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Korovkin Shadows and Korovkin Systems in C(S)-Spaces*

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INTRODUCTION

The theory of Korovkin sets has attracted widespread attention in recent years (several dozens of papers have been published in the last decade), due to the combination of the simplicity of the conditions and the power of their implications. The scope of the theory, which was originally established (by Bohman and Korovkin) (see e.g. [14]) for positive linear functionals and operators on the spaces $C[a, b]$ of continuous functions on $[a, b]$ and $C_{2\pi}$ of continuous periodic functions, has been significantly enlarged.

The setting for a general Korovkin-type theorem is a class $\mathscr{L}$ of mappings from a set X to a bitopological space $(Y, \mathscr{T}_1, \mathscr{T}_2)$, a class $\mathscr{D}$ of nets in $\mathscr{L}$, and a subclass $\mathscr{L}_0 \subset \mathscr{L}$.

A set $F \subset X$ is called a Korovkin-set (or Korovkin-system) with respect to $\{\mathscr{L}, X, Y, \mathscr{T}_1, \mathscr{T}_2, \mathscr{D}, \mathscr{L}_0\}$ when for each net $\{T_\alpha\} \in \mathscr{D}$, and $T_0 \in \mathscr{L}_0$, the relation

$$T_\alpha x \xrightarrow{\mathscr{T}_1} T_0 x \qquad \textit{for every } x \in F \tag{1.1}$$

entails

$$T_\alpha x \xrightarrow{\mathscr{T}_2} T_0 x \qquad \textit{for every } x \in X. \tag{1.2}$$

A newly defined concept related to the previous one is the concept of a "Korovkin shadow" (see, e.g. [13], [5] and [25]):

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Given any $F \subset X$, the K-shadow (Korovkin shadow) of F with respect to $\{\mathscr{L}, X, Y, \mathscr{T}_1, \mathscr{T}_2, \mathscr{D}, \mathscr{L}_0\}$ is the largest subset $G \subset X$ for which (1.1) *entails*

$$T_\alpha x \xrightarrow{\mathscr{T}_2} T_0 x \qquad \textit{for every } x \in G \tag{1.3}$$

for all nets $\{T_\alpha\} \in \mathscr{D}$.

Thus, each set is a K-set (Korovkin set) for its shadow, and a set F is a K-set for X if and only if X is the K-shadow of F.

It is our intent to analyze properties of K-shadows in several naturally arising circumstances and explore the interrelationship between such shadows.

In the present paper we confine attention to two closely related cases.

(1) $X = Y = C(S)$, where S is a compact Hausdorff space and $C(S)$ is the space of continuous real valued functions on S.

$\mathscr{L} = \mathscr{L}^+ =$ the class of bounded, positive linear operators

$\mathscr{L}_0 = \{I\} =$ the identity operator.

(2) $X = C(S)$, $Y = \mathscr{R}$ (= the reals)

$\mathscr{L} = \mathscr{M}^+ =$ the class of positive linear functionals

$\mathscr{L}_0 = \{\nu_{s_0}; s_0 \in S\}$, where ν_{s_0} is the evaluation functional at the point s_0.

Since we have not specified the topologies $\mathscr{T}_1$ and $\mathscr{T}_2$, and may on the other hand impose restrictions on the types of nets to be considered, the basic structures outlined above give rise to several natural settings, defining (possibly) different K-shadows of a given $F \subset C(S)$:

The following notation will be used:

$\mathscr{N}$ — the class of all nets in the given space

$\mathscr{N}_B$ — the class of uniformly bounded nets in the given space

$\mathscr{S}$ — the class of all sequences in the given space

$\mathscr{S}_B$ — the class of uniformly bounded sequences in the given space

$\mathscr{J}$ — the class of all constant nets in the given space

u — uniform convergence

p — pointwise convergence.

DEFINITION 0.1. With these notations, the various K-shadows can be described via the following list of statements. Each statement defines the corresponding K-shadow of F as the set of all g for which the implication holds

(1) $(\mathscr{L}^+, \mathscr{N}, u, u)$; $\{T_\alpha\} \in \mathscr{N}$, $T_\alpha f \rightrightarrows f$ for all $f \in F \Rightarrow T_\alpha g \rightrightarrows g$ (uniformly)

(2) $(\mathscr{L}^+, \mathscr{S}, u, u)$; $\{T_n\} \in \mathscr{S}$, $T_n f \rightrightarrows f$ for all $f \in F \Rightarrow T_n g \rightrightarrows g$.

The statements

(3) $(\mathscr{L}^+, \mathscr{N}_B, u, u)$

(4) $(\mathscr{L}^+, \mathscr{S}_B, u, u)$

are similarly written, with $\mathscr{N}_B$, $\mathscr{S}_B$ replacing $\mathscr{N}$ and $\mathscr{S}$, respectively.

(5) $(\mathscr{L}^+, \mathscr{N}, u, p)$; $\{T_\alpha\} \in \mathscr{N}$, $T_\alpha f \rightrightarrows f$ for all $f \in F \Rightarrow T_n g \to g$ (pointwise).

The statements

(6) $(\mathscr{L}^+, \mathscr{S}, u, p)$

(7) $(\mathscr{L}^+, \mathscr{N}_B, u, p)$

(8) $(\mathscr{L}^+, \mathscr{S}_B, u, p)$

are written in an analogous fashion. The following quartet (9)–(12) obtains via similar variations on relation (9)

(9) $(\mathscr{L}^+, \mathscr{N}, p, p)$; $\{T_\alpha\} \in \mathscr{N}$, $T_\alpha f \to f$ for all $f \in F \Rightarrow T_\alpha g \to g$

(10) $(\mathscr{L}^+, \mathscr{S}, p, p)$

(11) $(\mathscr{L}^+, \mathscr{N}_B, p, p)$

(12) $(\mathscr{L}^+, \mathscr{S}_B, p, p)$.

The following statements involve functionals; here T_0 is a fixed evaluation functional.

(13) $(\mathscr{M}^+, \mathscr{N})$; $\{\mu_\alpha\} \in \mathscr{N}$, $\mu_\alpha f \to T_0 f$ for all $f \in F \Rightarrow \mu_\alpha g \to T_0 g$.

The statements (14)–(16) are similarly written

(14) $(\mathscr{M}^+, \mathscr{S})$

(15) $(\mathscr{M}^+, \mathscr{N}_B)$

(16) $(\mathscr{M}^+, \mathscr{S}_B)$

(17) $\{\mathscr{L}^+, \mathscr{I}\}$; T, $Tf \equiv f$ for all $f \in F \Rightarrow Tg \equiv g$

(18) $\{\mathscr{M}^+, \mathscr{I}\}$; μ, $\mu f = f(s_0)$ for all $f \in F \Rightarrow \mu g = g(s_0)$.

In the first section we establish the implication relations between the various statements. This is equivalent to establishing inclusion relations between the corresponding shadows.

In the second section we examine the relationship between the shadow of hF and the shadow of F, where h is a nonnegative function.

In Section 3 we prove that certain familiar subspaces of $C(S)$ (including, among others, subspaces of dimension $\leqslant 2$ and Grothendieck subspaces) cast no $(\mathscr{M}^+, \mathscr{I})$-shadow, i.e., the $(\mathscr{M}^+, \mathscr{I})$-shadow is identical with the subspace. We conclude also that every Banach space is isometric to an $(\mathscr{M}^+, \mathscr{I})$-shadow.

In Section 4 we show that the shadows containing the function 1 are, isometrically, all the spaces $A(K)$ of affine continuous functions on a compact convex K, and derive other equivalent characterizations.

We establish, in Section 5, intrinsic characterizations of a K-subspace F in terms of the structure of the unit ball of the dual to the Banach space F. There are two different characterizations, one for K-subspaces containing the function 1, and the other for general K-subspaces.

Section 6 is devoted to a study of K-shadows of finite subsets. In particular, we obtain a lower bound on the codimension of shadows of the spaces of non-Tchebycheffian triplets.

Section 7 contains some observations concerning the minimal cardinality of finite K-sets. It is proved that there exists a minimal finite dimensional K-subspace (a K-subspace containing no proper K-subspace) whose dimension is larger than the minimal cardinality. Several conjectures are made about minimal K-subspaces.

Almost all sections leave some natural questions unanswered. We formulate some of these as problems, with the hope that in such an active field, they will be solved soon.

1. The Basic Relationship between K-Shadows

We start by establishing the basic implication relation between the statements (1)–(18) in the general $C(S)$ case. Then we proceed to analyze the additional implications that are present under various assumptions on the space or on the set F. We note that implications between statements are equivalent to inclusion relations (in the same direction) between the corresponding K-shadows. In most cases, we present the implications in the form of charts, as this seems to be the most economical way.

THEOREM 1.1. *Let $X = C(S)$, where S is a compact Hausdorff space, and let $\mathscr{L}^+$, $\mathscr{M}^+$ be the classes of bounded positive linear operators from $C(S)$ into itself and positive linear functionals, respectively. Let the statements be defined by Definition* 0.1. *Then the relations between the statements depicted in Fig.* 1 *are valid.*

THEOREM 1.2. *Let the general structure be as in Theorem* 1.

(a) *If* span F (= *the linear subspace spanned by* F) *contains a positive function then the relations depicted in Fig.* 2 *are valid.*

(b) *If the weak topology induced by F on S is first countable* (*for example, if S is first countale or if* dim span $F \leqslant \aleph_0$), *then in addition to the implications of Fig.* 1, *we have*

$$(\mathscr{L}^+, \mathscr{S}_B, u, p) \Rightarrow (\mathscr{M}^+, \mathscr{S}_B)$$
$$(\mathscr{L}^+, \mathscr{S}, u, p) \Rightarrow (\mathscr{M}^+, \mathscr{S})$$

Hence, the relations depicted in Fig. 3 *are valid.*

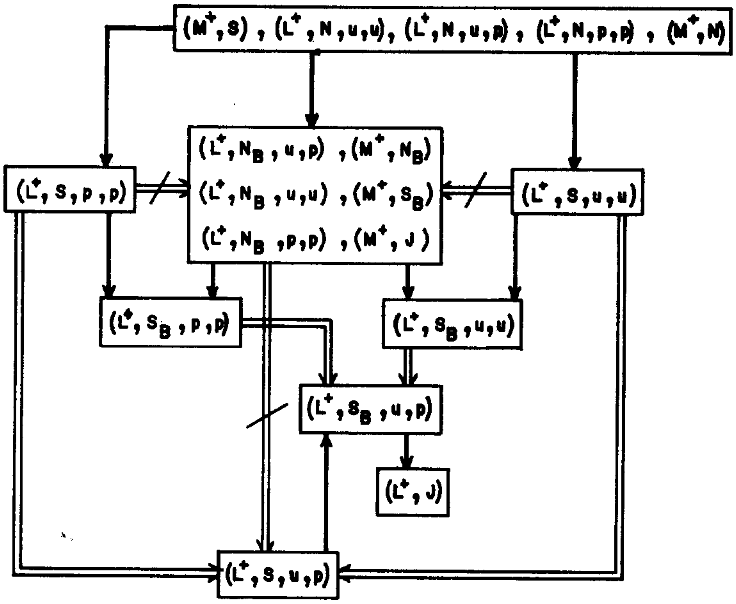


FIG. 1. The implications in the general $C(S)$ case. *Remarks*: (1) $\rightarrow$ means that the inverse implication is false. (2) $\not\Rightarrow$ means that the implication is false. (3) All statements in a given box are equivalent.

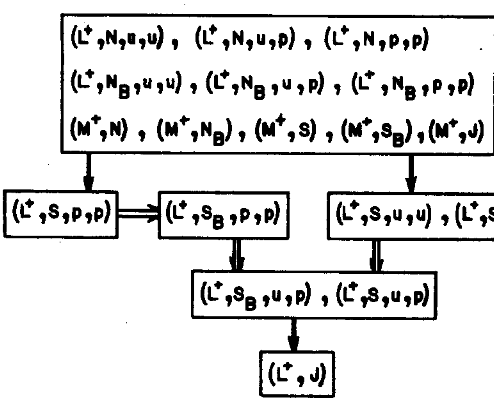


FIG. 2. The implications in the case span F contain a positive function.

(c) *If the weak topology induced by F on S is first countable and* span F *contains a positive function, then all statements except* $(\mathscr{L}^+, \mathscr{J})$ *are equivalent.*

Remark 1.3. Note that if F is $(\mathscr{M}^+, \mathscr{J})$-Korovkin (i.e., if the $(\mathscr{M}^+, \mathscr{J})$ shadow is X), then, by Šaškin's result [20] span F contains a positive function, so that (a) is valid.

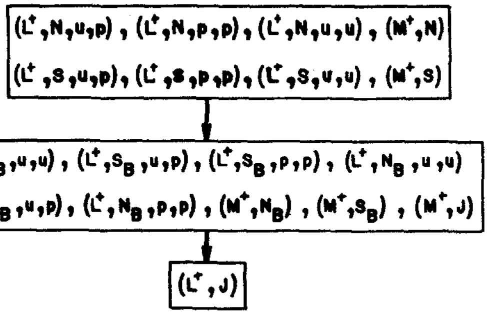
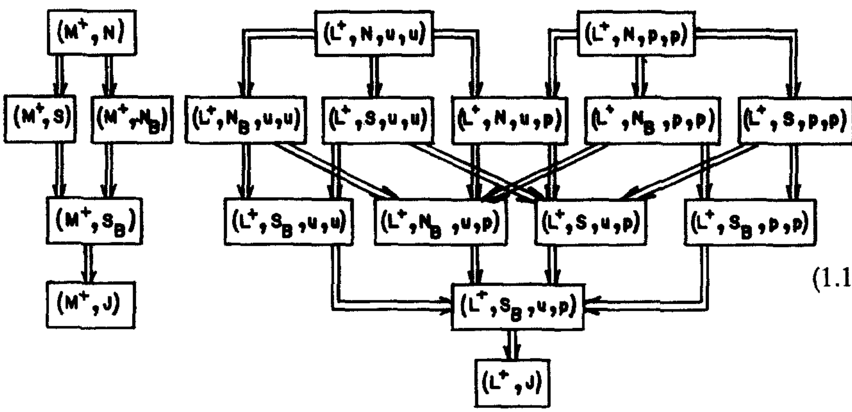


FIG. 3. The implications for the case where the weak topology induced by F on S is first countable.

We prove Theorem 1.1, and use the additional information, in the course of the proof, to prove Theorem 1.2.

Proof. The following implications are straightforward and do not necessitate any discussion.



(1.1)

We first set out to prove that

$$
\begin{aligned}
(\mathscr{M}^+, \mathscr{N}) &\Rightarrow (\mathscr{L}^+, \mathscr{N}, u, u) \\
(\mathscr{M}^+, \mathscr{N}_B) &\Rightarrow (\mathscr{L}^+, \mathscr{N}_B, u, u)
\end{aligned} \tag{1.2}
$$

We need the following simple lemma, which will be used again in the sequel.

LEMMA 1.4. (a) *The net $\{f_\alpha; \alpha \in A\}$ in $C(S)$ converges uniformly to f if and only if the following statement is valid: If $\{g_\beta; \beta \in E\}$ is any subnet of the given net such that the corresponding net in S, $\{s_\beta; \beta \in E\}$ satisfies*

$$s_\beta \to s \tag{1.3}$$

we have

$$g_\beta(s_\beta) \to f(s). \tag{1.4}$$

(b) *If S is sequentially compact, the analogous result for sequences and subsequences is valid.*

Proof (*of Lemma* 1.4). If the convergence is uniform, then for any subnet satisfying (1.3) the validity of (1.4) is a corollary of the inequality

$$| g_\beta(s_\beta) - f(s)| \leqslant | g_\beta(s_\beta) - f(s_\beta)| + | f(s_\beta) - f(s)| .$$

Assume now that the convergence is not uniform. Then there exists an $\epsilon > 0$ such that for all $\alpha \in A$ there exist a $\gamma(\alpha) \in A$, $\gamma(\alpha) \geqslant \alpha$, and an $s_\alpha \in S$ such that

$$| f_{\gamma(\alpha)}(s_\alpha) - f(s_\alpha)| \geqslant \epsilon. \tag{1.5}$$

Let $g_\alpha = f_{\gamma(\alpha)}$. Then $\{g_\alpha ; \alpha \in A\}$ is a subnet. Choose a convergent subnet $\{s_\beta ; \beta \in E\}$ from $\{s_\alpha ; \alpha \in A\}$, $s_\beta \to s$. Using (1.5) we have

$$\overline{\lim} \, | g_\beta(s_\beta) - f(s)| \geqslant \epsilon$$

contradicting (1.4). This completes the proof of (a).

(b) The proof is analogous, and can be found in [16]. Q.E.D.

Using the lemma, we proceed to prove (1.2). It suffices to prove the first relation, as the second is completely analogous. Assume $(\mathscr{M}^+, \mathscr{N})$ for g, and let $\{T_\alpha ; \alpha \in A\}$ be a net such that $T_\alpha f \rightrightarrows f$ for all $f \in F$. Let $\{T_\beta ; \beta \in E\}$ be a subnet such that $s_\beta \to s$. Then, by the lemma,

$$T^*{}_\beta \nu_{s_\beta}(f) = T_\beta f(s_\beta) \to f(s) \qquad \text{for all } f \in F. \tag{1.6}$$

Since $T^*{}_\beta \nu_{s_\beta}$ is a positive linear functional, we deduce from (1.6) and $(\mathscr{M}^+, \mathscr{N})$ that

$$T_\beta g(s_\beta) = T^*{}_\beta \nu_{s_\beta}(g) \to g(s). \tag{1.7}$$

Since this is true for all subnets T_β satisfying (1.3), another application of the lemma yields now $T_\alpha g \rightrightarrows g$. Q.E.D.

The next step is the proof that

$$\begin{aligned} (\mathscr{L}^+, \mathscr{N}, u, p) &\Rightarrow (\mathscr{M}^+, \mathscr{N}) \\ (\mathscr{L}^+, \mathscr{N}_B, u, p) &\Rightarrow (\mathscr{M}^+, \mathscr{N}_B). \end{aligned} \tag{1.8}$$

Assume $(\mathscr{L}^+, \mathscr{N}, u, p)$ for g. Suppose now that $(\mathscr{M}^+, \mathscr{N})$ does not hold. Then there exists a net $\{\mu_\alpha ; \alpha \in A\}$ such that

$$\mu_\alpha f \to f(s_0) \qquad \text{for all } f \in F$$

but

$$\mu_\alpha(g) \not\to g(s_0). \tag{1.9}$$

Let $\{W_\beta; \beta \in E\}$ be a basis for the neighborhood system of the F-topology at s_0, directed by inclusion. Consider the net $\{T_{\alpha,\beta}; (\alpha, \beta) \in A \times E\}$ of positive operators on $C(S)$, defined by

$$T_{\alpha,\beta}\phi = (1 - u_\beta)\phi + \mu_\alpha(\phi)\, u_\beta \tag{1.10}$$

where $u_\beta \in C(S)$ is such that

$$0 = u_\beta(z) \leqslant u_\beta(x) \leqslant u_\beta(s_0) = 1, \qquad z \in S\backslash W_\beta, \quad x \in S.$$

With this choice, for $z \in S\backslash W_\beta$ we have

$$T_{\alpha,\beta}\phi(z) = \phi(z), \qquad \text{for all } \alpha \in A, \quad \phi \in C(S). \tag{1.11}$$

If $f \in F$, choose β_0 such that for $\beta \geqslant \beta_0$

$$|f(s_0) - f(x)| < \epsilon/2, \qquad \text{for all } x \in W_\beta .$$

Using (1.9) we choose next α_0 such that for $\alpha \geqslant \alpha_0$

$$|\mu_\alpha(f) - f(s_0)| < \frac{\epsilon}{2} .$$

Summing up, we have, for $(\alpha, \beta) \geqslant (\alpha_0, \beta_0)$ and $f \in F$,

$$\begin{aligned} |T_{\alpha,\beta}f(x) - f(x)| &= |u_\beta(x)|\,|f(x) - \mu_\alpha(f)| \leqslant |f(x) - \mu_\alpha(f)| \\ &\leqslant |f(x) - f(s_0)| + |f(s_0) - \mu_\alpha(f)| < \epsilon, \qquad \text{for all } x \in W_\beta . \end{aligned}$$

Taken together with (1.11), this implies the uniform convergence $T_{\alpha,\beta}f \rightrightarrows f$.

Using $(\mathscr{L}^+, \mathscr{N}, u, p)$ we conclude that

$$T_{\alpha,\beta}g(s_0) \to g(s_0). \tag{1.12}$$

However, $T_{\alpha,\beta}g(s_0) = \mu_\alpha(g)$, so that (1.12) is inconsistent with (1.9).

We have thus proved the first relation of (1.8). The same proof establishes the second relation; we need only observe that

$$\| T_{\alpha,\beta} \| \leqslant \| \mu_\alpha \| + 1$$

so that if $\mu_\alpha \in \mathscr{N}_B$ then $T_{\alpha,\beta} \in \mathscr{N}_B$ as well.

Proof of (b) *of Theorem* 1.2. If the F-topology is first countable, then we replace (1.10) by the diagonal sequence

$$T_n\phi = (1 - u_n)\phi + \mu_n(\phi)\; u_n$$

and the rest of the proof proceeds mutatis mutandis, yielding

$$\begin{aligned}(\mathscr{L}^+, \mathscr{S}, u, p) &\Rightarrow (\mathscr{M}^+, \mathscr{S})\\ (\mathscr{L}^+, \mathscr{S}_B, u, p) &\Rightarrow (\mathscr{M}^+, \mathscr{S}_B),\end{aligned} \tag{1.13}$$

completing the proof of (c).

The proof of the equivalences of the first box in Fig. 1 will be complete once we prove

$$(\mathscr{M}^+, \mathscr{N}) \Rightarrow (\mathscr{L}^+, \mathscr{N}, p, p). \tag{1.14a}$$

Suppose $(\mathscr{M}^+, \mathscr{N})$ holds for g, and assume $\{T_\alpha\}$ is a net such that $T_\alpha f \to f$ pointwise for all $f \in F$. Define, for each $x \in S$, the functional $\mu_{\alpha,x} = T_\alpha \nu_x$. We have

$$\mu_{\alpha,x} f = T_\alpha \nu_x(f) = T_x f(x) \to f(x).$$

Hence, by $(\mathscr{M}^+, \mathscr{N})$,

$$T_\alpha g(x) = \mu_{\alpha,x} g \to g(x), \qquad \text{for all } x \in X. \qquad \text{Q.E.D.}$$

The same proof establishes also

$$(\mathscr{M}^+, \mathscr{N}_B) \Rightarrow (\mathscr{L}^+, \mathscr{N}_B, p, p) \tag{1.14b}$$

$$(\mathscr{M}^+, \mathscr{S}) \Rightarrow (\mathscr{L}^+, \mathscr{S}, p, p) \tag{1.14c}$$

$$(\mathscr{M}^+, \mathscr{S}_B) \Rightarrow (\mathscr{L}^+, \mathscr{S}_B, p, p). \tag{1.14d}$$

We prove now that

$$(\mathscr{M}^+, \mathscr{J}) \Rightarrow (\mathscr{M}^+, \mathscr{N}_B) \tag{1.15}$$

Assume that $(\mathscr{M}^+, \mathscr{J})$ holds for g, and suppose that $\{\mu_\alpha\}$ is a net such that $\| \mu_\alpha \| \leqslant K$ for all α and

$$\mu_\alpha f \to f(s_0), \qquad \text{for all } f \in F. \tag{1.16}$$

By w^*-compactness, there exists a convergent subnet $\mu_\beta \xrightarrow{w^*} \mu$. Using (1.16), we conclude that for every $f \in F$

$$\mu f = f(s_0).$$

Hence, by $(\mathscr{M}^+, \mathscr{J})$, $\mu g = g(s_0)$, i.e.,

$$\mu_\beta g \to g(s_0) \tag{1.17}$$

Since (1.17) is valid for all convergent subnets of $\{\mu_\alpha\}$, we deduce that

$$\mu_\alpha g \to g(s_0) \qquad \text{Q.E.D.}$$

The only implication in Fig. 1 that has not yet been proved is

$$(\mathcal{M}^+, \mathcal{S}) \Rightarrow (\mathcal{M}^+, \mathcal{N}).$$

Let $\{\mu_\alpha, \alpha \in A\}$ be a net such that $\mu_\alpha \geqslant 0$ and $\mu_\alpha(f) \to f(s_0)$ for all $f \in F$. Assume that $(\mathcal{M}^+, \mathcal{N})$ does not hold for g, i.e. $\mu_\alpha(g) \not\to g(s_0)$. Passing to a subnet if necessary, we may assume that $|\mu_\alpha(g) - g(s_0)| \geqslant \epsilon > 0$ for all α.

There are two possibilities: (a) For some n, $\{\alpha \in A; |\mu_\alpha| \leqslant n\}$ is cofinal in A (recall that H is cofinal iff for all $\alpha \in H$ there exists a $\beta \in H$, $\beta > \alpha$). Then there exists a bounded subnet $\{\mu_\beta; \beta \in B\}$ with $\mu_\beta(f) \to f(s_0)$ for all $f \in F$ but $\mu_\beta(g) \not\to g(s_0)$. This means that $(\mathcal{M}^+, \mathcal{N}_B)$ does not hold for g. Since $(\mathcal{M}^+, \mathcal{S}) \Rightarrow (\mathcal{M}^+, \mathcal{N}_B)$ we conclude that $(\mathcal{M}^+, \mathcal{S})$ does not hold for g.

(b) There exists a sequence $\alpha_n \in A$ such that $\|\mu_\alpha\| > n$ whenever $\alpha > \alpha_n$. Then $\{\alpha_n, n \in N\}$ is cofinal in A and $\{\mu_{\alpha_n}; n \in N\}$ is a subnet of $\{\mu_\alpha; \alpha \in A\}$. Thus, we have $\mu_{\alpha_n}(f) \to f(s_0)$ for all $f \in F$ while $\mu_{\alpha_n}(g) \not\to g(s_0)$, so that $(\mathcal{M}^+, \mathcal{S})$ does not hold for g. Q.E.D.

Proof of (a) *of Theorem* 1.2. Let $f_0 \in F$ satisfy $f_0 > 0$. Since f_0 is continuous, we have

$$0 < m \leqslant f_0 \leqslant M < \infty.$$

Let $\{T_\alpha\}$ be a net such that $T_\alpha f \rightrightarrows f$ for all $f \in F$. Then $m \| T_\alpha \| = m T_\alpha(1) \leqslant T_\alpha f_0 \rightrightarrows f_0$, and we have

$$\overline{\lim} \| T_\alpha \| \leqslant M/m.$$

Hence, for a net such that $T_\alpha f \rightrightarrows f$, $\mathcal{N} = \mathcal{N}_B$. Similarly, $\mathcal{S} = \mathcal{S}_B$, and the same results hold for functionals. This yields the additional implications necessary for Fig. 2.

Part (c) of Theorem 1.2 follows easily.

We now turn our attention to the negative results in Figures 1–3. These will be established by constructing appropriate counterexamples.

(a) We show that

$$(\mathcal{L}^+, \mathcal{J}) \not\Rightarrow (\mathcal{L}^+, \mathcal{S}_B, u, p) \tag{1.18}$$

Let $S = [0, 1]$, $F = \{1, \cos 2\pi t, \sin 2\pi t\}$, $g(t) = t$. Suppose T is any positive linear operator $C(S) \to C(S)$, such that $Tf = f$ for all $f \in F$. We prove that $Tg = g$.

Let $Tg(t) = h(t)$, $t \in [0, 1]$, and define for any $\phi \in C[0, 1]$

$$\psi(t) = \phi(t) - [\phi(1) - \phi(0)]\, t.$$

Then, $\psi(0) = \psi(1)$, so that by the trigonometric Korovkin theorem [14], $T\psi = \psi$. Hence, we have

$$T\phi(t) - [\phi(1) - \phi(0)]\, h(t) = \phi(t) - [\phi(1) - \phi(0)]\, t$$

or,

$$T\phi(t) = \phi(t) + [\phi(1) - \phi(0)]\, [h(t) - t].$$

Since $T \geqslant 0$, it follows that the coefficients of $\phi(1)$ and of $\phi(0)$ have to be non-negative, i.e., that $h(t) = t$ for $0 < t < 1$. By continuity of $Tg(t) = h(t)$, this extends to the endpoints, i.e., $Tg = g$. Hence, $(\mathscr{L}^+, \mathscr{J})$ holds for g.

Choose now $\mu(\phi) = [\phi(0) + \phi(1)]/2$. Then $\mu f = f(1)$ for all $f \in F$, $\mu g \neq g(1)$. Hence $(\mathscr{M}^+, \mathscr{J})$ does not hold for g. Noting that in the metric case $(\mathscr{M}^+, \mathscr{J})$ is equivalent to $(\mathscr{L}^+, \mathscr{S}_B, u, p)$, we conclude that (1.18) is established.

The space S in this counterexample is metric, and span F contains a positive function, so that the counterexample serves to establish (1.18) for Figures 1, 2, 3 simultaneously.

(b) We next show that

$$(\mathscr{M}^+, \mathscr{J}) \not\Rightarrow (\mathscr{M}^+, \mathscr{S}) \tag{1.19}$$

Let $S = [0, 1]$, $F = \{t^2, t^3, t^4\}$, $g(t) \equiv t$. Let μ be any positive linear functional such that for some $t_0 \in [0, 1]$

$$\mu(t^i) = t_0{}^i, \qquad i = 2, 3, 4.$$

We then have $\mu[t^2(t - t_0)^2] = 0$, so that μ is supported at $\{0, t_0\}$, and necessarily $\mu = \alpha\nu_0 + \beta\nu_{t_0}$. Hence $\mu(t) = t_0$, so that $(\mathscr{M}^+, \mathscr{J})$ holds for g.

Taking

$$\mu_n = n\nu_{1/n} + \nu_{t_0}$$

we have $\mu_n f \to f(t_0)$ for all $f \in F$, but $\mu_n(g) = 1 + t_0 \not\to t_0 = g(t_0)$. Hence, $(\mathscr{M}^+, \mathscr{S})$ does not hold for g.

The space S is metric, so that the counterexample serves for Figures 1 and 3, and in fact completes the proof of all relations in Fig. 3.

Taking into account the equivalences of Fig. 3, we can now deduce that in the general $C(S)$ case:

$$\begin{aligned} (\mathscr{L}^+, \mathscr{S}_B, u, u) &\not\Rightarrow (\mathscr{L}^+, \mathscr{S}, u, u) \\ (\mathscr{L}^+, \mathscr{S}_B, u, p) &\not\Rightarrow (\mathscr{L}^+, \mathscr{S}, u, p) \\ (\mathscr{L}^+, \mathscr{S}_B, p, p) &\not\Rightarrow (\mathscr{L}^+, \mathscr{S}, p, p) \\ (\mathscr{M}^+, \mathscr{J}) &\not\Rightarrow (\mathscr{L}^+, \mathscr{S}, u, p) \end{aligned} \tag{1.20}$$

(c) Finally, we show that

$$(\mathscr{L}^+, \mathscr{S}, u, u) \not\Rightarrow (\mathscr{M}^+, \mathscr{J}) \tag{1.21}$$

$$(\mathscr{L}^+, \mathscr{S}, p, p) \not\Rightarrow (\mathscr{M}^+, \mathscr{J}). \tag{1.22}$$

We treat both cases together. The example is based on a particular case of a class considered by Kitto and Wulbert [13].

Let S consist of two disjoint copies of the ordinal space $[0, \omega_1]$, where ω_1 is the first uncountable ordinal, with the order topology: $[0, \omega_1] \times \{1\}$ and $[0, \omega_1] \times \{2\}$. Let $F = \{f; f(\omega_1, 1) = f(\omega_1, 2)\}$. Suppose $\{T_n\}$ is any sequence in $\mathscr{S}$, such that $T_n f \rightrightarrows f$ for all $f \in F$ for (1.21), or $T_n f(s) \to f(s)$ for all $f \in F$, $s \in S$ for (1.22).

We show that the statements on the left hand side are valid for the function

$$g = \chi_{[0,\omega_1]\times\{1\}}$$

where $\chi_{[0,\omega_1]\times\{1\}}$ is the characteristic function of the set $[0, \omega_1] \times \{1\}$ i.e.,

$$\begin{aligned} g(\beta, i) &= 1 \qquad \beta \in [0, \omega_1], \quad i = 1 \\ g(\beta, i) &= 0 \qquad \text{otherwise.} \end{aligned}$$

Let $\mu^n_{\alpha,i} = T^*{}_n \nu_{\alpha,i}$, $c^n_{\alpha\, i} = \mu^n_{\alpha,i} g$ ($\alpha \in [0, \omega_1]$, $i = \{1, 2\}$). Starting with the pointwise convergence, we have to prove

$$c^n_{\alpha,i} \to \delta_{1,i} \qquad \text{for all } \alpha \leqslant \omega_1, \quad i = 1, 2. \tag{1.23}$$

Passing to a subsequence, we may assume that $c^n_{\alpha,i} \to c_{\alpha,i} \in [0, \infty]$. Let $\alpha < \omega_1$, and observe that

$$0 \leqslant T_n(g - \chi_{[0,\alpha]\times\{1\}})(\alpha, 1) = c^n_{\alpha,1} - \mu^n_{\alpha,1}(\chi_{[0,\alpha]\times\{1\}}) \to c_{\alpha,1} - 1$$

(since $\chi_{[0,\alpha]\times 1} \in F$). Hence

$$c_{\alpha,1} \geqslant 1, \qquad \text{for all } \alpha < \omega_1 \tag{1.24}$$

On the other hand,

$$0 \leqslant T_n(2 - \chi_{[0,\alpha]\times\{1\}} - g)(\alpha, 1) = \mu^n_{\alpha,1}(2 - \chi_{[0,\alpha]\times\{1\}}) - c^n_{\alpha,1} \to 1 - c_{\alpha,1}.$$

Hence, $c_{\alpha,1} \leqslant 1$, which taken together with (1.24), implies that

$$c_{\alpha,1} = 1 \qquad \text{for } \alpha < \omega_1. \tag{1.25}$$

Consider next $c_{\alpha,2}$. Let $\alpha < \omega_1$ and observe that

$$0 \leqslant T_n(1 - \chi_{[0,\alpha]\times\{2\}} - g)(\alpha, 2) = \mu^n_{\alpha,2}(1 - \chi_{[0,\alpha]\times\{2\}}) - c^n_{\alpha,2} \to 0 - c_{\alpha,2}.$$

Hence,

$$c_{\alpha,2} = 0 \qquad \text{for } \alpha < \omega_1 \tag{1.26}$$

establishing (1.23) for $\alpha < \omega_1$.

A suitable limiting procedure extends this to $\alpha = \omega_1$, and we have (1.23) for all α, establishing the pointwise convergence.

We next establish the uniform convergence. If $T_n g \not\rightrightarrows g$, then there exists an $\epsilon > 0$ and a subsequence $\{\alpha_n\} \subset [0, \omega_1)$ such that for $i = 1$ or for $i = 2$

$$| T_n g(\alpha_n, i) - g(\alpha_n, i)| \geqslant \epsilon, \qquad \text{for all } n. \tag{1.27}$$

Let $\alpha = \sup \alpha_n$ and define the function f_i, $i = 1, 2, 3$ by

$$f_1 = \chi_{[0,\alpha)\times\{1\}}, \qquad f_2 = 2 - \chi_{[0,\alpha)\times\{1\}}, \qquad f_3 = 1 - \chi_{[0,\alpha)\times\{2\}}$$

if $\alpha > \alpha_n$ for all n, and by

$$f_1 = \chi_{[0,\alpha]\times\{1\}}, \qquad f_2 = 2 - \chi_{[0,\alpha]\times\{1\}}, \qquad f_3 = 1 - \chi_{[0,\alpha]\times\{2\}},$$

if $\alpha \in \{\alpha_n\}$. Obviously $f_i \in F$, $i = 1, 2, 3$, so that for the same ϵ as before there exists an N such that for all $n > N$,

$$\| T_n f_i - f_i \| < \epsilon, \qquad i = 1, 2, 3. \tag{1.28}$$

Consider first $i = 1$. Letting $n > N$ and using the positivity of T_n, we have

$$\begin{aligned} 0 \leqslant T_n(g - f_1)\ (\alpha_n, 1) &= T_n g(\alpha_n, 1) - T_n f_1(\alpha_n, 1) < T_n g(\alpha_n, 1) - 1 + \epsilon \\ &= T_n g(\alpha_n, 1) - g(\alpha_n, 1) + \epsilon \end{aligned}$$

i.e.,

$$g(\alpha_n, 1) - T_n g(\alpha_n, 1) < \epsilon. \tag{1.29}$$

Similarly, we find

$$\begin{aligned} 0 \leqslant T_n(f_2 - g)\ (\alpha_n, 1) &= T_n f_2(\alpha_n, 1) - T_n g(\alpha_n, 1) < 1 + \epsilon - T_n g(\alpha_n, 1) \\ &= \epsilon + g(\alpha_n, 1) - T_n g(\alpha_n, 1) \end{aligned}$$

yielding

$$T_n g(\alpha_n, 1) - g(\alpha_n, 1) < \epsilon.$$

This inequality, taken together with (1.29) demonstrates that (1.27) is impossible for $i = 1$.

Consider next $i = 2$, and let $n \geqslant N$. A similar estimate yields

$$0 \leqslant T_n(f_3 - g)\ (\alpha_n, 2) < \epsilon - T_n g(\alpha_n, 2) = \epsilon + g(\alpha_n, 2) - T_n g(\alpha_n, 2).$$

This shows that (1.27) is impossible for $i = 2$ as well, establishing the uniform convergence. Thus $(\mathscr{L}^+, \mathscr{S}, u, u)$ holds for g.

On the other hand, choosing the functional $\mu\phi = [\phi(\omega_1, 1) + \phi(\omega_1, 2)]/2$, we find that $\mu f = f(\omega_1, 1)$ for all $f \in F$, but $\mu g \neq g(\omega_1, 1)$, so that $(\mathscr{M}^+, \mathscr{I})$ does not hold for g. Q.E.D.

In this example span F contains a positive function. Hence, relations (1.21) complete the proof of Fig. 2.

Taking into account the equivalences of Fig. 2, we conclude that in the general $C(S)$ case:

$$\begin{aligned} &(\mathscr{L}^+, \mathscr{S}, u, u) \not\Rightarrow (\mathscr{L}^+, \mathscr{N}) \\ &(\mathscr{L}^+, \mathscr{S}_B, u, u) \not\Rightarrow (\mathscr{M}^+, \mathscr{I}) \\ &(\mathscr{L}^+, \mathscr{S}_B, u, p) \not\Rightarrow (\mathscr{M}^+, \mathscr{I}). \end{aligned} \tag{1.30}$$

These relations complete the proof of Fig. 1, and, ipso facto, of Theorems 1.1 and 1.2.

REMARK 1.5. If the linear subspace F of $C(S)$ contains 1, then another formulation is: G is the $(\mathscr{M}^+, \mathscr{I})$-shadow of F iff S is the relative Choquet boundary of F with respect to G. (This is due to Franchetti [7], see also [24]).

PROBLEM 1.6. Complete the charts in Figures 1 and 2.

2. THE SHADOWS OF hF

In investigating the shadows of a system F, one may encounter a situation where the given system bears a close relationship to a system whose shadows are known. For example, considering the shadows of $\{t, t^2, t^3\}$ one observes that the system is closely related to the familiar system $\{1, t, t^2\}$. The relation between the shadows of such systems might then provide the desired information.

These considerations motivate this section in which we investigate the relationship between the shadows of hF and the shadows of F, where h is nonnegative.

Notation. (1) Given a function $h \in C(S)$, let $I_Z(h)$ be the set of functions vanishing at all points where h vanishes, i.e.,

$$I_Z(h) = \{g \in C(S); [h(s) = 0 \Rightarrow g(s) = 0]\}. \tag{2.1}$$

(2) The subset of $I_Z(h)$ containing all functions g such that g/h is bounded on S, is denoted by

$$I_{Z,B}(h) = \{g \in I_Z(h), \text{there exists an } M = M(g) \text{ such that } |g| \leqslant Mh\}. \tag{2.2}$$

The first theorem deals with $(\mathscr{M}^+, \mathscr{I})$-shadows.

THEOREM 2.1. *Let S, $C(S)$ and F be as in Section 0, and let h be a nonnegative function of $C(S)$. Let $G_{\mathscr{I}}$ be the $(\mathscr{M}^+, \mathscr{I})$-shadow of F and $H_{\mathscr{I}}$ the $(\mathscr{M}^+, \mathscr{I})$-shadow of hF. Then, we have*

(a) $$hG_{\mathscr{I}} \subset H_{\mathscr{I}} \subset I_Z(h) \tag{2.3}$$

(b) *If $G_{\mathscr{I}} = C(S)$ (i.e., F is an $(\mathscr{M}^+, \mathscr{I})$-Korovkin set for $C(S)$), then*

$$H_{\mathscr{I}} = I_Z(h) \tag{2.4}$$

Proof. (a) We start with the left hand inclusion of (2.2). Let $g \in G_{\mathscr{F}}$ and let $\mu \in \mathscr{M}^+$ satisfy

$$\mu(hf) = h(s_0) f(s_0) \qquad \text{for all } f \in F. \tag{2.5}$$

We have to prove that

$$\mu(hg) = h(s_0)\, g(s_0) \tag{2.6}$$

If $h(s_0) > 0$, define $\tilde{\mu}(\phi) = \mu(h\phi)/h(s_0)$, for $\phi \in C(S)$. By (2.5) we have $\tilde{\mu}(f) = f(s_0)$ for all $f \in F$. Since $g \in G_{\mathscr{F}}$, we conclude that $\tilde{\mu}(g) = g(s_0)$, i.e. (2.5) is satisfied.

If $h(s_0) = 0$, choose s_1 such that $h(s_1) > 0$ and define

$$\tilde{\mu}(\phi) = \frac{\mu(h\phi) + h(s_1)\,\phi(s_1)}{h(s_1)}, \qquad \text{for all } \phi \in C(S).$$

Then $\tilde{\mu}(f) = f(s_1)$ for all $f \in F$ and hence $\tilde{\mu}(g) = g(s_1)$, implying that $\mu(hg) = 0 = g(s_0)\, h(s_0)$.

For the proof of the right hand inclusion, let $s_0 \in S$, $g \in C(S)$ be such that $h(s_0) = 0$, $g(s_0) \neq 0$. Then $\mu = 2\nu_{s_0}$ satisfies

$$\mu(hf) = 2h(s_0) f(s_0) = 0 = h(s_0) f(s_0) \qquad \text{for all } f \in F$$

while $\mu(g) = 2g(s_0) \neq g(s_0)$. Hence $g \notin H_{\mathscr{F}}$.

Note that the positivity of h was not used for the last proof.

(b) Note that $hC(S)$ is dense in $I_Z(h)$, and that $H_{\mathscr{F}}$, as an $(\mathscr{M}^+, \mathscr{F})$-shadow, is a closed subspace. Hence, appealing to (2.3), we conclude that all inclusions turn into equalities. Q.E.D.

We next establish similar results for other shadows, and first recall the following familiar definition:

DEFINITION 2.2. A set $F \subset C(S)$ is called a *strict Korovkin set* if for each $s_0 \in S$ there exists a function $\psi_{s_0} \in \operatorname{span} F$ such that $\psi_{s_0}(s_0) = 0$, $\psi_{s_0}(s) > 0$ for all $s \neq s_0$.

THEOREM 2.3. *Let S, $C(S)$, F and h be as above. Then the following relations are valid.*

(a) $$(\mathscr{L}^+, \mathscr{N}, p, p)\text{-}\mathit{shadow\ of}\ hC(S) \subset I_{Z,B}(h). \tag{2.7a}$$

If S is first countable, then we have also

$$(\mathscr{L}^+, \mathscr{S}, p, p)\text{-}\mathit{shadow\ of}\ hC(S) \subset I_{Z,B}(h). \tag{2.7b}$$

(b) *If F is a strict Korovkin set, then*

$$(\mathscr{M}^+, \mathscr{N})\text{-}\mathit{shadow\ of}\ hF = I_{Z,B}(h). \tag{2.8}$$

Remark. Note that $(\mathscr{M}^+, \mathscr{N})$-shadow $= (L^+, \mathscr{N}, p, p)$-shadow always by Theorem 1.1.

Proof. (a) Suppose $g \notin I_{Z,B}(h)$. Then there exists a sequence $\{s_n\} \subset S$ such that $|g(s_n)|/h(s_n) = \rho_n \to \infty$. Let s_0 be an accumulation point of $\{s_n\}$, and let $\{V_\alpha\}$ be a basis for the neighborhood system of s_0. Let u_α be the Urysohn function corresponding to V_α, i.e.

$$0 = u_\alpha(z) \leqslant u_\alpha(s) \leqslant u_\alpha(s_0) = 1, \qquad \text{for all } z \in S \backslash V_\alpha, \quad s \in S.$$

Define the net of positive linear operators $\{T_\alpha\}$ by

$$T_\alpha \phi(s) = [1 - u_\alpha(s)]\, \phi(s) + u_\alpha(s) \left[\phi(s_0) + \frac{\phi(s_{n(\alpha)})}{h(s_{n(\alpha)}) \sqrt{\rho_{n(\alpha)}}}\right], \qquad \phi \in C(S) \tag{2.9}$$

where $n(\alpha)$ is chosen so that $s_{n(\alpha)} \in V_\alpha$.

Observe that

$$T_\alpha(h\phi)(s) = h\phi(s), \qquad \text{for all } s \in S \backslash V_\alpha$$

and

$$|T_\alpha(h\phi)(s) - h\phi(s)| = |u_\alpha(s)| \left| (h\phi)(s) - (h\phi)(s_0) - \frac{\phi(s_{n(\alpha)})}{\sqrt{\rho_{n(\alpha)}}} \right| \to 0, \qquad \text{for } s \in V_\alpha.$$

Hence $T_\alpha(h\phi) \to h\phi$ for all $\phi \in C(S)$; on the other hand

$$|T_\alpha(g)(s_0) - g(s_0)| = \frac{|g(s_{n(\alpha)})|}{\sqrt{\rho_{n(\alpha)}} \cdot h(s_{n(\alpha)})} = \sqrt{\rho_{n(\alpha)}} \to \infty$$

so that g does not belong to the $(\mathscr{L}^+, \mathscr{N}, p, p)$-shadow of $hC(S)$. The proof of (2.7b) is almost identical.

(b) In view of (2.7a), we have to prove that each $l(x) \in I_{Z,B}(h)$ is in the $(\mathscr{M}^+, \mathscr{N})$ shadow of hF. Let μ_α be a net of positive linear functionals such that

$$\mu_\alpha(hf) \to h(s_0) f(s_0) \qquad \text{for all } f \in F. \tag{2.10}$$

We prove that if $|l(s)| \leqslant Mh(s)$ for all s, then

$$\mu_\alpha(l) \to l(s_0). \tag{2.11}$$

Since F is a strict Korovkin system, it contains a strictly positive function (in fact, if $s_1 \neq s_2$, $\psi_{s_1} + \psi_{s_2}$ is such a function); denote it by $u(s)$. We may assume $u(s) \geqslant 1$.

Distinguish now between two cases:

If $h(s_0) = 0$, then $l(s_0) = 0$. Applying the positive linear functionals μ_α to

$$-Mu(s)\,h(s) \leqslant l(s) \leqslant Mu(s)\,h(s)$$

we have

$$-M\mu_\alpha(uh) \leqslant \mu_\alpha(l) \leqslant M\mu_\alpha(uh).$$

Both extremes tend to 0 in view of (2.10). Hence $\mu_\alpha(l) \to 0 = l(s_0)$.

If $h(s_0) > 0$, the function l/hu is continuous at s_0, so that for every $\epsilon > 0$ there exists a neighborhood V_ϵ of s_0 in which

$$\left| \frac{l(s)}{h(s)\,u(s)} - \frac{l(s_0)}{h(s_0)\,u(s_0)} \right| < \frac{\epsilon}{h(s_0)\,u(s_0)}$$

or

$$\left| l(s) - l(s_0)\frac{u(s)\,h(s)}{u(s_0)\,h(s_0)} \right| < \frac{\epsilon u(s)\,h(s)}{u(s_0)\,h(s_0)}. \tag{2.12}$$

Let $\psi_{s_0} \in F$ be as in Definition 2.2 and let $m_\epsilon = \min\{\psi(s);\, s \in S \backslash V_\epsilon\}$. Then, for $s \in S \backslash V_\epsilon$

$$\left| l(s) - \frac{l(s_0)\,u(s)\,h(s)}{h(s_0)\,u(s_0)} \right| \leqslant Mh(s) + M\,\|\,u\,\|\,h(s) \leqslant M'h(s)\,\psi_{s_0}(s)$$

where $M' = [(1 + \|\,u\,\|)\,M]/m_\epsilon$. Combining this with (2.12), we have

$$\left| l(s) - \frac{l(s_0)\,u(s)\,h(s)}{h(s_0)\,u(s_0)} \right| < \epsilon\frac{u(s)\,h(s)}{u(s_0)\,h(s_0)} + M'h(s)\,\psi_{s_0}(s) \qquad \text{for all } s \in S. \tag{2.13}$$

Applying μ_α to both sides we have

$$\left| \mu_\alpha(l) - \frac{l(s_0)}{h(s_0)\,u(s_0)}\,\mu_\alpha(hu) \right| \leqslant \frac{\epsilon\mu_\alpha(hu)}{u(s_0)\,h(s_0)} + M'\mu_\alpha(h\psi_{s_0}).$$

Making use of (2.10), we conclude that $\mu_\alpha(h\psi_{s_0}) \to 0$ and $\mu_\alpha(hu) \to u(s_0)\,h(s_0)$, so that

$$\mu_\alpha(l) \to l(s_0). \qquad \text{Q.E.D.}$$

Applying these results to the special case mentioned at the beginning of the section, we have

COROLLARY 2.4. *Let* $s = [0, 1]$. *Then*

(a) *The* $(\mathscr{M}^+, \mathscr{J})$ *shadow of* $\{t, t^2, t^3\}$ *is the set of all functions vanishing at* 0.

(b) *The* $(\mathscr{M}^+, \mathscr{N})$ *shadow of* $\{t, t^2, t^3\}$ *is the set of all functions* l *such that* $|\,l(t)| \leqslant M_l t$ *on* $[0, 1]$.

The natural extension would consist of relaxing the condition on h. We leave this as an open problem:

PROBLEM 2.5. What can be said about the shadow of hF when h has sign changes?

Henceforward we discuss only the $(\mathscr{M}^+, \mathscr{J})$ shadow, so that "the shadow" is to be interpreted as "the $(\mathscr{M}^+, \mathscr{J})$ shadow".

3. SUBSPACES OF $C(S)$ WHICH CAST NO SHADOW

In examining shadows one discerns two extreme cases: (a) The case where the shadow of F is the whole space $C(S)$. Then F is a Korovin set (K-set). (b) The case where the shadow of F is only span F, i.e., no shadow is cast.

Whereas (a) has been extensively explored, (b) has received no attention in the literature. We propose to show in this section that in several nontrivial cases F casts no shadow. Clearly, we may assume that F is a subspace throughout this section.

We observe that the subspaces which cast no shadow are themselves shadows. Hence they are characterized by a system of conditions of the form

$$\mu_\alpha(f) = \nu_\alpha(f), \qquad \alpha \in A$$

where ν_α is an evaluation functional and μ_α is a positive functional.

Denote the set of such measures by $\mathscr{M}_\nu^+$, and the annihilator of a measure λ_α by $(\lambda_\alpha)_\perp$. Thus $F = \bigcap_\alpha (\lambda_\alpha)_\perp$ for $\alpha \in A$.

LEMMA 3.1. *The condition $af(s_1) + bf(s_2) + cf(s_3) = 0$ is equivalent to a condition of the form $\lambda(f) = 0$, $\lambda \in \mathscr{M}_\nu^+$.*

Proof. If $a = b = c = 0$ the condition is void, and can be considered as formally equivalent to $\lambda(f) = 0$ where $\lambda = \nu_s - \nu_s = 0$.

Otherwise, we may assume $a \geqslant b \geqslant c$, $a = 1$, $bc \geqslant 0$. If $c \geqslant 0$, we can write the condition as $(\mu_{s_1} - \nu_{s_1})(f) = 0$ where $\mu_{s_1} = 2\nu_{s_1} + b\nu_{s_2} + c\nu_{s_3}$. If $c < 0$, we can write it as $(\tilde{\mu}_{s_1} - \nu_{s_1})(f) = 0$, where $\tilde{\mu}_{s_1} = (-b)\,\nu_{s_2} + (-c)\,\nu_{s_3}$.

LEMMA 3.2. *Let $\{s_\beta; \beta \in B\}$ be a net in S. Then the condition*

$$f(s_\beta) \to f(s_0) \tag{3.1}$$

is equivalent to a condition of the form

$$f \in \bigcap_\alpha (\lambda_\alpha)_\perp, \qquad \lambda_\alpha \in \mathscr{M}_\nu^+. \tag{3.2}$$

Proof. We observe that $f(s_\beta) \to f(s_0)$ iff for every subnet $\{s_\gamma; \gamma \in E\}$ of $\{s_\beta; \beta \in B\}$ and every Banach limit on $m(E)$ (the Hahn–Banach extension of the limit functional), we have $\Lambda(f(s_\gamma)) = f(s_0)$, for all γ. We need add only that $\mu_\gamma f = \Lambda\,(f(s_\gamma))$ is obviosuly a positive linear functional on $C(S)$.

We are now ready to state the main result of this section.

THEOREM 3.3. *The following classes of subspaces of $C(S)$ cast no shadow:*

(i) *All subspaces of dimension $\leqslant 2$.*

(ii) *All Grothendieck subspaces, i.e. the subspaces characterized by*

$$\{f \in C(S); f(s_1^\alpha) = c_\alpha f(s_2^\alpha), \alpha \in A\} \tag{3.3}$$

where $\{s_1^\alpha\}$, $\{s_2^\alpha\}$ are nets in S.

(iii) *The subspaces of the form $C_0(S, S_0) = \{f \in C(S); f(s) = 0$ for all $s \in S_0\}$ where $S_0 \subset S$.*

(iv) *Closed subalgebras.*

(v) *Subspaces admitting a positive projection and, in particular, subspaces containing* 1 *and admitting a norm-*1 *projection.*

(vi) *Subspaces consisting of all functions that are continuous with respect to some weaker topology.*

(vii) *When S is a convex set in a linear space, the subspace $A(S)$ of affine functions in $C(S)$.*

(viii) *Intersections of subspaces from the classes* (i)–(vi).

Proof. (i) If $\dim F = 0$ then $F = \{0\} = \cap\{(\nu_s - 0)_\perp; s \in S\}$. If $\dim F = 1$ then $F = \operatorname{span}\{f\}$ where $f \not\equiv 0$. Fix a point s_0 with $f(s_0) \neq 0$, and observe that

$$\operatorname{span}\{f\} = \bigcap_{s\in S} \left\{g; g(s) = \frac{g(s_0)}{f(s_0)} f(s)\right\} = \bigcap_{s\in S} \left(\nu_s - \frac{f(s)}{f(s_0)} \nu_{s_0}\right)_\perp.$$

If $\dim F = 2$ then $F = \operatorname{span}\{f, g\}$ where f and g are linearly independent. Fix s_1, s_2 such that $f(s_1)\, g(s_2) - f(s_2)\, g(s_1) \neq 0$. Observe that

$$\operatorname{span}\{f, g\} = \bigcap_{s\in S} \left\{h; \begin{vmatrix} f(s_1) & f(s_2) & f(s) \\ g(s_1) & g(s_2) & g(s) \\ h(s_1) & h(s_2) & h(s) \end{vmatrix} = 0\right\}$$

and note that by Lemma 3.1, the condition in the brackets is equivalent to $\lambda_s(f) = 0$, where $\lambda_s \in \mathscr{M}_\nu^+$. Hence, $\operatorname{span}\{f, g\} = \bigcap_{s\in S} (\lambda_s)_\perp$ completing the proof of (i).

(ii) This is an immediate corollary of Lemma 3.1.

(iii) This is a consequence of the observation that $C_0(S, S_0) = \bigcap_{s\in S_0} (\nu_s)_\perp$.

(iv) This is a particular case of (ii), by a well known corollary of the Stone–Weierstrass theorem.

(v) If P is a positive projection, then

$$PC(S) = \bigcap_{s \in S} \{f; (Pf)(s) = f(s)\}$$
$$= \bigcap_{s \in S} (\mu_s - \nu_s)_\perp, \qquad \text{where} \qquad \mu_s(f) = Pf(s).$$

(vi) This is an immediate corollary of Lemma 3.2.

(vii) We note that $A(S)$ may be written as

$$A(S) = \bigcap_{\substack{s_1, s_2 \in S \\ \alpha \in (0,1)}} \{f; f(\alpha s_1 + (1 - \alpha) s_2) = \alpha f(s_1) + (1 - \alpha) f(s_2)\}$$

and by Lemma 3.1, the condition in the brackets can be written as $\lambda_\beta(f) = 0$, where $\lambda_\beta \in \mathscr{M}_\nu^+$, $\beta = \beta(s_1, s_2, \alpha)$. Hence $A(S) = \bigcap_\beta (\lambda_\beta)_\perp$.

(viii) This is self-evident.

COROLLARY 3.4. (a) *Every Banach space E is isometric to a shadow.*

(b) *Every separable Banach space is isometric to a shadow in* $C[0, 1]$.

Proof. Consider the canonical embedding of E in $C(B^*)$, where B^* is the unit ball of E^* in its weak*-topology. In this embedding $E = A_0(B^*) = \{f; f \in A(B^*), f(0) = 0\}$, so that we may use (vii) of the previous theorem.

(b) In this case B^* is metrizable. Hence $C(B^*)$ is a closed subalgebra in its natural embedding in $C(\Delta)$, where Δ is the Cantor set. The class $C(\Delta)$ admits a positive projection in its canonical embedding in $C[0, 1]$ (by linear interpolation).

COROLLARY 3.5. *Let $C(S)$ be non-trivial. Then, for each n, there exists an n-dimensional shadow in $C(S)$.*

Proof. Choose n distinct points $s_1, \ldots, s_n$ and functions $f_1, \ldots, f_n \in C(S)$ such that

$$f_i(s_j) = \delta_{ij}, \qquad i, j = 1, \ldots, n,$$
$$\sum_{i=1}^{n} f_i = 1, \qquad 0 \leqslant f_i \leqslant 1, \quad i = 1, \ldots, n.$$

Then the n dimensional subspace spanned by $(f_1, \ldots, f_n)$ admits a norm-1 projection, and by (v) of the previous theorem it is a shadow.

4. THE CHARACTERIZATION OF REGULAR SHADOWS

Corollary 3.4 demonstrates that there can be no intrinsic characterization of shadows, unless we impose some extra conditions. This we propose to do in this section. We introduce the concept of a regular shadow and characterize Banach spaces isometric to such shadows.

DEFINITION 4.1. (a) A shadow containing the function 1 will be called a *regular shadow.*

(b) A Korovkin subspace containing the function 1 will be called a regular K-subspace.

THEOREM 4.2. *Let E be a Banach space. Then the following statements are equivalent:*

(i) *E is linearly isometric to a space $A(T)$ of all affine continuous functions on some compact convex set T in a locally convex space, with the supremum norm.*

(ii) *E is linearly isometric to a regular shadow F in some $C(S)$, where S is a compact Hausdorff space.*

(iii) *E is linearly isometric to a regular subspace H (i.e., a subspace containing 1), in some $C(S)$, where S is a compact Hausdorff space.*

(iv) *The unit ball B^* of the dual space E^* has a face Q such that*

$$B^* = \operatorname{conv}[Q \cup (-Q)] \tag{4.1}$$

where conv A *denotes the convex hull of A.*

Proof. The implication (i) $\Rightarrow$ (ii) follows directly from part (vii) of Theorem 3.3.

The implication (ii) $\Rightarrow$ (iii) is trivial.

We now prove that (iii) $\Rightarrow$ (iv). We may assume that $E = H$ of (iii). Define the set Q by $Q = \{\phi \in B^*; \phi(1) = 1\}$. Then Q is a w^*-compact face of B^*. Let ρ be the restriction map from the unit ball Σ of $[C(S)]^*$ to B^*. Note that ρ is onto by the Hahn–Banach theorem.

If $\phi \in \operatorname{ext} B^*$ (the set of extreme points of B^*) then $\rho^{-1}\phi$ is an extreme point of Σ. However, we know that

$$\operatorname{ext} \Sigma = \{\pm\nu_s;\ s \in S\}$$

so that $\rho^{-1}\phi = \pm\nu_s$, and therefore $\phi(1) = \pm\nu_s(1) = \pm 1$. By the definition of Q it follows that $\phi \in Q \cup (-Q)$. Since ϕ was an arbitrary point of ext B^* we conclude that

$$\operatorname{ext} B^* \subset Q \cup (-Q).$$

Invoking the Krein–Milman theorem, we have

$$B^* = \overline{\operatorname{conv}}^{w^*}(\operatorname{ext} B^*) = \overline{\operatorname{conv}}^{w^*}[Q \cup (-Q)] = \operatorname{conv}[Q \cup (-Q)],$$

establishing (4.1).

The last step is to prove that (iv) $\Rightarrow$ (i). Let $T = Q$ with its w^*-topology, and associate with each $x \in E$ the function $\hat{x}$ defined by $\hat{x}(q) = q(x)$, for all $q \in Q$. Obviously $\hat{x}$ is continuous and affine. Furthermore, $x \to \hat{x}$ is linear, and

$$\begin{aligned} \| \hat{x} \| &= \sup\{|9(x)|; q \in Q\} = \sup\{| \phi(x)|; \phi \in Q \cup (-Q)\} \\ &= \sup\{| \phi(x)|; \phi \in \operatorname{conv}[Q \cup (-Q)]\} \end{aligned}$$

so that by (4.1)

$$\| \hat{x} \| = \sup\{| \phi(x)|; \phi \in B^*\} = \| x \| .$$

Hence, $x \to \hat{x}$ is an isometry from E into $A(T)$. We have yet to show that the mapping is onto.

Suppose f is an arbitrary function of $A(T) = A(Q)$. We extend it to an affine function $\hat{f}$ on B^* (using (4.1)) by setting

$$\hat{f}[\alpha p + (1 - \alpha)(-q)] = \alpha f(p) - (1 - \alpha) f(q), \qquad \alpha \in [0, 1], \quad p, q \in Q. \tag{4.2}$$

We show that (4.2) is well defined on $Q \cup (-Q)$. Indeed, suppose

$$\alpha p + (1 - \alpha)(-q) = \beta p' + (1 - \beta)(-q'). \tag{4.3}$$

Noting that each $\phi \in Q$ satisfies $\phi(1) = 1$, we conclude that $2\alpha - 1 = 2\beta - 1$, or, $\alpha = \beta$. Substituting into (4.3) we find that

$$\alpha p + (1 - \alpha) q' = \alpha p' + (1 - \alpha) q.$$

These are points of Q, and f is linear there, so that

$$\alpha f(p) + (1 - \alpha) f(q') = \alpha f(p') + (1 - \alpha) f(q)$$

or,

$$\alpha f(p) - (1 - \alpha) f(q) = \alpha f(p') - (1 - \alpha) f(q').$$

Furthermore,

$$\hat{f}(0) = \hat{f}(\tfrac{1}{2}p - \tfrac{1}{2}p) = \tfrac{1}{2}f(p) - \tfrac{1}{2}f(p) = 0.$$

Extend now $\hat{f}$ to a linear functional on E^* (by homogeneity) which is then w^*-continuous. Then $\hat{f} = \hat{x}$ for some $x \in E$, and since f was an arbitrary element of $A(T)$ we have demonstrated that $x \to \hat{x}$ is onto. Q.E.D.

The ideas in this section enable us to prove simply the following characterizations of $A(T)$ spaces, which were obtained by various authors using more intricate methods.

PROPOSITION 4.3. *Let E be a Banach space: then the following statements are equivalent to* (i)–(iv) *of Theorem* 4.2:

(v) *The unit ball B of E contains a point e such that every 2-dimensional subspace of E containing e intersects B in a parallelogram one of whose vertices is e.*

(vi) *There exist a proper convex cone* $K \subset E$ *and a point* $e \in B$ *such that*

$$B = (e - K) \cap (-e + K).$$

(vii) *E is a partially ordered normed space with a point e such that*

$$\| x \| \leqslant 1 \Leftrightarrow -e \leqslant x \leqslant e.$$

Proof. We start by proving the equivalence of (iii) of Theorem 4.2 and (v) of our Theorem. For the implication (iii) $\Rightarrow$ (v) choose $e = 1$ and note that if $f \in C(S)$ is not a constant, then span$\{1, f\}$ is isometric to l_∞^2. For the converse implication, let $S = \{\phi \in B^*; \phi(e) = 1\}$ endowed with its w^*-topology. The natural embedding of E in $C(S)$ is an isometry. If $\| x \| = 1$, then one of the segments $[x, e]$, $[x, -e]$ lies on the unit sphere. Extend it to the supporting hyperplane $-\phi^{-1}(1)$ in the first case, and $\phi^{-1}(-1)$ in the second. Then $\varphi \in S$ and $\hat{e}(\phi) = 1$.

We next prove that (v) of Theorem 4.3 and (vi) of Theorem 4.3 are equivalent. Starting with (v), we choose K such that $e - K$ is the cone with vertex e generated by all the segments $[e, x]$ lying on the unit sphere. Conversely, assuming (vi), let $\| x \| = 1$. Then $[e, x]$ is either on the boundary of $e - K$, or on the boundary of $-e + K$. In both cases it is on the unit sphere.

The equivalence of (vi) and (vii) is immediate. Q.E.D.

Remark. The equivalence Th. 4.2(i) $\Leftrightarrow$ Th. 4.2(iii) $\Leftrightarrow$ Th. 4.3(vii) are found in Lacey [15]. The equivalence Th. 4.2(i) $\Leftrightarrow$ Th. 4.3(v) is due to Taylor [21].

PROPOSITION 4.4. *Every Banach space E is isomorphic to a regular shadow.*

Proof. We embed E in a $C(S)$ space. Let $f_0 \in E$, $\| f_0 \| = 1$ be arbitrary. We show that it can be interchanged with 1 by an autoisomorphism (in fact, with norm $\leqslant 5$).

Assume, without loss of generality, that $-1 \leqslant m = \min f_0 \leqslant \max f_0 = 1$. Since span$\{1, f_0\}$ is isometric to l_∞^2, it admits a norm-1 projection P in $C(S)$. Define the autoisomorphism ϕ by $\phi(f) = (I - P) f + \psi P(f)$, where $\psi(\alpha + \beta f_0) = \beta + \alpha f_0$. Since $\| \psi \| = 2 + m$, it easily follows that $\| \phi \| \leqslant 5$.

Remark. In view of the equivalence of (i) and (ii) in Theorem 4.2 this corollary is equivalent to a theorem of Behrends [2].

The next result is a simple corollary of Theorem 3.3(iv).

PROPOSITION 4.5. *Let* $\phi: S \to T$ *be a continuous map of the compact Hausdorff space S onto the compact Hausdorff space T, and let* $\phi^0: C(T) \to C(S)$ *be the canonical embedding defined by* $(\phi^0 f)(s) = f[\phi(s)]$. *Let F be a subset of* $C(T)$. *Then, we have*

$$\mathit{shadow}(\phi^0 F) = \phi^0(\mathit{shadow}\, F).$$

We close this section with two results about special types of regular shadows:

PROPOSITION 4.6. *Let $F \subset C(S)$. The shadow of $\{f^i; f \in F, i = 0, 1, 2\}$ is the set of all $g \in C(S)$ which are constant on each constancy set of F.*

Proof. Suppose $\mu \geqslant 0$ satisfies, for some $s_0 \in S$

$$\int f^i(s)\, d\mu(s) = f^i(s_0), \qquad \text{for all } f \in F, \quad i = 0, 1, 2. \tag{4.4}$$

Using this equality for $i = 0$ we conclude that μ is a probability measure. Using relation (4.4) again we conclude that Schwartz's inequality

$$f^2(s_0) = \left(\int f(s)\, d\mu(s)\right)^2 \leqslant \int f^2(s)\, d\mu(s) = f^2(s_0)$$

turns into an equality, for all $f \in F$. Hence, each $f \in F$ is constant almost everywhere with respect to μ. Thus, μ is supported at a constancy set A of F. The set A must contain s_0, and we easily conclude that $\int g(s)\, d\mu(s) = g(s_0)$, for each g that is constant on A. Since this has to hold for each constancy set, the proof is complete.

Similar considerations yield.

PROPOSITION 4.7. *Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of positive numbers, and let $F = \{f_n\}_{n=1}^\infty$ be a sequence of functions of $C(S)$. Assuming that $\sum_{n=1}^\infty \alpha_n f_n^2$ converges, we have:*

$$\textit{shadow}\left(F \cup \left\{1, \sum_{n=1}^{\infty} \alpha_n f_n^2\right\}\right) = \{g \in C(S);\ g \textit{ is constant on each constancy set of } F\}.$$

Propositions 4.6 and 4.7 generalize results of Franchetti [8], Freud [9] and Grossman [10].

5. THE CHARACTERIZATION OF KOROVKIN SUBSPACES

Following the discussion in the last chapter of the characterization of regular Korovkin shadows, we obtain an intrinsic characterization of Korovkin subspaces. It is illuminating (and somewhat surprising) to find that the two characterizations are quite similar.

THEOREM 5.1. *Let E be a Banach space. The following statements are equivalent:*

(i) *E is linearly isometric to a space $A(K)$ of affine continuous functions on a compact convex set K in a locally convex space, with* ext *K closed in K.*

(ii) *E is linearly isometric to a regular Korovkin subspace F in some C(S), where S is a compact Hausdorff space.*

(iii) *The unit ball B^* of the dual space E^* has a face Q such that* $\operatorname{ext} Q$ *is w^*-closed and* $B^* = \operatorname{conv}[Q \cup (-Q)]$.

Moreover, K of (i) *and Q of* (iii) *are affinely homeomorphic and S of* (ii) *is homeomorphic to* ext Q.

Proof. We start by proving that (i) $\Rightarrow$ (ii). We may assume that $E = A(K)$. Let now $S = \operatorname{ext} K$.

If $f \in A(K)$ we let $\tilde{f}$ be its restriction to S. Clearly, $\tilde{f} \in C(S)$. The set of restrictions F

$$F = \{\tilde{f} \in C(S); f \in A(K)\}$$

contains 1. We proceed to show that F is a Korovkin subspace. Let $s_0 \in S$ be arbitrarily fixed, and let $\mu \geqslant 0$ be a positive linear functional on $C(S)$ such that

$$\mu(\tilde{f}) = \tilde{f}(s_0) = f(s_0), \qquad \text{for all } \tilde{f} \in F. \tag{5.1}$$

Since F contains 1, it follows that μ corresponds to a probability measure on S. Let $y \in K$ be the barycenter of μ (which exists by compactness and convexity of K), i.e.

$$\mu(f) = f(y), \qquad \text{for all } f \in A(K). \tag{5.2}$$

Since $A(K)$ trivially separates K, (4.1) and (4.2) imply that $y = s_0$. Since $s_0 \in \operatorname{ext} K = S$, it follows that $y \in S$ so that $f(y) = \tilde{f}(y)$, and (4.2) implies that

$$\mu(\tilde{f}) = \tilde{f}(s_0) \qquad \text{for all } \tilde{f} \in C(S).$$

so that F is indeed a K-set.

We next prove that (ii) $\Rightarrow$ (iii). We may assume $E = F$. Let $\sigma\colon S \to B^*$ be the canonical mapping, i.e., $(\sigma s)(x) = x(s)$. This is a homeomorphism of S onto $\operatorname{ext} Q$, where $Q = \{\phi \in B^*, \phi(1) = 1\}$ as in Section 4.

Indeed, $\operatorname{ext} Q \subset \sigma(S)$ as in Theorem 4.2. If $\operatorname{ext} Q \neq \sigma(S)$, then there exists $s \in S$, $\alpha \in (0, 1)$, and $p, q \in Q$ such that $\sigma(s) = \alpha p + (1 - \alpha) q$. We extend now p, q to probability measures λ, μ on S, satisfying $\alpha\lambda + (1 - \alpha)\mu = \nu_s$ on F. Since this contradicts the Korovkin property for F, we conclude that $\sigma(S) = \operatorname{ext} Q$. Thus, $\operatorname{ext} Q$ is w^*-closed. The proof that $B^* = \operatorname{conv}[Q \cup (-Q)]$ is identical to the corresponding proof in Theorem 4.2.

The proof that (iii) $\Rightarrow$ (i) is essentially identical to the corresponding proof in Theorem 4.2.

Since $A(K)$ determines K (up to affine homeomorphism) it follows that K and Q determine each other and S uniquely. Q.E.D.

COROLLARY 5.2. *If μ is a σ-finite measure with an atom, then $L_1(\mu)$ is isometric to a Korovkin subspace.*

Proof. It is easily checked that the set of extreme points of the unit ball of $L_\infty(\mu) = [L_1(\mu)]^*$ is w^*-closed. If A_0 is an atom for μ, define $x_0 = \mu(A_0)^{-1} \chi_{A_0}$. Then

$$Q = \{\phi \in L_\infty(\mu); \|\phi\| = 1 = \phi(x_0)\}$$

satisfies the conditions of statement (iii) of Theorem 5.1.

Recalling that where K is a simplex, ext K is closed if and only if $A(K)$ is a lattice (see [1]), we now pose the following question.

PROBLEM 5.3. Does there exist an order-norm characterization of the spaces $A(K)$, where K is a compact convex set with ext K closed?

In general, a Korovkin subspace may not contain the function 1, but it has to contain a positive function. This can be deduced from the $(\mathscr{M}^+, \mathscr{J})$-condition and the Hahn–Banach theorem (see [5]). This motivates the following definition:

DEFINITION 5.4. A subspace of $C(S)$ is called *quasiregular* if it contains a positive function.

LEMMA 5.5. *A Banach space E is linearly isometric to a quasiregular subspace in some $C(S)$, where S is a compact Hausdorff space, if and only if there exists an $x_0 \in E$ such that*

$$\text{ext } B^* \subset Q_{x_0} \cup (-Q_{x_0}) \tag{5.3}$$

where $Q_{x_0} = \{\phi \in B^*; \phi(x_0) \geqslant 1\}$.

Proof. Assume $E = F$ is quasiregular, and let $0 < p \in E$. Set $x_0 = p/(\min p)$. Let ρ be the restriction map from the unit ball Σ of $[C(S)]^*$ to B^*. If $\phi \in \text{ext } B^*$ then (as in Theorem 4.2) $\rho^{-1}\phi = \pm\nu_s$ for some $s \in S$. Hence,

$$|\phi(p)| = |p(s)| \geqslant \min p$$

or

$$|\phi(x_0)| = \frac{|\phi(p)|}{\min p} \geqslant 1, \qquad \text{i.e.,} \qquad \phi \in Q_{x_0} \cup (-Q_{x_0}).$$

Conversely, assuming (5.3), we associate with each $x \in E$ the functional $\hat{x}$, $\hat{x}(q) = q(x)$ defined on Q.

We then have, using (5.3),

$$\begin{aligned} \|x\| &= \max\{|\phi(x)|; \phi \in B^*\} = \max\{|\phi(x); \phi \in \text{ext } B^*\} \\ &= \max\{|\hat{x}(q)|, q \in Q\}, \end{aligned}$$

implying that $x \to \hat{x}$ is a linear isometry of E into $C(Q)$; the image contains the function $\hat{x}_0$ which is $\geqslant 1$ on Q, i.e., it is quasiregular. Q.E.D.

PROPOSITION 5.6. *The following classes of Banach spaces are not isometric to a Korovkin subspace:*

(a) *Any Banach space E for which 0 is in the w^*-closure of the extreme points of B^*.*

(b) *$C_0(T)$ spaces, where T is locally compact, but not compact.*

(c) *$L_1(\mu)$ spaces where μ is an atomless σ-finite measure.*

(d) *Smooth spaces of dimension at least 2.*

Proof. (a) and (c) follow immediately from Lemma 5.5. (b) is a special case of (a). We set to prove (d). Let $x_0 \in E$ be as in Lemma 5.5. Take $\phi \in E^*$ such that $\|\phi\| = 1$, $\phi(x_0) = 0$. Applying the Bishop–Phelps theorem (see [3]) we obtain a function $\psi \in E^*$ and $x \in E$ such that $\|\psi\| = \psi(x) = \|x\| = 1$ and $\|\psi - \phi\| < 1$. We now have

$$|\psi(x_0)| \leqslant |\phi(x_0)| + \|\psi - \phi\| < 1$$

so that $\psi \notin Q_{x_0} \cup (-Q_{x_0})$. By Lemma 5.5 it follows that $\psi \notin \operatorname{ext} B^*$. Thus, there exist an $\alpha \in (0, 1)$, and $\phi_1, \phi_2 \in B^*$, $\phi_1 \neq \phi_2$, for which $\psi = \alpha\phi_1 + (1 - \alpha)\phi_2$. This implies $\phi_1(x) = \phi_2(x) = 1$, contradicting the assumption of smoothness of E.

THEOREM 5.6. *The following statements are equivalent for a Banach space E:*

(i) *E is isometric to a Korovkin subspace of some $C(S)$.*

(ii) *For some $x_0 \in E$ there exists a w^*-closed $\Phi \subset \{\phi \in B^*; \phi(x_0) \geqslant 1\}$ satisfying:*

(a) *Φ is a boundary of E (i.e., $\|x\| = \sup\{|\phi(x)|; \phi \in \Phi\}$, for all $x \in E$).*

(b) *The radial projection τ of Φ onto $\{\phi; \phi(x_0) = 1\}$ is a one-to-one mapping of Φ onto* ext $\overline{\operatorname{conv}}^{w^*} \tau\Phi$.

Proof. (i) $\Rightarrow$ (ii). We may assume that $E \subset C(S)$ is a Korovkin subspace. Let $x_0 \in E$ be a positive function. Consider now the isomorphism $Tx = x/x_0$ from E onto another Korovkin subspace, F. Let $Q \subset F^*$ be as in (iii) of Theorem 5.1. Considering the elements of E^* as members of F^*, we see that now $\phi \in Q \Leftrightarrow \{\phi(x_0) = 1, \phi \geqslant 0\}$. Let $\Phi = \sigma(S)$, where σ is the canonical mapping. For $\phi \in \Phi$, the radial projection $\tau\phi = \phi/\phi(x_0)$ is in Q. Furthermore, $\tau\Phi$ is w^*-compact, hence it contains ext $\overline{\operatorname{conv}}^{w^*} \tau\Phi$. However, $\tau\sigma$ is the canonical embedding corresponding to F, so that τ is in fact a homeomorphism of Φ onto ext Q and $Q = \overline{\operatorname{conv}}^{w^*} \tau\Phi$.

(ii) $\Rightarrow$ (i). Let $S = \Phi$ in its w^*-topology and let $\sigma\colon E \to C(S)$ be the canonical embedding. Let $Tx = x/x_0$, $F = TE$ as above. The conditions (ii) show that $\tau\sigma$ carries S homeomorphically onto ext $\overline{\operatorname{conv}}^{w^*} \tau\Phi \subset Q$. If $\phi \in \operatorname{ext} Q$ we have $\phi \in \tau\sigma(S) = \tau\Phi$ as above. By Theorem 5.1 it follows now that F is a Korovkin subspace, and therefore E is a Korovkin subspace as well.

EXAMPLES. (1) The simplest example of a regular Korovkin subspace is $F = \operatorname{span}\{1, \cos x, \sin x\}$ in $C[0, 2\pi]$. A straightforward computation shows that F is linearly isometric to $E = (R \oplus l_2^2)$ (i.e. $\| \alpha + \beta \cos x + \gamma \sin x \| = | \alpha | + \sqrt{\beta^2 + \gamma^2}$). The dual E^* is $(R \oplus l_2^2)_\infty$, and the unit ball B^* is the "tomato can" (Fig. 5.1).

(2) Set

$$Q = \operatorname{conv}[\{x^2 + y^2 = 1, z = 0\} \cup \{(1, 0, 1)\} \cup \{+1, 0, -1)\}]$$

(see Fig. 5.2). Here $\operatorname{ext} Q$ is not closed, but $A(Q)$ is regular. By Theorem 5.1, $A(Q)$ is not linearly isometric to a regular Korovkin subspace. However, by Theorem 4.2, $A(Q)$ is linearly isometric to a regular shadow.

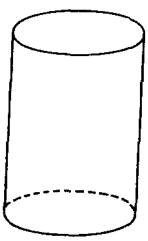


FIG. 5.1

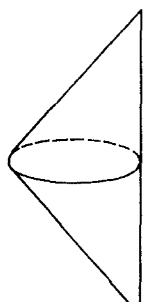


FIG. 5.2

6. SHADOWS OF FINITE SUBSETS IN $C(S)$

The power and novelty of Korovkin's ideas is naturally best apparent when the test system is finite. Finite K-sets and their characterizations have therefore attracted much attention (see e.g. [14], [9], [20] and part of [5]). In this section we devote our attention to shadows of finite subsets. We will prove a general characterization theorem, and then examine in detail the structure of shadows of triplets which fail to be Tchebycheff systems. In particular, we obtain complete results for the codimension of the shadow of $\{f_1, f_2, f_3\}$, when span $\{f_1, f_2, f_3\}$ is quasiregular.

Notation. Let $\{f_0, ..., f_n\}$ be a finite set. We denote the span of this set by $[f_0, ..., f_n]$.

We start with the adaptation to shadows of Šaškin's [20] useful characterization of finite K-systems.

THEOREM 6.1. *Let $f_0, \ldots, f_n$ be in $C(S)$. Then f_0 is in the shadow of $\{f_1, \ldots, f_n\}$ if and only if the following statement is valid for all m:*

whenever $s_1, \ldots, s_m \in S$ and $\alpha_1, \ldots, \alpha_m \geqslant 0$ are such that

$$f_i(s_0) = \sum_{j=1}^{m} \alpha_j f_i(s_j), \qquad i = 1, \ldots, m \tag{6.1}$$

then the equality holds also for $i = 0$.

Moreover, it suffices that the statement hold for $m \leqslant n + 2$, and in the special case where S has at most $n + 1$ components even $m = n + 1$ would suffice.

Proof. The necessity is obvious, since the statement can be represented as $f_0 \in \lambda_\perp$, where $\lambda = \sum_{j=1}^{m} \alpha_j \nu_{s_j} - \nu_{s_0}$ belongs to $\mathscr{M}_\nu^+ \cap [f_1, \ldots, f_n]_\perp$.

We now prove sufficiency. Assume the statement holds for all m, and let $\mu \geqslant 0$ and s_0 be such that

$$\mu(f_i) = f_i(s_0), \qquad i = 1, \ldots, m. \tag{6.2}$$

Consider the canonical embedding σ of S in $[f_0, \ldots, f_n]^*$ defined by $(\sigma s)(f_i) = f_i(s)$, $i = 0, \ldots, n$. σ is a continuous function from S to $[f_0, \ldots, f_n]^*$, so that σS is compact. Since $[f_0, \ldots, f_n]^*$ is finite dimensional, $\mathrm{conv}(\sigma S)$ is also compact.

Define $\rho = \mu \circ \sigma^{-1}/\| \mu \|$. This is a probability measure on σS and is therefore represented by some point $y_0 \in \mathrm{conv}(\sigma S)$. Hence, there exist $s_1, \ldots, s_m \in S$ and $\alpha_1, \ldots, \alpha_m \geqslant 0$ with $\sum_{j=1}^{m} \alpha_j = 1$ such that

$$\sum_{j=1}^{m} \alpha_j \sigma(s_j) = y_0. \tag{6.3}$$

Note that by Caratheodory's theorem, we can have (6.3) with $m \leqslant n + 2$, and if S has at most $n + 1$ components, $m \leqslant n + 1$.

Using (6.3), we have

$$\rho(\hat{f}_i) = \hat{f}_i(y_0) = \sum_{j=1}^{m} \alpha_j f_i(s_j), \qquad \text{for } i = 0, 1, \ldots, n,$$

where $\hat{f}_i$ is the canonical image of f_i in $[f_0, \ldots, f_n]^{**}$, defined by $\hat{f}_i(y) = y(f_i)$ if $y \in [f_0, \ldots, f_n]^*$. Using now (6.2), we have

$$f_i(s_0) = \mu(f_i) = \| \mu \| \rho(\hat{f}_i) = \sum_{j=1}^{n} \| \mu \| \alpha_j f_i(s_j), \qquad i = 1, \ldots, n.$$

Since the statement is assumed valid, we conclude that

$$f_0(s_0) = \sum_{j=1}^{n} \| \mu \| \alpha_j f_0(s_j) = \| \mu \| \rho(\hat{f}_0) = \mu(f_0). \qquad \text{Q.E.D.}$$

PROBLEM 6.2. Is it sufficient to assume the statement for $m \leqslant N$, where N is smaller than $n + 2$?

We now turn to the study of the smallest subspaces that cast shadows, namely the spans of triplets $\{f_1, f_2, f_3\}$.

It is classical (cf. [14]) that the following statements are equivalent for $F = \{f_1, f_2, f_3\} \subset C[a, b]$

(i) F is a Korovkin system

(ii) F is a strict Korovkin system

(iii) F is a Tchebycheff system (i.e. the determinant of $\| f_i(s_j) \|_{i,j=1}^{3}$ does not vanish for distinct s_j's).

(iv) $\{f_1, f_2, f_3\}$ is a 3-dimensional Tchebycheff subspace (i.e., each $f \in [f_1, f_2, f_3] \backslash \{0\}$ has at most two distinct zeros in $[a, b]$.

The following theorem deals with a relaxation of the conditions in (iv).

THEOREM 6.3. *If $[f_1, f_2, f_3] \subset C(S)$ contains a function $f \neq 0$ possessing m distinct roots in S, then the codimension of F in $C(S)$ is at least $m - 2$.*

Proof. With no loss of generality we may assume that $f_3(s_i) = 0$, $i = 1,..., m$. Then, the canonical image $(\sigma s_1, ..., \sigma s_m)$, lies in the plane $x_3 = 0$ in $R^3 = [f_1, f_2, f_3]^*$. Note that for any 3 points in the plane, either one of them lies in the convex cone, with vertex 0, generated by the other two, or 0 lies in the convex hull of the three points. Keeping σs_1 and σs_2 fixed and varying all the rest, we obtain in this way $m - 2$ linearly independent relations of the form $\sum_1^3 \alpha_j \sigma s_{i_j} = 0, \sum_1^3 | \alpha_i | > 0$ which are equivalent to a linearly independent set of $m - 2$ annihilating measures of the form $\sum_1^3 \alpha_j v_{s_{i_j}} = 0$. Q.E.D.

Remark 6.4. No similar bound in the opposite direction exists even for $C[a, b]$ (except for $m = 2$, of course). This is evident from the next example. Let $F = \{1, t \sin t, t \cos t\}$ in $C[0, 2\pi]$ (which is distinct from $C_{2\pi}$– the space of periodic functions).

It is clear that no element of F can have more than 3 zeros; hence, for all f, $m \leqslant 3$.

On the other hand, consider the canonical image of $[0, 2\pi]$ in $R_3 = (\operatorname{span} F)^*$. This is a planar spiral in the plane $x_1 = 1$. There are infinitely many lines intersecting the graph at three points, and each such line yields an annihilating measure. Explicitly, we have the infinite family of linearly independent annihilating measures given by

$$\frac{\pi + t}{\pi + 2t} \nu_t + \frac{t}{\pi + 2t} \nu_{\pi+t} - \nu_0, \qquad 0 < t \leqslant \pi.$$

Thus, the shadow of F has infinite codimension in $C[0, 2\pi]$, despite the fact that $m \leqslant 3$.

Remark 6.5. The geometric argument used here is not valid for higher dimensions, but this is not surprising, since a Korovkin system of n elements, $n > 3$, does not have to be a Tchebycheff system.

For the special case of a quasiregular set $\{f_1, f_2, f_3\}$ in $C[a, b]$, the information is much more extensive.

THEOREM 6.6. *If $[f_1, f_2, f_3] \subset C[a, b]$ contains a positive function, then the codimension of its shadow in $C[a, b]$ is 0, 1 or ∞.*

Proof. Suppose there exists a positive function. With no loss of generality we may assume that $f_1 > 0$. Consider the canonical image of $\sigma[a, b]$ and its radial projection $\rho\sigma[a, b]$ on the plane $x_1 = 1$, (i.e. $(\rho\sigma[a, b], \sigma[a, b], 0)$ are on one line and $\rho\sigma[a, b]$ is on the plane $x_1 = 1$).

There are three possibilities:

(i) $\rho\sigma(t)$ is one to one (i.e., $\rho\sigma[a, b]$ is a simple Jordan curve) and $\rho\sigma[a, b]$ is convex (i.e., no line intersects it at more than 2 points). This is equivalent to F being a K-system.

(ii) $\rho\sigma[a, b]$ is a convex, simple *closed* Jordan curve (i.e., $\rho\sigma(a) = \rho\sigma(b)$ is the only point covered twice). Then $\nu_a - \nu_b$ is the only measure in $[f_1, f_2, f_3]^\perp \cap \mathscr{M}_\nu^+$, so that the shadow has codimension 1. This is for example the case $F = \{1, \cos t, \sin t\}$ in $C[0, 2\pi]$, where $\sigma[0, 2\pi]$ is the circle, and thus the shadow consists of all functions such that $f(0) = f[2\pi]$, i.e., all periodic functions.

(iii) There exists a line that contains 3 points $\rho\sigma(s_i)$. Then by continuity, there are infinitely many lines with this property and each line corresponds to an annihilating measure. This is exemplified by $(1, t \cos t, t \sin t)$ of Remark 6.4.

Remark 6.7. For each $n > 0$ there exist shadows with codimension n. Indeed, this follows from Theorem 2.1 by taking h with n zeros in $[a, b]$ (e.g., the shadow of $(t - \frac{1}{3})(t - \frac{2}{3})$, $(t - \frac{1}{3})(t - \frac{2}{3})\, t$, $(t - \frac{1}{3})(t - \frac{2}{3})\, t^2$ in $C[0, 1]$, has codimension 2). Note that in this case $\sigma[a, b]$ passes through the origin.

7. MINIMAL KOROVKIN SETS AND SUBSPACES

It is well known (cf. [23]) that there exists an infinite dimensional K-subspace such that no finite dimensional subspace thereof is a K-subspace for $C(S)$.

Indeed, the subspace of all piecewise linear functions in $C[0, 1]$ is such a subspace.

This raises the question whether a similar situation prevails where the given K-subspace is finite dimensional. We will show in this section that the answer is in the affirmative.

DEFINITION 7.1. Let S be given. The minimal cardinality of a Korovkin set for $C(S)$ is called the Korovkin-cardinality (K-cardinality) of S, and denoted by $k(S)$.

We note that if S is a metric space, $k(S)$ is a topological invariant of S ([5]). We observe also that if $k(S) \leqslant \aleph_0$, then S is metrizable, and that for a metric space $k(S)$ can be determined in terms of the minimal dimension of the sphere into which S can be topologically embedded ([5]).

We now introduce two new concepts.

DEFINITION 7.2. Let F be a Korovkin set for $C(S)$. If no proper subset of F is a K-set for $C(S)$ then F is called a *minimal K-set.*

DEFINITION 7.3. Let $\bar{F}$ be a K-subspace for $C(S)$. If no proper subspace of $\bar{F}$ is a K-subspace for $C(S)$ then $\bar{F}$ is called a *minimal K-subspace.*

The following theorem demonstrates that the concepts are distinct.

THEOREM 7.4. *Let F be a K-set for $C(S)$. If span F is a minimal K-subspace then F is a minimal K-set. On the other hand, there exists a minimal K-set F such that span F is not a minimal K-subspace.*

Proof. The first part of the theorem is obvious. For the second part, consider $C[0, 2\pi]$ and $F = \{1, t, \cos t, \sin t\}$. The set F is a minimal K-set for $C[0, 2\pi]$. Indeed, any subset of two functions does not cast any shadow (Theorem 3.3). Checking 3-function subsets, we easily observe that none of the triplets is a Tchebycheff system (T-system) on $[0, 2\pi]$ ($\{1, \cos t, \sin t\}$ is a T-system on the circle but not on $[0, 2\pi]$). Invoking Korovkin's theorem [14], we conclude that these are not K-sets.

On the other hand, span F is not a minimal K-subspace, since span $[1, \cos t, t - \sin t]$ is a K-subspace. This is a corollary of the fact that $\{1, \cos t, t - \sin t\}$ is a strict K-set, as we will presently show, by constructing for each $t_0 \in [0, 2\pi]$ a function $\psi_{t_0}(t)$ such that $\psi_{t_0}(t) \geqslant 0$ for all t, with equality only if $t = t_0$. The functions are given by: $\psi_0(t) = t - \sin t$, $\psi_{2\pi}(t) = 4\pi - (1 - \cos t) - 2(t - \sin t)$, while for $t_0 \in (0, 2\pi)$ we define $\psi_{t_0}(t) = (\cos t - \cos t_0)(1 - \cos t_0) + (\sin t_0)[t - \sin t - (t_0 - \sin t_0)]$. It is a matter of simple computation to verify that $\psi_{t_0}(t)$ satisfies the required conditions. Q.E.D.

The last theorem shows that the concept of a minimal subspace is more restrictive. We now prove:

THEOREM 7.5. *The subspace* $[1, \sqrt{t}, \cos t, \sin t]$ *is a minimal K-subspace for* $C[0, 2\pi]$.

COROLLARY 7.6. *There exists a minimal K-subspace in* $C[0, 2\pi]$ *whose cardinality is larger than the K-cardinality of* $[0, 2\pi]$.

Proof. (a) We note first that $\{1, \sqrt{t}, \cos t, \sin t\}$ is a K-system for $C[0, 2\pi]$. This can be proved in two ways, both of which are instructive:

(1) Let f be an arbitrary function of $C[0, 2\pi]$ and consider ψ defined by

$$\psi(t) = f(t) - [f(2\pi) - f(0)] \sqrt{t} / \sqrt{2\pi}.$$

Then $\psi(0) = \psi(2\pi) = f(0)$, so that $\psi \in C_{2\pi}$ and therefore by Korovkin's trigonometric theorem [14], ψ belongs to the shadow of $\{1, \cos t, \sin t\}$; this implies that f belongs to the shadow of $\{1, \sqrt{t}, \cos t, \sin t\}$.

(2) We make use of the observation that a T-system on $[a, b]$ is a strict K-system on this interval (this follows from a theorem of Krein on the zeros of T-systems (cf. [5]). Hence it suffices to show that our set is a T-system on $[0, 2\pi]$. A close scrutiny easily confirms that no line $a + b\sqrt{t}$ can intersect $\cos(t + \alpha)$ (α-arbitrarily chosen) at more than three points, i.e., no linear combination of the form $a + b\sqrt{t} + c \cos t + d \sin t$ vanishes more than 3 times. This is equivalent to the statement that the set is a T-system.

We now proceed to show that no proper subspace of $[1, \sqrt{t}, \cos t, \sin t]$ is a K-subspace. Two dimensional subspaces cannot be K-subspaces, and a 3-dimensional subspace can be a K-subspace if and only if it is a T-subspace [14].

Hence, it suffices to show that there exists no 3 dimensional T-subspace. We have to check the following possibilities (where A can take any value).

(a) $[\cos t + A \sin t, \sqrt{t}, 1]$
(b) $[\cos t + A\sqrt{t}, \sin t, 1]$
(c) $[\cos t + A, \sin t, \sqrt{t}]$
(d) $[\cos t, \sin t + A\sqrt{t}, 1]$
(e) $[\cos t, \sin t + A, \sqrt{t}]$
(f) $[\cos t, 1 + A\sqrt{t}, \sin t]$
(g) $[\sin t, \sqrt{t}, 1]$.

In each case, we have to exhibit a function in the span with three zeros.

Cases b, c, g, f are taken care of easily, by noting that $\sin t$ has three zeros in $[0, 2\pi]$.

(a) $[\cos t + A \sin t, \sqrt{t}, 1]$.

If $A \neq 0$, consider $\cos t + A \sin t - 1 = f(t)$. Then $f(0) = f(2\pi) = 0$ $f'(0) = f'(2\pi) \neq 0$. Hence there must be an intermediate zero. If $A = 0$ consider $\cos t - 1 + \sqrt{t}/\sqrt{\pi} = f(t)$. Then $f(0) = 0$, $f(\epsilon) \sim \sqrt{\epsilon}/\sqrt{\pi} > 0$ for small ϵ, $f(\pi) = -2 + 1 = -1 < 0$, $f(2\pi) = \sqrt{2} > 0$. Hence there exists a zero in $(0, \pi)$ and one in $(\pi, 2\pi)$.

(d) $[\cos t, \sin t + A\sqrt{t}, 1]$, $A \neq 0$ [($A = 0$ is subsumed in (b))]. Consider

$$\sin t + A\sqrt{t} - A\sqrt{\pi}(1 - \cos t) = f(t)$$
$$f(0) = 0, \qquad f(\epsilon) \sim A\sqrt{\epsilon}$$
$$f(\pi) = -A\sqrt{\pi}$$
$$f(2\pi) = -A\sqrt{2\pi}$$

hence $(f(\epsilon), f(\pi), f(2\pi))$ has two sign changes, or two zeros. Q.E.D.

(e) $[\cos t, \sin t + A, \sqrt{t}]$, $A \neq 0$ [$A = 0$ is subsumed in (c)]. Consider

$$\sin t + A - A\cos t = f(t),$$
$$f(0) = 0, \qquad f(2\pi) = 0, \qquad f'(0) = f'(2\pi) = 1.$$

Hence there must exist an intermediate zero. Q.E.D.

The last topic demonstrates anew the relevance of the structural properties of T-subspaces to the study of K-subspaces. The structure of T-subspaces has been examined by several authors in recent years. Nemeth [17] and Hadeler [11] established conditions for the existence of an $n - 1$ dimensional T-subspace for a given n-dimensional T-subspace of $C[a, b]$. Concrete examples of an n-dimensional T-subspace possessing no $n - 1$ dimensional T-subspace have been previously given only for $n = 3$ (by Volkov [22] and Hadeler [11]). The existence of an example for an arbitrary n can be derived from Zielke's recent result [26].

The foregoing discussion leads us to formulate some conjectures and open problems.

CONJECTURE 7.7. Consider $C[a, b]$. For each $n \geqslant 3$ there exists a minimal K-subspace of dimension n.

CONJECTURE 7.8. Consider $C[a, b]$. For each $n \geqslant 3$ there exists a minimal K-set of cardinality n.

Note that the validity of both conjectures for $n = 3, 4$ has been established.

It is known that a T-system is a strict K-system. We suspect that the following is a partial converse.

CONJECTURE 7.9. Consider $C[a, b]$, and let $\{f_1, ..., f_n\}$ be a minimal strict K-system. Then it is a T-system.

The following two problems concern minimal cardinality systems:

PROBLEM 7.10. Find a characterization of minimal cardinality K-systems in $C(S)$.

The answer is known only for $S = [a, b]$, and the question is of interest even for $S = [a, b] \times [a, b]$.

PROBLEM 7.11. Suppose $\{f_1, ..., f_n\}$ is a minimal cardinality K-set on $C(S)$. Is the codimension of the shadow of every subset infinite?

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