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k-kernels in k-transitive and k-quasi-transitive digraphs

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ABSTRACT

Let D be a digraph, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D, respectively. A subset N of $V(D)$ is *k-independent* if for every pair of vertices $u, v \in N$, we have $d(u, v), d(v, u) \geq k$; it is *l-absorbent* if for every $u \in V(D) - N$ there exists $v \in N$ such that $d(u, v) \leq l$. A *(k, l)-kernel* of D is a k-independent and l-absorbent subset of $V(D)$. A *k-kernel* is a *$(k, k-1)$-kernel*.

A digraph D is *transitive* if for every path (u, v, w) in D we have $(u, w) \in A(D)$. This concept can be generalized as follows, a digraph D is *quasi-transitive* if for every path (u, v, w) in D, we have $(u, w) \in A(D)$ or $(w, u) \in A(D)$. In the literature, beautiful results describing the structure of both transitive and quasi-transitive digraphs are found that can be used to prove that every transitive digraph has a k-kernel for every $k \geq 2$ and that every quasi-transitive digraph has a k-kernel for every $k \geq 3$.

We introduce three new families of digraphs, two of them generalizing transitive and quasi-transitive digraphs respectively; a digraph D is *k-transitive* if whenever $(x_0, x_1, \ldots, x_k)$ is a path of length k in D, then $(x_0, x_k) \in A(D)$; *k-quasi-transitive digraphs* are analogously defined, so (quasi-)transitive digraphs are 2-(quasi-)transitive digraphs. We prove some structural results about both classes of digraphs that can be used to prove that a k-transitive digraph has an n-kernel for every $n \geq k$; that for even $k \geq 2$, every k-quasi-transitive digraph has an n-kernel for every $n \geq k+2$; that every 3-quasi-transitive digraph has k-kernel for every $k \geq 4$. Also, we prove that a k-transitive digraph has a k-king if and only if it has a unique initial strong component and that a k-quasi-transitive digraph has a $(k+1)$-king if and only if it has a unique initial strong component.

1. Introduction

We will denote by D a finite digraph without loops or multiple arcs in the same direction, with vertex set $V(D)$ and arc set $A(D)$. All walks, paths and cycles will be considered to be directed. For undefined concepts and notation, we refer the reader to [2,6].

The *out-neighborhood* $N^+(v)$ of a vertex v is the set $\{u \in V(D) : (v, u) \in A(D)\}$. The *out-degree* $d^+(v)$ of a vertex v is defined as $d^+(v) = |N^+(v)|$. Definitions of *in-neighborhood* and *in-degree* of a vertex v are analogously given. If X and Y are subsets of $V(D)$, an *XY-path* (resp. *XY-arc*) will be a path (resp. arc) whose initial vertex belongs to X and terminal vertex belongs to Y. In particular, if $X = \{u\}$, we write uY-path (resp. uY-arc) instead of $\{u\}Y$-path (resp. $\{u\}Y$-arc); likewise for $Y = \{v\}$. If $\mathscr{C} = (x_0, x_1, \ldots, x_n)$ is a walk and there are integers i and j such that $0 \leq i < j \leq n$, then $x_i\mathscr{C}x_j$ will denote the subwalk $(x_i, x_{i+1}, \ldots, x_{j-1}, x_j)$ of $\mathscr{C}$. Union of walks will be denoted by concatenation or with $\cup$.

A digraph is *strongly connected* (or strong) if for every pair of vertices $u, v \in V(D)$, there exists a uv-path. A *strong component* (or component) of D is a maximal strong subdigraph of D. The *condensation* of D is the digraph $D^\star$ with $V(D^\star)$

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equal to the set of all strong components of D, and such that $(S, T) \in A(D^\star)$ if and only if there is an ST-arc in D. Clearly, $D^\star$ is an acyclic digraph, and thus it has vertices of out-degree equal to zero and vertices of in-degree equal to zero. A *terminal component* of D is a strong component T of D such that $d^+_{D^\star}(T) = 0$. An *initial component* of D is a strong component S of D such that $d^-_{D^\star}(S) = 0$.

A *tournament* is an orientation of a complete graph. An *m-partite tournament* is an orientation of a complete m-partite digraph. A *biorientation* of the graph G is a digraph obtained from G by replacing each edge $\{x, y\} \in E(G)$ by either the arc (x, y) or the arc (y, x) or the pair of arcs (x, y) and (y, x). A *semicomplete* digraph is a biorientation of a complete graph. A *semicomplete m-partite digraph* is a biorientation of a complete m-partite graph. A *complete* digraph on n vertices is a digraph obtained from K_n replacing each edge $xy \in E(K_n)$ by the arcs (x, y) and (y, x).

Let D be a digraph with vertex set $V(D) = \{v_1, \dots, v_n\}$ and $H_1, \dots, H_n$ a family of vertex disjoint digraphs. The *composition* $D[H_1, \dots, H_n]$ of digraphs D and $H_1, \dots, H_n$ is the digraph having $\bigcup_{i=1}^n V(H_i)$ as its vertex set and arc set $\bigcup_{i=1}^n A(H_i) \cup \left\{(u, v) : u \in V(H_i), v \in V(H_j), (v_i, v_j) \in A(D)\right\}$. The *dual* (or *converse*) of D, $\overleftarrow{D}$ is the digraph with vertex set $V(\overleftarrow{D}) = V(D)$ and such that $(u, v) \in A(\overleftarrow{D})$ if and only if $(v, u) \in A(D)$.

In the abstract, (k, l)-kernels and k-kernels have been already defined. A *kernel* is a 2-kernel. Kernels have been widely studied, a nice survey on the subject is [7]. Chvátal proved in [8] that recognizing digraphs that have a kernel is an NP-complete problem, so finding sufficient conditions for a digraph to have a kernel or finding large families of digraphs with kernel have been a very prosperous line of investigation explored by many authors. In [18], Borowiecki and Kwaśnik introduced the concept of (k, l)-kernels generalizing the notion of kernel. The notion of solution set of a digraph is dual to the notion of kernel, a *(k, l)-solution* is defined to be a k-independent, l-dominating set of vertices. It is clear that a digraph D has a (k, l)-kernel if and only if $\overleftarrow{D}$ has a (k, l)-solution. In [21,22,25,26] the existence of (k, l)-kernels in some products of digraphs is studied. Galeana-Sánchez and Hernández Cruz recently studied the existence of k-kernels in some large families of digraphs, *e.g*, quasi-transitive digraphs and pre-transitive digraphs [10], cyclically k-partite digraphs [11] and multipartite tournaments [12].

A *k-king* (resp. a *k-serf*) is a $(k + 1)$-solution (resp. $(k + 1)$-kernel) consisting of a single vertex. The problem of finding k-kings in digraphs have been largely studied for some classes of digraphs, the principal are multipartite tournaments and multipartite semicomplete digraphs [15–17,19,20,23], but also Bang-Jensen and Huang explored this problem for quasi-transitive digraphs in [3], proving in particular that a quasi-transitive digraph has a 3-king if and only if it has one unique terminal strong component.

Bang-Jensen and Huang proved in [4] a very nice structural characterization of quasi-transitive digraphs in terms of compositions of digraphs. This characterization can be used to prove that, for instance, the longest path and cycle problems are polynomial time solvable in this family, or to characterize Hamiltonian and traceable quasi-transitive digraphs. Also, simple sufficient conditions have been found for quasi-transitive digraphs to have a kernel in [13] and, in [10], it was proved that quasi-transitive digraphs have k-kernel for every integer $k \geq 3$. So, quasi-transitive digraphs is a good family to verify the behavior of difficult problems. We introduce k-quasi-transitive digraphs as a natural generalization of quasi-transitive digraphs, and prove that some of the existing results for quasi-transitive digraphs can be generalized. This leads us to conjecture that other results and properties of quasi-transitive digraphs can be generalized as well.

In this paper we prove that every k-transitive digraph has an n-kernel for every $n \geq k$, which was to be expected from the case when $k = 2$, but maybe the most interesting family of the three we introduce is the k-quasi-transitive digraphs. We prove that if k is even, D is a k-quasi-transitive digraph and $u, v \in V(D)$ such that $d(u, v) \geq k + 2$, then $d(v, u) = 1$; this result is used to prove that for even k, every k-quasi-transitive digraph has an n-kernel for every $n \geq k + 2$ and that a k-quasi-transitive digraph has a $(k+1)$-king if and only if it has a unique initial strong component, generalizing the result of Bang-Jensen and Huang. For odd k, we have that if D is a k-quasi-transitive digraph, $u, v \in V(D)$ such that $d(u, v) = n \geq k+2$ then, $d(v, u) = 1$ if n is odd and $d(v, u) \leq 2$ if n is even; this result is used to prove that, with an additional hypothesis, for odd k, k-quasi-transitive digraphs have n-kernel for $n \geq k + 2$. We also prove that every 3-quasi-transitive strong digraph has a k-kernel for every $k \geq 4$.

2. *k*-path-transitive digraphs

We begin this work introducing a quite simple family of digraphs that will be used as a tool to prove some results in the next sections.

Definition 2.1. A digraph D is called *k-path-transitive* if whenever there are a uv-path of length less than or equal to k and a vw-path of length less than or equal to k, then there exists a uw-path of length less than or equal to k.

Lemma 2.2. *A digraph D is k-path-transitive if and only if whenever $u, v \in V(D)$ and there exists a uv-path in D, then $d(u, v) \leq k$.*

Proof. First let D be a k-path-transitive digraph, $u, v \in V(D)$ two arbitrary distinct vertices and $\mathscr{C} = (u = x_0, x_1, \dots, x_n = v)$ a uv-path in D. We will prove by induction on n that $d(u, v) \leq k$. If $n \leq k$ then we are done. Let us assume that the result holds for every $m < n$ and consider the uv-path $\mathscr{C}$ of length $n \geq k + 1$. Clearly (x_0, x_1) is an x_0x_1-path of length $\leq k$ and $x_1 \cdots x_{k+1}$ is an x_1x_{k+1}-path of length less than or equal to k, then, by the k-path-transitivity of D there must exist an

x_0x_{k+1}-path of length $\leq k$, let us say, $\mathscr{C}'$. So $\mathscr{C}' \cup x_{k+1}\mathscr{C}x_n$ is an x_0x_n-path of length less than n and by induction hypothesis it follows that $d(u, v) \leq k$.

Now, let D be a digraph such that whenever $u, v \in V(D)$ and there exists a uv-path in D, then $d(u, v) \leq k$. Let $\mathscr{C}$ and $\mathscr{D}$ be uv and vw-paths of length less than or equal to k, then $\mathscr{C} \cup \mathscr{D}$ is a uw-path in D so $d(u, w) \leq k$ and a uw-path of length less than or equal to k exists. □

To prove the next theorem we need the definition of another generalization of the concept of kernel. If D is a digraph, a subset $S \subseteq V(D)$ is *independent by paths* if whenever $u, v \in S$ then there does not exist uv-paths neither vu-paths in D; and it is *absorbent by paths* if whenever $u \in V(D) - S$, there exists a vertex $v \in S$ such that a uv-path exists in D; the set S is a *kernel by paths* if it is both independent and absorbent by paths. Berge proved that every digraph has a kernel by paths, a proof of this fact can be consulted in [5], and as an obvious dual result it can be derived that every digraph has a solution by paths.

Theorem 2.3. *If D is a k-path transitive digraph then D has an n-kernel for every $n \geq k + 1$.*

Proof. It suffices to choose a kernel by paths of D, let us say N, we affirm than N is also an n-kernel. It is clearly n-independent for every $n \geq k$ because N is independent by paths. Now, let $u \in V(D) - N$ be an arbitrary vertex in the complement of N, then there is a uv-path for some $v \in N$, because N is absorbent by paths, but in virtue of Lemma 2.2, there is also a uv-path of length less than or equal to k, so N is $(n - 1)$-absorbent for every $n \geq k + 1$. Thence, N is an n-kernel for D. □

The particular case of k-kings is considered in the next theorem.

Theorem 2.4. *Let D be a k-path transitive digraph, then D has a k-king if and only if D has a unique initial strong component. Moreover, every vertex in the unique initial strong component of D is a k-king.*

Proof. If D has a k-king v, then the component that contains v is clearly the unique initial strong component of the digraph. If D has a unique initial strong component, is suffices to choose any vertex in such component, this vertex is a solution by paths and hence a k-king in virtue of Lemma 2.2. □

3. *k*-transitive digraphs

The next definition generalizes the definition of transitive digraphs.

Definition 3.1. A digraph D is *k-transitive* if whenever $\mathscr{C} = (x_0, x_1, \ldots, x_k)$ is a path of length k in D, then $(x_0, x_k) \in A(D)$.

Lemma 3.2. *Let $k \geq 2$ be an integer. If D is a k-transitive digraph, then D is $(k - 1)$-path-transitive.*

Proof. Let $u, v \in V(D)$ be arbitrary distinct vertices and let $\mathscr{C} = (u = x_0, x_1, \ldots, x_n = v)$ be a uv-path. We will prove by induction on n that $d(u, v) \leq k - 1$. If $n \leq k - 1$ then we are done. So let us assume that $n \geq k$, then, by the k-transitivity of D, since $x_0\mathscr{C}x_k$ is a path of length k in D, $(x_0, x_k) \in A(D)$, so $(x_0, x_k) \cup x_k\mathscr{C}x_n$ is a uv-path of length strictly less than n, we can derive from the induction hypothesis that $d(u, v) \leq k - 1$. The result follows from the principle of mathematical induction and Lemma 2.2. □

Just like the transitive case, the k-transitive case is very simple to analyze, at least the obvious generalization of the theorem that affirm that if D is a 2-transitive digraph, then D has a 2-kernel, which can be found in [5].

Theorem 3.3. *Let $k \geq 2$ be an integer. If D is a k-transitive digraph, then D has an n-kernel for every $n \geq k$.*

Proof. It follows immediately from Lemma 3.2 and Theorem 2.3. □

And once again, the particular case of k-kings.

Theorem 3.4. *Let D be a k-transitive digraph, then D has a $(k - 1)$-king if and only if D has a unique initial strong component. Moreover, every vertex in the unique initial strong component of D is a $(k - 1)$-king.*

Proof. It is clear from Lemma 3.2 and Theorem 2.4. □

Let us make the rather obvious observation that a digraph D is k-transitive if and only if $\overleftarrow{D}$ is k-transitive, so every result for k-kernels has a dual for k-solutions, and the same is true for k-kings and k-serfs.

Thus, since our main interest is to find families of digraphs with k-kernel, we only present a simple exploration of both the k-path-transitive and k-transitive digraphs, but considering the rich structure of transitive digraphs, a lot of questions arise concerning the structure of both strong and non-strong k-transitive digraphs. It is clear that transitive strong digraphs are complete digraphs, and that the condensation of a transitive digraphs is again a transitive digraph. However, this is not true for k-transitive digraphs, k-transitive strong digraphs are not complete digraphs and the condensation of a k-transitive digraph is not k-transitive, but k-path-transitive. So is a natural question to ask if k-transitive digraphs have a nice structural characterization. At least is easy to observe that for every $k \geq 2$, a k-transitive strong digraph have diameter $\leq k - 1$. Also, what happens to the n-kernels for $n \leq k$ in k-transitive digraphs, is k the least integer such that every k-transitive digraph has a k-kernel? We think that these are two interesting problems.

4. *k*-quasi-transitive digraphs

Among the families we introduce in this work, k-quasi-transitive digraph seem to be the most interesting one. At least for us, the most intuition-defying results were obtained for this family.

Definition 4.1. A digraph D is called *k-quasi-transitive* if, whenever $(x_0, x_1, \ldots, x_k)$ is a path of length k, then $(x_0, x_k) \in A(D)$ or $(x_k, x_0) \in A(D)$.

From the definition above it is clear that a quasi-transitive digraph in the usual sense is a 2-quasi-transitive digraph. Also, 3-quasi-transitive digraphs have been studied in [1,9,24].

Lemma 4.2. *Let D be a digraph, then D is a k-quasi-transitive digraph if and only if $\overleftarrow{D}$ is a k-quasi-transitive digraph.*

Proof. It suffices to observe that, if $(x_0, \ldots, x_k)$ is a path in D, then $(x_k, \ldots, x_0)$ is a path in $\overleftarrow{D}$, and in either case $(x_0, x_k) \in A(D)$ or $(x_k, x_0) \in A(D)$. $\square$

Proceeding as Bang-Jensen in the study of quasi-transitive digraphs we propose the following lemmas.

Lemma 4.3. *Let $k \in \mathbb{N}$ be an even natural number, D a k-quasi-transitive digraph and $\mathscr{C} = (x_0, x_1, \ldots, x_{k+3})$ a path such that $d(x_0, x_{k+3}) = k+3$ and $(x_{k+3}, x_1) \in A(D)$, then $(x_{k+3}, x_{k-2i}) \in A(D)$ for every $0 \le i \le \frac{k}{2}$. In particular $(x_{k+3}, x_0) \in A(D)$.*

Proof. By induction on i. For the base case, let $i = 0$, then $(x_{k+3}, x_1) \cup x_1 \mathscr{C} x_k$ is clearly an $x_{k+3}x_k$-path of length k. Since D is k-quasi-transitive then $(x_{k+3}, x_k) \in A(D)$ or $(x_k, x_{k+3}) \in A(D)$, but $d(x_0, x_{k+3}) = k+3$, so $(x_k, x_{k+3}) \notin A(D)$ and therefore $(x_{k+3}, x_k) \in A(D)$.

For the induction step, let us assume that $(x_{k+3}, x_{k-2i}) \in A(D)$ for every $0 \le i < n \le \frac{k}{2}$. Clearly $\mathscr{C}' = (x_{k+3}, x_{k-2(n-1)}) \cup x_{k-2(n-1)} \mathscr{C} x_k \cup (x_k, x_0) \cup x_0 \mathscr{C} x_{k-2n}$ is a path of length k, and therefore $(x_{k+3}, x_{k-2n}) \in A(D)$ or $(x_{k-2n}, x_{k+3}) \in A(D)$, but since $d(x_0, x_{k+3}) = k+3$, then $(x_{k-2n}, x_{k+3}) \notin A(D)$ which implies that $(x_{k+3}, x_{k-2n}) \in A(D)$.

The desired result now follows from the principle of mathematical induction. $\square$

Lemma 4.4. *Let $k \in \mathbb{N}$ be an even natural number, D a k-quasi-transitive digraph and $u, v \in V(D)$ such that a uv-path exists. Then:*

1. *If $d(u, v) = k$, then $d(v, u) = 1$.*
2. *If $d(u, v) = k+1$, then $d(v, u) \le k+1$.*
3. *If $d(u, v) \ge k+2$, then $d(v, u) = 1$.*

Proof. 1. Let $\mathscr{C} = (u = x_0, x_1, \ldots, x_k = v)$ be a path in D that realizes the distance from u to v. Since D is k-quasi-transitive, then $(u, v) \in A(D)$ or $(v, u) \in A(D)$, but $d(u, v) = k$, so $(u, v) \notin A(D)$, therefore $(v, u) \in A(D)$.

2. Let $\mathscr{C} = (u = x_0, x_1, \ldots, x_k, x_{k+1} = v)$ be a path in D that realizes the distance from u to v. From the k-quasi-transitivity and the fact that $d(u, v) = k+1$ it follows that $(x_{k+1}, x_1), (x_k, x_0) \in A(D)$. Observe that $(x_{k+1}, x_1) \cup x_1 \mathscr{C} x_k \cup (x_k, x_0)$ is a vu-path of length $k+1$ implying $d(v, u) \le k+1$.
3. By induction on $n = d(u, v)$. Let $\mathscr{C} = (u = x_0, x_1, \ldots, x_n = v)$ be a path in D. If $n = k+2$, then by the k-quasi-transitivity of D we have that $(x_k, x_0), (x_n, x_2) \in A(D)$. Now, let $\mathscr{C}' = (x_n, x_2) \cup x_2 \mathscr{C} x_k \cup (x_k, x_0)$ be a path in D. It is clear that the length of $\mathscr{C}'$ is k, and by the k-quasi-transitivity of D it follows that $(x_n, x_0) \in A(D)$. So the base case holds.

   If $n = k+3$, then by the base case and Lemma 4.3, we have that $d(v, u) = 1$. So we can assume that $n > k+3$ and by the induction hypothesis we can deduce that $(x_n, x_2) \in A(D)$. It is clear that $\mathscr{C}' = (x_n, x_2) \cup x_2 \mathscr{C} x_k \cup (x_k, x_0)$ is a vu-path of length k. From the k-quasi-transitivity of D we can deduce that $(v, u) \in A(D)$.

The result now follows from the principle of mathematical induction. $\square$

Lemma 4.5. *Let $k \in \mathbb{N}$ be an odd natural number, D a k-quasi-transitive digraph and $u, v \in V(D)$ such that a uv-path exists. Then:*

1. *If $d(u, v) = k$, then $d(v, u) = 1$.*
2. *If $d(u, v) = k+1$, then $d(v, u) \le k+1$.*
3. *If $d(u, v) = n \ge k+2$ with n odd, then $d(v, u) = 1$.*
4. *If $d(u, v) = n \ge k+3$ with n even, then $d(v, u) \le 2$.*

Proof. 1. As in Lemma 4.4.

2. As in Lemma 4.4.
3. Will be proved along with (4).

4. By induction on $n = d(u, v)$. For the case $n = k + 2$ the proof is as in Lemma 4.4. So, to complete the base case let us consider the case $n = k + 3$. Let $\mathscr{C} = (u = x_0, x_1, \ldots, x_n = v)$ be a path. By the case $n = k + 2$ we know that $(x_n, x_1) \in A(D)$ and, clearly $(x_n, x_1) \cup x_1\mathscr{C}x_k$ is an x_nx_k-path of length k, so by the k-quasi-transitivity of D, we conclude that $(x_n, x_k) \in A(D)$. Also from (1) we know that $(x_k, x_0) \in A(D)$, so (x_n, x_k, x_0) is a vu-path of length 2 and $d(v, u) \leq 2$.
   For the induction step let us assume that $n > k + 3$ and that $\mathscr{C} = (u = x_0, x_1, \ldots, x_n = v)$ is a path. If n is odd, then by induction hypothesis $(x_n, x_2) \in A(D)$, also we know that $(x_k, x_0) \in A(D)$, so $(x_n, x_2) \cup x_2\mathscr{C}x_k \cup (x_k, x_0)$ is an x_nx_0-path of length k. By the k-quasi-transitivity of D we have $(x_n, x_0) \in A(D)$. If n is even, then by induction hypothesis $(x_n, x_1) \in A(D)$. So $(x_n, x_1) \cup x_1\mathscr{C}x_k$ is an x_nx_k-path of length k and it follows from the k-quasi-transitivity of D that $(x_n, x_k) \in A(D)$. Once again, (x_n, x_k, x_0) is a path of length 2 and therefore $d(v, u) \leq 2$. This completes the induction step and the result follows from the principle of mathematical induction. □

Our next lemma also resembles a result obtained by Bang-Jensen in the study of quasi-transitive digraphs, although we are unable to characterize k-quasi-transitive non-strong digraphs, a nice behavior is observed in the condensation of a k-quasi-transitive digraph. We will say that a vertex u is *k-absorbed* (*k-dominated*) by the vertex v if $d(u, v) \leq k$ ($d(v, u) \leq k$).

Definition 4.6. If D is a digraph, A and B are strong components of D, we denote by $A \overset{k}{\to} B$ the fact that every vertex of B is k-dominated by every vertex of A.

Definition 4.7. Let D be a digraph. The k condensation of D is the digraph $D_k^\star$ such that $V(D_k^\star)$ is the set of strong components of D, and if A and B are strong components of D, then $(A, B) \in A(D_k^\star)$ if and only if there is an AB-path of length less than or equal to k in D.

Lemma 4.8. *Let D be a k-quasi-transitive digraph. If $A \neq B$ are strong components of D such that there exists an AB-path in D, then $A \overset{k-1}{\to} B$.*

Proof. Since there exists a AB-path in D, then for every $u \in V(A)$ and $v \in V(B)$ a uv-path exists. By Lemmas 4.4 and 4.5 it must be the case that $d(u, v) \leq k - 1$ for if not, there would exist a vu-path, which cannot happen because A and B are distinct strong components of D and a AB-path already exists. □

Lemma 4.9. *Let D be a k-quasi-transitive digraph, then the condensation $D^\star$ of D is k-path-transitive. Also, the $(k - 1)$-condensation $D_{k-1}^\star$ of D is transitive.*

Proof. Let D be a k-quasi-transitive digraph and $A, B \in V(D^\star)$ be two strong components of D such that there is a AB-path in D. Then, by Lemma 4.8, $A \overset{k-1}{\to} B$ and since $d_{D^\star}(A, B) \leq d_D(A, B)$ we have that $d_{D^\star}(A, B) \leq k - 1$. It follows from Lemma 2.2 that $D^\star$ is k-path-transitive.
   Now, let A, B and C be strong components of D such that $(A, B), (B, C) \in A(D_{k-1}^\star)$, then by Lemma 4.8, there exists an AC-path in D, and again by the same lemma, $A \overset{k-1}{\to} C$, thus $(A, C) \in A(D_{k-1}^\star)$. □

Let us remark that not only the $(k - 1)$-condensation of D is transitive, also we can think of D it in terms of some kind of "composition" over its $(k - 1)$-condensation in the next way. In virtue of Lemmas 4.8 and 4.9, $(u, v) \in A(D_{k-1}^\star)$ if and only if $u \overset{k-1}{\to} v$ in D. And clearly for $k = 2$, D is just a quasi-transitive digraph in the usual sense and Lemmas 4.8 and 4.9 are those obtained by Bang-Jensen stating that if $A \neq B$ are strong components of D such that there is an arc from A to B, then $A \to B$, and that any non-strong quasi-transitive digraph is a composition of strong quasi-transitive digraphs over a non-strong transitive digraph (its condensation).
   The next few lemmas are oriented to prove that every k-quasi-transitive digraph has a $(k + 2)$-kernel with even k. Also a sufficient condition will be stated for the same result to hold with odd k.

Lemma 4.10. *Let $k \geq 2$ be an integer and D be a k-quasi-transitive digraph. Then, for every integer $n \geq 2$ there does not exist a cycle $\mathscr{C}$ of length n in D such that, with at most one exception, for every* arc $(x, y) \in A(\mathscr{C})$ *holds that $d(y, x) \geq k + 1$.*

Proof. Let us proceed by induction on n and by contradiction in both the base case and the induction step. For the case $n \leq k + 1$ the result holds trivially.
   For the induction step let $n \geq k+2$ be an integer and $\mathscr{C} = (x_0, x_1, \ldots, x_n, x_0)$ a cycle of length n with the desired property. If there is an arc $(x, y) \in A(\mathscr{C})$ such that $d(y, x) \leq k$ we can assume without loss of generality that it is the arc (x_1, x_2), if there is no such arc the argumentation is the same. Since our only exception is the arc (x_1, x_2), then $d(x_1, x_0) \geq k + 1$, but D is k-quasi-transitive and $(x_0, x_1, \ldots, x_k)$ is a path of length k; if $(x_k, x_0) \in A(D)$ we would have a contradiction because $(x_1, \ldots, x_k, x_0)$ would be an x_1x_0-path of length k, so $(x_0, x_k) \in A(D)$ and therefore $\mathscr{C}' = (x_0, x_k) \cup x_k\mathscr{C}x_0$ is a cycle in which every arc $(x, y) \in A(\mathscr{C}')$, with the possible exception of (x_0, x_k), fulfills that $d(y, x) \geq k + 1$. But the length of $\mathscr{C}'$ is less than the length of $\mathscr{C}$, which is equal to n, and, by induction hypothesis, there are no cycles with this property and with length less than n, so a contradiction arises from the assumption of the existence of $\mathscr{C}$. We conclude that no such cycle of length n exists. □

Lemma 4.11. *Let $k \geq 2$ be an integer and D be a k-quasi-transitive digraph, then there exists a vertex $v \in V(D)$ such that whenever $(v, u) \in A(D)$, then $d(u, v) \leq k$.*

Proof. We will proceed by contradiction. Let us assume that for every vertex $v \in V(D)$ there exists an arc $(v, u) \in V(D)$ such that $d(u, v) \geq k + 1$. Then, since the subdigraph H of D induced by these arcs has $\delta^+(H) \geq 1$, then there exist a cycle $\mathscr{C}$ in D such that for every arc $(v, u) \in A(\mathscr{C})$, $d(u, v) \geq k + 1$, which clearly results in a contradiction by Lemma 4.10. □

For our next lemma we need an additional definition. If D is a digraph, a subset S of $V(D)$ is a *k-semikernel* of D if S is independent and for every vertex $v \in V(D) - S$ such that a vertex s exists in S with $d(s, v) \leq k - 1$, then there exists a vertex $s' \in S$ such that $d(v, s') \leq k - 1$.

Lemma 4.12. *Let $k \geq 2$ be an even integer and let D be a k-quasi-transitive digraph, then D has a $(k + 2)$-semikernel consisting in a single vertex.*

Proof. By Lemma 4.11 we can choose a vertex $v \in V(D)$ such that for every arc $(v, u) \in A(D)$, $d(u, v) \leq k$. So let $u \in V(D)$ be a vertex such that $2 \leq d(v, u) \leq k + 1$. It cannot happen that $d(u, v) \geq k + 2$, because this would imply by Lemma 4.4 that $d(v, u) = 1$, but $2 \leq d(v, u)$, so $d(u, v) \leq k + 1$ and thus $\{v\}$ is a $(k + 2)$-semikernel of D. □

A problem arose while working with the odd case since we could not find a good analog for Lemma 4.12 because, although almost the same proof can be done, we cannot assure that once we have chosen a vertex v such that for every arc (v, u) it follows that $d(u, v) \leq k + 1$, if we choose a vertex u such that $d(v, u) = 2$ then it will be the case that $d(u, v) \leq k + 1$ like in the even case.

So a weaker analog of Lemma 4.12 will be proposed and proved.

Lemma 4.13. *If $k \geq 3$ is an odd integer and D is a k-quasi-transitive digraph such that at least one vertex $v \in S = \{u \in V(D)\colon (u, w) \in A(D)$ implies that $d(w, u) \leq k + 1\}$ is such that whenever $d(v, x) = 2$ then $d(x, v) \leq k + 1$ then $\{v\}$ is a $(k + 2)$-semikernel for D.*

Proof. By Lemma 4.11 the set S is non-empty and also there is a vertex $v \in S$ such that whenever $d(v, x) = 2$ then $d(x, v) \leq k + 1$. So let $u \in V(D)$ be a vertex such that $3 \leq d(v, u) \leq k + 1$. It cannot happen that $d(u, v) \geq k + 2$, because this would imply by Lemma 4.4 that $d(v, u) \leq 2$, but $3 \leq d(v, u)$, so $d(u, v) \leq k + 1$ and thus $\{v\}$ is a $(k + 2)$-semikernel of D. □

At this point we have two possible courses of action. The one we will not follow is to prove directly that, for even k, whenever a k-quasi-transitive digraph has a $(k + 2)$-semikernel then it has a $(k + 2)$-kernel; this can be achieved by considering a $\subseteq$-maximal $(k + 2)$-semikernel and proving by means of contradiction that it is $(k + 1)$-absorbent. But, even though it is a more efficient way to prove this fact, this would not give any information about the structure of the $(k + 2)$-kernel. So, we will use a couple of lemmas (including Lemma 4.9) that will help us to know how a $(k + 2)$-kernel look like, we will begin proving the strong case.

Lemma 4.14. *Let D be a k-quasi-transitive strong digraph. If D has a non-empty $(k + 2)$-semikernel S, then S is a $(k + 2)$-kernel of D.*

Proof. Let $S \subseteq V(D)$ be a $(k + 2)$-semikernel for D and $N^-_{k+1}(S)$ the set of all vertices in D which are $(k + 1)$-absorbed by at least one vertex of S. Define $T := V(D) - \left(S \cup N^-_{k+1}(S)\right)$. If $T = \varnothing$, then S is a $(k + 2)$-kernel of D. If $T \neq \varnothing$, then we can consider a vertex $v \in T$ which, by the definition of T, is not $(k + 1)$-absorbed by S, but since D is strong, there exists a vS-path. Let $u \in S$ be a vertex such that $d(v, u) = d(v, S)$, then $d(v, u) \geq k + 2$ because $v \notin N^-_{k+1}(S)$, but from Lemmas 4.4 and 4.5 it can be derived that $d(u, v) \leq 2$. This fact, altogether with the second $(k + 2)$-semikernel condition implies that $v \in N^-_{k+1}(S)$ which results in a contradiction. Since the contradiction arises from assuming that $T \neq \varnothing$, we can conclude that $T = \varnothing$ and then S is a $(k + 2)$-kernel for D. □

The main results of the section and its consequences are now stated.

Theorem 4.15. *Let $k \geq 2$ be an even integer and let D be a k-quasi-transitive strong digraph, then D has an n-kernel for every $n \geq k + 2$.*

Proof. By Lemma 4.12, D has a $(k + 2)$-semikernel N consisting in a single vertex, but by Lemma 4.14, N is indeed a $(k + 2)$-kernel of D. But since N has only one vertex, then N is n-independent for every $n \geq k + 2$, and since it is $(k + 1)$-absorbent, then it is $(n - 1)$-absorbent for every $n \geq k + 2$, so N is an n-kernel for every $n \geq k + 2$. □

Theorem 4.16. *Let $k \geq 2$ be an even integer and let D be a k-quasi-transitive digraph, then D has an n-kernel for every $n \geq k + 2$.*

Proof. In virtue of Lemmas 4.9 and 4.15, it suffices to choose a subset $N \subseteq V(D)$ consisting in an n-kernel for every terminal component of D, this set will be n-independent for every $n \in \mathbb{Z}^+$ because every such n-kernel consist in a single vertex and terminal components are path-independent. Also N will be $(k + 1)$-absorbent because every n-kernel is inside its component and every vertex of D not in a terminal component is $(k - 1)$-absorbed by every vertex in some terminal component. □

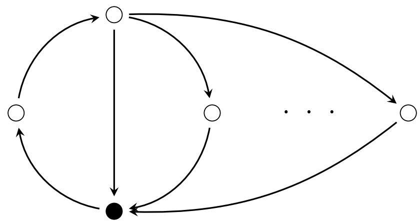

Fig. 1. The third family of 3-quasi-transitive digraphs.

Corollary 4.17. *Let $k \geq 2$ be an even integer and let D be a k-quasi-transitive digraph.*

1. *D has a $(k+1)$-serf if and only if it has a unique terminal component.*
2. *D has a $(k+1)$-king if and only if it has a unique initial component.*

Proof. Let us prove the first assertion and the second one will follow immediately from Lemma 4.2.

If D has finite out-radius, then D has a unique terminal strong component, so it suffices to pick a $(k+2)$-kernel there $\{v\}$. The result is clear from recalling that $\{v\}$ is a $(k+1)$-absorbing set, so v is the $(k+1)$-serf. □

At this point we want to remark that for every $k \geq 2$ we can find a k-quasi-transitive digraph that does not have a k-kernel, that is to say, the cycle of length $k+1$, C_{k+1}, which also is an example of a k-quasi-transitive digraph with a k-king rather than a $(k+1)$-king. Nevertheless we have been unable to find a k-quasi-transitive digraph that does not have a $(k+1)$-kernel, so the question remain open, and since every quasi-transitive digraph has a 3-kernel we are inclined to state the next conjecture.

Conjecture 4.18. *If $k \geq 2$ is an even integer and D is a k-quasi-transitive strong digraph, then D has a $(k+1)$-kernel.*

It suffices to consider the strong case, in virtue of Lemma 4.9 this would imply that every k-quasi-transitive strong digraph has a $(k+1)$-kernel.

Next, we deal with the odd case again. The following two theorems are analogs of Theorems 4.15 and 4.16.

Theorem 4.19. *Let $k \geq 3$ be an odd integer and let D be a k-quasi-transitive strong digraph such that at least one vertex $v \in S = \{u \in V(D)\colon (u,w) \in A(D)$ implies that $d(w,u) \leq k+1\}$ is such that whenever $d(v,x) = 2$ then $d(x,v) \leq k+1$, then D has an n-kernel for every $n \geq k+2$.*

Theorem 4.20. *Let $k \geq 3$ be an odd integer and let D be a k-quasi-transitive digraph such that at least one vertex $v \in S = \{u \in V(D)\colon (u,w) \in A(D)$ implies that $d(w,u) \leq k+1\}$ is such that whenever $d(v,x) = 2$ then $d(x,v) \leq k+1$, then D has an n-kernel for every $n \geq k+2$.*

Despite this fact, we were actually able to work out an odd case. As we have mentioned before, Galeana-Sánchez, Goldfeder and Urrutia successfully characterized the 3-quasi-transitive strong digraphs, their theorem goes as follows.

Theorem 4.21 (*Galeana-Sánchez et al. [9]*). *If D is a 3-quasi-transitive strong digraph, then D is either a semicomplete digraph or a bipartite semicomplete digraph or a digraph of the family depicted below.*

The dots in Fig. 1 indicate that any number of intermediate vertices can be added respecting the direction of the arcs.

Also, the authors of the present work have proved in [12] that every m-partite tournament has a k-kernel for every $m \geq 2$, $k \geq 4$. The exact same proof can be used to prove the next theorem.

Theorem 4.22 (*Galeana-Sánchez and Hernández-Cruz [12]*). *Every m-partite semicomplete digraph has a k-kernel for every $m \geq 2$, $k \geq 4$.*

Finally, it is a well known result that every semicomplete digraph has a k-kernel for every $k \geq 2$.

In view of this results we can deduce the following.

Theorem 4.23. *If D is a 3-quasi-transitive strong digraph, then D has a k-kernel for every $k \geq 4$.*

Proof. In virtue of Theorem 4.21, D is either a semicomplete digraph, or a bipartite semicomplete digraph, or a digraph of the third family depicted in Fig. 1. If D is semicomplete, then D has a k-kernel for every $k \geq 2$. If D is a bipartite semicomplete digraph, then by Theorem 4.22 it has a k-kernel for every $k \geq 4$. If D is a digraph of the third family, then it suffices to pick the filled vertex in Fig. 1; that vertex is clearly a k-kernel for every $k \geq 3$. □

Theorem 4.24. *If D is a 3-quasi-transitive digraph, then D has a k-kernel for every $k \geq 4$.*

Proof. Let $k \geq 4$ be an integer. Let $\{S_i\}_{i=i}^{k}$ be the set of terminal strong components of D and $N_i \subseteq S_i$ a k-kernel for S_i, $1 \leq i \leq k$, which exists by Theorem 4.23. It is clear from Lemma 4.9 that $N = \bigcup_{i=1}^{k} N_i$ is a k-kernel for D. The set N is clearly k-independent since each N_i is, and they are contained in terminal components. Also, every vertex not in a terminal component is 2-absorbed by every vertex in some terminal component. □

We can get again a corollary about k-kings and k-serfs.

Corollary 4.25. *Let D be a 3-quasi-transitive digraph and let $n \geq 2$ be an integer.*

- *D has an n-king if and only if D has finite in-radius and the terminal strong component of D has an n-king.*
- *D has an n-serf if and only if D has finite out-radius and the initial strong component of D has an n-serf.*

Proof. The proof is analog to the proof of Corollary 4.17. □

We would like to point out that it follows from Theorem 4.21 and Corollary 4.25 that a 3-quasi-transitive digraph with finite out-radius (in-radius) which initial (terminal) strong component is not a bipartite semicomplete digraph always have a 2-king (2-serf). Sufficient conditions for the existence of n-kings in the case when the digraph does not have a 2-king (2-serf) can be obtained from the extensive bibliography (*e.g.* [15–17,19,20,23]) about kings in multipartite semicomplete digraphs.

Recalling that the cycle of length 4 has no 3-kernel, the result of Theorem 4.24 is as good as it gets, resembling the case when $k = 2$. So, considering that from the case $k = 2$ we conjectured that for even k, every k-quasi-transitive digraph has a $(k + 1)$-kernel, we have two conjectures on the matter for the odd case.

Conjecture 4.26. *If $k \geq 3$ is an odd integer and D is a k-quasi-transitive strong digraph, then D has a $(k + 2)$-kernel.*

Conjecture 4.27. *If $k \geq 3$ is an odd integer and D is a k-quasi-transitive strong digraph, then D has a $(k + 1)$-kernel.*

The former would match the results obtained for the even case in this work, while the latter would match the results obtained for the case $k = 3$ for every odd integer.

5. Conclusions

Three new families of digraphs were introduced, k-path-transitive, k-transitive and k-quasi-transitive digraphs, the second and third of them generalizing transitive and quasi-transitive digraphs respectively. Altogether with this definition, some results were proved that help in a first instance to describe the structure of such digraphs, resembling analogous existing results for the original classes. Also, these results where useful to prove the existence of k-kernels in the three aforementioned families, nonetheless, the problem of existence of k-kernels was no completely solved and Conjectures 4.18, 4.26 and 4.27 were proposed.

The existence of k-kings in the three families were also boarded, obtaining some necessary and sufficient conditions for their existence, but we think that the structure of the families is rich enough to ask for the exact number of k-kings, as it has been done for multipartite semicomplete digraphs. In particular, we propose the problem of generalizing the next result, due to Bang-Jensen and Huang [3], to k-quasi-transitive digraphs. Corollary 4.17 is a partial generalization of 1. of the following theorem.

Theorem 5.1. *Let D be a quasi-transitive digraph. Then we have*

1. *D has a 3-king if and only if it has a finite out-radius.*
2. *If D has a 3-king, then the following holds:*
   (a) *Every vertex in D of maximum out-degree is a 3-king.*
   (b) *If D has no vertex of in-degree zero, then D has at least two 3-kings.*
   (c) *If the unique initial strong component of D contains at least three vertices, then D has at least three 3-kings.*

A graph is a *comparability graph* if it admits a transitive orientation. Quasi-transitive digraphs were introduced in [14] by Ghouila-Houri to characterize comparability graphs as those graphs that admit a quasi-transitive orientation, so, every asymmetrical quasi-transitive digraph can be reoriented into a asymmetrical transitive digraph. Another problem is to generalize the result of Ghouila-Houri.

Problem 5.2. Is it true that a graph G can receive a k-transitive orientation if and only if G can receive a k-quasi-transitive orientation?

And just like this couple of problems proposed, other problems originally solved for transitive or quasi-transitive digraphs can be generalized, maybe considering the even k case first, to k-transitive and k-quasi-transitive digraphs. As a matter of fact, although the introduced families are a fertile ground to work with dominating or absorbing structures, we think that, considering the properties of transitive and quasi-transitive digraphs, they may be also a good class to work another problems such as maximum length paths and cycles or Hamiltonicity.

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