigma - 1$, we can use Remark 2.2. $\square$

3.4. The Hermite–Riesz transforms: $\mathcal{R}_i$ *and* $\mathcal{R}_{ij}$

The first order Hermite–Riesz transforms are given by

$$\mathcal{R}_i = A_i H^{-1/2}, \quad 1 \leqslant |i| \leqslant n.$$

These operators were first introduced and studied by Thangavelu [14]. The second order Hermite–Riesz transforms are (see [8,12])

$$\mathcal{R}_{ij} = A_i A_j H^{-1}, \quad 1 \leqslant |i|, |j| \leqslant n.$$

Using Hermite series expansions it is easy to check that the first and second order Hermite–Riesz transforms are symmetric operators in $\mathcal{S}$ and that they map $\mathcal{S}$ into $\mathcal{S}$ continuously.

Taking $\sigma = 1/2$ in Theorem 3.8 we get

Theorem 3.9. *If $u \in C_H^{0,\alpha}$, $0 < \alpha \leqslant 1$, then, for all $1 \leqslant |i| \leqslant n$, we have $\mathcal{R}_i u \in \mathcal{S}' \cap C(\mathbb{R}^n)$, and*

$$\mathcal{R}_i u(x) = \int_{\mathbb{R}^n} \big(u(z) - u(x)\big) A_i F_{-1/2}(x,z)\,dz + u(x) A_i H^{-1/2} 1(x), \quad x \in \mathbb{R}^n.$$

Using the properties of the kernel of the second order Riesz transform $R_{ij}(x,z) = A_i A_j F_{-1}(x,z)$ (see Lemma 5.8 in Section 5), it is easy to get a pointwise description of $\mathcal{R}_{ij} f$, $f \in \mathcal{S}$. Hence, we can apply Proposition 2.1, with $\gamma = 0$ and $\varepsilon = \alpha$, to have the following theorem.

Theorem 3.10. *If $u \in C_H^{0,\alpha}$, $0 < \alpha \leqslant 1$, then, for all $1 \leqslant |i|, |j| \leqslant n$, we have $\mathcal{R}_{ij} u \in \mathcal{S}' \cap C(\mathbb{R}^n)$, and*

$$\mathcal{R}_{ij} u(x) = \int_{\mathbb{R}^n} \big(u(z) - u(x)\big) R_{ij}(x,z)\,dz + u(x) A_i A_j H^{-1} 1(x), \quad x \in \mathbb{R}^n.$$

4. Proofs of the main results

4.1. Regularity properties of the Hermite–Riesz transforms

As we already said in the Introduction, a study of the action of the Hermite–Riesz transforms in the Hölder spaces $C_H^{k,\alpha}$ is needed.

Theorem 4.1. *The Hermite–Riesz transforms $\mathcal{R}_i$ and $\mathcal{R}_{ij}$, $1 \leqslant |i|, |j| \leqslant n$, are bounded operators on the spaces $C_H^{0,\alpha}$: if $u \in C_H^{0,\alpha}$, for some $0 < \alpha < 1$, then $\mathcal{R}_i u, \mathcal{R}_{ij} u \in C_H^{0,\alpha}$, and*

$$\|\mathcal{R}_i u\|_{C_H^{0,\alpha}} + \|\mathcal{R}_{ij} u\|_{C_H^{0,\alpha}} \leqslant C \|u\|_{C_H^{0,\alpha}}.$$

Proof. By Lemmas 5.6, 5.7, and 5.10 of Section 5, and Theorem 3.9, the result for $\mathcal{R}_i$ can be deduced applying Proposition 2.3, with $\gamma = 0$ and $\varepsilon = \alpha$.

Let us consider the operator $\mathcal{R}_{ij}$, for some $j \in \{1, \ldots, n\}$. Then, by Remark 3.4,

$$\mathcal{R}_{ij} = A_i A_j H^{-1} = A_i \big(A_j H^{-1/2}\big) H^{-1/2} = A_i \big[(H+2)^{-1/2} A_j\big] H^{-1/2} = A_i (H+2)^{-1/2} \circ \mathcal{R}_j.$$

Therefore, it is enough to prove that $A_i (H+2)^{-1/2}$ is a continuous operator on $C_H^{0,\alpha}$. When $f \in \mathcal{S}$, we can write

$$A_i (H+2)^{-1/2} f(x) = \int_{\mathbb{R}^n} \big(f(z) - f(x)\big) A_i F_{2,-1/2}(x,z)\,dz + f(x) A_i (H+2)^{-1/2} 1(x),$$

where

$$F_{2,-1/2}(x,z) = \frac{1}{\Gamma(1/2)} \int_0^1 \left(\frac{1-s}{1+s}\right) G_{t(s)}(x,z)\,d\mu_{-1/2}(s),$$

and

$$(H+2)^{-1/2}1(x)=\int_{\mathbb{R}^n}F_{2,-1/2}(x,z)\,dz.$$

Following the proof of Lemmas 5.6 and 5.7 given in Section 5, it can be checked that the kernel $A_iF_{2,-1/2}(x,z)$ and the function $(H+2)^{-1/2}1(x)$ share the same size and smoothness properties than the kernel $A_iF_{-1/2}(x,z)$ and the function $H^{-1/2}1(x)$ stated in the mentioned lemmas (the details are left to the reader). Thus, as a consequence of the results of Section 2, $A_i(H+2)^{-1/2}:C_H^{0,\alpha}\to C_H^{0,\alpha}$ continuously. Therefore $\mathcal{R}_{ij}$ is a bounded operator on $C_H^{0,\alpha}$, when $j\in\{1,\dots,n\}$.

Note that

$$\mathcal{R}_{ij}=\partial^2_{x_i,x_j}H^{-1}+x_j\partial_{x_i}H^{-1}+x_i\partial_{x_j}H^{-1}+x_ix_jH^{-1}+\delta_{ij}H^{-1},$$

which, at the level of kernels, means that

$$R_{ij}(x,z)=\partial^2_{x_i,x_j}F_{-1}(x,z)+x_j\partial_{x_i}F_{-1}(x,z)+x_i\partial_{x_j}F_{-1}(x,z)+x_ix_jF_{-1}(x,z)+\delta_{ij}F_{-1}(x,z).$$

By the estimates given in Lemmas 5.6, 5.8 and 5.9 of Section 5, we can apply the statements of Section 2 to show that the operators $x_i\partial_{x_j}H^{-1}$, $x_ix_jH^{-1}$ and H^{-1} are bounded on $C_H^{0,\alpha}$. Hence, $\partial^2_{x_i,x_j}H^{-1}$ maps $C_H^{0,\alpha}$ into $C_H^{0,\alpha}$ continuously. Observe now that the operator $\mathcal{R}_{i,-j}$, for $j\in\{1,\dots,n\}$, can be written as

$$\mathcal{R}_{i,-j}=-\partial^2_{x_i,x_j}H^{-1}+x_j\partial_{x_i}H^{-1}-x_i\partial_{x_j}H^{-1}+x_ix_jH^{-1}+\delta_{ij}H^{-1}.$$

The observations above give the conclusion for $\mathcal{R}_{i,-j}$, $j\in\{1,\dots,n\}$. □

For technical reasons we have to consider the first order adjoint Hermite–Riesz transforms, that are defined by

$$\mathcal{R}_i^*f(x)=H^{-1/2}A_if(x)=\int_{\mathbb{R}^n}F_{-1/2}(x,z)(A_if)(z)\,dz,\qquad f\in\mathcal{S},\ x\in\mathbb{R}^n,\ 1\leqslant|i|\leqslant n.$$

Theorem 4.2. *The operators $\mathcal{R}_i^*$, $1\leqslant|i|\leqslant n$, are bounded operators on $C_H^{0,\alpha}$, $0<\alpha<1$.*

Proof. Observe that, if $1\leqslant i\leqslant n$, then, by Remark 3.4, $R^*_{-i}=H^{-1/2}A_{-i}=A_{-i}(H+2)^{-1/2}$. This operator already appeared in the proof of Theorem 4.1, and there we showed that it is a bounded operator on $C_H^{0,\alpha}$.

On the other hand, $R^*_{-i}=-H^{-1/2}\partial_{x_i}+H^{-1/2}x_i$. But by Lemmas 5.6, 5.7, and 5.9 of Section 5, and Proposition 2.3, we can see that the operator $f\mapsto H^{-1/2}x_if$, initially defined on $\mathcal{S}$, maps $C_H^{0,\alpha}$ into itself continuously. Therefore, we obtain the same conclusion for the operator $f\mapsto R^*_{-i}f-H^{-1/2}x_if=-H^{-1/2}\partial_{x_i}f$. Consequently, $R_i^*=H^{-1/2}A_i=H^{-1/2}\partial_{x_i}+H^{-1/2}x_i$ is a bounded operator on $C_H^{0,\alpha}$. □

4.2. Proof of Theorem A

We start with (A1). By recalling the results in Lemmas 5.4 and 5.5, if we put $\gamma = 2\sigma < 1$ and $\varepsilon = \alpha - 2\sigma$ in Proposition 2.3, we get the conclusion.

Consider now (A2). Using Remark 3.4 and Theorem 3.6, we have

$$H^\sigma u \in C_H^{1,\alpha-2\sigma} \iff A_i H^\sigma u, A_{-i} H^\sigma u \in C_H^{0,\alpha-2\sigma}$$
$$\iff (H+2)^\sigma A_i u, (H-2)^\sigma A_{-i} u \in C_H^{0,\alpha-2\sigma}.$$

By Theorem 3.5, together with Lemmas 5.4 and 5.5, we can apply Proposition 2.3, with $\gamma = 2\sigma < 1$ and $\varepsilon = \alpha - 2\sigma$, in order to get $(H+2)^\sigma : C_H^{0,\alpha} \to C_H^{0,\alpha-2\sigma}$ continuously, and then $\|(H+2)^\sigma A_i u\|_{C_H^{0,\alpha-2\sigma}} \leqslant C\|A_i u\|_{C_H^{0,\alpha}} \leqslant C\|u\|_{C_H^{1,\alpha}}$. Applying Theorem 3.6, Lemmas 5.4 and 5.5, and Proposition 2.3, we get $\|(H-2)^\sigma A_{-i} u\|_{C_H^{0,\alpha-2\sigma}} \leqslant C\|u\|_{C_H^{1,\alpha}}$. Thus, $\|H^\sigma u\|_{C_H^{1,\alpha-2\sigma}} \leqslant C\|u\|_{C_H^{1,\alpha}}$.

Let us prove (A3). We can write

$$H^\sigma = H^{\sigma-1/2} \circ H^{-1/2} \circ H = H^{\sigma-1/2} \circ \frac{1}{2}\sum_{i=1}^n \big(\mathcal{R}_{-i}^* A_{-i} + \mathcal{R}_i^* A_i\big),$$

where $\mathcal{R}_{\pm i}^*$ are the adjoint Hermite–Riesz transforms, that are bounded operators on $C_H^{0,\alpha}$ (Theorem 4.2). Consequently,

$$\frac{1}{2}\sum_{i=1}^n \big(\mathcal{R}_{-i}^* A_{-i} u + \mathcal{R}_i^* A_i u\big) =: v \in C_H^{0,\alpha}.$$

Now we distinguish two cases. If $\sigma - 1/2 > 0$, then $0 < \alpha - 2(\sigma - 1/2) < 1$ by hypothesis, so we can apply (A1) to obtain that $H^{\sigma-1/2} v \in C_H^{0,\alpha-2\sigma+1}$, and $\|H^\sigma u\|_{C_H^{0,\alpha-2\sigma+1}} \leqslant C\|u\|_{C_H^{1,\alpha}}$. If $\sigma - 1/2 < 0$, then $0 < \alpha + 2(-\sigma + 1/2) < 1$, and we will get $H^{-(-\sigma+1/2)} v \in C_H^{0,\alpha-2\sigma+1}$, and $\|H^\sigma u\|_{C_H^{0,\alpha-2\sigma+1}} \leqslant C\|u\|_{C_H^{1,\alpha}}$, as soon as we have proved Theorem B(B1). If $\sigma = 1/2$, the result just follows from the boundedness of the adjoint Hermite–Riesz transforms on $C_H^{0,\alpha}$, Theorem 4.2.

By iteration of (A1), (A2) and (A3), and using Remark 3.4 and Theorems 3.5 and 3.6, we can derive (A4). The rather cumbersome details are left to the interested reader.

4.3. Proof of Theorem B

To prove (B1) note that, if $\alpha + 2\sigma \leqslant 1$ then $0 < \sigma < 1/2$. Let us write

$$H^{-\sigma} u(x_1) - H^{-\sigma} u(x_2) = \int_{\mathbb{R}^n} \big[u(z) - u(x_1)\big]\big[F_{-\sigma}(x_1, z) - F_{-\sigma}(x_2, z)\big]\,dz$$
$$+ u(x_1)\big[H^{-\sigma}1(x_1) - H^{-\sigma}1(x_2)\big].$$

By Lemma 5.7, the second term above is bounded by $C[u]_{M^\alpha}|x_1 - x_2|^{\alpha+2\sigma}$. Split the remaining integral on $B = B(x_1, 2|x_1 - x_2|)$ and on B^c. By Lemma 5.6,

$$\int_B \left|u(z)-u(x_1)\right| F_{-\sigma}(x_1,z)\,dz \leqslant C[u]_{C^{0,\alpha}} \int_B \frac{|x_1-z|^\alpha}{|x_1-z|^{n-2\sigma}}\,dz = C[u]_{C^{0,\alpha}}|x_1-x_2|^{\alpha+2\sigma}.$$

Let $B' = B(x_2, 4|x_1-x_2|)$. Then, by the triangle inequality,

$$\begin{aligned}
&\int_B \left|u(z)-u(x_1)\right| F_{-\sigma}(x_2,z)\,dz \\
&\quad\leqslant C[u]_{C^{0,\alpha}} \int_{B'} \frac{|x_1-z|^\alpha}{|x_2-z|^{n-2\sigma}}\,dz \\
&\quad\leqslant C[u]_{C^{0,\alpha}} \left[|x_1-x_2|^\alpha \int_{B'} \frac{1}{|x_2-z|^{n-2\sigma}}\,dz + \int_{B'} |x_2-z|^{\alpha-n+2\sigma}\,dz\right] \\
&\quad= C[u]_{C^{0,\alpha}}|x_1-x_2|^{\alpha+2\sigma}.
\end{aligned}$$

Denote by $\widetilde{B}$ the ball with center x_2 and radius $|x_1-x_2|$. Note that, for $z \in \widetilde{B}^c$, $|z-x_1| < 2|z-x_2|$. Then, apply Lemma 5.6 to get

$$\begin{aligned}
&\int_{B^c} \left|u(z)-u(x_1)\right|\left|F_{-\sigma}(x_1,z)-F_{-\sigma}(x_2,z)\right| dz \\
&\quad\leqslant C[u]_{C^{0,\alpha}}|x_1-x_2| \int_{\widetilde{B}^c} \frac{|z-x_2|^\alpha}{|z-x_2|^{n+1-2\sigma}} e^{-\frac{|x-z|^2}{C}}\,dz \\
&\quad\leqslant C[u]_{C^{0,\alpha}}|x_1-x_2|^{\alpha+2\sigma}.
\end{aligned}$$

Thus, $[H^{-\sigma}u]_{C^{0,\alpha+2\sigma}} \leqslant C\|u\|_{C_H^{0,\alpha}}$. For the decay, we put

$$H^{-\sigma}u(x) = \int_{\overline{B}} \big(u(z)-u(x)\big)F_{-\sigma}(x,z)\,dz + \int_{\overline{B}^c} \big(u(z)-u(x)\big)F_{-\sigma}(x,z)\,dz + u(x)H^{-\sigma}1(x),$$

where $\overline{B} = B(x, \frac{1}{1+|x|})$. We have

$$\int_{\overline{B}} \left|u(z)-u(x)\right| F_{-\sigma}(x,z)\,dz \leqslant C[u]_{C^{0,\alpha}} \int_{\overline{B}} \frac{|z-x|^\alpha}{|x-z|^{n-2\sigma}}\,dz \leqslant [u]_{C^{0,\alpha}} \frac{C}{(1+|x|)^{\alpha+2\sigma}},$$

and

$$\left|\int_{\overline{B}^c} \big(u(z)-u(x)\big)F_{-\sigma}(x,z)\,dz + u(x)H^{-\sigma}1(x)\right| \leqslant \int_{\overline{B}^c} \left|u(z)\right| F_{-\sigma}(x,z)\,dz + 2\left|u(x)\right|\left|H^{-\sigma}1(x)\right|.$$

To estimate the very last integral, we can proceed as we did for the integral part of the operator T in the proof of Proposition 2.3, splitting the integral in annulus (the details are left to the reader). By Lemma 5.7, $|u(x)H^{-\sigma}1(x)| \leqslant C[u]_{M^\alpha}(1+|x|)^{-(\alpha+2\sigma)}$. This concludes the proof of Theorem B(B1).

In order to prove (B2), we observe that, by using the boundedness of the first order Hermite–Riesz transforms on $C_H^{0,\alpha}$ (Theorem 4.1), we get

$$\left\|A_i H^{-\sigma} u\right\|_{C_H^{0,\alpha+2\sigma-1}} = \left\|\mathcal{R}_i H^{-\sigma+1/2} u\right\|_{C_H^{0,\alpha+2\sigma-1}} \leqslant C\left\|H^{-\sigma+1/2} u\right\|_{C_H^{0,\alpha+2\sigma-1}} \leqslant C\|u\|_{C_H^{0,\alpha}},$$

where in the last inequality we applied Theorem A(A1) if $-\sigma+1/2>0$, and the case (B1) just proved above if $-\sigma+1/2<0$. The case $\sigma=1/2$ is contained in Theorem 4.1.

Under the hypotheses of (B3), we have to prove that $A_iA_jH^{-\sigma}u$ belongs to $C_H^{0,\alpha+2\sigma-2}$. But $A_iA_jH^{-\sigma}u=\mathcal{R}_{ij}H^{1-\sigma}u$. Therefore, Theorem A(A1) and Theorem 4.1 give the result.

5. Computational lemmas

Lemma 5.1. *For each positive number a, let*

$$\psi_{s,z}^a(x)=e^{-a[s|x+z|^2+\frac{1}{s}|x-z|^2]}, \quad x,z\in\mathbb{R}^n,\ s\in(0,1).$$

Then,

$$\psi_{s,z}^a(x)\leqslant e^{-\frac{a}{4}|x||x-z|}e^{-\frac{a}{4}\frac{|x-z|^2}{s}}. \tag{5.1}$$

Proof. We have

$$\psi_{s,z}^a(x)\leqslant e^{-\frac{a}{2}\frac{|x-z|^2}{s}}e^{-\frac{a}{2}[s|x+z|^2+\frac{1}{s}|x-z|^2]}\leqslant e^{-\frac{a}{2}\frac{|x-z|^2}{s}}e^{-\frac{a}{2}|x-z||x+z|}.$$

The first inequality above is obvious. For the second one, we argue as follows: if $|x+z|\leqslant|x-z|$ then it is clearly valid; when $|x-z|<|x+z|$ we minimize the function $\theta(s)=\frac{a}{2}[s|x+z|^2+\frac{1}{s}|x-z|^2]$, $s\in(0,1)$, to get $\theta(s)\geqslant\frac{a}{2}|x-z||x+z|$. To obtain the desired estimate let us first assume that $x\cdot z>0$. Then $|x+z|\geqslant|x|$, and $e^{-\frac{a}{2}\frac{|x-z|^2}{s}}e^{-\frac{a}{2}|x-z||x+z|}\leqslant e^{-\frac{a}{4}\frac{|x-z|^2}{s}}e^{-\frac{a}{4}|x||x-z|}$. If $x\cdot z\leqslant 0$, then $|x-z|\geqslant|x|$, and $e^{-\frac{a}{2}\frac{|x-z|^2}{s}}e^{-\frac{a}{2}|x-z||x+z|}\leqslant e^{-\frac{a}{4}\frac{|x-z|^2}{s}}e^{-\frac{a}{4}\frac{|x||x-z|}{s}}\leqslant e^{-\frac{a}{4}\frac{|x-z|^2}{s}}e^{-\frac{a}{4}|x||x-z|}$. Thus, (5.1) follows. □

Remark 5.2. Note that Lemma 5.1 gives the estimate $G_{t(s)}(x,z)\leqslant C(\frac{1-s}{s})^{n/2}e^{-\frac{|x||x-z|}{C}}e^{-\frac{|x-z|^2}{Cs}}$, $s\in(0,1)$, $x,z\in\mathbb{R}^n$, which appeared in Lemma 5.10 of [13].

Lemma 5.3. *Let $\eta,\rho\in\mathbb{R}$. Then, for all $x,z\in\mathbb{R}^n$,*

$$\int_0^1\left(\frac{1-s}{s}\right)^{n/2}\frac{1}{s^\eta}e^{-C[s|x+z|^2+\frac{1}{s}|x-z|^2]}\,d\mu_\rho(s)\leqslant Ce^{-\frac{|x||x-z|}{C}}e^{-\frac{|x-z|^2}{C}}\cdot I_{\eta,\rho}(x,z),$$

where

$$I_{\eta,\rho}(x,z)=\begin{cases}\frac{1}{|x-z|^{n+2\eta+2\rho}}, & \text{if } n/2+\eta+\rho>0,\\ 1+\log(\frac{C}{|x-z|^2})\chi_{\{\frac{C}{|x-z|^2}>1\}}(x-z), & \text{if } n/2+\eta+\rho=0,\\ 1, & \text{if } n/2+\eta+\rho<0.\end{cases}$$

Proof. By (5.1), we get

$$\begin{aligned}&\int_0^1\left(\frac{1-s}{s}\right)^{n/2}\frac{1}{s^\eta}e^{-C[s|x+z|^2+\frac{1}{s}|x-z|^2]}\,d\mu_\rho(s)\\ &\leqslant Ce^{-\frac{|x||x-z|}{C}}\left(\int_0^{1/2}\frac{1}{s^{n/2+\eta+\rho}}e^{-\frac{|x-z|^2}{Cs}}\,\frac{ds}{s}+e^{-\frac{|x-z|^2}{C}}\int_{1/2}^1(1-s)^{n/2}\,d\mu_\rho(s)\right)\\ &=Ce^{-\frac{|x||x-z|}{C}}\left(\frac{C}{|x-z|^{n+2\eta+2\rho}}\int_{\frac{|x-z|^2}{C}}^\infty r^{n/2+\eta+\rho}e^{-2r}\,\frac{dr}{r}+Ce^{-\frac{|x-z|^2}{C}}\right)\\ &\leqslant Ce^{-\frac{|x||x-z|}{C}}\left(\frac{e^{-\frac{|x-z|^2}{C}}}{|x-z|^{n+2\eta+2\rho}}\int_{\frac{|x-z|^2}{C}}^\infty r^{n/2+\eta+\rho}e^{-r}\,\frac{dr}{r}+e^{-\frac{|x-z|^2}{C}}\right)\\ &=Ce^{-\frac{|x||x-z|}{C}}\cdot I.\end{aligned}$$

If $n/2+\eta+\rho>0$, we have

$$I\leqslant\frac{e^{-\frac{|x-z|^2}{C}}}{|x-z|^{n+2\eta+2\rho}}\int_0^\infty r^{n/2+\eta+\rho}e^{-r}\,\frac{dr}{r}+e^{-\frac{|x-z|^2}{C}}\leqslant\frac{C}{|x-z|^{n+2\eta+2\rho}}e^{-\frac{|x-z|^2}{C}}.$$

Suppose now that $n/2+\eta+\rho\leqslant 0$. Consider two cases: if $\frac{|x-z|^2}{C}\geqslant 1$, then,

$$\begin{aligned}I&\leqslant e^{-\frac{|x-z|^2}{C}}\left[\frac{1}{|x-z|^{n+2\eta+2\rho}}\int_1^\infty r^{n/2+\eta+\rho}e^{-r}\,\frac{dr}{r}+1\right]\\ &=Ce^{-\frac{|x-z|^2}{C}}\left[\frac{C}{|x-z|^{n+2\eta+2\rho}}+1\right]\leqslant C,\end{aligned}$$

and, when $\frac{|x-z|^2}{C}<1$, we have

$$I \leqslant e^{-\frac{|x-z|^2}{C}}\left[\frac{1}{|x-z|^{n+2\eta+2\rho}}\left(\int_{\frac{|x-z|^2}{C}}^{1} r^{n/2+\eta+\rho}\,\frac{dr}{r}+C\right)+1\right]$$

$$\leqslant Ce^{-\frac{|x-z|^2}{C}}\cdot\begin{cases}1+\log(\frac{C}{|x-z|^2}), & \text{if } n/2+\eta+\rho=0,\\ 1, & \text{if } n/2+\eta+\rho<0.\end{cases} \qquad \square$$

Lemma 5.4. *Denote by $\mathcal{F}$ any of the kernels $F_\sigma(x,z)$ (defined in* (1.3)*), or $F_{\pm 2k,\sigma}(x,z)$ (given in Theorems* 3.5 *and* 3.6*). Then,*

$$\left|\mathcal{F}(x,z)\right| \leqslant \frac{C}{|x-z|^{n+2\sigma}}e^{-\frac{|x||x-z|}{C}}e^{-\frac{|x-z|^2}{C}}, \tag{5.2}$$

for all $x,z\in\mathbb{R}^n$, and

$$\left|\mathcal{F}(x_1,z)-\mathcal{F}(x_2,z)\right| \leqslant \frac{C|x_1-x_2|}{|x_2-z|^{n+1+2\sigma}}e^{-\frac{|z||x_2-z|}{C}}e^{-\frac{|x_2-z|^2}{C}}, \tag{5.3}$$

for all $x_1,x_2\in\mathbb{R}^n$ such that $|x_1-z|>2|x_1-x_2|$.

Proof. Let us first consider $\mathcal{F}=F_\sigma$. The estimate in (5.2) is already stated in [13, Lemma 5.11]. Nevertheless, we can prove it here quickly by using, in (3.5), Lemma 5.3, with $\eta=0$ and $\rho=\sigma$. To get (5.3), we observe that, by the Mean Value Theorem,

$$\left|G_{t(s)}(x_1,z)-G_{t(s)}(x_2,z)\right| \leqslant C|x_1-x_2|\left(\frac{1-s}{s}\right)^{n/2}\frac{1}{s^{1/2}}e^{-\frac{1}{8}[s|\xi+z|^2+\frac{1}{s}|\xi-z|^2]}, \tag{5.4}$$

for some $\xi=(1-\lambda)x_1+\lambda x_2$, $\lambda\in[0,1]$. Then, by Lemma 5.3, with $\eta=1/2$ and $\rho=\sigma$,

$$\begin{aligned}\left|F_\sigma(x_1,z)-F_\sigma(x_2,z)\right| &\leqslant |x_1-x_2|\sup_{\{\xi=(1-\lambda)x_1+\lambda x_2:\ \lambda\in[0,1]\}}\left|\nabla_x F_\sigma(\xi,z)\right|\\ &\leqslant C|x_1-x_2|\sup_\xi\int_0^1\left(\frac{1-s}{s}\right)^{n/2}\frac{1}{s^{1/2}}e^{-\frac{1}{8}[s|\xi+z|^2+\frac{1}{s}|\xi-z|^2]}\,d\mu_\sigma(s)\\ &\leqslant C|x_1-x_2|\sup_\xi\frac{1}{|\xi-z|^{n+1+2\sigma}}e^{-\frac{|z||\xi-z|}{C}}e^{-\frac{|\xi-z|^2}{C}}\\ &\leqslant C\frac{|x_1-x_2|}{|x_2-z|^{n+1+2\sigma}}e^{-\frac{|z||x_2-z|}{C}}e^{-\frac{|x_2-z|^2}{C}},\end{aligned}$$

where in the last inequality we used that $|\xi-z|\geqslant\frac{1}{2}|x_2-z|$, since $|x_1-z|>2|x_1-x_2|$. In a similar way we can prove both estimates for $\mathcal{F}=F_{2k,\sigma}$, because $0\leqslant F_{2k,\sigma}(x,z)\leqslant F_\sigma(x,z)$, and the details are left to the reader. Note that, by (3.8)–(3.9), up to a multiplicative constant we have

$$\begin{aligned}&\left|F_{-2k,\sigma}(x,z)\right| \\ &= \left| \int_0^{1/2} \left(\frac{1+s}{1-s}\right)^k G_{t(s)}(x,z)\,d\mu_\sigma(s) + \int_{1/2}^1 \left(\frac{1+s}{1-s}\right)^k \left[G_{t(s)}(x,z) - \phi_{2k}(x)\right] d\mu_\sigma(s) \right| \\ &\leqslant C\left[F_\sigma(x,z) + e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}} \int_{1/2}^1 \left(\frac{1-s}{1+s}\right)^{n/2} d\mu_\sigma(s)\right],\end{aligned}$$

and therefore, (5.2) is valid for $F_{-2k,\sigma}$. By (5.4),

$$\begin{aligned}&\int_0^{1/2} \left(\frac{1+s}{1-s}\right)^k \left|G_{t(s)}(x_1,z) - G_{t(s)}(x_2,z)\right| d\mu_\sigma(s) \\ &\quad \leqslant C|x_1-x_2| \sup_\xi \int_0^1 \left(\frac{1-s}{s}\right)^{n/2} \frac{1}{s^{1/2}} e^{-\frac{1}{8}[s|\xi+z|^2 + \frac{1}{s}|\xi-z|^2]}\,d\mu_\sigma(s).\end{aligned} \tag{5.5}$$

Recall the definition of $M_r(x,z)$ given in (3.7). It can be checked that

$$\left|\frac{d^k}{dr^k}\nabla_x M_r(x,z)\right| \leqslant Ce^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}, \quad r \in (0,1/3).$$

Thus, by Taylor's formula,

$$\left|\nabla_x\left[G_{t(s)}(x,z) - \phi_{2k}(x,z,s)\right]\right| \leqslant C\left(\frac{1-s}{1+s}\right)^{k+n/2} e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}, \quad s \in (1/2,1), \tag{5.6}$$

and, consequently, when $|x_1-z| > 2|x_1-x_2|$,

$$\begin{aligned}&\int_{1/2}^1 \left(\frac{1+s}{1-s}\right)^k \left|\left(G_{t(s)}(x_1,z) - \phi_{2k}(x_1,z,s)\right) - \left(G_{t(s)}(x_2,z) - \phi_{2k}(x_2,z,s)\right)\right| d\mu_\sigma(s) \\ &\quad \leqslant C|x_1-x_2| \sup_\xi \int_{1/2}^1 \left(\frac{1+s}{1-s}\right)^k \left|\nabla_x\left[G_{t(s)}(\xi,z) - \phi_{2k}(\xi,z,s)\right]\right| d\mu_\sigma(s) \\ &\quad \leqslant C|x_1-x_2| \sup_\xi e^{-\frac{|z||\xi-z|}{C}} e^{-\frac{|\xi-z|^2}{C}} \leqslant C|x_1-x_2| e^{-\frac{|z||x_2-z|}{C}} e^{-\frac{|x_2-z|^2}{C}}.\end{aligned} \tag{5.7}$$

Pasting estimates (5.5) and (5.7), (5.3) follows for $\mathcal{F} = F_{-2k,\sigma}$. □

Lemma 5.5. *Denote by $\mathcal{B}$ any of the functions B_σ or $B_{\pm 2k,\sigma}$ defined in* (1.3) *and in Theorems* 3.5 *and* 3.6. *Then $\mathcal{B} \in C^\infty(\mathbb{R}^n)$ and, for all $x \in \mathbb{R}^n$,*

$$\left|\mathcal{B}(x)\right| \leqslant C\left(1+|x|^{2\sigma}\right), \quad \text{and} \quad \left|\nabla\mathcal{B}(x)\right| \leqslant C \begin{cases} |x|, & \text{if } |x| \leqslant 1, \\ |x|^{2\sigma-1}, & \text{if } |x| > 1. \end{cases} \tag{5.8}$$

Proof. The first inequality in (5.8) for the case $\mathcal{B} = B_\sigma$ is contained in [13, Lemma 5.11]. The identity

$$e^{-tH}1(x) = \int_{\mathbb{R}^n} G_t(x,z)\,dz = \frac{1}{(2\pi \cosh 2t)^{n/2}} e^{-\frac{\tanh 2t}{2}|x|^2}$$

(stated in [8]) and Meda's change of parameters (3.2) give

$$B_\sigma(x) = \frac{1}{\Gamma(-\sigma)} \int_0^1 \left[\left(\frac{1-s^2}{2\pi(1+s^2)}\right)^{n/2} e^{-\frac{s}{1+s^2}|x|^2} - 1\right] d\mu_\sigma(s).$$

We differentiate under the integral sign to see that $B_\sigma \in C^\infty(\mathbb{R}^n)$, and

$$\begin{aligned} \left|\nabla B_\sigma(x)\right| &= \left|\frac{2x}{\Gamma(-\sigma)} \int_0^1 \frac{s}{1+s^2}\left(\frac{1-s^2}{2\pi(1+s^2)}\right)^{n/2} e^{-\frac{s}{1+s^2}|x|^2}\,d\mu_\sigma(s)\right| \\ &\leqslant C|x| \int_0^1 s e^{-\frac{s}{2}|x|^2}\,d\mu_\sigma(s) =: \widetilde{I}(x). \end{aligned}$$

By (3.6),

$$\widetilde{I}(x) \leqslant C|x|\left[\int_0^{1/2} e^{-\frac{s}{2}|x|^2}\frac{ds}{s^\sigma} + e^{-\frac{|x|^2}{C}} \int_{1/2}^1 d\mu_\sigma(s)\right] = C|x|^{2\sigma-1} \int_0^{\frac{|x|^2}{4}} e^{-r}\frac{dr}{r^\sigma} + C|x|e^{-\frac{|x|^2}{C}}.$$

If $|x| \leqslant 1$,

$$\int_0^{\frac{|x|^2}{4}} e^{-r}\frac{dr}{r^\sigma} \leqslant \int_0^{\frac{|x|^2}{4}} \frac{dr}{r^\sigma} = C|x|^{2-2\sigma},$$

and, if $|x| > 1$,

$$\int_0^{\frac{|x|^2}{4}} e^{-r}\frac{dr}{r^\sigma} \leqslant \int_0^{1/4} \frac{dr}{r^\sigma} + \int_{1/4}^{\frac{|x|^2}{4}} e^{-r}\,dr = C - e^{-\frac{|x|^2}{C}} \leqslant C.$$

Hence, (5.8) with $\mathcal{B} = B_\sigma$ is proved.

We can write

$$B_{2k,\sigma}(x)=\frac{1}{\Gamma(-\sigma)}\int_0^1\left[\left(\frac{1-s}{1+s}\right)^k\left(\frac{1-s^2}{2\pi(1+s^2)}\right)^{n/2}-1\right]e^{-\frac{s}{1+s^2}|x|^2}\,d\mu_\sigma(s)$$
$$+\frac{1}{\Gamma(-\sigma)}\int_0^1\big(e^{-\frac{s}{1+s^2}|x|^2}-1\big)\,d\mu_\sigma(s)=:I+II.$$

The bounds for I and II can be deduced as in the proof of Lemma 5.11 of [13]. We give the calculation for completeness. For both terms we use (3.6) and the Mean Value Theorem. That is,

$$|I|\leqslant C\int_0^{1/2}\left|\left(\frac{1-s}{1+s}\right)^k\left(\frac{1-s^2}{2\pi(1+s^2)}\right)^{n/2}-1\right|\frac{ds}{s^{\sigma+1}}+\int_{1/2}^1 d\mu_\sigma(s)\leqslant C\int_0^{1/2}s\,\frac{ds}{s^{1+\sigma}}+C=C.$$

For II we have to consider two cases. Assume first that $|x|^2\leqslant 2$. Then,

$$|II|\leqslant C\int_0^{1/2}\big|e^{-\frac{s}{1+s^2}|x|^2}-1\big|\,\frac{ds}{s^{1+\sigma}}+\int_{1/2}^1 d\mu_\sigma(s)\leqslant C\int_0^{1/2}|x|^2s\,\frac{ds}{s^{1+\sigma}}+C\leqslant C.$$

In the case $|x|^2>2$,

$$|II|\leqslant |x|^2\int_0^{\frac{1}{|x|^2}}s\,\frac{ds}{s^{1+\sigma}}+\int_{\frac{1}{|x|^2}}^1 d\mu_\sigma(s)\leqslant |x|^2\int_0^{\frac{1}{|x|^2}}s^{-\sigma}\,ds+\int_{\frac{1}{|x|^2}}^1\frac{ds}{(1-s)(-\log(1-s))^{1+\sigma}}$$
$$=C|x|^{2\sigma}+C\left[-\log\left(1-\frac{1}{|x|^2}\right)\right]^{-\sigma}\leqslant C|x|^{2\sigma},$$

since $-\log(1-s)\sim s$, as $s\sim 0$. On the other hand, observe that $|\nabla B_{2k,\sigma}(x)|\leqslant\widetilde{I}(x)$. Thus, (5.8) follows with $\mathcal{B}=B_{2k,\sigma}$.

When $\mathcal{B}=B_{-2k,\sigma}$,

$$B_{-2k,\sigma}(x)=\frac{1}{\Gamma(-\sigma)}\int_0^{1/2}\left[\left(\frac{1+s}{1-s}\right)^k\left(\frac{1-s^2}{2\pi(1+s^2)}\right)^{n/2}e^{-\frac{s}{1+s^2}|x|^2}-1\right]d\mu_\sigma(s)$$
$$+\frac{1}{\Gamma(-\sigma)}\int_{1/2}^1\left[\left(\frac{1+s}{1-s}\right)^k\int_{\mathbb{R}^n}\big(G_{t(s)}(x,z)-\phi_{2k}(x,z,s)\big)\,dz-1\right]d\mu_\sigma(s)$$
$$=III+IV.$$

If we write III as

$$\frac{1}{\Gamma(-\sigma)}\int_0^{1/2}\left[\left(\frac{1+s}{1-s}\right)^k\left(\frac{1-s^2}{2\pi(1+s^2)}\right)^{n/2}-1\right]d\mu_\sigma(s)+\frac{1}{\Gamma(-\sigma)}\int_0^{1/2}\left(e^{-\frac{s}{1+s^2}|x|^2}-1\right)d\mu_\sigma(s),$$

then we can handle these two terms as we did for I and II above to get $|III|\leqslant C(1+|x|^{2\sigma})$. By (3.8)–(3.9), $|IV|\leqslant C$. For the gradient of $B_{-2k,\sigma}$, similar estimates to those used for ∇B_σ can be applied for the term $\nabla_x III$. Finally, (5.6) implies that $|\nabla_x IV|\leqslant C$. The proof is complete. □

The following lemma contains a small refinement of the estimate for the kernel $F_{-\sigma}(x,z)$ given in [4, Proposition 2].

Lemma 5.6. *Take $\sigma\in(0,1]$. Then, for all $x,z\in\mathbb{R}^n$,*

$$0\leqslant F_{-\sigma}(x,z)\leqslant C\begin{cases}\frac{1}{|x-z|^{n-2\sigma}}e^{-\frac{|x||x-z|}{C}}e^{-\frac{|x-z|^2}{C}}, & \text{if } n>2\sigma,\\ e^{-\frac{|x||x-z|}{C}}e^{-\frac{|x-z|^2}{C}}[1+\log(\frac{C}{|x-z|^2})\chi_{\{\frac{C}{|x-z|^2}>1\}}(x-z)], & \text{if } n=2\sigma,\\ e^{-\frac{|x||x-z|}{C}}e^{-\frac{|x-z|^2}{C}}, & \text{if } n<2\sigma.\end{cases}\tag{5.9}$$

If $\mathrm{F}(x,z)$ denotes any of the kernels $\nabla_x F_{-\sigma}(x,z)$, $x_iF_{-\sigma}(x,z)$ or $z_iF_{-\sigma}(x,z)$, then

$$\left|\mathrm{F}(x,z)\right|\leqslant C\begin{cases}\frac{1}{|x-z|^{n+1-2\sigma}}e^{-\frac{|x||x-z|}{C}}e^{-\frac{|x-z|^2}{C}}, & \text{if } n>2\sigma-1,\\ e^{-\frac{|x||x-z|}{C}}e^{-\frac{|x-z|^2}{C}}[1+\log(\frac{C}{|x-z|^2})\chi_{\{\frac{C}{|x-z|^2}>1\}}(x-z)], & \text{if } n=2\sigma-1.\end{cases}\tag{5.10}$$

Moreover, when $|x_1-z|\geqslant 2|x_1-x_2|$,

$$\begin{aligned}&\left|F_{-\sigma}(x_1,z)-F_{-\sigma}(x_2,z)\right|\\&\quad\leqslant C|x_1-x_2|\begin{cases}\frac{1}{|x_2-z|^{n+1-2\sigma}}e^{-\frac{|z||x_2-z|}{C}}e^{-\frac{|x_2-z|^2}{C}}, & \text{if } \sigma\neq 1,\\ e^{-\frac{|z||x_2-z|}{C}}e^{-\frac{|x_2-z|^2}{C}}[1+\log(\frac{C}{|x-z|^2})\chi_{\{\frac{C}{|x-z|^2}>1\}}(x-z)], & \text{if } \sigma=1,\end{cases}\end{aligned}\tag{5.11}$$

and,

$$\left|\mathrm{F}(x_1,z)-\mathrm{F}(x_2,z)\right|\leqslant C\frac{|x_1-x_2|}{|x_2-z|^{n+2-2\sigma}}e^{-\frac{|z||x_2-z|}{C}}e^{-\frac{|x_2-z|^2}{C}}.\tag{5.12}$$

Proof. By (3.2),

$$F_{-\sigma}(x,z)=\frac{1}{\Gamma(\sigma)}\int_0^1 G_{t(s)}(x,z)\,d\mu_{-\sigma}(s)=C\int_0^1\left(\frac{1-s^2}{s}\right)^{n/2}e^{-\frac{1}{4}[s|x+z|^2+\frac{1}{s}|x-z|^2]}\,d\mu_{-\sigma}(s).\tag{5.13}$$

Then apply Lemma 5.3, with $\eta = 0$ and $\rho = -\sigma$, to get (5.9). Differentiation with respect to x inside the integral in (5.13) gives

$$\left|\nabla_x F_{-\sigma}(x,z)\right| \leqslant C \int_0^1 \left(\frac{1-s}{s}\right)^{n/2} \frac{1}{s^{1/2}} e^{-\frac{1}{8}[s|x+z|^2+\frac{1}{s}|x-z|^2]}\, d\mu_{-\sigma}(s), \tag{5.14}$$

and then Lemma 5.3, with $\eta = 1/2$ and $\rho = -\sigma$, implies (5.10) with $\mathrm{F}(x,z) = \nabla_x F_{-\sigma}(x,z)$. Take $x, z \in \mathbb{R}^n$. If $x \cdot z \geqslant 0$, then $|x| \leqslant |x+z|$ and, in this situation, $|x| F_{-\sigma}(x,z)$ is bounded by the RHS of (5.14). If $x \cdot z < 0$, we have $|x| \leqslant |x-z|$, and in this case

$$|x| F_{-\sigma}(x,z) \leqslant C|x-z| e^{-\frac{1}{8}|x-z|^2} \int_0^1 \left(\frac{1-s}{s}\right)^{n/2} e^{-\frac{1}{8}[s|x+z|^2+\frac{1}{s}|x-z|^2]}\, d\mu_{-\sigma}(s).$$

Therefore, by Lemma 5.3, we obtain (5.10) for $\mathrm{F}(x,z) = x_i F_{-\sigma}(x,z)$. The same reasoning applies to $\mathrm{F}(x,z) = z_i F_{-\sigma}(x,z)$, since $|z| \leqslant |z-x| + |x|$. To derive (5.11), we follow the proof of (5.3) in Lemma 5.4, with $-\sigma$ in the place of σ, and we use Lemma 5.3. Estimate (5.12) for $\mathrm{F}(x,z) = \nabla_x F_{-\sigma}(x,z)$ can be deduced by using the Mean Value Theorem and Lemma 5.3, since

$$\begin{aligned}\partial^2_{x_i,x_j} F_{-\sigma}(x,z) = {} & \frac{1}{\Gamma(\sigma)} \int_0^1 \left(\frac{1-s^2}{4\pi s}\right)^{n/2} e^{-\frac{1}{4}[s|x+z|^2+\frac{1}{s}|x-z|^2]} \\ & \times \left[\left(-\frac{s}{2}(x_i+z_i) - \frac{1}{2s}(x_i-z_i)\right)\left(-\frac{s}{2}(x_j+z_j) - \frac{1}{2s}(x_j-z_j)\right) \right. \\ & \left. + \delta_{ij}\left(-\frac{s}{2} - \frac{1}{2s}\right)\right] d\mu_{-\sigma}(s)\end{aligned}$$

gives that

$$\left|D^2_x F_{-\sigma}(x,z)\right| \leqslant C \int_0^1 \left(\frac{1-s}{s}\right)^{n/2} \frac{1}{s} e^{-C[s|x+z|^2+\frac{1}{s}|x-z|^2]}\, d\mu_{-\sigma}(s). \tag{5.15}$$

Similar ideas can also be used to prove (5.12) when $\mathrm{F}(x,z)$ is either $x_i F_{-\sigma}(x,z)$ or $z_i F_{-\sigma}(x,z)$. We skip the details. □

Lemma 5.7. *The function $H^{-\sigma}1$ belongs to the space $C^\infty(\mathbb{R}^n)$, and*

$$\left|H^{-\sigma}1(x)\right| \leqslant \frac{C}{(1+|x|)^{2\sigma}}, \quad \textit{and} \quad \left|\nabla H^{-\sigma}1(x)\right| \leqslant \frac{C}{(1+|x|)^{1+2\sigma}}.$$

Proof. Observe that (3.2) applied to (3.11) gives

$$H^{-\sigma}1(x)=\frac{1}{\Gamma(\sigma)}\int_0^1 e^{-t(s)H}1(x)\,d\mu_{-\sigma}(s)=C\int_0^1\left(\frac{1-s^2}{1+s^2}\right)^{n/2}e^{-\frac{s}{1+s^2}|x|^2}\,d\mu_{-\sigma}(s). \tag{5.16}$$

Since

$$\left|\nabla_x e^{-t(s)H}1(x)\right|=2|x|\frac{s}{1+s^2}\left(\frac{1-s^2}{2\pi(1+s^2)}\right)^{n/2}e^{-\frac{s}{1+s^2}|x|^2}\leqslant Cs^{1/2}e^{-\frac{s}{C}|x|^2}, \tag{5.17}$$

differentiation inside the integral sign in (5.16) is justified. By repeating this argument we obtain $H^{-\sigma}1\in C^\infty(\mathbb{R}^n)$. To study the size of $H^{-\sigma}1$, note that we can restrict to the case $|x|>1$ because $H^{-\sigma}1$ is a continuous function. By (3.6), we have

$$\int_0^{1/2}\left(\frac{1-s^2}{1+s^2}\right)^{n/2}e^{-\frac{s}{1+s^2}|x|^2}\,d\mu_{-\sigma}(s)\leqslant C\int_0^{1/2}e^{-\frac{s}{C}|x|^2}\frac{ds}{s^{1-\sigma}}=C|x|^{-2\sigma}\int_0^{\frac{|x|^2}{2C}}e^{-r}\frac{dr}{r^{1-\sigma}}\leqslant C|x|^{-2\sigma},$$

and

$$\int_{1/2}^1\left(\frac{1-s^2}{1+s^2}\right)^{n/2}e^{-\frac{s}{1+s^2}|x|^2}\,d\mu_{-\sigma}(s)\leqslant Ce^{-C|x|^2}\int_{1/2}^1\frac{(1-s)^{n/2-1}}{(-\log(1-s))^{1-\sigma}}\,ds=Ce^{-C|x|^2}.$$

Plugging these two estimates into (5.16) we get the bound for $H^{-\sigma}1$. For the growth of the gradient, we can use (5.17) and similar estimates as above to obtain the result. □

Lemma 5.8. *For $1\leqslant|i|,|j|\leqslant n$, denote by $\mathrm{R}(x,z)$ any of the kernels $\partial^2_{x_i,x_j}F_{-1}(x,z)$, $x_i\partial_{x_j}F_{-1}(x,z)$ or $x_ix_jF_{-1}(x,z)$. Then*

$$\left|\mathrm{R}(x,z)\right|\leqslant\frac{C}{|x-z|^n}e^{-\frac{|x||x-z|}{C}}e^{-\frac{|x-z|^2}{C}}, \tag{5.18}$$

and, when $|x_1-z|\geqslant2|x_1-x_2|$,

$$\left|\mathrm{R}(x_1,z)-\mathrm{R}(x_2,z)\right|\leqslant C\frac{|x_1-x_2|}{|x_2-z|^{n+1}}e^{-\frac{|z||x_2-z|}{C}}e^{-\frac{|x_2-z|^2}{C}}. \tag{5.19}$$

As a consequence, the kernel of the second order Hermite–Riesz transforms $R_{ij}(x,z)=A_iA_jF_{-1}(x,z)$ also satisfies these size and smoothness estimates.

Proof. We put $\sigma=1$ in (5.15) and we use Lemma 5.3, with $\eta=1$ and $\rho=-1$, to obtain the desired estimate for $D^2_xF_{-1}$. From (5.14),

$$\left|x_i \partial_{x_j} F_{-1}(x,z)\right| \leqslant C|x| \int_0^1 \left(\frac{1-s}{s}\right)^{n/2} \frac{1}{s^{1/2}} e^{-\frac{1}{8}[s|x+z|^2+\frac{1}{s}|x-z|^2]}\, d\mu_{-1}(s). \tag{5.20}$$

If $|x| \leqslant 2$, then Lemma 5.3, with $\eta = 1/2$ and $\rho = -1$, applied to (5.20) gives

$$\left|x_i \partial_{x_j} F_{-1}(x,z)\right| \leqslant \frac{C}{|x-z|^{n-1}} e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}.$$

Assume that $|x| > 2$ in (5.20). Consider first the case $|x| < 2|x-z|$, then by Lemma 5.3,

$$\begin{aligned}\left|x_i \partial_{x_j} F_{-1}(x,z)\right| &\leqslant C \int_0^1 \left(\frac{1-s}{s}\right)^{n/2} e^{-C[s|x+z|^2+\frac{1}{s}|x-z|^2]}\, d\mu_{-1}(s) \\ &\leqslant \frac{C}{|x-z|^{n-2}} e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}.\end{aligned}$$

In the other case, namely $|x| \geqslant 2|x-z|$, we use the fact that $|x| > 2$ to see that $|x+z|^2 = 2|x|^2 - |x-z|^2 + 2|z|^2 > |x|^2$. Hence,

$$\begin{aligned}\left|x_j \partial_{x_j} F_{-1}(x,z)\right| &\leqslant C \int_0^1 \left(\frac{1-s}{s}\right)^{n/2} \frac{1}{s^{1/2}} e^{-C[s|x+z|^2+\frac{1}{s}|x-z|^2]}\, d\mu_{-1}(s) \\ &\leqslant \frac{C}{|x-z|^{n}} e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}.\end{aligned}$$

Collecting terms we have (5.18) for $\mathrm{R}(x,z) = x_j \partial_{x_j} F_{-1}(x,z)$. Finally, to obtain (5.18) with $\mathrm{R}(x,z) = x_i x_j F_{-1}(x,z)$, we note that by (5.13),

$$\left|x_i x_j F_{-1}(x,z)\right| \leqslant C|x|^2 \int_0^1 \left(\frac{1-s}{s}\right)^{n/2} e^{-\frac{1}{8}[s|x+z|^2+\frac{1}{s}|x-z|^2]}\, d\mu_{-1}(s),$$

and we consider the cases $|x| \leqslant 2$ and $|x| > 2$ as before. In the second situation, we assume first that $|x| \leqslant 2|x-z|$ and, then, that $|x| \geqslant 2|x-z|$ (which implies $|x| \leqslant |x+z|$), and we use the method of the proof given for $x_j \partial_{x_i} F_{-1}$ above.

To prove (5.19) we can use the Mean Value Theorem and Lemma 5.1 (see the proof of (5.3) and (5.12)). We omit the details. □

Lemma 5.9. *Denote by* $\mathrm{K}(x,z)$ *any of the functions* $|x|^{2\sigma} F_{-\sigma}(x,z)$, $|z|^{2\sigma} F_{-\sigma}(x,z)$, $0 < \sigma \leqslant 1$, *or the kernel* $x_i \partial_{x_j} F_{-1}(x,z)$. *Then*

$$\sup_x \int_{\mathbb{R}^n} \left|\mathrm{K}(x,z)\right| dz \leqslant C.$$

Proof. Consider the function $|x|^{2\sigma}F_{-\sigma}(x,z)$. If $|x|\leqslant 2$ then, by (5.9), $|x|^{2\sigma}\int_{\mathbb{R}^n}F_{-\sigma}(x,z)\,dz\leqslant C$. If $|x|>2$, we consider two regions of integration: $|x|<|x-z|$ and $|x|\geqslant|x-z|$. In the first region, by Lemma 5.3,

$$|x|^{2\sigma}F_{-\sigma}(x,z)\leqslant C\int_0^1\left(\frac{1-s}{s}\right)^{n/2}\frac{1}{s^{-\sigma}}e^{-C[s|x+z|^2+\frac{1}{s}|x-z|^2]}\,d\mu_{-\sigma}(s)\leqslant C\Phi(x-z),$$

with $\Phi\in L^1(\mathbb{R}^n)$. To study the second region of integration, namely $|x|\geqslant|x-z|$, we use the fact that $|x+z|>|x|$ and we split the integral defining $F_{-\sigma}$ into two intervals: $(0,1/2)$ and $(1/2,1)$. To estimate the part of the integral over the interval $(0,1/2)$ we note that, by using (3.6) and three different changes of variables, we have

$$\begin{aligned}
\int_{|x|\geqslant|x-z|}\int_0^{1/2}G_{t(s)}(x,z)\,d\mu_{-\sigma}(s)\,dz &\leqslant C\int_{|x|\geqslant|x-z|}\int_0^{1/2}\frac{1}{s^{n/2}}e^{-\frac14[s|x|^2+\frac1s|x-z|^2]}\frac{ds}{s^{1-\sigma}}\,dz\\
&=C|x|^{n-2\sigma}\int_{|x|\geqslant|x-z|}\int_0^{\frac{|x|^2}{2}}\frac{1}{r^{n/2}}e^{-\frac14[r+\frac1r|x|^2|x-z|^2]}\frac{dr}{r^{1-\sigma}}\,dz\\
&=C|x|^{-2\sigma}\int_{|x|^2\geqslant|w|}\int_0^{\frac{|x|^2}{2}}\frac{1}{r^{n/2}}e^{-\frac14[r+\frac1r|w|^2]}\frac{dr}{r^{1-\sigma}}\,dw\\
&=C|x|^{-2\sigma}\int_0^{|x|^2}\int_0^{\frac{|x|^2}{2}}\frac{1}{r^{n/2}}e^{-\frac14[r+\frac{\rho^2}{r}]}\frac{dr}{r^{1-\sigma}}\,\rho^{n-1}\,d\rho\\
&\leqslant C|x|^{-2\sigma}\int_0^\infty\frac{e^{-\frac r4}}{r^{n/2-\sigma}}\left[\int_0^\infty e^{-\frac{\rho^2}{4r}}\rho^n\frac{d\rho}{\rho}\right]\frac{dr}{r}\\
&=C|x|^{-2\sigma}\left[\int_0^\infty e^{-\frac r4}r^\sigma\frac{dr}{r}\right]\left[\int_0^\infty e^{-t}t^{n/2}\frac{dt}{t}\right]=C|x|^{-2\sigma}.
\end{aligned}$$

The integral over the interval $(1/2,1)$ is bounded by

$$|x|^{2\sigma}\int_{1/2}^1(1-s)^{n/2}e^{-C|x|^2}e^{-\frac{|x-z|^2}{C}}\,d\mu_{-\sigma}(s)\leqslant Ce^{-\frac{|x-z|^2}{C}}\in L^1\big(\mathbb{R}^n\big).$$

Hence we get the conclusion for $\mathrm{K}(x,z)=|x|^{2\sigma}F_{-\sigma}(x,z)$. To prove the result for the function $|z|^{2\sigma}F_{-\sigma}(x,z)$, observe that $|z|^{2\sigma}\leqslant C(|z-x|^{2\sigma}+|x|^{2\sigma})$, so we can apply the estimates above. When $\mathrm{F}(x,z)=x_i\partial_{x_j}F_{-1}(x,z)$ we can argue as we did for $|x|^{2\sigma}F_{-\sigma}(x,z)$ above, because of (5.20). □

Lemma 5.10. *For all* $1\leqslant|i|\leqslant n$, *and* $0<r_1<r_2\leqslant\infty$,

$$\sup_x\left|\int\limits_{r_1<|x-z|\leqslant r_2}A_iF_{-1/2}(x,z)\,dz\right|\leqslant C,$$

where $C>0$ *is independent of* r_1 *and* r_2.

Proof. By estimate (5.10) given in Lemma 5.6, it is enough to consider $r_2<1$. From Lemma 5.9, with $\sigma=1/2$, we have that $\int_{\mathbb{R}^n}x_iF_{-1/2}(x,z)\,dz\leqslant C$. We can write

$$\int\limits_{r_1<|x-z|<r_2}\partial_{x_i}F_{-1/2}(x,z)\,dz=\int\limits_{r_1<|x-z|<r_2}I(x,z)\,dz+\int\limits_{r_1<|x-z|<r_2}II(x,z)\,dz,$$

where

$$I(x,z)=\frac{1}{\Gamma(1/2)}\int_0^1\left(\frac{1-s^2}{4\pi s}\right)^{n/2}e^{-\frac{1}{4}[s|x+z|^2+\frac{1}{s}|x-z|^2]}\left(-\frac{s}{2}(x_i+z_i)\right)d\mu_{-1/2}(s).$$

Lemma 5.3 shows that

$$\left|I(x,z)\right|\leqslant C\int_0^1\left(\frac{1-s}{s}\right)^{n/2}\frac{1}{s^{-1/2}}e^{-\frac{1}{4}[s|x+z|^2+\frac{1}{s}|x-z|^2]}\,d\mu_{-1/2}(s)\leqslant\Phi(x-z),$$

for some integrable function Φ. To deal with $II(x,z)$, we consider the integral

$$\widetilde{II}(x,z)=\frac{1}{\Gamma(1/2)}\int_0^1\left(\frac{1-s^2}{4\pi s}\right)^{n/2}e^{-\frac{1}{4}[s|2x|^2+\frac{1}{s}|x-z|^2]}\frac{-(x_i-z_i)}{2s}\,d\mu_{-1/2}(s),$$

which verifies

$$\left|\int\limits_{r_1<|x-z|<r_2}\widetilde{II}(x,z)\,dz\right|=0.$$

Therefore, by applying the Mean Value Theorem and some argument parallel to the one used in the proof of Lemma 5.3, we have

$$\left|II(x,z) - \widetilde{II}(x,z)\right| \leqslant C \int_0^1 \left(\frac{1-s}{s}\right)^{n/2} \frac{1}{s^{1/2}} e^{-\frac{|x-z|^2}{Cs}} \left|e^{-\frac{1}{4}s|x+z|^2} - e^{-\frac{1}{4}s|2x|^2}\right| d\mu_{-1/2}(s)$$

$$\leqslant C \int_0^1 \left(\frac{1-s}{s}\right)^{n/2} e^{-\frac{|x-z|^2}{Cs}} d\mu_{-1/2}(s) \leqslant \Psi(x-z),$$

for some $\Psi \in L^1(\mathbb{R}^n)$. □

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References

[1] R.F. Bass, Regularity results for stable-like operators, J. Funct. Anal. 257 (2009) 2693–2722.
[2] B. Bongioanni, E. Harboure, O. Salinas, Weighted inequalities for negative powers of Schrödinger operators, J. Math. Anal. Appl. 348 (2008) 12–27.
[3] B. Bongioanni, E. Harboure, O. Salinas, Riesz transforms related to Schrödinger operators acting on *BMO* type spaces, J. Math. Anal. Appl. 357 (2009) 115–131.
[4] B. Bongioanni, J.L. Torrea, Sobolev spaces associated to the harmonic oscillator, Proc. Indian Acad. Sci. Math. Sci. 116 (2006) 1–24.
[5] L. Caffarelli, S. Salsa, L. Silvestre, Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian, Invent. Math. 171 (2008) 425–461.
[6] L. Caffarelli, L. Silvestre, The Evans–Krylov theorem for non local fully non linear equations, Ann. of Math., in press, arXiv:0905.1339v1.
[7] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Classics Math., Springer, Berlin, 2002.
[8] E. Harboure, L. de Rosa, C. Segovia, J.L. Torrea, L^p-dimension free boundedness for Riesz transforms associated to Hermite functions, Math. Ann. 328 (2004) 653–682.
[9] R. Radha, S. Thangavelu, Multipliers for Hermite and Laguerre Sobolev spaces, J. Anal. 12 (2004) 183–191.
[10] L. Silvestre, PhD thesis, The University of Texas at Austin, 2005.
[11] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, Comm. Pure Appl. Math. 60 (2007) 67–112.
[12] K. Stempak, J.L. Torrea, Higher Riesz transforms and imaginary powers associated to the harmonic oscillator, Acta Math. Hungar. 111 (2006) 43–64.
[13] P.R. Stinga, J.L. Torrea, Extension problem and Harnack's inequality for some fractional operators, Comm. Partial Differential Equations 35 (2010) 2092–2122.
[14] S. Thangavelu, Lectures on Hermite and Laguerre Expansions, Math. Notes, vol. 42, Princeton Univ. Press, Princeton, NJ, 1993.
[15] S. Thangavelu, On regularity of twisted spherical means and special Hermite expansions, Proc. Indian Acad. Sci. Math. Sci. 103 (1993) 303–320.