The Morse–Sard theorem for Sobolev spaces in a borderline case

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Abstract

We extend the Morse–Sard theorem to mappings $u$ belonging to the Sobolev class $W^{n,n}(\mathbb{R}^n, \mathbb{R})$ with $n \geq 2$ under mild regularity assumptions on the critical set of $u$.

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Résumé

Nous prolongeons le théorème de Morse–Sard aux fonctions dans l’espace de Sobolev $W^{n,n}(\mathbb{R}^n, \mathbb{R})$ où $n \geq 2$ avec faibles hypothèses de régularité sur le ensemble des points critiques de $u$.

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1. Introduction

The formulation of the problem related to the Morse–Sard theorem is quite simple: let $\Omega$ be a domain in $\mathbb{R}^n$, $u: \Omega \rightarrow \mathbb{R}^m$ a differentiable map and $C_u$ the critical set of $u$ (i.e. the set of points $x \in \Omega$ such that $Du(x)$ is not of maximum rank). What can we say about the “size” of the image set $u(C_u)$? How “much” and what type of differentiability one has to assume on $u$ for $u(C_u)$ to be a set of $(m$-dimensional) measure zero? The problem has its origin in the thirties and in spite of its easy formulation, it shows itself in its depth when Whitney [24] in 1935 provided an example of a $C^1$ function $u: \mathbb{R}^2 \rightarrow \mathbb{R}$ non-constant on a connected unrectifiable arc. Some
years later, Morse [15], in the scalar case and Sard [22] in the vectorial one, gave optimal results in the setting of $C^k$ functions.

**Theorem 1.1 (Sard).** Let $\Omega \subset \mathbb{R}^n$, let $0 < m < n$ be an integer and let $u \in C^k(\Omega, \mathbb{R}^m)$. Then $\mathcal{H}^m(u(C_u)) = 0$ if $k \ge n - m + 1$.

Whitney’s example is enlightening since underlines the main tools of the problem: the differentiability of the function and the rectifiability of the critical set. In his paper, Whitney posed the following question: how far from rectifiable must be a closed set to be a critical set for a function $u$ on which $u$ is not constant? Answers to this question were given by Sard [23] and Norton [20] in terms of Hausdorff dimension of the critical set. Here we recall just one of the results obtained by Norton [20, Theorem 2] in the setting of Hölder differentiable function $C^{k,\alpha}$.

**Theorem 1.2 (Norton).** Let $n,m$ be positive integers with $n > m$, $0 \leq \alpha \leq 1$ and let $u \in C^{k,\alpha}(\mathbb{R}^n, \mathbb{R}^m)$. If $\mathcal{H}^{k+\alpha+m-1}(C_u) = 0$, then $\mathcal{H}^m(u(C_u)) = 0$.

Later Bates [2] improved this result dropping away the assumption on the critical set.

**Theorem 1.3 (Bates).** Let $n,m$ be positive integers with $n > m$ and let $u \in C^{n-m,1}(\mathbb{R}^n, \mathbb{R}^m)$. Then $\mathcal{H}^m(u(C_u)) = 0$.

Further, Bates completed the picture showing that Theorem 1.3 cannot be improved in the setting of $C^k$ and $C^{k,\alpha}$ functions [3]. This result induced several authors to consider new class of functions as Norton who extended the Morse–Sard theorem to the class of $C^{k,Zygmund}$ functions [21] and L. De Pascale who consider the problem in the setting of Sobolev function [6] (see also [10] for a different approach and [5] for results in Riemannian manifolds).

**Theorem 1.4 (De Pascale).** Let $n,m$ be positive integers with $n > m$, let $p > n$, $k = n - m + 1$ and let $u \in W^{k,p}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)$. Then $\mathcal{H}^m(u(C_u)) = 0$.

A fundamental role in De Pascale’s paper is played by the $N_0$-property.

**Definition 1.5.** Let $u \in C^1(\Omega, \mathbb{R}^m)$. We say that $u$ has the $N_0$-property if

$N \subset C_u, \quad \mathcal{H}^n(N) = 0 \quad \Rightarrow \quad \mathcal{H}^m(u(N)) = 0.$

As pointed out by De Pascale, it is enough to prove that $u \in W^{k,p}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ satisfies $N_0$-property.

In fact if $u \in W^{k,p}(\Omega, \mathbb{R}^m)$, for every $\epsilon > 0$ there exist a closed set $K_\epsilon \subset \Omega$ and a function $u_\epsilon \in C^k(\Omega, \mathbb{R}^m)$ such that $D^j u(x) = D^j u_\epsilon(x)$ for any $x \in K_\epsilon$, $j \leq k$ with $\mathcal{H}^n(\Omega \setminus K_\epsilon) < \epsilon$. Therefore we can write $C_u = \bigcup_{h \in \mathbb{N}}(C_u \cap K_{\frac{1}{h}}) \cup N$ where $\mathcal{H}^n(N) = 0$ and, by Theorem 1.1, we obtain $\mathcal{H}^m(u(C_u \cap K_{\frac{1}{h}})) = \mathcal{H}^m(u(C_u \cap K_{\frac{1}{h}})) = 0$. Hence, the Morse–Sard theorem holds if $u$ satisfies $N_0$-property.

We summarize this observation in the following lemma.

**Lemma 1.6.** Let $n,m$ be positive integers with $n > m$, let $k = n - m + 1$ and let $u \in W^{k,p}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)$. If $u$ has the $N_0$-property, then $\mathcal{H}^m(u(C_u)) = 0$. 
In this paper we extend the Morse–Sard theorem to the Sobolev space $W^{n,n}(\Omega)$ where $\Omega$ is a bounded open subset of $\mathbb{R}^n$ with $n \geq 2$. The result is obtained under an assumption of regularity of the set of condensation point of the critical set which allows to include the Ahlfors regular space and some type of self-similar sets. More precisely we prove the following theorem.

**Theorem (Main result).** Let $u \in W^{n,n}(\Omega)$ such that $C_u^*$ is a 1-weak regular set. Then $H^1(u(C_u)) = 0$.

The most of the results we have mentioned above have been obtained focusing on the behavior of the function $u$ near its critical set: the main theorem which has been adapted by many authors is the Morse Criticality Theorem. In our proof we give an estimate of the oscillation of $u$ by using a Poincaré-type inequality involving a capacity and imbedding and trace theorems in the setting of Sobolev spaces on metric spaces defined by Hajłasz [12]. As in the De Pascale’s paper the estimate allows us to show that $u$ satisfies the $N_0$-property. Capacity also plays an important role in the proof, besides it’s essential in the case $n = 2$ to give a correct statement of the theorem since a function $u \in W^{2,2}(\mathbb{R}^2)$ is defined up to a set of capacity zero. At this end we prove that a function in a suitable Sobolev space transforms sets of capacity zero in sets of measure zero. This result is a direct consequence of the generalization of Eilenberg inequality to Sobolev space (Lemma 3.1).

2. Definitions and preliminary results

2.1. Notations and definitions

Throughout the paper $n$ is an integer such that $n \geq 2$ and $\Omega$ is a bounded, open subset of $\mathbb{R}^n$.

- $B(x,r)$ is the open ball of center $x$ and radius $r > 0$; $Q(x,r)$ is the open cube of center $x$ with edge length $r > 0$ and sides parallel to coordinate axes.
- If $E \subset \mathbb{R}^n$, $E^*$ will be the set of all condensation points of $E$.
- If $(X, \mu)$ is a measurable space, $A$ is a $\mu$-measurable subset of $X$ and $f \in L^1(\mu)$, the average value of $f$ over $A$ is denoted by $f_A = \frac{1}{\mu(A)} \int_A f \, d\mu$.
- If $u \in W^{k,p}(\Omega)$ and $1 \leq j \leq k$ is an integer, we denote by $D^j u$ the vector whose components are all the distributional derivatives of order $j$ of $u$. If $j = 1$ we omit the index.

2.2. Capacity and Hausdorff measure

Let $\alpha \geq 0$. We recall the Caratheodory’s construction of a generic $\alpha$-dimensional Borel measure in $\mathbb{R}^n$. Let $\alpha \geq 0$ and let $\mathcal{F}$ be a family of subset of $\mathbb{R}^n$. We define

$$
\psi^\alpha(A) = \lim_{\delta \to 0^+} \psi^\alpha_\delta(A), \quad A \subset \mathbb{R}^n
$$

where

$$
\psi^\alpha_\delta(A) = \inf \left\{ \sum_{i \in \mathbb{N}} \left[ \text{diam}(B_i) \right]^\alpha : A \subset \bigcup_{i \in \mathbb{N}} B_i, \ B_i \in \mathcal{F}, \ \text{diam}(B_i) \leq \delta \right\}.
$$

The limit exists since $\psi^\alpha_\delta$ is non-decreasing as $\delta$ decreases. If $\mathcal{F} = \mathcal{B}(\mathbb{R}^n)$ is the family of the Borel set, we denote $\psi^\alpha := \mathcal{H}^\alpha$ and we called it $\alpha$-dimensional Hausdorff measure. If $\mathcal{F}$ is the family of half-open dyadic cubes in $\mathbb{R}^n$, we denote $\psi^\alpha := \mathcal{N}^\alpha$ and we called it $\alpha$-dimensional
net measure. Although the two measures are equivalent, we introduce the net measure since, sometimes, it is easier to handle than Hausdorff measure.

- If \( N \subset \mathbb{R}^n \), \( \dim_{\text{net}}(N) \) will be the Hausdorff dimension of \( N \) that is \( \dim_{\text{net}}(N) := \inf(\alpha \geq 0: \mathcal{H}^\alpha(N) = 0) \). Besides we denote by \( \mathcal{L}^n \) the Lebesgue measure in \( \mathbb{R}^n \).

Now we introduce some types of capacities we largely use in the sequel. The first is equivalent to the well-known Bessel capacity [1, Proposition 2.3.13].

**Definition 2.1.** Let \( \alpha \in \mathbb{N}, p \geq 1 \), \( K \subset \mathbb{R}^n \) be a compact. Then
\[
C_{\alpha,p}(K) = \inf \{ \|v\|_{W^{\alpha,p}(\mathbb{R}^n)}^p: v \in C_0^\infty(\mathbb{R}^n), \ v \geq 1 \text{ on } K \}.
\]

We refer to the book of Adams and Hedberg for the extension of this definition to the case of open set and finally to arbitrary set [1, Definitions 2.2.2, 2.2.3] and for the basic properties; here we just recall one of them we frequently use in the sequel: if \( E \subset \mathbb{R}^n \) and \( C_{1,n}(E) = 0 \) then \( \dim_{\text{net}}(E) = 0 \) [1, Theorem 5.1.13].

- We say that a property is true \((\alpha, p)\)-quasieverywhere \((\alpha, p)\text{-q.e.}) if it holds for every \( x \) except those belonging to a set \( N \) with \( C_{\alpha,p}(N) = 0 \).
- If \( u \in W^{\alpha,p}(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \) we denote
\[
\tilde{u}(x) = \lim_{r \to 0^+} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} u(y) \, dy.
\]

The function \( \tilde{u} \) is defined \((\alpha, p)\text{-q.e.} \) and it is an \((\alpha, p)\text{-quasicontinuous representative of } u \) [1, Chap. 6.1].

Now we are able to define another type of capacity [18, §10.3.2].

**Definition 2.2.** Let \( x \in \mathbb{R}^n, r > 0 \) and \( K \subset \mathbb{R}^n \) be a compact. We define
\[
\text{Cap}_{x,r}(K) = \inf \left\{ \int_{Q(x,2r)} \|\nabla v\|^n \, dz: v \in W^{1,n}_0(Q(x,2r)), \ \tilde{v}(z) = 1, \ (1, n)\text{-q.e. on } K \cap Q(x,r) \right\}.
\]

Finally we recall the definition of Sobolev spaces on metric spaces introduced by Hajłasz [12] by using a metric characterization of ordinary Sobolev spaces.

**Definition 2.3.** Let \((X, d, \mu)\) be a metric space \((X, d)\) equipped with a Borel measure \( \mu \). Assume that \( \text{diam } X < \infty \) and \( \mu(X) < \infty \). If \( p > 1 \) the Sobolev space \( W^{1,p}(X, d, \mu) \) is defined as follows

\[
W^{1,p}(X, d, \mu) = \{ f \in L^p(X, \mu): \text{there exists } g \in L^p(X, \mu) \text{ such that } |f(x) - f(y)| \leq d(x, y)(g(x) + g(y)) \text{ holds } \mu \text{ a.e.} \}.
\]

The space \( W^{1,p}(X, d, \mu) \) is equipped with the seminorm
\[
\|f\|_{L^{1,p}} = \inf \{ \|g\|_{L^p}: g \text{ satisfies the inequality in the definition} \}.
\]
3. Main results

In the sequel we deal with functions in Sobolev space. If \( u \in W^{1,p}(\Omega) \) we will refer to the \((1, p)\)-quasicontinuous representative of \( u \). If \( u \in W^{n,p}(\Omega) \), we will always refer to the Hölder continuous representative of \( u \). We recall that, by Sobolev imbedding, if \( n \geq 3 \) then \( u \in C^{n-2} \). Besides we denote

\[
C_u = \left\{ x \in \Omega : \lim_{r \to 0^+} \frac{1}{r^n} \int_{B(x,r)} \|Du(z)\| \, dz = 0 \right\}.
\]

Therefore, if \( n \geq 3 \) \( C_u \) coincides with the classical set of critical points; in the case \( n = 2 \) the limit of the average of \( Du \) exists \((1, 2)\)-quasieverywhere [1, Theorem 6.2.1] therefore outside a set \( N \) with \( \dim_H(N) = 0 \). We prove that the image set \( u(N) \) is negligible. At this end we show the following generalization of Eilenberg inequality to Sobolev space.

**Lemma 3.1.** Suppose \( m \leq d \leq n < p \), \( A \subset \Omega \subset \mathbb{R}^n \), and \( u \in W^{1,p}(\Omega, \mathbb{R}^m) \). Then

\[
\int_{\mathbb{R}^m} \mathcal{H}^{d-m} (A \cap u^{-1}(y)) \, d\mathcal{H}^m(y) \leq c \|u\|_{W^{1,p}(\Omega, \mathbb{R}^m)}^{m} \mathcal{H}^{\frac{pd-nm}{p-m}} (A)^{\frac{p-m}{p}}
\]

where \( \int^* \) denotes the upper integral and \( c \) is a constant depending only on \( n, m, p \) and \( d \).

**Proof.** For sake of simplicity we set \( q = \frac{pd-nm}{p-m} \). Let \( A \subset \Omega \) such that \( \mathcal{H}^q(A) < \infty \). By definition of net measure, for every \( i \in \mathbb{N} \) there exists, at most, countable family of open dyadic cubes \( \{Q^i_h\}_h \) such that \( A \subset \bigcup_{h \in \mathbb{N}} Q^i_h \), \( \text{diam}(Q^i_h) < \frac{1}{i} \) and \( N^q(A) > \sum_{h \in \mathbb{N}} (\text{diam}(Q^i_h))^q - \frac{1}{i} \). We observe that one can choose a disjoint family of such cubes. Since \( p > n \), by Sobolev imbedding theorem, it follows that

\[
\|u(y) - u(x)\| \leq c_1 \|u\|_{W^{1,p}(Q^i_h, \mathbb{R}^m)} \|y - x\|^{\frac{p-n}{p}}
\]

for every \( x, y \in Q^i_h \) where \( c_1 = c_1(p, n) \). Therefore we obtain

\[
diam u(Q^i_h) \leq c_1 \|u\|_{W^{1,p}(Q^i_h, \mathbb{R}^m)} (\text{diam}(Q^i_h))^{\frac{p-n}{p}}. \tag{3.1}
\]

By definition of Hausdorff measure

\[
\mathcal{H}^{d-m}_{\frac{1}{i}}(A \cap u^{-1}(y)) \leq \sum_{h \in \mathbb{N}} [(\text{diam}(Q^i_h))^{d-m} C_{i,h}(y)]
\]

for every \( y \in \mathbb{R}^m \), where \( C_{i,h} \) is the characteristic function of \( u(Q^i_h) \). We infer from Fatou lemma and isodiametric inequality that

\[
\int_{\mathbb{R}^m} \mathcal{H}^{d-m} (A \cap u^{-1}(y)) \, d\mathcal{H}^m(y) \leq \liminf_{i \to \infty} \int_{\mathbb{R}^m} \sum_{h \in \mathbb{N}} [(\text{diam}(Q^i_h))^{d-m} C_{i,h}(y)] \, d\mathcal{H}^m(y)
\]

\[
= \liminf_{i \to \infty} \sum_{h \in \mathbb{N}} [(\text{diam}(Q^i_h))^{d-m} \mathcal{H}^m(u(Q^i_h))] \]

\[
\leq \liminf_{i \to \infty} c_2(m) \sum_{h \in \mathbb{N}} [(\text{diam}(Q^i_h))^{d-m} \text{diam}(u(Q^i_h))^m].
\]
By (3.1) and Hölder inequality, we obtain
\[
\int_{\mathbb{R}^m} \mathcal{H}^{d-m}(A \cap u^{-1}(y)) \, d\mathcal{H}^m(y) 
\leq \liminf_{i \to \infty} c_3(p, n, m) \sum_{h \in \mathbb{N}} \left[ (\text{diam } Q_h^i)^{d-m} \|u\|_{W^{1,p}(\Omega, \mathbb{R}^m)}^m \left( \frac{\text{diam } Q_h^i}{p^{m-n}} \right) \right] 
= \liminf_{i \to \infty} c_3(p, n, m) \sum_{h \in \mathbb{N}} \left[ \|u\|_{W^{1,p}(\Omega, \mathbb{R}^m)}^m \left( \frac{\text{diam } Q_h^i}{p^{m-n}} \right)^{\frac{d-p}{p}} \right] 
\leq c_3(p, n, m) \liminf_{i \to \infty} \left[ \sum_{h \in \mathbb{N}} \|u\|_{W^{1,p}(\Omega, \mathbb{R}^m)}^p \left( \frac{\text{diam } Q_h^i}{p^{m-n}} \right) \right] \left[ \sum_{h \in \mathbb{N}} (\text{diam } Q_h^i)^q \right]^{\frac{p-m}{p}} 
\leq c_3(p, n, m) \|u\|_{W^{1,p}(\Omega, \mathbb{R}^m)}^m \liminf_{i \to \infty} \left[ N^q(A) + \frac{1}{i} \right]^{\frac{p-m}{p}} 
\leq c_4(p, n, m, d) \|u\|_{W^{1,p}(\Omega, \mathbb{R}^m)}^m \mathcal{H}^q(A) \frac{p-m}{p}. 
\]

If we set \(d = m\) in the previous inequality we obtain the following lemma as a trivial consequence.

**Lemma 3.2.** Suppose \(m \leq n < p\) and \(u \in W^{1,p}(\Omega, \mathbb{R}^m)\). Then \(\mathcal{H}^m(u(N)) = 0\) for every \(N \subset \Omega\) such that \(\mathcal{H}^m(p^{m-n}) = 0\).

**Remark.** As a particular case we observe that if \(N \subset \Omega\) is such that \(C_{1,n}(N) = 0\) then \(\dim_{\mathcal{H}}(N) = 0\) and \(\mathcal{H}^m(u(N)) = 0\).

The main result of the paper will be proved under a regularity assumption on the set of the critical points. Here are the definitions.

**Definition 3.3.** Let \(K \subset \mathbb{R}^n\) be a compact set and \(0 < s \leq t\). We say that \(K\) is \((t, s)\)-regular if there exists a Borel measure \(\mu\) supported on \(K\) with \(\mu(K) < \infty\) and there exist constants \(a, b > 0\) such that

(i) \(\mu(B(x, r)) \leq ar^s\) for all \(x \in \mathbb{R}^n\) and \(r > 0\),

(ii) \(br^t \leq \mu(B(x, r))\) for all \(x \in K\) and \(r < \text{diam } K\).

**Remark.** Since \(\mu(K) < \infty\), if \(0 < s_1 \leq s_2 \leq t_2 \leq t_1\) and \(K\) is \((t_2, s_2)\)-regular then \(K\) is \((t_1, s_1)\)-regular.

**Definition 3.4.** Let \(K \subset \mathbb{R}^n\) be a compact set. We say that \(K\) is \(1\)-weak regular if there exists \(d > 1\) such that \(K\) is \((n, d)\)-regular.

**Remark.** Conditions (i) and (ii) suggest that these definitions of regularity are related to the Hausdorff dimension of the set. In fact if \(K\) is \((t, s)\)-regular then \(s \leq \dim_{\mathcal{H}}(K) \leq t\) [14, Chap. 8].
A particular case of regular sets are the strictly $q$-regular (or Ahlfors $q$-regular) sets that is every metric measure space $(K, d, \mu)$ satisfying
\[ c_1 r^q \leq \mu(B(x, r) \cap K) \leq c_2 r^q \]
for all $x \in K$ and $r < \text{diam} K$ and constants $c_i > 0$. For a more detailed analysis about these sets we refer to [7] and [8]. We just recall as examples two well-known self-similar sets: the von Koch curve which is strictly $\log 4 / \log 3$-regular and the Cantor-like square set in $\mathbb{R}^2$ [19, p. 34]: a purely unrectifiable 1-dimensional set. The regularity condition we have introduced in Definition 3.4 is less restrictive than strictly $q$-regularity and requires that $K$ supports a measure $\mu$ such that, for every $x \in K$, the function $h_x(r) = \mu(B(x, r))$ satisfies a wide “growth” condition.

Finally we remark that if there exists $s > 0$ such that $H^s(K) > 0$, by Frostman’s lemma [17, Theorem 8.8], there exists $\mu$ a Radon measure supported on $K$ such that condition (i) of Definition 3.3 is satisfied.

In the following lemma we use the fractional maximal operator
\[ M_{\lambda}^\mu g(x) = \sup_{r < R} \left\{ r^{\lambda - n} \int_{B(x, r)} |g(z)| \, dz \right\}. \]

**Lemma 3.5.** Let $1 < d < n - 1 + \frac{1}{n}$, $K$ be an $(n, d)$-compact regular set, $x \in K$ and $r > 0$. Then there exists $c = c(n, d, \mu(K)) > 0$ such that $\text{Cap}_{x, r}(K) > c$.

**Proof.** Let $v \in W_{1,n}^1(Q(x, 2r))$ and let $\bar{v}$ be its canonical extension to $\mathbb{R}^n$. If $\lambda$ satisfies $1 - \frac{d}{n} < \lambda \leq n - d$, for every $h > 0$ the operator
\[ M_{\lambda}^\mu : L^n(\mathbb{R}^n) \to L^{\frac{n}{\lambda - n}}(K, \mu) \]
is bounded [13, Theorem 5]. By choosing $\lambda = 1 - \frac{1}{n}$ we can apply Theorem 4.7.2 in [25] and there exists $c_1 = c_1(n, d)$ such that
\[ \int_{Q(x, 2r)} \|Dv\|^n \, dx = \|D\bar{v}\|^n_{L^n(\mathbb{R}^n)} \geq c_1 \|M_{\lambda}^\mu(\|D\bar{v}\|)\|^n_{L^d(K, \mu)} \geq c_1 \|\bar{v}\|^n_{L^{1,d}}. \]
By the regularity of $K$ we can use an imbedding theorem [12, Theorem 6]. Since $dn > n$ there exists $c_2 = c_2(n, d)$ such that
\[ \|\bar{v}\|^n_{L^{1,d}} \geq c_2 \mu(K)^{1 - 1/n} \|\bar{v} - \bar{v}_K\|^n_{L^\infty}. \]
We recall that $\bar{v}_K = \frac{1}{\mu(K)} \int_K \bar{v} \, d\mu = \frac{1}{\mu(K)} \int_K \bar{v} \, d\mu := \bar{v}_K$. Now we suppose that $v = 1$, $(1, n)$-quasieverywhere on $K \cap Q_r$. Then
\[ \|\bar{v} - \bar{v}_K\|_{L^\infty} \geq \max\{|1 - v_K|, |v_K|\} \geq \frac{1}{2} \]
for every $r$. Therefore
\[ \int_{Q(x, 2r)} \|Dv\|^n \, dx \geq c_3 \mu(K)^{1 - 1/n} \]
for every \( v \in W^{1,n}_0(Q_{2r}) \) such that \( v = 1 \), \((1,n)\)-q.e. on \( K \cap \overline{Q_r} \), where \( c_3 = c_3(n,d) \) and the thesis follows. \( \square \)

Now we are able to prove the main result of the paper.

**Theorem 3.6.** Let \( u \in W^{n,n}_0(\Omega) \) such that \( C_u^\ast \) is a 1-weak regular set. Then \( \mathcal{H}^1(u(C_u)) = 0 \).

**Proof.** First we consider a Lusin type regularization of \( u \) [4, Theorem 1.2]. For every \( m \in \mathbb{N} \) there exist a closed subset \( F_m \subset \Omega \) and \( w_m \in C^{n-1}(\Omega) \cap W^{n,n}_0(\Omega) \) such that \( C_{1,n}(\Omega \setminus F_m) < \frac{1}{m} \) and \( D^j u(x) = D^j w_m(x) \) for any \( x \in F_m \) and any \( j \leq n - 1 \). Now let \( K_m = C_u \cap F_m \) and let \( K_m^\ast \) be the set of condensation points of \( K_m \). We have that \( A_m := K_m \setminus K_m^\ast \) is at most numerable and \( C_u = \bigcup_{m \in \mathbb{N}} (K_m^\ast \cup A_m) \cup N \) where \( N \subset \Omega \) is such that \( C_{1,n}(N) = 0 \). Clearly \( \mathcal{H}^1(u(A_m)) = 0 \) and, by remark following Lemma 3.2 we obtain \( \mathcal{H}^1(u(N)) = 0 \). Therefore it is enough to prove that \( \mathcal{H}^1(u(K_m^\ast)) = \mathcal{H}^1(u(w_m(K_m^\ast))) = 0 \) for every \( m \in \mathbb{N} \). In the case \( n \geq 3 \) we observe that \( Du = D^2 u = \cdots = D^{n-2} u = 0 \) and \( Dw_m = D^2 w_m = \cdots = D^{n-1} w_m = 0 \) on \( K_m^\ast \). Therefore in this case we may assume that \( u \in C^{n-1}(\Omega) \cap W^{n,n}_0(\Omega) \) and \( D^j u = 0 \) on \( C_u^\ast \) for any \( j \leq n - 1 \).

Now we argue as in the proof of the Hölder continuity of functions \( u \in W^{1,p} \) with \( p > n \) (see [9]).

Let \( x, y \in \Omega, r = \|x - y\| \) and \( B = B(x, r) \cap B(y, r) \).

We use the following inequality [11, Lemma 7.16]

\[
\|u(z) - u_S\| \leq \frac{d^n}{\mathcal{L}^n(S)} \int_Q \|Du(t)\| d^n t
\]

(3.2)

which holds for almost every \( z \) belonging to the convex set \( Q \) where \( d = diam Q \) and \( S \subset Q \). Since \( u \) is continuous and by Lemma 3.11.3 in [25], we obtain that (3.2) holds for every \( z \in Q \).

Then there exists \( c_1 > 0 \) (independent of \( r \) and \( x \)) such that

\[
\|x - u(y)\| \leq \|x - u_B\| + \|u(y) - u_B\|
\]

\[
\leq c_1 \left( \int_{B(x,r)} \frac{\|Du(z)\|}{\|x - z\|^n} dz + \int_{B(y,r)} \frac{\|Du(z)\|}{\|y - z\|^n} dz \right).
\]

(3.3)

We consider the first integral. Applying \( n - 1 \) times the Gauss–Green theorem (see the proof of Lemma 3.1.2 in [25]), there exist \( c_i = c_i(n) \) such that

\[
\int_{B(x,r)} \frac{\|Du(z)\|}{\|z - x\|^n} dz \leq c_2 \left[ \int_{B(x,r)} \frac{\|D^2 u(z)\|}{\|z - x\|^{n-2}} dz + \frac{1}{r^{n-2}} \int_{\partial B(x,r)} \|Du(z)\| d\sigma(z) \right]
\]

\[
\leq c_3 \left[ \int_{B(x,r)} \|D^n u(z)\| dz + \sum_{j=1}^{n-1} \frac{1}{r^{n-1-j}} \int_{\partial B(x,r)} \|D^j u(z)\| d\sigma(z) \right].
\]

Further in the proof we denote by \( c_i \) positive constants independent of \( x, y \) and \( r \). Now we use a trace theorem. By Theorem A.1, there exists \( c_4 = c_4(n) \) such that, for every \( j \leq n - 1 \), we obtain

\[
\int_{\partial B(x,r)} \|D^j u(z)\| d\sigma(z) \leq c_4 \left[ \frac{1}{r^j} \int_{B(x,r)} \|D^j u(z)\| dz + \int_{B(x,r)} \|D^{j+1} u(z)\| dz \right].
\]
Hence
\[
\int_{B(x,r)} \frac{\|D\!u(z)\|}{\|z-x\|^{n-1}} \, dz \leq c_4 \int_{B(x,r)} \|D^n u(z)\| \, dz
\]
\[
+ c_5 \sum_{j=1}^{n-1} \left[ \frac{1}{r^{n-j}} \int_{B(x,r)} \|D^j u(z)\| \, dz \right]
\]
\[
+ \frac{1}{r^{n-j-1}} \int_{B(x,r)} \|D^{j+1} u(z)\| \, dz \right].
\]
(3.4)

Now let \( x \in C_u^n \) and \( M = \{ v \in W^{1,n}(Q(x, 4r)) : \tilde{v}(z) = 0 \text{ for } z \in C_u^n \} \). The components of \( D^j u \) belong to \( M \) if \( j \) is any integer such that \( 1 \leq j \leq n-1 \). Since \( C_u^n \) is a regular set, by Lemma 3.5, \( \text{Cap}_{x,r}(C_u^n) > 0 \). Then, by a Poincaré type inequality [18, Theorem 10.3.3], there exists \( c_6 > 0 \) such that
\[
\|D^j u(z)\|_{L^1(B(x,r))} \leq c_6 \frac{r^n}{\text{Cap}_{x,r}(C_u^n)^{\frac{1}{n}}} \|D^{j+1} u\|_{L^n(Q(x,2r))}
\]
(3.5)

for every \( j = 1, \ldots, n-1 \). Besides, applying again \( n-j \) times the same inequality we obtain
\[
\|D^{j+1} u\|_{L^n(Q(2r))} \leq c_7 \frac{r^{n-j}}{\text{Cap}_{x,r}(C_u^n)^{\frac{n-j-1}{n}}} \|D^n u\|_{L^n(Q(2r))}.
\]

By Lemma 3.5 we have \( \text{Cap}_{x,r}(C_u^n) > c_9 > 0 \). Therefore (3.5) yields
\[
\|D^j u(z)\|_{L^1(B(x,r))} \leq c_{10} r^{2n-j-1} \|D^n u\|_{L^n(Q(2r))}
\]
and from (3.4) we obtain
\[
\int_{B(x,r)} \frac{\|D\!u(z)\|}{\|z-x\|^{n-1}} \, dz \leq c_1 \int_{B(x,r)} \|D^n u(z)\| \, dz + c_{11} r^{n-1} \|D^n u\|_{L^n(Q(2r))}.
\]

Finally by Hölder and Young’s inequalities, we have
\[
\int_{B(x,r)} \frac{\|D\!u(z)\|}{\|z-x\|^{n-1}} \, dz \leq c_3 \int_{B(x,r)} \|D^n u(z)\| \, dz + c_{12} r^n + c_{13} \int_{Q(x,2r)} \|D^n u(z)\|^{\frac{n}{n-1}} \, dz
\]
\[
\leq c_3 \int_{B(x,r)} \|D^n u(z)\|^{\frac{n}{n-1}} \, dz + c_{12} r^n + c_{13} \int_{B(x,\sqrt{n}r)} \|D^n u(z)\|^{n} \, dz. \quad (3.6)
\]

Now, following Maly and Martio [16], we define
\[
\Omega_1 = \left\{ x \in \Omega : \text{ess lim}_{r \to 0} r^{\frac{n}{n+2}} \int_{\partial B(x,r)} \|D^n u(y)\|^{\frac{n}{n+2}} \, d\sigma(y) \leq 2^{n+2} \right\}
\]
and let \( \Omega_0 = \Omega \setminus \Omega_1 \).
Let \( x \in \Omega_0 \). Then there exists \( \delta_x > 0 \) such that for a.e. \( r \in (0, \delta_x) \) \[
\int_{B(x, r)} \| D^n u(y) \| dy \leq 2^{-n-2} r \int_{\partial B(x, r)} \| D^n u(y) \| d\sigma(y). \tag{3.7}
\]
We fix \( \rho < \frac{\delta_x}{2} \) and integrate (3.7) over the interval \([\rho, 2\rho]\). We get \[
\omega(\rho) := \int_{B(x, \rho)} \| D^n u(y) \| dy \leq 2^{-n-1} \int_{B(x, 2\rho)} \| D^n u(y) \| dy. \tag{3.8}
\]
Let \( m \) be an integer such that \( \delta_x 2^m \leq \rho \leq \delta_x 2^m - 1 \).

Therefore \[
\omega(\rho) \leq \omega \left( \frac{\delta_x}{2^m} \right) \leq \left( \frac{1}{2^{n+1}} \right)^{m-1} \omega(\delta_x) \leq \left( \frac{2}{\delta_x} \right)^{n+1} \left( \frac{\delta_x}{2^m} \right)^{n+1} \omega(\delta_x) \leq \left( \frac{2}{\delta_x} \right)^{n+1} \omega(\delta_x) \rho^{n+1}.
\]

Hence, if \( \rho \leq \delta'_{x} := (\frac{\delta_x}{2})^{n+1} \), we obtain \[
\int_{B(x, \rho)} \| D^n u(y) \| dy \leq \rho^n \| D^n u(z) \|_{L^n(\Omega)} \tag{3.9}
\]
Now we consider \( x \in \Omega_1 \). Then there exists \( \eta_x > 0 \) such that for a.e. \( r \in (0, \eta_x) \) \[
\frac{\int_{\partial B(x, r)} \| D^n u(y) \| d\sigma(y)}{\int_{B(x, r)} \| D^n u(y) \| dy} \leq \frac{2^{n+3}}{r}
\]
that is, following the notation introduced in (3.8), \[
\frac{\omega'(r)}{\omega(r)} \leq \frac{2^{n+3}}{r} \tag{3.10}
\]
for a.e. \( r \in (0, \eta_x) \). Now, let \( a > 1 \) and \( \rho \leq \frac{\eta_x}{a} \). Integrating (3.10) over the interval \([\rho, a\rho]\) we obtain \[
\log \left( \frac{\omega(a\rho)}{\omega(\rho)} \right) \leq 2^{n+3} \log a
\]
that is \[
\omega(a\rho) \leq c_{15}(a, n) \omega(\rho). \tag{3.11}
\]
Therefore, in any case, if \( x \in C_u^* \) and \( r < \frac{1}{\sqrt{n}} \min \{ \eta_x, \delta'_{x} \} \), by (3.6) we have \[
\int_{B(x, r)} \frac{\| Du(z) \|}{\| z - x \|^{n-1}} dz \leq c_{14} \int_{B(x, r)} \| D^n u(z) \| dz + c_{15} r^n. \tag{3.12}
\]
Analogously, if \( y \in C_u^* \) we obtain an inequality similar to (3.6) and we get
\[
\int_{B(y,r)} \frac{\|Du(z)\|}{\|z - y\|^{n-1}} \, dz \leq c_3 \int_{B(y,r)} \|D^n u(z)\|^n \, dz + c_{12} r^n + c_{13} \int_{Q(y,2r)} \|D^n u(z)\|^n \, dz
\]
\[
\leq c_3 \int_{B(x,2r)} \|D^n u(z)\|^n \, dz + c_{12} r^n + c_{13} \int_{B(x,2\sqrt{n}r)} \|D^n u(z)\|^n \, dz.
\]

Then, for \( r < \frac{1}{2\sqrt{n}} \min\{\eta_x, \delta_x'\} \) we have
\[
\int_{B(y,r)} \frac{\|Du(z)\|}{\|z - x\|^{n-1}} \, dz \leq c_{16} \int_{B(x,r)} \|D^n u(z)\|^n \, dz + c_{17} r^n.
\]

From (3.3), (3.12) and (3.13) it follows the main inequality
\[
|u(x) - u(y)| \leq c_{18} \left[ \int_{B(x,r)} \|D^n u(z)\|^n \, dz + r^n \right],
\]
which holds if \( x, y \in C_u^n \) and \( r = \|x - y\| < \frac{1}{2\sqrt{n}} \min\{\eta_x, \delta_x'\} \).

Finally we prove that \( u \) satisfies the \( N_0 \)-property. Let \( N \subset C_u \) such that \( \mathcal{H}^n(N) = 0 \) and consider an open set \( A \) such that \( N \subset A \subset \Omega \). Since \( C_u \setminus C_u^n \) is at most numerable, we may assume that \( N \subset C_u^n \). Let \( r(x) = \frac{1}{10} \min\{\frac{x}{\sqrt{n}}, \delta_x'\} \text{dist}(x, \partial A) \) and the collection of closed balls \( B = \{B(x,r) : x \in K, r < r(x)\} \). By a well-known covering theorem [17, Theorem 2.1], there exists a countable sequence of disjoint balls \( \{B(x_i, r_i)\} \) belonging to \( B \) such that
\[
N \subset \bigcup_{i \geq 1} B(x_i, 5 r_i).
\]

From (3.14) we obtain
\[
\mathcal{H}^1(u(N)) \leq c_{19} \sum_{i \in \mathbb{N}} \text{diam}(u(B(x_i, 5r_i) \cap N))
\]
\[
\leq c_{20} \sum_{i \in \mathbb{N}} \left[ \int_{B(x_i, 5r_i)} \|D^n u(z)\|^n \, dz + r_i^n \right].
\]

Hence (3.9) and (3.11) lead to
\[
\mathcal{H}^1(u(N)) \leq c_{21} \sum_{i \in \mathbb{N}} \left[ \int_{B(x_i, r_i)} \|D^n u(z)\|^n \, dz + r_i^n \right]
\]
\[
\leq c_{22} \int_{\bigcup_{i \geq 1} B(x_i, r_i)} \left( 1 + \|D^n u(z)\|^n \right) \, dz \leq c_{22} \int_A \left( 1 + \|D^n u(z)\|^n \right) \, dz.
\]

Since \( A \) is an arbitrary open set and \( \mathcal{H}^0(N) = 0 \), letting \( \mathcal{H}^0(A) \to 0 \), it follows that \( \mathcal{H}^1(u(N)) = 0 \). Finally, by Lemma 1.6, we have \( \mathcal{H}^1(u(C_u)) = 0 \). \( \square \)

Appendix A

In this appendix we prove the trace theorem on ball for Sobolev functions which has been used in the proof of the main result. The theorem is already known but we haven’t found a bibliographic reference with the explicit dependence of the multiplicative constant on the radius.
**Theorem A.1.** Let $p > 1$, $u \in W^{1,p}(\Omega)$, $x \in \Omega$ and $r < \text{dist}(x, \partial \Omega)$. Then

$$\int_{\partial B(x,r)} \|u(z)\| d\sigma(z) \leq \frac{n}{r} \int_{B(x,r)} \|u(z)\| dz + \int_{B(x,r)} \|Du(z)\| dz.$$ 

**Proof.** For sake of simplicity, we may suppose that $x = 0$. Let $u \in C^1(\Omega)$ and $\eta \in \mathbb{R}^n$ such that $\|\eta\| = 1$. Then

$$|u(r\eta)|r^n = \int_0^r \frac{d}{dt} |u(t\eta)|t^n dt = n \int_0^r (|u(t\eta)|t^{n-1}) dt,$$

$$+ \int_0^r \eta \cdot [Du^+(t\eta) - Du^-(t\eta)]t^n dt \leq n \int_0^r |u(t\eta)|t^{n-1} dt,$$

$$+ \int_0^r \|Du^+(t\eta) - Du^-(t\eta)\| t^n dt,$$

$$= n \int_0^r |u(t\eta)|t^{n-1} dt + \int_0^r \|Du(t\eta)\| t^n dt.$$ 

Integrating with respect to $\eta$ we obtain

$$r \int_{\partial B(0,1)} |u(r\eta)|t^{n-1} d\sigma(\eta) \leq n \int_0^r \int_{\partial B(0,1)} |u(t\eta)t^{n-1}| d\sigma(\eta) dt,$$

$$+ r \int_0^r \int_{\partial B(0,1)} \|Du(t\eta)t^{n-1}\| d\sigma(\eta) dt,$$

that is

$$\int_{\partial B(0,r)} \|u(z)\| d\sigma(z) \leq \frac{n}{r} \int_{B(0,r)} \|u(z)\| dz + \int_{B(0,r)} \|Du(z)\| dz. \quad (A.1)$$

If $u \in W^{1,p}(\Omega)$ there exists a sequence of mollified functions $u_k$ such that $\|u_k - u\|_{W^{1,p}_{\text{loc}}(\Omega)} \to 0$ and $u_k(x) \to u(x)$ provide $x$ being a Lebesgue point of $u$ [25, Theorem 1.6.1]. Since $u \in W^{1,p}(\Omega)$ we have that $(1, p)$-quasievery point is a Lebesgue point [1, Theorem 6.2.1] therefore $u_k \to u$, $\mathcal{H}^{n-p}$-quasieverywhere in $\Omega$ [1, Theorem 5.1.15]. Applying (A.1) to the mollified functions and passing to the limits we obtain the thesis. 

### References


