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On an Explicit Duck Solution and Delay in the Fitzhugh–Nagumo Equation

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The existence of delay in the FitzHugh–Nagumo equation was proved by J. Su. It is shown that an explicit duck solution and delay exist in this equation under certain conditions with respect to the coefficients by using the E. Benoît's criterion.

Key Words: FitzHugh–Nagumo equation, constrained systems, delayed phenomena, duck solutions.

1. INTRODUCTION

In the early 1960s, FitzHugh [5] and Nagumo *et al.* [7] proposed simplified systems which contain the main qualitative features of the original Hodgkin–Huxley system [6] in 1952. These systems describe the generation and propagation of the nerve impulse along the giant axon of the squid. The above systems, so-called the FitzHugh–Nagumo (FHN) equations for the space clamped segment of the axon, have the autonomous form

$$\begin{cases} dv/dt = -\rho(v) - w + I, \\ dw/dt = b(v - \gamma w), \qquad \rho(v) = v(v-1)(v-a), \end{cases} \tag{1.1}$$

where $0 < a < 1/2$, b and γ are positive constants. Here $v(t)$ denotes the potential difference at the time t across the membrane of the axon and $w(t)$ represents a recovery current which is often taken to be the sum of all ion flows [6]. Furthermore, I is an injected electric current on the membrane. It is a control or bifurcation parameter. The first of (1.1) expresses Kirchhoff's law applied to the membrane; the second relates the recovery

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current with the potential. From biophysical considerations, it is reasonable to restrict γ so that

$$\gamma_1 = 1/\gamma - (1-a+a^2)/3 > 0, \tag{1.2a}$$

$$\gamma_2 = (1-a+a^2)/3 - b\gamma > 0. \tag{1.2b}$$

FitzHugh [5] and Baer *et al.* [2] have pointed out that the coefficient b in the FHN equation is a very small constant. Some authors [13–15] have put $b = \varepsilon$ directly in this equation.

Taking account of this fact, we put

$$b = c\varepsilon, \tag{1.3}$$

where $\varepsilon > 0$ is a very small constant.

In addition, putting $I = I_0 + \varepsilon t$, the equation is newly formulated. Here I_0 is a constant.

The reason for putting these conditions is the following.

We consider the relation between b and ε in the formula $I = I_0 + \varepsilon t$. Strictly speaking, we may put $b = c\varepsilon_1$ and $I = I_0 + \varepsilon_2 t$. The point is that whether we should consider ε_1 and ε_2 are independent or not. In the asymptotic expansion for the solution, Baer *et al.* [2] have considered ε_1 and ε_2 are dependent each other ($\varepsilon = O(b^{3/2})$) in Section 2 and 4. They also have referred to $\varepsilon = O(b)$ and obtained some results though they have not stated the existence of duck solutions. In this background, we also assume that ε_1 and ε_2 are dependent each other and form the simplest assumption $\varepsilon_1 = \varepsilon_2 = \varepsilon$. We think that we will deal with some important cases of the FHN equation.

As the bifurcation parameter I varies very slowly and b satisfies (1.3), the system (1.1) becomes the non-autonomous form

$$\begin{cases} \varepsilon\, dv/dI = -\rho(v) - w + I, \\ dw/dI = c(v - \gamma w). \end{cases} \tag{1.4}$$

Furthermore, by changing the coordinates, the system (1.4) becomes the following autonomous form,

$$\begin{cases} dX/dI = 1, \\ dY/dI = c(Z - \gamma Y), \\ dZ/dI = (-\rho(Z) - Y + X)/\varepsilon, \end{cases} \tag{1.5}$$

where the conditions in (1.1) and (1.2) are still satisfied. In 1983, Benoît [3] proved the existence of a duck solution under the condition that "a pseudo singular point is a saddle point," see Section 3. In the following sections, it will be seen that the system (1.5) has a duck solution and delay by using the results of Benoît.

2. DELAYED PHENOMENA IN THE FHN EQUATION

In the system (1.1), if the current I is kept constant, (1.2a) ensures that the system (1.1) has a unique steady state solution $(v_0(I), w_0(I))$. This solution called the frame solution is determined by the equation

$$F(v, w) = \begin{pmatrix} -\rho(v) - w \\ bv - b\gamma w \end{pmatrix} = \begin{pmatrix} -I \\ 0 \end{pmatrix}. \tag{2.1}$$

The frame solution $(v_0(I), w_0(I))$ is uniquely determined by the bifurcation parameter I, since F in (2.1) is diffeomorphic. Its components $v_0(I)$ and $w_0(I)$ increase as I increases.

The linearized system of (1.1) for the frame solution is

$$\begin{cases} dv/dt = -\rho'(v_0(I))\, v - w, \\ dw/dt = bv - b\gamma w, \end{cases} \qquad \rho'(v) = d\rho/dv, \tag{2.2}$$

where $v = v - v_0(I)$, $w = w - w_0(I)$. The stability of the frame solution is determined by the two eigenvalues of the Jacobian matrix

$$M = \begin{pmatrix} -\rho'(v_0(I)) & -1 \\ b & -b\gamma \end{pmatrix}. \tag{2.3}$$

There exist I_- and I_+, where $I_- < (a+1)/3 < I_+$ such that whenever $I < I_-$ or $I > I_+$, the eigenvalues of M have negative real parts, i.e., the frame solution is stable and the eigenvalues of M have positive real parts if $I_- < I < I_+$, i.e., the frame solution is unstable.

Assume that the current I which is treated as a bifurcation parameter varies very slowly as the time goes by. Moreover, for simplicity, assume that the current $I(t)$ has the form of

$$I = I(t) = I_0 + \varepsilon t, \tag{2.4}$$

where $\varepsilon > 0$ is a very small constant and I_0 is a constant such that $I_0 < I_-$. Using I as an independent variable, the system (1.1) becomes the non-autonomous form:

$$\begin{cases} \varepsilon\, dv/dI = -\rho(v) - w + I, \\ \varepsilon\, dw/dI = bv - b\gamma w. \end{cases} \tag{2.5}$$

Note that the conditions of ρ in (1.1) and γ_1, γ_2 in (1.2) are still satisfied. If I is an independent bifurcation parameter, the following phenomenon will occur. The solution of (2.5) with the initial conditions,

$$v(I_0) = v_0(I_0), \qquad w(I_0) = w_0(I_0), \qquad I(0) = I_0, \tag{2.6}$$

stays close to its steady state solution until I reaches I_-, and then jumps away from the steady state solution shortly after I increases and passes I_-.

However, if I is as in (2.4), this phenomenon does occur but with delay. From a Hopf bifurcation structure, as I increases through I_-, the solution of (2.5) would turn to the large amplitude oscillations. That is, such a critical point is observed, but the value I_q of I at which it occurs is considerably delayed beyond the value I_-. In 1989, Baer, Erneux and Rinzel [2] proceeded an extensive computational experiment of the FHN equation for the delayed phenomena (or simply delay) and they began to consider the corresponding mathematical problem.

In 1993, Su [11] provided a rigorous proof of the results conjectured in [2] and [10] by considering the linearized equation of the system (2.5) for the steady state solution $(v(I_0), w(I_0))$. He showed that the solution of (2.5) starting from any point near the steady state solution at $I_0 < I_-$ stays near the steady one until I reaches $I_q > I_-$. Furthermore, in case that I_0 is close to I_-, a description of how the solution moves from the steady state solution to become a large amplitude solution after $I > I_q$ was given. The general case of it was considered by Neishtadt [8] and Arnold [1].

3. BENOÎT'S THEOREM CONCERNING CONSTRAINED SYSTEMS

First we consider a constrained system,

$$\begin{cases} dx/dt = f(x, y, z), \\ dy/dt = g(x, y, z), \\ h(x, y, z) = 0, \end{cases} \tag{3.1}$$

where f, g and h are defined in R^3.

The system (3.1) satisfies the following conditions:

(A1) f and g are of class C^1, and h is of class C^2,

(A2) the set $S = \{(x, y, z) \in R^3 : h(x, y, z) = 0\}$ is a 2-dimensional differentiable manifold and the set S intersects the set $T = \{(x, y, z) \in \mathrm{R}^3 : \partial h(x, y, z)/\partial z = 0\}$ transversely so that the set $PL = \{(x, y, z) \in S : \partial h(x, y, z)/\partial z = 0\}$ is a 1-dimensional differentiable manifold, (A3) either the value of f or that of g is nonzero at any point $p \in PL$.

Let $(x(t), y(t), z(t))$ be a solution of (3.1), then the following equation holds by differentiating $h(x, y, z)$ with respect to the time t:

$$h_x(x, y, z) f(x, y, z) + h_y(x, y, z) g(x, y, z) + h_z(x, y, z)\, dz/dt = 0, \quad (3.2)$$

where

$$h_\alpha(x, y, z) = \partial h(x, y, z)/\partial \alpha, \qquad \alpha = x, y, z.$$

The above system becomes the system

$$\begin{cases} dx/dt = f(x, y, z), \\ dy/dt = g(x, y, z), \\ dz/dt = -\{h_x(x, y, z) f(x, y, z) + h_y(x, y, z) g(x, y, z)\}/h_z(x, y, z), \end{cases} \quad (3.3)$$

where $(x, y, z) \in S \backslash PL$.

Remark. The system (3.1) coincides with the system (3.3) at any point $p \in S \backslash PL$.

Secondly in order to study the system (3.3), we consider the newly revised system:

$$\begin{cases} dx/dt = -h_z(x, y, z) f(x, y, z), \\ dy/dt = -h_z(x, y, z) g(x, y, z), \\ dz/dt = h_z(x, y, z) f(x, y, z) + h_y(x, y, z) g(x, y, z) \end{cases} \quad (3.4)$$

Note that the system (3.4) is well defined at any point of R^3. Therefore, the system (3.4) is well defined indeed at any point of PL.

Compare the solutions of (3.4) with those of (3.1) on $S \backslash PL$. The solutions of (3.4) coincide with those of (3.1) except the velocity and the orientation when they start from the same initial points. Thus each phase path is quite the same, that is, they have the very same orbits.

DEFINITION 3.1. A singular point of (3.4) is called a *pseudo singular point* of (3.1) and a set of the pseudo singular points PS is denoted as

$$PS = \{(x, y, z) \in PL : h_x(x, y, z) f(x, y, z) + h_y(x, y, z) g(x, y, z) = 0\}. \quad (3.5)$$

Moreover, the following conditions (A4) and (A5) are assumed.

(A4) For any $(x, y, z) \in S$, either the following holds

$$h_y(x, y, z) \neq 0, \qquad h_x(x, y, z) \neq 0,$$

that is, the surface S can be expressed as $y=\phi(x, z)$, or $x=\psi(y, z)$ in the neighborhood of PL. When $y=\phi(x, z)$, the following system, which restricts the system (3.4) on the surface S, is obtained using (A4):

$$\begin{cases} dx/dt = -h_z(x, \phi(x, z), z) f(x, \phi(x, z), z), \\ dz/dt = h_x(x, \phi(x, z), z) f(x, \phi(x, z), z) + h_y(x, \phi(x, z), z) g(x, \phi(x, z), z). \end{cases} \tag{3.6}$$

(A5) All the singular points of (3.6) are nondegenerate, that is, the matrix induced from the linearized system of (3.6) at a singular point has two nonzero eigenvalues. Note that all the points contained in PS are the singular points of (3.6). When $x=\psi(y, z)$, a similar equation is obtained in the same manner.

Definition 3.2. If the two eigenvalues λ_1, λ_2 mentioned in (A5) have the property that $\lambda_1 < 0 < \lambda_2$, a pseudo singular point of (3.1) is called a pseudo singular saddle point.

Thirdly, we consider the new system

$$\begin{cases} dx/dt = f(x, y, z), \\ dy/dt = g(x, y, z), \\ \varepsilon_n \, dz/dt = h(x, y, z), \end{cases} \tag{Bn}$$

where f, g and h are the same as system (3.1) and $\varepsilon_n \to 0$ as $n \to \infty$. For a fixed sufficiently large n, (Bn) devides the surface $S\backslash PL$ into two parts: one is an attractive region and the other is a repulsive region.

A solution which starts from a neighborhood of the attractive part goes rapidly toward S perpendicularly and then goes slowly along S. A solution which starts from a neighborhood of repulsive part leaves from S rapidly unless it starts at $p \in S\backslash PL$. Now let us introduce a standard duck solution defined on systems (Bn), $n=1, 2, \ldots$.

Definition 3.3. A standard duck solution, or simply a standard duck, on the systems (Bn), $n=1, 2, \ldots,$ is defined as follows.

It is a sequence of solutions $(x_n(t), y_n(t), z_n(t))$ of (Bn), $n=1, 2, \ldots,$ such that

(1) the solution $(x_n(t), y_n(t), z_n(t))$ is defined for $t \in (c_n, d_n)$,

(2) there are two closed disjoint subintervals $[c'_n, d'_n]$ and $[c''_n, d''_n]$ of (c_n, d_n) in which for any $t \in [c'_n, d'_n]$, the point $(x_n(t), y_n(t), z_n(t))$ lies in a sufficiently small neighbourhood of the attractive part of S, and for any

$t \in [c_n'', d_n'']$, the point $(x_n(t), y_n(t), z_n(t))$ lies in a sufficiently small neighbourhood of the repulsive part of S, and

(3) as $n \to \infty$, the solution curves $(x_n(t), y_n(t), z_n(t))$, $t \in (c_n, d_n))$ converge to a curve C of the finite length in S, and the curve C is devided into two parts C', C''. The arc C' belongs to the attractive part of S, the arc C'' belongs to the repulsive part and the lengths of C', C'' are not zeroes.

Now we consider the following system in a non-standard version,

$$\begin{cases} dx/dt = f(x, y, z), \\ du/dt = g(x, y, z), \\ \varepsilon\, dz/dt = h(x, y, z), \end{cases} \tag{B}$$

where ε is infinitesimally small. Here a definition of a non-standard duck solution is given as follows.

DEFINITION 3.4. A solution $(x(t), y(t), z(t))$ of the system (B) is called a non-standard duck solution, or simply a non-standard duck, if there are standard $t_1 < t_0 < t_2$ such that

(1) $^*(x(t_0), y(t_0), z(t_0)) \in S$, where $^*(X)$ denotes the standard part of X,

(2) for $t \in (t_1, t_0)$ the segment of the trajectory $(x(t), y(t), z(t))$ is infinitesimally close to the attracting part of the slow curves,

(3) for $t \in (t_0, t_2)$, it is infinitesimally close to the repelling part of the slow curves, and

(4) the attracting and repelling parts of the solution curve are not infinitesimally small.

Zvonkin and Shubin [12] proved the following theorem concerning a standard duck and a non-standard duck.

THEOREM 3.1. *There exists a standard duck on the systems* (Bn), $n = 1, 2, \ldots$, *if and only if there exists a non-standard duck on the system* (B).

Benoît [3] investigated the relations between the system (3.1) and the systems (Bn), $n = 1, 2, \ldots$. By introducing a method of a non-standard analysis (Nelson's version [9]) on each system (Bn), he got a result, which is essentially the same as the following theorem.

THEOREM 3.2 (Benoît). *If the system* (3.1) *has a pseudo saddle point, then the systems* (Bn), $n = 1, 2, \ldots$, *have a standard duck solution.*

4. A DUCK SOLUTION IN THE FHN EQUATION

Now we return to the system (1.1). Let ε be a small positive constant, and let ε_n, $n = 1, 2, \ldots$, be positive constants such that $0 < \varepsilon_n < \varepsilon$ and $\varepsilon_n \to 0$ as $n \to \infty$. As mentioned in Section 2, we put $I = I_0 + \varepsilon_n t$ and $b = c\varepsilon_n$ for each n.

By using I as an independent variable, the system (1.1) becomes the following new systems:

$$\begin{cases} \varepsilon_n \, dv/dI = -\rho(v) - w + I, \\ dw/dI = c(v - \gamma w). \end{cases} \tag{Dn}$$

Then the following theorem is obtained.

THEOREM 4.1. *The systems* (Dn), $n = 1, 2, \ldots$, *have delay for sufficiently small* ε.

Proof. If ε_n is sufficiently small, the condition (1.2) holds. So, we can conclude that the delayed phenomenon occurs by the main theorem of Su [11].

By changing the coordinates $I = X$, $w = Y$ and $v = Z$, the system (Dn) becomes system (En) for each n:

$$\begin{cases} dX/dI = 1, \\ dY/dI = c(Z - \gamma Y), \\ dZ/dI = (-\rho(Z) - Y + X)/\varepsilon_n. \end{cases} \tag{En}$$

Consider the following constrained system induced from the system (En):

$$\begin{cases} dX/dI = 1, \\ dY/dI = c(Z - \gamma Y), \\ -\rho(Z) - Y + X = 0. \end{cases} \tag{4.1}$$

The conditions (A1)–(A5) in Sect. 2 are satisfied on the system (4.1). The condition (A1) holds, since $f(X, Y, Z) = 1$, $g(X, Y, Z) = c(Z - \gamma Y)$ and $h(X, Y, Z) = -\rho(Z) - Y + X$ are all analytic. Since the set S is expressed as

$$S = \{(X, Y, Z) \in \mathbf{R}^3 : Y = -\rho(Z) + X\}, \tag{4.2}$$

the condition (A4) holds. It is obvious that S is a 2-dimensional differentiable manifold. Put the set

$$T = \{(X, Y, Z) \in \mathbf{R}^3 : 3Z^2 - 2(a+1)Z + a = 0\}. \tag{4.3}$$

Then the set T is a differentiable manifold and S intersects T transversely. The condition (A2) holds, since $PL = S \cap T$. Also the condition (A3) holds, since $dX/dI = 1 \neq 0$ at any point in R^3. The system (4.1) restricted to the surface S is described as the following equation by using the local coordinates (X, Z),

$$\begin{cases} dX/dI = \rho'(Z), \\ dZ/dI = 1 + c\gamma X - c(Z + \gamma\rho(Z)). \end{cases} \tag{4.4}$$

Let (X_0, Z_0) be a singular point in the system (4.4), then the following equation holds:

$$\begin{cases} \rho'(Z_0) = 0, \\ 1 + c\gamma X_0 - c(Z_0 + \gamma\rho(Z_0)) = 0. \end{cases} \tag{4.5}$$

Consider the linearized system for (X_0, Z_0) in the system (4.4) as follows:

$$\begin{cases} d\tilde{X}/dI = \rho''(Z_0)\tilde{Z}, \\ d\tilde{Z}/dI = c\gamma\tilde{X} - c\tilde{Z}, \end{cases} \tag{4.6}$$

where $\tilde{X} = X - X_0$, $\tilde{Z} = Z - Z_0$. As $\rho'(Z) = 0$ has the following two solutions

$$Z_{0\pm} = \frac{a + 1 \pm \sqrt{a^2 - a + 1}}{3},$$

system (4.4) has two singular points; (X_{0+}, Z_{0+}) and (X_{0-}, Z_{0-}).

The linearized system for (X_{0+}, Z_{0+}) in system (4.4) is given by the following equation:

$$\begin{cases} dX/dI = 2\sqrt{a^2 - a + 1}Z, \\ dZ/dI = c\gamma X - cZ. \end{cases} \tag{4.7}$$

Similarly, the linearized system for (X_{0-}, Z_{0-}) is given by the following equation:

$$\begin{cases} dX/dI = -2\sqrt{a^2 - a + 1}Z, \\ dZ/dI = c\gamma\gamma X - cZ. \end{cases} \tag{4.8}$$

Since the eigenvalues associated with the linearized systems (4.7) and (4.8) are not zeroes, the condition (A5) holds.

THEOREM 4.2. *There exists a duck solution and delay on the systems* (En), $n = 1, 2, \ldots$.

Proof. Our purpose is to show that the system (4.7) has a saddle point. By the Benoît's results, it follows that there is a duck solution on the set of the systems (En), $n = 1, 2, \ldots$, in this situation. The characteristic equation for the Jacobian matrix at the singular point of the system (4.7) is given by

$$\lambda^2 + c\lambda - 2c\gamma\sqrt{a^2 - a + 1} = 0. \tag{4.9}$$

The solutions λ_1, λ_2 of (4.9), which are the eigenvalues associated with (4.7) are

$$\lambda_i = \frac{-c + (-1)^i \sqrt{c^2 + 8c\gamma\sqrt{a^2 - a + 1}}}{2}, \qquad i = 1, 2. \tag{4.10}$$

It follows immediately $\lambda_1 < 0 < \lambda_2$. This implies that the system (4.7) has a saddle point. This completes the proof.

5. AN EXPLICIT EXPRESSION OF DUCK SOLUTIONS

In this section, we consider the FHN equation which contains not only a pseudo singular saddle point but also a pseudo singular node point. In this case the existence of duck solutions is shown explicitly. When we apply the "local model" theory by E. Benoît [4] on the system B in Sect. 3, we need the following two conditions in order to obtain the explicit duck solutions:

(1) $f(O) \simeq h(O) \simeq h_y(O) \simeq h_z(O) \simeq 0$,

(2) $g(O) \not\simeq 0$, $h_x(O) \not\simeq 0$, $h_{zz}(O) \not\simeq 0$, where $O = (0, 0, 0)$.

In the FHN Eq. (1.5), exchanging the variable X to Y, the next system is obtained:

$$\begin{cases} dX/dt = c(Z - \gamma X), \\ dY/dt = 1, \\ \varepsilon\, dZ/dt = -\rho(Z) - X + Y. \end{cases} \tag{5.1}$$

Note that this system does not satisfy the above conditions (1) and (2) by itself.

THEOREM 5.1. *By a suitable coordinate transformation, the system* (5.1) *is transformed to satisfy the conditions* (1) *and* (2).

Proof. Let $P_o = (X_o, Y_o, Z_o)$ be one of the two pseudo singular points of the system (5.1). Applying the affine transformation (5.2) of the coordinate

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X - X_o \\ Y - Y_o \\ Z - Z_o \end{pmatrix}, \tag{5.2}$$

the following sytstem (5.3) is obtained:

$$\begin{cases} dx/dt = -c\gamma(2x+y) + cz, \\ dy/dt = c\gamma(2x+y) - cz + 1, \\ \varepsilon\, dz/dt = -x - \rho(z + Z_o) + \rho(Z_o). \end{cases} \tag{5.3}$$

It is clear that the system (5.3) satisfies these conditions (1) and (2).

Moreover, on the system (5.3) we make the following coordinate transformations (5.4a) and (5.4b) successively:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha^2 u \\ \alpha v \\ \alpha w \end{pmatrix}, \tag{5.4a}$$

$(\alpha \simeq 0,\ \varepsilon/\alpha^2 \simeq 0)$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} h_x(o)\, h_{zz}(o)\, u/2 + (h_{yy}(o)\, h_{zz}(o) - h_{yz}(o)^2)\, v^2/4 \\ v/g(o) \\ -h_{yz}(o)\, v/2 - h_{zz}(o)\, w/2 \end{pmatrix}. \tag{5.4b}$$

Then the system (5.5) is obtained,

$$\begin{cases} dX/dt = pY + qZ + \xi(X, Y, Z), \\ dY/dt = 1 + \eta(X, Y, Z), \\ \delta dZ/dt = -(Z^2 + X) + \zeta(X, Y, Z), \end{cases} \tag{5.5}$$

where

$$p = g(o)\, h_x(o)(f_y(o)\, h_{zz}(o) - f_z(o)\, h_{yz}(o))/2 + g(o)^2\, (h_{yy}(o)\, h_{zz}(o) - h_{yz}(o)^2)/2,$$

$$q = -h_x(o)\, f_z(o), \qquad \delta = \varepsilon/\alpha^2.$$

Here $\xi(X, Y, Z)$, $\eta(X, Y, Z)$ and $\zeta(X, Y, Z)$ are in finitesimal when X, Y and Z are limited. By the "local model" theory, the existence of the explicit duck solutions of the system (5.1) is equivalent to those of the system

$$\begin{cases} dX/dt = pY + qZ, \\ dY/dt = 1, \\ \delta dZ/dt = -(Z^2 + X), \end{cases} \tag{5.6}$$

where $p = (-1)^i\, c\gamma\sqrt{a^2 - a + 1}$, $i = 1, 2$, $q = c$.

Restricting the system (5.6) on the surface $-(Z^2 + X) = 0$, the linearized system (5.7) is obtained,

$$\begin{cases} dY/dt = Z, \\ dZ/dt = -(pY + cZ)/2. \end{cases} \tag{5.7}$$

If we choose $p = -c\gamma\sqrt{a^2 - a + 1}$ so that the system (5.6) has a pseudo singular saddle point, then the characteristic equation with respect to the system (5.7) is the following as the same as (4.9):

$$2\lambda^2 + c\lambda - c\gamma\sqrt{a^2 - a + 1} = 0. \tag{5.8}$$

THEOREM 5.2. *Let* λ_i $(i = 1, 2)$ *be two solutions of the Eq.* (5.8), *the explicit duck solutions* $\gamma_{\lambda_i}(t)$ *of the system* (5.1) *are obtained as follows:*

$$\gamma_{\lambda_i}(t) = (-\lambda_i^2 t^2 - \delta\lambda_i,\ t,\ \lambda_i t), \qquad i = 1, 2.$$

COROLLARY 5.3. *If we choose* $p = c\gamma\sqrt{a^2 - a + 1}$ *and* $c > 8\gamma\sqrt{a^2 - a + 1}$ *so that the system* (5.6) *has a pseudo singular node point, then there are explicit duck solutions* $\gamma_{\mu_i}(t)$:

$$\gamma_{\mu_i}(t) = (-\mu_i^2 t^2 - \delta\mu_i,\ t,\ \mu_i t), \qquad i = 1, 2.$$

Here μ_i $(i = 1, 2)$ *are the solutions of the following characteristic equation with respect to the linearized Eq.* (5.7)

$$2\mu^2 + c\mu + c\gamma\sqrt{a^2 - a + 1} = 0. \tag{5.9}$$

Remark. It is known that there may be another kind of duck solutions in this system. See [4].

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