VANISHING OF WHITEHEAD TORSION IN DIMENSION FOUR

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In topology and geometry it is often useful or important to recognize n-dimensional manifolds that are isomorphic to products \( X \times [0, 1] \) of a compact manifold \( X \) with the unit interval. The \( s \)-cobordism theorem of Barden, Mazur and Stallings (see [13, 17]) states that a compact \((n + 1)\)-manifold \( W \) of dimension \( n + 1 \geq 6 \) with boundary \( \partial W = \partial M_0 \cup \partial M_1 \) is a product \( M_0 \times [0, 1] \) if and only if \( W \) is an \( h \)-cobordism (i.e., \( W \) is homotopically a product) and a certain algebraic invariant \( \tau(W; M_0) \) called the Whitehead torsion is trivial (for \( n = 4 \) the results of [8, 9] yield a topological version of the \( s \)-cobordism theorem for a large class of the fundamental groups). The Whitehead torsion invariant takes values in an abelian group called the Whitehead group of \( M \) that depends only on the fundamental group and is denoted by \( Wh(\pi_1(W)) \). The vanishing condition on \( \tau(W; M_0) \) is essential because the Whitehead group is nonzero in many cases and every element can be realized as the torsion of some \( h \)-cobordism \((W^{n+1}; M_0, M_1)\) for \( n \geq 4 \); a proof in the case \( n \geq 5 \) appears in [13], and the case \( n = 4 \) is treated in [2]. On the other hand, our understanding of the case \( n = 3 \) is still quite limited. For example, the following realization question from [16, Problem 4.9] is still open.

**Problem.** Does there exist a 4-dimensional \( h \)-cobordism with nontrivial Whitehead torsion?

In studying the structure of 4-dimensional \( h \)-cobordisms \((W^4; M_0, M_1)\), questions concerning the structure of the 3-manifolds \( M_0 \) and \( M_1 \) arise naturally. For example, it is appropriate to ask if \( M_0 \) and \( M_1 \) are homeomorphic. Unfortunately, questions of this type are generally beyond the reach of existing techniques; the Poincaré Conjecture is a special case. Therefore, some assumptions on \( M_0 \) and \( M_1 \) are necessary to obtain a tractable version of the realization question. In this paper we assume that the 3-dimensional manifolds under consideration are compact, oriented, unbounded, and geometric in the sense of W. Thurston (see [44] or P. Scott's expository article [36]). A closed oriented geometric 3-manifold has a decomposition into nonoverlapping bounded 3-manifolds that have geometric structures; there are eight types of structures associated to various Lie groups such as the isometry groups of 3-manifolds with constant curvature. Thurston's well known Geometrization Conjecture states that all oriented closed 3-manifolds are geometric (compare [44] and [36], Section 6, p. 484). This paper is devoted to a proof of the following theorem.
THEOREM. Let $(W^4; M_0, M_1)$ be a 4-dimensional h-cobordism between oriented geometric 3-manifolds $M_0$ and $M_1$. Then $(W^4; M_0, M_1)$ is an s-cobordism.

COROLLARY. Every h-cobordism between 3-dimensional linear space forms is an s-cobordism.

Remark. One can use the unique decomposition of oriented 3-manifolds into a connected sum of primes (cf. [11]) and the decomposition results of W. Jaco, P. Shalen, and K. Johannson [14, 15] to show that a closed oriented 3-manifold is geometric if and only if it is a finite connected sum of (possibly empty) families of hyperbolic manifolds, Seifert manifolds, and Haken manifolds (see [36, pp. 482–484], especially the first conjecture on p. 484). Our methods depend heavily upon this characterization.

The proof of the theorem has two main steps. First the case of a finite fundamental group is considered. The main result here is Theorem 3.1 which says that h-cobordisms between 3-dimensional linear space forms are s-cobordisms. Most of the algebraic machinery used in this paper is needed in order to handle this case. The methods of topological surgery theory play the important role in the proof of Theorem 3.1. Of course here one takes an advantage of Freedman’s results (see [9]) which enable us to do topological surgery on 4-manifolds with finite fundamental groups.

Next the general case is treated, i.e., h-cobordisms between arbitrary oriented geometric 3-manifolds. It turns out that there is a serious difficulty here because the fundamental groups in this case are generally outside the range of groups for which four-dimensional surgical techniques are known to be applicable. To illustrate this consider one of the simplest examples. Let $X_1$ be a 3-dimensional spherical linear space form with nontrivial Whitehead group $Wh(\pi_1(X_1))$, and let $X_2$ be Haken. Let $(W; M_0, M_1)$ be an h-cobordism where $M_0 = M_1 = X_1 \neq X_2$. Although $Wh(\pi_1(X_2)) = 0$ (cf. [48]) and $Wh(\pi_1(W)) \approx Wh(\pi_1(W_1)) \oplus Wh(\pi_1(W_2)) \approx Wh(\pi_1(X_1))$, this does not really help to analyze the torsion of $(W; M_0, M_1)$. The difficulty here arises because $\pi_1(W) \approx \pi_1(X_1) \ast \pi_1(X_2)$ is in general outside of the range of groups for which four-dimensional surgical techniques are known to work. Consequently there is no reason to believe that $(W; M_0, M_1)$ can be split (as a connected sum along an arc) into two h-cobordisms $(W_1; X_1, X_1)$ and $(W_2; X_2, X_2)$. This in turn implies that the triviality of the Whitehead torsion for an h-cobordism between two copies of $X_1$ can not be used directly to provide the needed conclusion concerning the Whitehead torsion of $(W; M_0, M_1)$. In order to overcome these difficulties we use the proper surgery theory of L. Taylor [42] and S. Maumary [24]. Specifically, we apply proper surgery to the infinite covering space of $W$ corresponding to the subgroup $\pi_1(X_1) \subset \pi_1(X_1) \ast \pi_1(X_2)$. Although the structure at infinity is more complicated than in many applications of proper surgery theory (e.g., the set of ends is a Cantor set), one has just enough geometric information to make some necessary computations of proper surgery obstruction groups. These computations allow one to establish the triviality of the Whitehead torsion for h-cobordisms between arbitrary geometric 3-manifolds.

Finally, although we have used the results of M. Freedman [8, 9] at certain points of our arguments, the proofs can be modified to be independent of [9] and [8]. This is explained further in the Final Remarks at the end of Section 5.

This paper is divided into five sections. In the first section we collect some observations concerning the homotopy equivalences of geometric 3-manifolds. The second section contains the necessary results on the Whitehead groups of finite groups. In section three we prove the triviality of the Whitehead torsion for h-cobordisms between 3-dimensional spherical linear space forms. Section four contains some geometric results needed for
application of the theory of proper Whitehead torsion and proper surgery theory to our problem. Finally section five contains the proof of our main result. The paper ends with remarks explaining how to modify the proof of the main theorem in order to avoid the use of Freedman's results.

Remarks:
1. The above Corollary together with the results of [3, 4] and [20, 21] provides a complete classification of 3-cobordisms between 3-dimensional space forms.
2. In contrast to higher dimensions, it is not possible to represent nonzero classes of Whitehead groups $Wh(Z)$ as Whitehead torsions of inertial 3-cobordisms $(W^4; M_0, M_1)$ with $M_0 \cong M_1$ and $\pi_1 = Z^k$. This follows directly from the methods of [18].
3. It is worthwhile to note that the realization problem for the $K_0$-analog of the $s$-cobordism theorem namely, Siebenmann's Ribbon Theorem (see [38]) has a nontrivial solution in many cases. That is, there are open 4-manifolds proper homotopy equivalent to $L \times \mathbb{R}$, where $L$ is an appropriate lens space, with nontrivial end invariants $\sigma(\sigma(L)) \in K_0(Z[\pi_1(L)])$; see [19].
4. Every 2- or 3-dimensional $h$-cobordism $(W; M_0, M_1)$ is an $s$-cobordism since $Wh(\pi_1) = 0$ in this case.

It appears that the techniques of proper surgery theory can be applied to a wider range of problems where four-dimensional manifolds with arbitrary fundamental groups are involved. We shall pursue this in future papers.

1. HOMOTOPY EQUIVALENCES OF GEOMETRIC 3-MANIFOLDS

In this section we shall collect some elementary observations that will be useful later in this article. We start with a short proof of the following.

Theorem 1.1. Let $f: M \to N$ be a simple homotopy equivalence of orientable geometric 3-manifolds. Then $f$ is homotopic to a homeomorphism.

Proof. We shall split the proof into two cases depending on whether or not the fundamental group is finite.

Case 1. The fundamental group $\pi_1(M)$ is finite. We claim that in this case one can assume that $M \cong N$. Indeed, the restriction on the fundamental group implies that both $M$ and $N$ are (linear) spherical space forms. If $\pi_1(M) = 0$ or $\mathbb{Z}_2$ then $M = N = S^3$ or $M = N = \mathbb{RP}^2$. If $\pi_1(M) \neq 0$ or $\mathbb{Z}_2$ and is abelian then both $M$ and $N$ are lens spaces and being simple homotopy equivalent they are homeomorphic (see [5] p. 100). If $\pi_1(M)$ is nonabelian then $M$ and $N$ up to homeomorphism are determined by the fundamental group (see [32], p. 113, and [43], p. 567 and the discussion beginning with the last two lines on p. 565) so we can assume $M = N$.

Without loss of generality we may also assume $f$ preserves some distinguished basepoint. Choose a free 4-dimensional representation $V$ such that $M = S(V)/G$, where $S(V)$ denotes the unit sphere and $G$ is the fundamental group of $M$; if we wish to emphasize this choice of $V$ and the polarization we shall write $M(V)$. The homotopy self-equivalence $f$ induces an automorphism $\phi$ of $\pi_1(M) \cong G$, and it follows that $f$ induces an equivariant simple homotopy equivalence $\tilde{f}$ from $S(\phi^*V)$ to $S(V)$ whose induced map of orbit spaces
$M(V) \equiv M(\varphi^* V) \to M(V)$ is homotopic to $f$; the map $M(V) = M(\varphi^* V)$ is just the standard identification of $G$-orbits in $S(V)$ with $G$-orbits in $S(\varphi^* V)$. But the existence of $f$ implies that the Reidemeister torsions of $\varphi^* V$ and $V$ are the same, which in turn implies that $\varphi^* V$ and $V$ are equivalent representations (compare [26] or [5] p. 100). The equivalence of representations yields a homeomorphism $h_0: M(V) \to M(\varphi^* V)$ such that the composite $M(V) \to M(\varphi^* V) \equiv M(V)$ induces $\varphi^{-1}$ on the fundamental group $G$. Thus the composite $f/G \circ h_0$ is a self-equivalence of $M(V)$ that induces the identity on the fundamental group $G$. But if $\pi_1 \neq 0$ or $\mathbb{Z}_2$ this implies that $f \circ h_0$ is homotopic to the identity, and from this we conclude that $f$ is homotopic to a homeomorphism.

Case 2. The fundamental group is infinite. Consider first the subcase in which $M$ and $N$ are prime and have infinite fundamental groups. In this subcase every homotopy equivalence is homotopic to a homeomorphism. If $M$ is Haken this is due to Waldhausen (see [47]). The hyperbolic case follows from the well known Mostow Rigidity Theorem. The case when $M$ (and $N$) are Seifert fibered is due to Scott (see [37]). It should be pointed out that the main result of [37] is stated in a somewhat different form. It asserts that if $N$ is Seifert fibered and $M$ is orientable and irreducible then $\pi_1(M) \cong \pi_1(N)$ implies that $M$ and $N$ are homeomorphic. The conclusion which we are using is however implicitly contained in the proof of the main result in [37]. Namely, one starts with a homotopy equivalence $f: N \to M$ and incompressibly immersed surface $S$ in $N$ ($S$ is a torus or a Klein bottle) with no triple points. Then one essentially deforms $f$ (up to homotopy) to obtain $f(S) \subset M$ to be immersed without triple points. This implies that $M$ is Seifert fibered. Now one deforms $f|S$ to be fiber preserving (the fibration induced on $S$ and $f(S)$ comes from $N$ and $M$ respectively) and then extends $f|S$ to fiber preserving homotopy equivalence between $N$ and $M$. It is easy to see that such a homotopy equivalence is homotopic to a homeomorphism.

Now let $f: M \to N$ be a simple homotopy equivalence and $M$, $N$ arbitrary orientable geometric 3-manifolds. Then both $M$ and $N$ can be decomposed into connected sums of prime geometric 3-manifolds. Applying the Hendriks–Laudenbach splitting theorem [12] we can represent $f$ as a connected sum of simple homotopy equivalences (these homotopy equivalences are homeomorphisms on the embedded $S^2$'s) and thus the general result follows from the special case for prime manifolds.

The assumption of a geometric structure on 3-manifolds was essential for the proof of Theorem 1.1. On the other hand without any such assumption we have the following stable result.

**Theorem 1.2.** Let $f: M \to N$ be a simple homotopy equivalence of closed oriented 3-manifolds. Then the map $f \times \text{id}: M \times S^2 \to N \times S^2$ is homotopic to a diffeomorphism.

**Proof.** Consider the map $\overline{f}: M \to N \times D^3$ given by the composition $M \xrightarrow{\sim} N \subset N \times D^3$. By Corollary 11.3.4 and Corollary 11.3.2 in [49], p. 120 and p. 118 respectively, we can approximate $\overline{f}$ by a smooth embedding. Let $j: M \to N \times D^3$ be such an embedding. Since $M$ is orientable (and hence has a trivial tangent bundle) then $j(M) \subset N \times D^3$ has a trivial normal bundle and thus we can extend $j$ to an embedding $j: M \times D^3 \to N \times D^3$. Now $N \times D^3 - \text{closure}(j(M \times D^3))$ forms a smooth $s$-cobordism $W$ between $M \times S^2$ and $N \times S^2$. By the $s$-cobordism theorem $W \cong M \times S^2 \times I$. Note that $W \xrightarrow{\text{def}} M \times S^2 \subset W \xrightarrow{r} N \times S^2$ is homotopic to $f \times \text{id}$, where $r$ is the natural deformation retraction. On the other hand if
\( g: W \to N \times S^2 \times I \) is a diffeomorphism then \( h \) is homotopic to
\[
M \times S^2 \subset W \xrightarrow{id \times h} N \times S^2
\]
which is a diffeomorphism. Consequently \( f \times id \) is homotopic to a diffeomorphism as asserted in Theorem 1.2.

Remarks:

1. The assumption in Theorem 1.1 that \( f \) is a simple homotopy equivalence is essential. There is a self homotopy equivalence of \( M = L(12, 1) \) which is not homotopic to a homeomorphism (\( f \) is not simple!). Similar examples exist for all lens spaces of the form \( L(4m, 1) \) where \( m \geq 2 \). Complete discussions of simple and nonsimple homotopy self-equivalences of lens spaces are given in [5, Section 29–30].

2. Theorems 1.1 and 1.2 were announced by Turaev in [45] and a proof of Theorem 1.1 was given in [46]. The above treatment of the finite fundamental group case can be used to clarify the corresponding part of the argument in [46].

In the course of proving our main theorem we shall need the automorphisms of 3-dimensional spherical space forms given by the following result:

**Proposition 1.3.** Let \( G = Q_8 \times \mathbb{Z}_n \) where \( Q_8 \) is the quaternionic group of order 8 and \( n \) is odd, and let \( M \) be a 3-dimensional spherical spaceform with fundamental group \( G \). Let \( T \) be the automorphism of \( Q_8 \) that cyclically permutes the standard ordered subset \( \{ i, j, k \} \) in \( Q_8 \). Then there is an orientation preserving homeomorphism \( h: M \to M \) such that \( h \) induces \( T \times id \) on \( \pi_1(M) \).

**Proof.** Let \( A: S^3 \to S^3 \) be the orthogonal map induced by the permutation matrix that sends 1 to itself and cyclically permutes the standard quaternion unit vectors \( i, j, k \), and define an \( A \)-invariant embedding of \( \mathbb{Z}_n \subset S^1 \subset S^3 \) by the formula
\[
e^{i\theta} \mapsto (\cos \theta) \cdot 1 + \frac{\sin \theta}{\sqrt{3}} \cdot (i + j + k).
\]
By the standard classification theorems for 4-dimensional linear representations of \( Q_8 \times \mathbb{Z}_n \), it follows that the free linear action of \( Q_8 \times \mathbb{Z}_n \) is equivalent to the standard action
\[
\Phi: Q_8 \times \mathbb{Z}_n \times S^3 \to S^3
\]
sending \((q, z; x)\) to the quaternion product \( qzx \), where \( z_n \) is embedded in \( S^3 \) by the composite of the \( A \)-invariant embedding above with an automorphism of \( \mathbb{Z}_n \). It then follows that \( A(qzx) = T(q)A(v)z \) for all \((q, z, v) \in Q_8 \times \mathbb{Z}_n \times S^3 \). Therefore \( A \) passes to a base-point preserving self-homeomorphism \( A_0 \) on \( M \) such that the induced map of fundamental groups is \( T \times id \) and the degree is +1.

2. **MORPHISMS OF WHITEHEAD GROUPS**

In order to prove the main result it is necessary to understand conjugation involutions and transfers on the Whitehead groups of 3-dimensional spherical spaceform groups. This information follows from the machinery developed by R. Oliver for studying \( SK_1(\mathbb{Z}[G]) \), where \( G \) is a finite group. We are grateful to Oliver for indicating how these results could be proved.
Recall that the standard conjugation involution on $Wh(G)$ is induced by the involution of the group ring $\mathbb{Z}[G]$ that takes every element of $G$ to its inverse (signs do not appear because we are working with oriented objects); specifically, if the matrix $A$ represents a class $X$ in $Wh(G)$ then $X^*$ is represented by the conjugate transpose of $A$.

**THEOREM 2.1.** Let $G \subset SO(4)$ be a finite group that can act freely on $S^3$. Then the conjugation involution $*: Wh(G) \to Wh(G)$ is the identity.

**Proof.** Here is the list of all possible groups satisfying the condition of Theorem 2.1. Descriptions of the first factors are listed in [20, p. 527].

1. $\mathbb{Z}_k$
2. $O(48) \times \mathbb{Z}_n$ ($O(48)$ is the binary octahedral group)
3. $SL(2, 5) \times \mathbb{Z}_n$
4. $Q(4k) \times \mathbb{Z}_n$
5. $T_{8k/3} \times \mathbb{Z}_n$
6. $D_{2k(2r+1)} \times \mathbb{Z}_n$.

In each case the order of the relevant cyclic group is prime to the order of the first factor.

It is known that the involution is trivial on $SK_1(G)$ and also on the quotient $Wh'(G) \approx Wh(G)/SK_1(G)$ (see [30], [40]). Let $Cl_1(G)$ be the kernel of natural surjection (see [28], p. 184)

$$SK_1(G) := SK_1(\mathbb{Z}G) \to \prod_p SK_1(\hat{\mathbb{Z}}_pG),$$

where $\hat{\mathbb{Z}}_p$ is the $p$-adic completion. Since $SK_1(G) \approx Cl_1(G) \approx (\mathbb{Z}_2)^t$, for some $t \geq 0$ (see [29] Theorem 6, p. 334), it will be enough to show that the involution is trivial if $G$ is 2-hyperelementary (cf. [40]).

From now on we assume $G$ is 2-hyperelementary. There are two possibilities.

(a) $G$ is metacyclic.

(b) $G \approx \mathbb{Z}_m \ast \mathbb{Q}^2$, where $\tau: Q^2 \to \text{Aut}(\mathbb{Z}_m)$ is some homomorphism.

If $G$ is metacyclic, then $SK_1(G) = 0$ (see [28]) and thus $Wh(G) \approx Wh'(G)$ with the trivial involution.

We will now consider the second case, in which $G \approx \mathbb{Z}_m \ast \mathbb{Q}^2$. The decomposition (see [29], p. 307)

$$\mathbb{Q}[\mathbb{Z}_m] \approx \prod_d \mathbb{Q} \varphi_d \quad (\varphi_d = \exp(2\pi i/d))$$

induces a homomorphism

$$\eta: \mathbb{Z}[G] \to \prod_m \mathbb{Z}[\varphi_m],$$

where $\mathbb{Z}[\varphi_m]$ is the induced twisted group ring (see [28], [40]). It turns out (see [30], Proposition 1.2) that $\eta$ induces an isomorphism

$$Cl_1(\mathbb{Z}[G]) \approx Cl_1\left(\prod_m \mathbb{Z}[\varphi_m]\right) = \prod_m Cl_1(\mathbb{Z}[\varphi_m]).$$

As a consequence Theorem 2.1 will be proved if we show that the boundary homomorphism

$$H_1(\mathbb{Z}_2; Cl_1(\mathbb{Z}[\varphi_m])) \to H_0(\mathbb{Z}_2; Cl_1(\mathbb{Z}[\varphi_m])).$$
induced by the corresponding short exact sequence of coefficient groups, is trivial.

For each \( d \mid m \) consider the \( t_d \), where

\[ t_d : \mathbb{Q}^2 \to \text{Aut}(\mathbb{Z}_d) \approx \text{Gal}(\mathbb{Q}(\xi_d)/\mathbb{Q}). \]

Note that if \( \ker(t_d) \neq \mathbb{Q}^2 \), then \( \text{Cl}_1(\mathbb{Z}_d[\mathbb{Q}^2])' = 0 \). This follows from the fact that \( SK_1(G) \) is generated by induction from elementary subgroups of \( G \) (see [29], Theorem 3, p. 327). In the case where \( \ker(t_d) \neq \mathbb{Q}^2 \) the only elementary subgroups are \( \mathbb{Q}^2 \) and cyclic subgroups and in both cases \( SK_1 \)-groups are trivial (cf. Theorem 4 in [28]).

Therefore we can assume \( t_d = 1 \), and we have to consider the untwisted case only.

If \( r > 1 \), then \( \text{Cl}_1(\mathbb{Z}_d[\mathbb{Q}^2]) \approx \text{Cl}_1(\mathbb{Z}_d[D(8)]) \), where \( D(8) \) is the dihedral group of order 8 (see [28]), and in order to prove Theorem 2.1 it is enough to show that the involution is trivial on \( K_1(\mathbb{Z}_d[H]) \), where \( d > 1 \) and \( H = Q(8) \) or \( D(8) \). Let \( a, b \) be the generators of \( H \) and \( z = [a, b] \) the central element of order 2. Put \( R := \mathbb{Z}_d \). The following commutative diagram is essentially contained in [28], p. 197 (cf. [30], p. 331):

\[
\begin{array}{ccc}
K_1(R[H], 1 - z) & \longrightarrow & K_1(R[H]) & \longrightarrow & K_1(R[\mathbb{Z}_2 \times \mathbb{Z}_2]) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \\
K_1(R[H]/\langle 1 + z \rangle, 2) & \longrightarrow & K_1(R[H]/\langle 1 + z \rangle) & \longrightarrow & K_1(R/2[\mathbb{Z}_2 \times \mathbb{Z}_2]) \\
\downarrow & & \downarrow \phi & & \downarrow \\
K_1(R[H]/\langle 1 + z \rangle, 2) & \longrightarrow & K_1(R[H]/\langle 4, 1 + z \rangle, 2) & \approx & R/2[\mathbb{Z}_2 \times \mathbb{Z}_2] \\
\downarrow & & \downarrow & & \downarrow \\
\text{Cl}_1(R[H]) & \approx & \mathbb{Z}_2 & \rightarrow & R/2 \\
\end{array}
\]

Here the bottom square defines the homomorphism \( \phi \) and the maps \( \text{Tr} \) and \( \varepsilon \) are defined in [28], p. 197; the definitions of the relative groups \( K_1(\cdots) \) appear in [27].

Let \( M \in GL_n(R[H]) \) be given. After multiplying by elementary matrices (if necessary) we may assume \( \alpha(M) \in GL_1(R[\mathbb{Z}_2 \times \mathbb{Z}_2]) \) (note that \( SK_1(R[\mathbb{Z}_2 \times \mathbb{Z}_2]) = 0 \); cf. [28]). Furthermore, by taking instead of \( M \) some odd order power (if necessary), we can get rid of odd order roots of unity in \( R \) and arrange for \( \alpha(M) \) to be symmetric (i.e. \( \alpha(M) = \alpha(M^t) \), where \( M^t \) is the conjugate transpose of \( M \)) and

\[ \alpha(M) = \text{diag}(u_0 + u_1 a + u_2 b + u_3 ab, 1, \ldots, 1) \]

where \( u_i \in R, \Sigma u_i \equiv 1 \pmod{2} \) and \( \bar{u}_i = u_i \).

Consider now

\[ \beta(M \cdot (\bar{M}^t)^{-1}). \]

First we note

\[ \beta(M(\bar{M}^t)^{-1}) = I + \beta((M - \bar{M}^t) \cdot (\bar{M}^t)^{-1}). \]

This leads to

\[ \text{trace}[\beta(M \cdot (\bar{M}^t)^{-1}) - I] \equiv [(v_0 - \bar{v}_0) + (v_1 + \bar{v}_1) a + (v_2 \pm \bar{v}_2) b + (v_3 \pm \bar{v}_3) ab] \cdot [u_0 + u_1 a + u_2 b + \cdots]^{-1} \]

(mod 4), where \( v_i \in R, v_i \equiv u_i \pmod{2}, a \) is the generator of order 4, and the sign \( \pm \) depends on the order of \( b, ab \), in \( H \) (i.e., whether \( H \approx Q(8) \) or \( H \approx D(8) \)).
Since $\Sigma u_i \equiv 1 \pmod{2}$ we have
\[
\epsilon \cdot \text{trace} \left( {\text{det} \left( M \cdot (M^t)^{-1} \right) - I} \right) \equiv \frac{1}{2} \left[ (v_0 - \bar{v}_0) + (v_1 + \bar{v}_1) + (v_2 \pm \bar{v}_2) + (v_3 \pm \bar{v}_3) \right] \pmod{2}.
\]

The last expression can be written as
\[
\frac{1}{2} \left[ -1 + v_0 + v_1 + v_2 + v_3 - 1 + v_0 - v_1 + v_2 + v_3 \right].
\]

However, in $\mathbb{Q}_d/\mathbb{Q}$ the mapping $Tr$ has the property
\[
Tr(\bar{\xi}) = Tr(x).
\]

This leads to
\[
\begin{align*}
Tr(-) &= Tr\left[ v_1 + v_2 + v_3 \right] \quad \text{for } H \approx Q(8) \\
Tr(-) &= Tr(v_1) \quad \text{for } H \approx D(8).
\end{align*}
\]

But for all $i$ we have $v_i = u_i = \bar{u}_i = \bar{v}_i \pmod{2R}$ and hence $Tr(v_i) \equiv 0 \pmod{2}$. By the diagram above, this implies
\[
\phi(\beta(M \cdot (M^t)^{-1})) = 0,
\]
where $\beta(M \cdot (M^t)^{-1})$ is considered as an element in $K_1(R[H]/\langle 1 + z \rangle, 2)$ (by the commutativity of the corresponding diagram).

This last equation implies
\[
[(M \cdot (M^t)^{-1}) = 0 \quad \text{in } Cl_1(R[H]).
\]

Since $\phi$ is onto (see [28]), this last equation implies that conjugation on $Wh(G)$ is trivial and the proof of Theorem 2.1 is finished.

**Complement to 2.1.** The above argument actually proves more: Let $G$ be a finite fundamental group of a 3-dimensional closed manifold $M$. Then the involution $* : Wh(G) \to Wh(G)$ is trivial.

To see this we note that $\pi_1(M)$ is 4-periodic and the results of [23] place some further restrictions on groups that can act freely on a homotopy $S^3$. These considerations imply that the hyperelementary subgroups of $\pi_1(M)$ are the same ones that arise in the proof of Theorem 2.1. Since Whitehead groups of finite groups are detected by their transfers to hyperelementary subgroups, the triviality assertion follows directly from Theorem 2.1. ■

We shall also need the following result:

**Theorem 2.2.** Let $Q$ be a quaternionic 2-group of order $\geq 16$, let $Q' \subset Q$ be the corresponding quaternionic subgroup of index 2, and let $n$ be an odd integer. Then the transfer map from $SK_1(\mathbb{Z}[Q \times \mathbb{Z}_n])$ to $SK_1(\mathbb{Z}[Q' \times \mathbb{Z}_n])$ is an isomorphism.

**Proof.** As noted in the proof of Theorem 2.1 the groups $SK_1(\mathbb{Z}[Q \times \mathbb{Z}_n])$ split into direct sums of groups $SK_1(R[Q])$ where $R$ runs through all cyclotomic rings $\mathbb{Z} \xi_d$ with $d | n$ and $\xi_d$ a primitive $d$-th root of unity; the analogous statement holds with $Q'$ replacing $Q$, and the splittings are compatible with respect to transfers. In particular, Theorem 11.10 of [31] reduces the proof of Theorem 2.2 to showing that the transfer map $SK_1(R[Q]) \to SK_1(R[Q'])$ is an isomorphism when $R$ is a ring of integers in a number field with no real embeddings. As noted in [31, p. 144, proof of Cor. 5.7] the map
$\mathcal{FS}_G, R_G(G) \rightarrow SK_i(G)$ defined in [31, Proposition 5.2(iii)] factors through the complex Artin cokernels $A_c(G)$ as defined in [31, p. 144], and in fact Proposition 5.2 and Lemma 14.3 of [31] imply that $\mathcal{FS}_G$ induces an isomorphism from $A_c(G)$ to $Cl_i(R[G])$ if $G = Q$ or $Q'$. The double coset formula shows that the Artin cokernels are functorial with respect to transfers, and therefore the isomorphism $A_c(G) \cong Cl_i(R[G])$ is natural with respect to transfer; thus the proof reduces to verifying that the transfer map $A_c(Q) \rightarrow A_c(Q')$ is an isomorphism. But this follows immediately because both groups have order two and their nonzero elements are represented by any of the nontrivial one-dimensional representations.

We shall also need an analog of Theorem 2.2 to cover an exceptional case.

**Proposition 2.3.** Let $O^*$ be the binary octahedral group, let $Q \subset O^*$ be the Sylow 2-subgroup $(\cong Q_16)$, and let $n$ be an odd integer prime to 3. Then the transfer map from $SK_1(Z[O^* \times \mathbb{Z}_n])$ to $SK_1(Z[Q \times \mathbb{Z}_n])$ is a monomorphism.

**Proof.** Since $SK_1(Z[G])$ is an elementary abelian 2-group for all groups $G$ under consideration (cf. the discussion of Section 1), a Dress induction argument as in [31, Chapter 11] shows that $SK_1(Z[O^* \times \mathbb{Z}_n])$ is determined by its transfers to 2-hyperelementary subgroups. However, the results of [31] show that $SK_1$ vanishes for all such subgroups except $Q \times \mathbb{Z}_n$ and subgroups of the latter. Therefore the transfer into $Q \times \mathbb{Z}_n$ must be a monomorphism.

### 3. PROOF OF THE MAIN RESULT FOR FINITE FUNDAMENTAL GROUPS

The following result is the main objective of this section:

**Theorem 3.1.** Let $(W, M_0, M_1)$ be an $h$-cobordism between 3-dimensional linear space forms. Then $(W, M_0, M_1)$ is an $s$-cobordism.

The first step in the proof of the above result is the following simple observation.

**Proposition 3.2.** Let $M$ be a 3-dimensional linear space form, let $(W, M, M)$ be an $h$-cobordism, and let $\tau: W \rightarrow M$ be a homotopy inverse to the inclusion of $\partial_0 W \cong M$. Then the Whitehead torsion $\tau(W, M) \in Wh(\pi_1(M))$ has order at most 2 and the composite homology equivalence

$$h : M \cong \partial_1 W \subset W \rightarrow \partial_0 W \cong M$$

is homotopic to a homeomorphism.

**Proof.** Let $\tau = \tau(W, M) \in Wh(\pi_1(M))$. We first note that $\tau$ can not be of infinite order. This is a consequence of the triviality of the involution $*: Wh(\pi_1(M)) \rightarrow Wh(\pi_1(M))$. Indeed, using the Reidemeister torsion (and its duality) together with the triviality of $*$ one shows (cf. [18]) that $\tau$ satisfies the equation $2\tau - 0$. Consequently we can assume that $\tau$ has order two (or is trivial) because the torsion part of $Wh(\pi_1(M))$ is an elementary abelian 2-group (cf. [29]). Let $h: M \rightarrow M$ be the map described in the statement of the proposition. Then the Whitehead torsion $\tau(h)$ is given by $\tau(h) = \tau - (1)2\tau = \tau + \tau = 2\tau = 0$, and hence $h$ is simple self-homotopy equivalence. By Theorem 1.1 we conclude that $h$ is homotopic to a homeomorphism (in fact, to a diffeomorphism).
Corollary 3.3. Let \((W; M, M)\) be as in Proposition 3.2. Then there is a homotopy equivalence of manifold triads \(f: (W; M, M) \to (M \times I; M \times \{0\}, M \times \{1\})\) that is a homeomorphism on the boundary. Furthermore, the Whitehead torsion of \(f\) satisfies \(\tau(f) = -\tau(W, \partial_0 W)\).

Proof. Let \(r: W \to \partial_0 W \cong M\) be a homotopy inverse to the inclusion; by the homotopy extension property for inclusions of boundary components we can assume that \(r|_M\) is the identity. On the other hand, by Proposition 3.2 we know that \(r|_\partial_1 W\) is homotopic to a homeomorphism, so by another application of the homotopy extension property we can also assume that \(r|_\partial_1 W\) is a homeomorphism. Let \(g: (W; M_0, M_0) \to (I, \{0\}, \{1\})\) be a continuous map. We define a homotopy equivalence of triads

\[
f: (W; M_0, M_0) \to (M_0 \times I; M_0, M_0)
\]

by \(f = (r, g)\). It follows immediately that \(f\) has all the desired properties.

The following result involving computations for Wall groups of finite groups will be needed in the proof of Theorem 3.1.

Lemma 3.4. Let \(G\) be a finite group that acts freely and linearly on \(S^3\). Then the forgetful map of Wall groups from \(L^*_1(G)\) to \(L^*_1(G)\) is onto.

Proof. If \(SK_1(\mathbb{Z}[G]) = 0\) then the result follows from the Rothenberg sequence because \(Wh(G)\) is torsion free and the involution induced by conjugation is trivial.

The results of [28–31] show that \(SK_1(\mathbb{Z}[G]) \neq 0\) and if and only if \(G\) contains a subgroup of the form \(\mathbb{Q}_2^k \times \mathbb{Z}\), where \(\mathbb{Q}_2^k\) is a quaternionic group of order \(2^k \geq 8\) and \(n\) is odd. If \(G\) is the fundamental group of a spherical spaceform, then this and the list in the proof of Theorem 2.1 imply that either \(G\) is a generalized tetrahedral group \(T(8 \cdot 3^k)\) or else \(G\) has the form \(\mathbb{Z}_n \times H\), where \(H \cong SO(4)\) has order prime to \(n\) and cannot be split as a nontrivial cartesian product.

It suffices to prove the lemma for the second class of groups. For if \(G \cong T(8 \cdot 3^k)\) then we have \(L_1^*(T(8 \cdot 3^k)) = L_1^*(SL(2, 3))\) by [22], p. 64, and hence \(L_1^*(T(8 \cdot 3^k))\) surjects onto \(L_1^*(T(8 \cdot 3^k))\). On the other hand, the map \(L_1^*(G) \to L_1^*(G)\) is onto for all finite groups \(G\) by the appropriate Rothenberg sequence because \(Wh(G)/SK_1(G)\) is torsion free and the involution induced by conjugation is trivial (compare the discussion in the preceding paragraph). Combining these we see that \(L_1^*(T(8 \cdot 3^k))\) surjects onto \(L_1^*(T(8 \cdot 3^k))\).

We shall now compare \(L_1^*(\mathbb{Z}_n \times H)\) to \(L_1^*(\mathbb{Z}_n \times H)\). This will be done in two steps.

First, the appropriate Rothenberg type exact sequence implies that

\[
L_1^*(\mathbb{Z}_n \times H) \to L_1^*(\mathbb{Z}_n \times H)
\]

where \(L_1^*(-)\) are the intermediate \(L\) groups considered in [51]. Indeed, the sequence has the form

\[
\cdots \to Wh(\mathbb{Z}_n \times H) \otimes \mathbb{Z}_2 \to L_1^*(\mathbb{Z}_n \times H) \to L_1^*(\mathbb{Z}_n \times H) \to 0.
\]

Let \(X := SK_1(\pi)\). Then it follows from [22, Theorem 3.6] that projection onto the second factor induces an isomorphism

\[
L_1^*(\mathbb{Z}_n \times H) \approx L_1^*(H)
\]

(and the inverse is inclusion onto the second factor). Let \(Y := X \oplus \{ \pm 1 \} \oplus \mathbb{Z}_n \times H^{ab}\), where \(\pi^{ab}\) denotes the abelianization of \(\pi\) (also recall that \(\pi^{ab} \cong H_1(\pi; \mathbb{Z})\)). By definition the intermediate Wall groups \(L_1^*(-)\) are merely the groups \(L_1^*(-)\).
Consider now the sequence (cf. [Sl], p. 4)
\[
\cdots \to L^1(\mathbb{Z}_a \times H) \to L^1(\mathbb{Z}_a \times H) \to H^1(Y/X) \to \cdots
\]
Now \(\mathbb{Z}_2 \oplus H^{ab} \to \mathbb{Z}_2 \oplus (\mathbb{Z}_n \times H)^{ab}\) induces an isomorphism
\[
H^1(\mathbb{Z}_2; \mathbb{Z}_2 \oplus H_1(H; \mathbb{Z})) \xrightarrow{\sim} H^1(\mathbb{Z}_2; \mathbb{Z}_2 \oplus \mathbb{Z}_n \oplus H_1(H; \mathbb{Z})).
\]
This and a 5-Lemma argument for the sequence
\[
\cdots \to L^1(\mathbb{Z}_a \times H) \to L^1(\mathbb{Z}_a \times H) \to H^1(Y/X) \to \cdots
\]
\[
\uparrow \quad \uparrow \quad \uparrow
\]
\[
\to L^1(H) \to L^1(H) \to H^1(Y/X) \to \cdots
\]
imply
\[
L^1(\mathbb{Z}_a \times H) \cong L^1(H) \quad \text{and hence}
\]
\[
L^1(\mathbb{Z}_a \times H) \cong L^1(H).
\]
Since \(SK_1(H) = 0\) and \(L' = L^2\), the last isomorphism implies that in the sequence
\[
\cdots L^1(\mathbb{Z}_a \times H) \xrightarrow{i^*} L^1(\mathbb{Z}_a \times H) \xrightarrow{A_1} H^1(SK_1(\mathbb{Z}_a \times H)) \xrightarrow{} \cdots
\]
the homomorphism \(A_1\) is trivial. Therefore if \(L^1(\mathbb{Z}_a \times H) \to L^1(\mathbb{Z}_a \times H)\) is onto then so is \(L^1(\mathbb{Z}_a \times H) \to L^1(\mathbb{Z}_a \times H)\).

**Proof of Theorem 3.1.** Let \((W; M_0, M_1)\) be an \(h\)-cobordism between 3-dimensional linear space forms. Let \(\tau = \tau(W; M_0)\) be its Whitehead torsion. We recall that without loss of the generality we can assume \(M_0 = M_1\) (cf. [26], p. 410). By Proposition 3.2 and the main result of [18] we know that \(\tau(W; M) = 0\) unless \(SK_1(\pi_1(M)) \neq 0\), so assume \(M\) is such a manifold. The choice of a homotopy equivalence \(f\) as in the Corollary 3.3 determines an element in the relative structure set \(S_{\text{Top}}(M \times I, \partial)\) as defined in [49, Chapter 10]. Let \(s(f) \in \mathcal{OP}_{\text{Top}}(M \times I, \partial)\) be the class in the structure set determined by \(f\), and let \(\eta(f) = \eta(f) \in [\Sigma M, G/\text{TOP}]\) be the associated normal invariant. Since \(G/\text{TOP}\) has the homotopy type of a product of Eilenberg–MacLane spaces through dimension 7, it follows that \(\eta(f)\) is given by a class in the group
\[
H^2(\Sigma M, \mathbb{Z}_2) \oplus H^4(\Sigma M, \mathbb{Z}).
\]

The second summand maps nontrivially into \(L^3(\pi_1(M))\) under the surgery obstruction \(\Theta\), and therefore we may view \(\eta(f)\) as an element of \(H^2(\Sigma M, \mathbb{Z}_2)\). If \(u \in H^2(\Sigma M; \mathbb{Z}_2)\) then as in [20, 21] there is homotopy self-equivalence \(h_u: M \times I \to M \times I\) whose restriction to the boundary is the identity and such that \(\eta(M \times I, h_u) = u\). Thus if \(f: W \to M \times I\) is a map of triads such that \(\eta(W \times f) = \nu\) we can glue together \((W, f)\) and \((M \times I, h_u)\) to get a new homotopy equivalence of triads \((W \cup M \times I \cong W \cup f \cup h_u)\) whose normal invariant is \(\nu - \nu = 0\). By the exactness of the surgery sequence
\[
\to L^3(\pi_1(M)) \to \mathcal{OP}_{\text{Top}}(M \times I, \partial) \to [\Sigma M; G/\text{TOP}] \xrightarrow{\Theta} L^3(\pi_1(M))
\]

it follows that \(s(f)\) comes from the action of \(L^3(\pi_1(M))\) on the unit element of the structure set. Since \(L^1(\mathbb{Z}_a \times H) \to L^1(\mathbb{Z}_a \times H)\) is onto by Lemma 3.4, it follows that \(s(f)\) in fact comes
from the action of an element in $L_1(\pi_1(M))$. This means that $f$ is $h$-cobordant to a map of triads $f^*\colon (W^*, M, M) \to (M \times I, M, M)$ such that $W^*$ is an $s$-cobordism. Therefore there is an $h$-cobordism $X^5$ with $\partial X^5 = W^* \cup - W$ and $\partial W^* \cap \partial W = M_0 \sqcup M_1$ (where $M_0, M_1 \cong M$). By the duality formula for Whitehead torsions of $h$-cobordisms we have $\tau(X, W^*) = \tau(X, W^*)^*$. Since $\tau(X, W^*) - \tau(X, W)$ by Theorem 2.1, it follows that $\tau(X, W^*) = \tau(X, W)$. Furthermore, we also have

$$\tau(W, M_0) = \tau(X, W^*) + \tau(W^*, M_0) - \tau(X, W).$$

Since $W^*$ is an $s$-cobordism, the middle term vanishes so that $\tau(W, M_0) = \tau(X, W^*) - \tau(X, W) = \tau(X, W) - 7(X, W) = 0$.

### 4. Covering Space Ends for Geometric 3-Manifolds

Since a closed oriented geometric 3-manifold $M$ is a connected sum of manifolds $M_i$ whose universal covering spaces are homeomorphic to $S^3$, $S^2 \times \mathbb{R}$, or $\mathbb{R}^3$, it is natural to study the universal covering $\tilde{M}$ in terms of the universal coverings of the pieces. If each $M_i$ is aspherical the methods of E. Bloomberg [1] and D. McCullough [25] provide a means for doing this. In this section we shall refine their ideas to obtain some required information about the ends of these universal covering spaces and certain of their quotients. Two standard references for ends of manifolds are H. Freudenthal's original article [10] and an article by F. Raymond [35].

Let $N$ be a geometric 3-manifold that cannot be decomposed as a nontrivial connected sum, and let $N_0 := N - D$ where $D$ is the interior of a smoothly embedded closed 3-disk in $N$. The work of [1] and [25] uses the universal coverings of manifolds such as $N_0$ extensively, and therefore it will be helpful to have convenient models for these objects. Specifically, the observations of [1, Subsection 2.2, pp. 71-72] can be extended as follows:

**Proposition 4.1.** Let $N$ and $N_0$ be as above, let $\tilde{N}_0$ be the universal covering of $N_0$, and let $e \in \mathbb{R}^3$ be a unit vector.

(i) If $N$ is aspherical, then $\tilde{N}_0$ is diffeomorphic to

$$D^3 - \left\{ \{0\} \cup \left( \bigcup_j \left\{ x : \left| x - \frac{2}{2j + 1} \right| < \left( \frac{1}{2j + 2} \right)^2 \right\} \right) \right\}.$$

(ii) If $N$ is a spherical spaceform and $|\pi_1|$ is the order of its fundamental group, then $\tilde{N}_0$ is diffeomorphic to $D^3 - E$, where $E$ is a union of the interiors of $|\pi_1| - 1$ disjoint closed smooth disks in the interior of $D^3$.

(iii) If $N = S^1 \times S^2$, then $\tilde{N}_0$ is diffeomorphic to $D^3 - E$, where

$$E := \left\{ \pm \frac{1}{2} \bigcup \left( \bigcup_j \left\{ x : \left| x - \frac{1}{2} e + \frac{1}{2j + 1} \right| < \left( \frac{1}{2j + 2} \right)^2 \right\} \right) \right\}.$$

**Explanation of terminology.** The set in (i) is $D^3$ minus the origin and a nicely positioned sequence of disks converging to the origin, and the set in (iii) is $D^3$ minus two points and two nicely positioned sequences of disks converging to $\pm \frac{1}{2} e$.

**Proof.** (Sketch.) We begin with (i). If $N$ is an aspherical geometric 3-manifold then its universal covering is isomorphic to $\mathbb{R}^3$, and therefore $\tilde{N}_0$ is $\mathbb{R}^3$ minus the interiors of a countably infinite, locally finite family of closed disks. By stereographic projection we may
also view $\tilde{N}_0$ as $D^3$ minus the origin and a corresponding infinite family of disks. Standard isotopy theorems allow us to move these disks to the family appearing in the statement of 3.1(i). The proof of (ii) is similar but simpler; in this case we know that the universal covering of $N$ is $S^3$, and therefore $\tilde{N}_0$ is merely $S^3$ with the interiors of $|\pi_1|$ closed disks removed. The proof of (iii) is also similar to that of (i). In this case the universal covering of $N$ is $S^2 \times \mathbb{R}$ and we can use the identification of the latter with $S^3$ minus two points to show that $\tilde{N}_0$ is $S^3$ minus two points and two infinite sequences of disks, one converging to each point.

Stereographic projection and isotopy considerations resembling those of (i) will then yield an identification of such a set with the specific example described in (iii).

For each $N$ as above take an identification of $\tilde{N}_0$ with the subset of $D^3$ given by (i)-(iii), and set $\partial_1 \tilde{N}_0$ equal to the component of $\partial \tilde{N}_0$ corresponding to $S^2 = \partial D^3$. For each $g \in \pi_1(N)$ set $\partial_g \tilde{N}_0$ equal to the translate of $\partial_1 \tilde{N}_0$ under the covering transformation associated to $g$; it follows that $\partial \tilde{N}_0$ is the disjoint union of the 2-spheres $\partial_i \tilde{N}_0$ where $g$ runs through all elements of the fundamental group.

If $M$ is a geometric 3-manifold that splits as a nontrivial connected sum then one can use the methods of [1, Subsection 2] and [25] together with 4.1 to show that the set of ends for the universal covering is an uncountable Cantor set and the Freudenthal end point compactification is homeomorphic to $S^3$. Since the ends form a Cantor set, they clearly cannot have boundary collars (a collared end, and more generally a tame end in the sense of [38], is always isolated). However, the ends have the best property one could expect under the circumstances: namely, they are all simply connected. Before proving this in Theorem 4.2, we shall recall the relevant definitions from [38].

Let $X$ be a noncompact connected (paracompact) topological $k$-manifold ($k > 0$), and let $e$ be an end of $X$. Following [38, p. 141] we say that $E$ has a stable fundamental group (or is stable at $E$) if there is a sequence of arcwise connected neighborhoods of $E$

$$U_1 \supset U_2 \supset \cdots$$

with $\cap_i U_i = \emptyset$ such that (with basepoints and base paths chosen) the sequence

$$\pi_1(U_1) \xleftarrow{f_1} \pi_1(U_2) \xleftarrow{f_2} \cdots$$

induces isomorphisms

$$\text{Im}(f_i) \cong \text{Im}(f_{i+1}) \cong \cdots$$

If this holds and $W_1 \supset W_2 \supset \cdots$ is any other sequence of neighborhoods whose closures have empty intersection, then by [38, pp. 11-14] $\lim \pi_1(W_i)$ will be isomorphic to $\lim \pi_1(U_i)$ equal to $\text{Im}(f_1)$, and therefore it is meaningful to set $\pi_1(e)$ equal to $\text{Im}(f_1)$ if $\pi_1$ is stable at $e$. We shall say that $X$ is simply connected at $e$ if $\pi_1$ is stable at $e$ and $\pi_1(e)$ is trivial.

**Theorem 4.2.** Let $M$ be a geometric 3-manifold that has a nontrivial decomposition as a connected sum, and let $e$ be an end of the universal covering space of $M$. Then $M$ is simply connected at $e$.

**Proof.** We shall first introduce some notation. Let $B \langle k \rangle$ be $S^3$ with the interiors of $k > 0$ closed smoothly embedded 3-disks removed, so that $\partial B \langle k \rangle$ is a disjoint union of $k$ copies of $S^2$. Denote these spheres by $\partial_i B \langle k \rangle$, where $1 \leq i \leq k$.

Express $M$ as a connected sum $M_1 \# \cdots \# M_k$, where $M_i$ is either $S^1 \times S^2$ or an irreducible geometric 3-manifold. Following [1] and [25] we can decompose the universal
covering $\tilde{M}$ into a union of pieces $B_x$ and $A_{(x,J_i)}$ where $x$ runs through all elements of $\pi_1(M)$, the integer $J \in \{1, \ldots, k\}$ satisfies

\[(\star) \text{ if } 1 \neq x \text{ and } x = g_1 \ldots g_n \text{ is the unique expansion of } x \text{ as a reduced word in the element of } \bigsqcup_i \pi_1(M_i), \text{ then } g_n \neq \pi_1(M_J).\]

and the following additional conditions hold:

(i) Each set $B_x$ is diffeomorphic to $B\langle k \rangle$, and each set $A_{(x,J_i)}$ is diffeomorphic to $\tilde{M}_{J_i}$.

(ii) All identifications between the pieces are given by the rules \( \partial_i A_{(x,J_i)} = \partial_j B_x \) and \( \partial_a A_{(x,J_i)} = \partial_j B_{xg} \), where $1 \neq g \in \pi_1(M_J)$ in the latter case.

Note that if $x = g_1 \ldots g_n$ is the unique expansion of $x$ as a reduced word, then $g_1 \ldots g_{n-1}$ is the unique expansion of $xg$ as a reduced word because $g_n \in \pi_1(M)$ for some $i \neq J$.

It follows that the ends of $\tilde{M}$ arise either from ends of the pieces $A_{(x,J)}$ (provided $\pi_1(M)$ is infinite) or from finite reduced words $g_1 g_2 \ldots$ (i.e., each finite segment is a reduced word). Given $x \in \pi_1(M)$ with a reduced word expansion $g_1 \ldots g_n$, define $T(x) \subset \tilde{M}$ to be the union of all $B_x$ and $A_{(x,J)}$ such that the reduced word expansion of $z$ begins with $g_1 \ldots g_n$.

As in [25] it follows that $T(x)$ is homotopy equivalent to a wedge of 2-spheres and in particular is simply connected. Furthermore, if $g_1 g_2 \ldots$ is a reduced infinite word, then the sequence \( \{ T(g_1 \ldots g_m) \}_{m=1}^{\infty} \) forms a cofinal set of simply connected neighborhoods of the corresponding end. Thus all such ends have stable fundamental groups that are trivial.

Now consider an end $\epsilon$ of $\tilde{M}$ arising from an end of some $A_{(x,J)}$. By Theorem 3.1 the associated end $\epsilon_0$ in $A_{(x,J)}$ has a cofinal set of neighborhoods of the form $N_\epsilon \cong D^3 - E_{\epsilon}$ where $E_{\epsilon}$ is the origin plus a sequence of pairwise disjoint disks converging to the origin. It follows that the end $\epsilon$ has a cofinal system of neighborhoods of the form

\[N_\epsilon^* = D^3 - E \bigcup (\cup T(xg))\]

when one takes all $T(xg)$'s such that $g_1 \in \pi_1(M_J)$ and $\partial_0 A_{(x,J)} \subset N_\epsilon$, and the sphere $\partial_0 A_{(x,J)}$ is identified to $\partial T(xg) = \partial_1 B_{xg}$. The methods of [25] also apply in this case to show that $N_\epsilon^*$ is simply connected. Thus every end of $\tilde{M}$ associated to an end of some $A_{(x,J)}$ also has a stable fundamental group that is trivial. 

Strictly speaking we need an analog of Theorem 4.2 for covering spaces of $M$ associated to finite subgroups of $\pi_1(M)$.

**Theorem 4.3.** Let $M$ satisfy the conditions of Theorem 4.2, and take $M'$ to be a covering space associated to some finite subgroup of $\pi_1(M)$. Then all the ends of $M'$ are simply connected.

**Proof.** We have already mentioned that the methods of [1, Subsection 2.2, pp. 70–71] show that the Freudenthal end point compactification of $\tilde{M}$ is homeomorphic to $S^3$. Furthermore, as in [1] it follows that the action of $\pi_1(M')$ by covering transformations extends to the end point compactification. We claim that this action of $\pi_1(M')$ is free. Since a finite group acts freely if and only if every cyclic subgroup of prime order acts freely, we may as well assume that $\pi_1(M')$ is cyclic of prime order; the action by covering transformations is orientation preserving because $M$ is orientable.

Since the fundamental group acts freely on the universal covering, it follows that the action of $\pi_1(M')$ on $S^3$ is free except possibly on the set of ends. On the other hand, by Smith theory the fixed point set of the action is either empty or a circle (recall the assumption that $\pi_1(M')$ is cyclic of prime order). Since the set of ends is zero-dimensional, it
cannot contain a circle of fixed points, and thus the fixed point set of the action must be empty.

Since the finite group \( \pi_1(M') \) acts freely on the end point compactification, each end \( \epsilon \) of \( \tilde{M} \subset S^3 \) has an invariant slice neighborhood that is equivariantly homeomorphic to a product \( \pi_1(M') \times \mathbb{R}^3 \) (in \( S^3 \)). It follows that all sufficiently small neighborhoods of \( \epsilon \) are disjoint from their translates under the action of \( \pi_1(M') \), and therefore all sufficiently small neighborhoods of ends in \( \tilde{M} \) map homeomorphically into \( M' \). This and Theorem 4.2 imply that every end of \( M' \) is simply connected.

\[
\begin{align*}
\text{5. PROOF OF THE MAIN RESULT: THE GENERAL CASE} & \\
\text{In this section we complete the proof of our main result:} & \\
\text{THEOREM. Let } (W; M_0, M_1) \text{ be a 4-dimensional } h\text{-cobordism between oriented geometric} & \\
\text{3-manifolds } M_0 \text{ and } M_1. \text{ Then } (W; M_0, M_1) \text{ is an } s\text{-cobordism.} & \\
\text{Suppose that } M \text{ is a geometric 3-manifold, and express } M \text{ as a connected sum} & \\
M = \bigoplus_{i} M_i \text{ where each } M_i \text{ is irreducible or } S^1 \times S^2. \text{ The fundamental group of } M \text{ is then a free product of the fundamental groups of the summands, and since the Whitehead groups of free products satisfy } & \\
Wh(H_1 \ast H_2) = Wh(H_1) \oplus Wh(H_2) \text{ (compare [40] or [5, Section 23]) we have a splitting} & \\
Wh(\pi_1(M)) = \bigoplus_i Wh(\pi_1(M_i)). & \\
\text{In fact, we can say considerably more because we are working with geometric 3-manifolds:} & \\
\text{PROPOSITION 5.0. Let } M = \bigoplus_i M_i \text{ be as above, and suppose that } M_1, \ldots, M_l \text{ are spherical} & \\
\text{spaceforms and } M_{l+1}, \ldots, M_k \text{ have infinite fundamental groups (by convention, if there are no} & \\
spaceform summands then } l = 0 \text{ and if there are no summands with infinite fundamental groups} & \\
\text{then } l = k. \text{ Then } Wh(\pi_1(M)) = 0 \text{ if } l = 0 \text{ and } Wh(\pi_1(M)) \cong Wh(\pi_1(M_1)) \oplus \cdots & \\
\text{if } l > 0. & \\
\text{Proof. It suffices to check that } Wh(\pi_1(N)) = 0 \text{ if } N \text{ has infinite fundamental group: i.e.,} & \\
\text{if } N \cong S^1 \times S^2, N \text{ is Haken, } N \text{ is Seifert fibered, or } N \text{ is hyperbolic. But} & \\
Wh(\pi_1(S^1 \times S^2)) = Wh(\mathbb{Z}) = 0 \text{ is well known (compare [5] or [26]), and } & \\
Wh(\pi_1(N)) = 0 \text{ if } N \text{ is Haken, Seifert fibered, or hyperbolic by results of Waldhausen [48], Plotnick [24],} & \\
\text{and Farrell and Jones [6] respectively.} & \\
\text{Notation. When we write } M = \bigoplus_i M_i, \text{ we shall assume that the fundamental groups of} & \\
\text{the first } l \text{ summands are finite and the fundamental groups of the remaining summands are} & \\
infinite (i.e., the same terminology as in Proposition 5.0). & \\
The first step in the proof of the main theorem is analogous to Proposition 3.2. & \\
\text{PROPOSITION 5.1. Let } M \text{ be a geometric 3-manifold, let } (W; M, M) \text{ be an } h\text{-cobordism, and} & \\
\text{let } r: W \to M \text{ be a homotopy inverse to the inclusion of } \partial_0 W \cong M. \text{ Then the Whitehead} & \\
torsion } \tau(W; M) \in Wh(\pi_1(M)) \text{ has order at most } 2 \text{ and the composite homotopy equivalence} & \\
h: M \cong \partial_1 M \subset W \to \partial_0 W \cong M & \\
is homotopic to a homeomorphism.} & \\
\end{align*}
\]
Remark. If $M_0$ and $M_1$ are $h$-cobordant geometric 3-manifolds then one can use the notion of generalized Reidemeister torsion as defined by V. Turaev [46] to show that $M_0 \approx M_1$ (see [45, Theorem 1.4]). Therefore our assumption that $\partial W \cong M \cup M$ is actually redundant (but a little more work is needed to show that the self-map $h$ is homotopic to a homeomorphism).

Proof. We adopt the same notation as in the statement and proof of Proposition 5.0. Let $M_0 = \partial_0 W$. Let $\tau = \tau(W; M_0) = (\tau_1, \ldots, \tau_t) \in \bigoplus_{i=1}^{t} Wh(\pi_1(M_i))$ be the Whitehead torsion of the $h$-cobordism $(W; M_0, M_0)$.

We claim first that none of the classes $\tau_i$ can have infinite order. This can be seen using Turaev's generalized Reidemeister torsion [45] in the same way that ordinary Reidemeister torsion was used in Section 3 when $M_0$ was a spherical spaceform. It follows that $\tau$ itself must have finite order, and since each group $SK_1(\pi_1(M_i))$ has exponent 1 or 2 the class $\tau$ satisfies $2\tau = 0$, exactly as in the case when $M_0$ is a spherical spaceform (see the proof of Proposition 3.2). Since we also have $\tau = \tau^*$, the final steps in the proof of Proposition 3.2 are valid in the present situation and show that $\tau = 0$.

In analogy with Section 3 we have the following:

Corollary 5.2. Let $(W; M, M)$ be as in Proposition 5.1. Then there is a homotopy equivalence of manifold triads $f: (W; M, M) \rightarrow (M \times I; M \times \{0\}, M \times \{1\})$ that is a homeomorphism on the boundary. Furthermore, the Whitehead torsion of $f$ satisfies $\tau(f) = -\tau(W; \partial_0 W)$.

An $h$-cobordism $(W; M, M)$ as in Proposition 5.1 and a choice of homotopy equivalence $f$ as in the Corollary jointly determine an element in the relative structure set $\mathcal{S}_{TOP}(M \times I, \partial)$ as defined in [49, Chapter 10]. If $\pi_1(M)$ is small then by the results of [9] and [8] this structure set fits into an exact surgery sequence

$$L^1(\pi_1(M)) \rightarrow \mathcal{S}_{TOP}(M \times I, \partial) \rightarrow [\Sigma M; G/TOP] \rightarrow L^0(\pi_1(M)).$$

In general one can still define the last two terms in the sequence, construct a monoid structure on $\mathcal{S}_{TOP}(M \times I, \partial)$ by gluing the relative structures $(W_0, f_0)$ and $(W_1, f_1)$ along $\partial_! W_0 \cong \partial_0 W_1$, show that $\eta$ is a monoid homomorphism with most of the same formal properties as in higher dimensional cases, and conclude that the composite $\Theta \eta$ is trivial.

Information about the normal invariant $\eta(W, f)$ often implies information about the Whitehead torsion $\tau(W; M)$. In order to state the most important relationship it is helpful to adopt some more notational conventions. As usual write $M \cong \#_i M_i$ and view $\pi_1(M)$ as the free product of the groups $\pi_1(M_i)$; we shall denote the covering space associated to the standard inclusion $\pi_1(M_i) \hookrightarrow \pi_1(M)$ by $q_i: M[i] \rightarrow M$. The relationship is the following:

Proposition 5.3. In the setting above, suppose it is possible to find a homotopy equivalence of triads $f$ that is a homeomorphism on the boundary and such that $q_i^* \eta(W, f) = 0$ (notation as above) for all $j$ such that $\pi_1(M_j)$ is finite, $SK_1(\mathbb{Z}[\pi_1(M_j)]) \neq 0$, and $H^2(\pi_1(M_j); \mathbb{Z}_2) \neq 0$. Then $(W; M, M)$ is an $s$-cobordism.

Remark. The results cited in Section 2 and the proof of Proposition 5.1 show that $j$ satisfies the technical conditions of Proposition 5.3 if and only if $M_j$ is a spherical
spaceform whose fundamental group is a product $H \times \mathbb{Z}_n$, where $H \subset S^3$ is either the binary octahedral group or a quaternionic group with $|H| = 0 \mod 8$ and $n$ is prime to $|H|$. The proof of Proposition 5.3 requires the following result.

**Lemma 5.4.** In the setting above, assume that all ends of $M[1]$ are simply connected. Let $(W, M, M')$ be an $h$-cobordism, express the Whitehead torsion of $(W, M)$ as $(\tau_1, \tau_2) \in \text{Wh}(\pi_1(M_1)) \oplus \text{Wh}(\pi_1(M_2)) \cong \text{Wh}(\pi_1(M))$, and let $W[1]$ be the covering space associated to $\pi_1(M_1)$. Then the following hold:

1. The triad $(W[1]; M[1], M'[1])$ is a proper $h$-cobordism in the sense of [39] where $M'[1]$ is the covering space associated to $\pi_1(M_1)$.
2. The proper Whitehead group $\text{Wh}(W[1])$ as defined in [39] or [7] is isomorphic to $\text{Wh}(\pi_1(M_1))$.
3. The proper Whitehead torsion of $(W[1]; M[1], M'[1])$ in the group $\text{Wh}(W[1])$ corresponds to $\tau_1$ under the isomorphism in (ii).

**Proof (Sketch)** Let $Q$ be a simply connected manifold with $\chi(Q) = 1$; for example, take $Q = CP^* \# S^3 \times S^3 \# S^3 \times S^3$. If we consider $W \times Q$ instead of $W$ we obtain an $h$-cobordism with the same Whitehead torsion and splitting properties in a higher dimension. Therefore without loss of generality we may assume that $M$ is at least 5-dimensional.

The first conclusion of the lemma follows immediately from the hypotheses. The proof of the second conclusion is based upon the following two exact sequences for $\text{Wh}(W[1])$ due to Siebenmann [39]

$$0 \to S_b(W[1]) \to \text{Wh}(W[1]) \to \overline{K}_n(\pi_1(\epsilon)) \to \overline{K}_n(\pi_1(W[1]))$$

$$\text{Wh}(\pi_1(\epsilon)) \to \text{Wh}(\pi_1(W[1])) \to S_b(W[1]) \to \text{Wh}(\pi_1(\epsilon)) \to 0.$$

Since the ends of $W[1]$ are simply connected and stable, it follows that $S_b(W[1]) \cong \text{Wh}(W[1])$ and $\text{Wh}(\pi_1(\epsilon)) = \text{Wh}(\pi_1(\epsilon')) = 0$. This implies

$$\text{Wh}(W[1]) \cong \text{Wh}(\pi_1(W[1])).$$

To prove the final assertion, observe that $W$ can be constructed from $h$-cobordisms over $M_1$ and $M_2$, by attaching handles away from the boundary $N = \partial M_1 = \partial M_2$; this can be done because $\text{Wh}(\pi_1(M)) \cong \text{Wh}(\pi_1(M_1)) \oplus \text{Wh}(\pi_1(M_2))$. Therefore the splitting of $M$ extends to a splitting of $W$ as (say) $W_1 \cup W_2$ where $\partial W_1 \cap \partial W_2 \cong N \times I$. The hypotheses imply that the inclusion $M_1 \subset M$ lifts to $M[1]$, and therefore the manifold $M[1]$ can be split as $M[1] \cong M_1 \cup_V V$ where $M_1$ projects homeomorphically down to $M$ and $V$ is simply connected with simply connected ends. There is a corresponding splitting $W[1] = W_1 \cup V^*$ where $W_1 \cap V^* = N \times I$. The Siebenmann exact sequences imply that the proper Whitehead group of $V$ is trivial, and hence the proper $s$-cobordism theorem of [39] implies that $V$ is a product (strictly speaking we are using a relative version of [39] for proper $h$-cobordisms over manifolds with compact boundaries such that the $h$-cobordism over the boundary is a product). It follows that $W[1]$ can be split as $W_1 \cup V \times I$. By the definitions of the maps in the Siebenmann sequences this means that the torsion of $(W[1], M[1])$ is equal to the torsion of $(W_1, M_1)$; but the latter is equal to $\tau_1$ by construction.

**Proof of Proposition 5.3.** The results of Section 3 imply that we need only consider the case where $M$ is a nontrivial connected sum. Let $\nu(h) \in \mathcal{P}_\text{top}^b(M \times I, \delta)$ be as above, and let

$$\eta_s(f) \in [\Sigma M, G/\text{TOP}] \cong H^2(\Sigma M; \mathbb{Z}_2) \oplus H^4(\Sigma M; \mathbb{Z})$$
be the associated normal invariant. The same considerations as before imply that $\eta(f)$ must lie in $H^2(\Sigma M; \mathbb{Z}_2)$. If we write $M \simeq \# M_i$ then we obtain a corresponding splitting

$$H^2(\Sigma M; \mathbb{Z}_2) \cong \bigoplus_i H^2(\Sigma M_i; \mathbb{Z}_2).$$

Our hypotheses imply that the $i$-th coordinate of $\eta(f)$ is zero if $\pi_1(M_i)$ is finite and $SK_1(\pi_1(M_i)) \neq 0$.

For each $i$ such that $\pi_1(M_i)$ is finite and $SK_1(\pi_1(M_i))$ is nonzero consider the lifting $f[i]: W[i] \to M[i] \times I$. This map is a proper homotopy equivalence of noncompact manifold triads such that the restrictions to the boundary components are homeomorphisms. Therefore we can use the proper surgery-theoretic machinery of S. Maumary [24] and L. Taylor [42] to obtain an associated element in a proper homotopy structure set

$$\mathcal{S}_{T/O}(M[i] \times I, \partial).$$

Strictly speaking we are outside the dimension range for which the results of [24] and [42] are formulated. However, in this case we still have a normal invariant homomorphism

$$\eta: \mathcal{S}_{T/O}(M[i] \times I, \partial) \to [\Sigma M[i], G/TOP] \cong H^2(M[i]; \mathbb{Z}_2)$$

(since $M[i]$ is noncompact the 4-dimensional term vanishes). Furthermore, standard arguments show that $\eta(f[I]) = q^*\eta(f)$, and therefore the hypotheses imply $\eta(f[I]) = 0$.

Let $F[i]$ be the product of $f[I]$ with the identity on $\mathbb{C}P^2$. Then $F[i]$ determines a class $s(F[i])$ in the structure set $\mathcal{S}_{T/O}(M[i] \times \mathbb{C}P^2 \times I, \partial)$, but we are now in the dimension range where [24] and [42] apply. In particular, there is an exact proper surgery sequence

$$\cdots \to L^h_1(M[i] \times \mathbb{C}P^2 \times I^2) \to \mathcal{S}_{T/O}(\bar{M}_i \times I \times \mathbb{C}P^2, \partial) \to [\Sigma \bar{M}_i \times \mathbb{C}P^2; G/TOP] \to \cdots$$

where all maps in sight are group homomorphisms. Standard considerations imply that $\eta(F[i]) = P\eta(f[I])$, where $P: M[i] \times \mathbb{C}P^2 \to M[i]$ is projection, and therefore we conclude that $\eta(F[i]) = 0$. By the exactness of the surgery sequence the class $s(F[i])$ comes from the action of $L^h_1(M[i] \times \mathbb{C}P^2 \times I^2)$ on the identity.

The group $L^h_1(M[i] \times \mathbb{C}P^2 \times I^2)$ can be computed via Maumary’s sequence (see [24]; compare [33, p. 251]) as follows. Let $K = M[i] \times \mathbb{C}P^2 \times I^2$ and let $K_1 \supset K_2 \supset \ldots$ be a sequence of neighborhoods of ends in $K$, so that each $K_i$ is cocompact and $\bigcap_{i=1}^{\infty} K_i = \emptyset$. Define $\Pi^h_i(K) := \prod_{i=1}^{\infty} L^h_i(\pi_1(K_i))$, $(q = p, h$ or $s)$, and let $1 - s: \Pi^h_i(K) \to L^h_i(\pi_1(K)) \oplus \Pi^h_{i-1}(K)$ be given by $(1 - s)(a_1, a_2, a_3, \ldots) = (-j_q(a_1), a_1 - j_q(a_2), \ldots)$, where $j_q$ is the map induced by the inclusion $j : K_i \subset K_{i-1}$ ($i \geq 1, K_0 = K$). Then the proper surgery group $L^h_1(K)$ fits into the following exact sequence:

$$\Pi^h_1(K) \xrightarrow{1 - s} L^h_1(\pi_1(K)) \oplus \Pi^h_{i-1}(K) \xrightarrow{1 - s} L^h_1(\pi_1(K)) \oplus \Pi^h_{i-1}(K).$$

It follows that $L^h_1(\pi_1(K)) \oplus \Pi^h_1(K) \to L^h_1(K)$ is onto. Since $\pi_1(K) = \pi_1(M[i]) = \pi_1(M_i)$, it follows that $L^h_1(\pi_1(K)) \oplus \Pi^h_1(K) \to L^h_1(\pi_1(K)) \oplus \Pi^h_1(K)$ is onto as in the proof of Lemma 3.4. If we combine this with the Rothenberg sequence for $L^h_2 \to L^h_3$, the algebraic determination of $Wh(W[i])$ (see [7], p. 510), and Theorem 4.3 we conclude that $L^h_1(K) \to L^h_1(K)$ is onto.

The same considerations as in the first paragraph of the proof now show that the proper Whitehead torsion $\tau(W[i] \times \mathbb{C}P^2; M[i] \times \mathbb{C}P^2)$ is trivial. Since $Wh(W[i]) = \ldots$
On the other hand, the results of Section 4 show that the conditions of Lemma 5.4 hold for the splitting $M \cong M, \# (\text{other terms})$, and thus Lemma 5.4 implies that $t(W, M)$ is the projection of $\tau(W, M)$ onto $Wh(\pi_1(M))$. Combining this with the previous observations we see that the projection of $\tau(W, M)$ onto $Wh(\pi_1(M))$ is trivial for every $i$ such that $\pi_1(M_i)$ is finite and $SK_1(\pi_1(M_i)) \neq 0$. Since Proposition 5.0 shows that the projections of $\tau(W, M)$ onto the other factors are trivial and $Wh(\pi_1(M))$ is the direct sum of the groups $Wh(\pi_1(M_i))$, this completes the proof.

**COMPLEMENT 5.5.** In the setting of Proposition 5.3, consider the normal invariant $\eta(W, f)$ as an element of $H^2(\Sigma M, Z_2) \cong \bigoplus_i H^2(\Sigma M_i, Z_2)$, and assume that for some particular value of $j$ the projection of $\eta(W, f)$ onto $H^2(\Sigma M_j, Z_2)$ is trivial. Then the projection of $\tau(W, M)$ onto $Wh(\pi_1(M_j))$ is also trivial.

This follows from the same type of argument employed in the proof of Proposition 5.2. The following result provides the means for dealing with $\eta(W, f)$ in the other cases:

**PROPOSITION 5.6.** Suppose that $M$ is a geometric 3-manifold of the form $\# M_i$, and assume that the summand $M_j$ satisfies one of the following conditions:

(i) $\pi_1(M_j)$ is infinite.

(ii) $\pi_1(M_j)$ is finite and $SK_1(\pi_1(M_j)) = 0$.

(iii) $H^1(\pi_1(M_j); Z_2) = 0$.

(iv) $\pi_1(M_j) \cong Q(8) \times \mathbb{Z}_n$ where $n$ is odd.

Then the projection of $\tau(W, M)$ onto $Wh(\pi_1(M_j))$ is trivial.

**Proof.** Once again we use the splitting $H^2(\Sigma M; Z_2) \cong \bigoplus_i H^2(\Sigma M_i; Z_2)$.

Suppose there is some homotopy equivalence of triads $f_0: W \rightarrow M \times I$ such that the $j$-th coordinate of $\eta(W, f_0)$ is nonzero for some $j$ such that $SK_1(\pi_1(M_j)) \neq 0$; the hypotheses and Complement 5.5 imply that $\eta(W, f_j)$ must hold. Let $J = \{j_1, \ldots, j_k\}$ be the set of all $j$ satisfying these conditions. Define a self-homeomorphism $h: M \rightarrow M$ such that $h$ is a connected sum of homeomorphisms $h_j$ where $h_j$ is given as in Proposition 1.3 if $j \in J$ and $h_j$ is the identity otherwise. The induced automorphism $h^*$ on $H^1(M; Z_2) \cong \bigoplus_j H^1(M_j; Z_2)$ then has the following properties:

(i) $h^*$ sends each summand $H^1(M_j; Z_2)$ into itself.

(ii) The associated self map of $H^1(M_j; Z_2)$ is the identity if $j \notin J$.

(iii) If $j \in J$ so that $H^1(M_j; Z_2) \cong Z_2$ then $h^*$ cyclically permutes the three nonzero classes in $Z_2 \oplus Z_2$.

We claim that $h^* \tau(f_0) = \tau(f_0)$; this holds because $\tau(f_0)$ lies in the subgroup of elements with finite order by Proposition 5.1 and $h^*$ is the identity on this subgroup by the discussion in the proof of Proposition 1.3. Let $f_1 = (h \times 1)f_0$ and let $f_2 = (h^* \times 1)f_0$ It follows that the Whitehead torsions satisfy $\tau(f_2) = \tau(f_1) = \tau(f_0) = -\tau(W, M)$. Furthermore, the normal
invariants satisfy $\eta(W,f_i) = h^* \eta(W,f_0)$ and $\eta(W,f_2) = h^* \eta(W,f_1) = h^{*2} \eta(W,f_0)$. Consider the class $u_3 \in \mathcal{S} \mathcal{H}_{top}(M \times I, \partial)$ formed by gluing $(W,f_0), (W,f_1)$, and $(W,f_2)$ end to end, and let $(W',f_3)$ be the representative constructed in this fashion. Since the torsions of the maps $f_i$ are all the same this means that $\tau(f_3) = 3 \tau(f_0)$; but $\tau(f_0)$ has order at most 2, so in fact we have $\tau(f_3) = \tau(f_0) = - \tau(W,M)$. Furthermore, since end to end gluing defines the group operation on the relative structure sets, it follows that the normal invariant $\eta(W',f_3)$ is merely $\eta(W,f_0) + h^* \eta(W,f_0) + h^{*2} \eta(W,f_0)$. On the other hand, by construction the self map $1 + h^* + h^{*2}$ on $H^1(M; \mathbb{Z}_2)$ is zero on $H^1(M; \mathbb{Z}_2)$ if $j \in J$ and the identity otherwise. Therefore we have replaced $(W,f_0)$ with a new $h$-cobordism and homotopy equivalence $(W,f_3)$ such that both $h$-cobordisms have the same boundaries and the same Whitehead torsions but $\eta(W',f_3)$ satisfies the hypotheses of Complement 5.5. Therefore the projection of $\tau(f_3)$ onto $Wh(\pi_1(M_j))$ is trivial. Since $\tau(f_3) = \tau(f_0)$ this shows that the projection of $\tau(W,M)$ onto $Wh(\pi_1(M_j))$ is trivial.

**Corollary.** Let $M = \# M_i$ be a geometric 3-manifold, let $(W; M, M)$ be an $h$-cobordism, and suppose that the summand $M_j$ is a spherical spaceform with fundamental group $\mathbb{Z} \times \mathbb{Z}_n$ where $n$ is odd. Then the projection of $\tau(W,M)$ onto $Wh(\pi_1(M_j))$ is trivial.

This follows the same sort of argument used in Proposition 5.5, the main difference being the replacement of Proposition 5.3 by Corollary 5.3A.

We are finally ready to prove the main result.

**Theorem.** Let $(W^4; M_0, M_1)$ be a 4-dimensional topological $h$-cobordism between oriented geometric 3-manifolds. Then $(W^4; M_0, M_1)$ is an $s$-cobordism.

**Proof.** By the remarks following the proof of Proposition 5.1 we know that $M_0 \simeq M_1$. Suppose there is an $h$-cobordism whose torsion $\tau := \tau(W,M)$ is nontrivial. As usual write $M \simeq M_1 \# \cdots \# M_n$ and $Wh(\pi_1(M_j)) \simeq \oplus Wh(\pi_1(M_i))$. Without loss of generality we may rearrange the summands so that the projection of $\tau$ onto $Wh(\pi_1(M_j))$ is nonzero. Denote this projection by $\tau_1$.

Since $\tau_1 \neq 0$ we know from Proposition 5.0 and the results mentioned in Section 1 that $\pi_1(M_j) \cong H \times \mathbb{Z}_n$ where $H$ contains a copy of $\mathbb{Z}_n$ and $n > 1$ is prime to $|H|$. Let $P_1 \to M_1$ be the covering associated to the inclusion $j: \mathbb{Z} \times \mathbb{Z}_n \to H \times \mathbb{Z}_n$, and denote the number of sheets in this covering by $k$. Then $M$ has a naturally associated finite covering of the form

$$M^* := P_1 \# k(\# \geq 2 M_i).$$

Let $W^*$ be the $h$-cobordism formed by taking the corresponding finite covering space for $W$. Then $\tau^* = \tau(W^*,M^*)$ is the image of $\tau$ under the associated transfer homomorphism; furthermore, if $\tau^j_1$ is the projection of $\tau^*$ onto $Wh(\pi_1(P_1))$, then $\tau^j_1$ is equal to the image of $\tau_1$ under the transfer determined by $j$. By the results of Section 2 (specifically, Theorem 2.2 and Proposition 2.3) we know that this transfer is a monomorphism on elements of finite order; since $\tau$ has finite order, it follows that $\tau^j_1$ must be nonzero. On the other hand, Proposition 5.6 implies that $\tau^j_1 = 0$. This contradiction arises from our assumption that $\tau$ was nonzero, and therefore we conclude that $\tau = 0$ as asserted in the theorem.

**Final Remarks.** In the proof of the main theorem 4-dimensional topological surgery theory was used. However, it is possible to give a proof that does not require the results of [9] or [8]. To explain this we first note that the 4-dimensional topological surgery was used only in the proof of Theorem 3.1. To avoid its use one can proceed as follows: Let
(W; M₀, M₁) be an arbitrary h-cobordism with finite fundamental group. We can assume that the Whitehead torsion τ(W; M₀) has order two. Consequently there is a homotopy equivalence f: (W; M₀, M₁) → (M × {0}; M × {0}, M × {1}) that is a homeomorphism on the boundary components. Purely homotopy theoretic computations as in [20] show that the normal invariant of f can be assumed to be trivial. In particular f× id(CP²) ∈ $\mathcal{S}_\text{Top}(M₀ × I × CP², ∂)$ also has the trivial normal invariant and hence $f× id(CP²) = Δ(u)$ for $u ∈ L¹(π₁(M₀))$ and $Δ: L¹(π₁(M₀)) → $ $\mathcal{S}_\text{Top}(M₀ × I × CP², ∂)$ is the map in the surgery sequence. This leads to the triviality of τ(W; M₀). The remaining surgery-theoretic arguments in this paper are entirely higher dimensional, and thus the main theorem does not depend on the results of [9] or [8].

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