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Compactness of solutions to the Yamabe problem. III

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Abstract

For a sequence of blow up solutions of the Yamabe equation on non-locally conformally flat compact Riemannian manifolds of dimension 10 or 11, we establish sharp estimates on its asymptotic profile near blow up points as well as sharp decay estimates of the Weyl tensor and its covariant derivatives at blow up points. If the Positive Mass Theorem held in dimensions 10 and 11, these estimates would imply the compactness of the set of solutions of the Yamabe equation on such manifolds.

Keywords: Yamabe problem; Compactness of solutions

1. Introduction

Let (M^n, g) be a compact, smooth, connected Riemannian manifold (without boundary) of dimension $n \geqslant 3$. The Yamabe conjecture has been proved through the works of Yamabe [19], Trudinger [18], Aubin [1] and Schoen [16]: the conformal class of g contains a metric of constant scalar curvature. Different proofs of the Yamabe conjecture in the case $n \leqslant 5$ and in the case (M, g) is locally conformally flat are given by Bahri and Brezis [5] and Bahri [4].

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Consider the Yamabe equation and its sub-critical approximations:

$$-\Delta_g u + c(n)R_g u = n(n-2)u^p, \qquad u > 0 \quad \text{on } M, \tag{1}$$

where $1 < p \leqslant \frac{n+2}{n-2}$, Δ_g is the Laplace–Beltrami operator associated with g, R_g is the scalar curvature of g, and $c(n) = \frac{(n-2)}{4(n-1)}$. Let

$$\mathcal{M}_p = \{u \in C^2(M) \mid u \text{ satisfies } (1)\}.$$

If (M, g) is locally conformally flat and is not conformally diffeomorphic to the standard sphere, Schoen [17] proved that for any $1 < 1 + \epsilon \leqslant p \leqslant \frac{n+2}{n-2}$ and any non-negative integer k,

$$\|u\|_{C^k(M,g)} \leqslant C, \quad \forall u \in \mathcal{M}_p, \tag{2}$$

where C is some constant depending only on (M, g), ϵ and k. The same conclusion has been proved to hold in dimension $n \leqslant 7$ for (M^n, g) which are not locally conformally flat, see Li and Zhang [13] and Marques [15]. See also the introduction of [13] where works of Li and Zhu [14], Li and Zhang [12], and Druet [9,10] for dimensions $n = 3, 4, 5$ are described. Extensive works on the problem and closely related ones can be found in [13] as well.

For $n = 8, 9$ and on (M^n, g) which are not locally conformally flat, [13] also contains sharp estimates on blow up solutions of (1) and sharp decay estimates of the Weyl tensor and its first covariant derivatives at blow up points. If the Positive Mass Theorem held in dimensions 8 and 9, these estimates would yield (2) for $n = 8, 9$. Soon after completing [13], we extended these sharp estimates to dimensions $n = 10, 11$ (see Theorem 1.1); however we have encountered some difficulty in extending such estimates to $n \geqslant 12$. Very interesting results have subsequently been obtained by Aubin in [2,3].

To study the compactness of solutions to the Yamabe equation, it is crucial to establish sharp estimates of blow up solutions. An important step is to find out the right asymptotic profile of blow up solutions near a blow up point. Our earlier work [13] strongly suggested such a profile in dimensions $n \geqslant 10$, which we describe below.

Let $\{u_k\}$ be a sequence of solutions to the Yamabe equation on (M^n, g) satisfying, for some $\bar{P}_k \in M$,

$$u_k(\bar{P}_k) := \max_M u_k \to \infty.$$

Assume that $g = g_{ij}(z)\,dz^i\,dz^j$ is already in conformal normal coordinates centered at $\bar{P}_k$.

For dimensions $n = 8, 9$, sharp estimates on blow up solutions and sharp decay estimates of the Weyl tensor and its covariant derivatives at blow up points were established in [13] through an iterative procedure. Due to this procedure, we expect to obtain enough estimates on the decay rates of the Weyl tensor and its covariant derivatives of appropriate order before making the next step in the iterative process, and therefore we can use

$$-\Delta u_k + c(n)R_g u_k = n(n-2)u_k^{\frac{n+2}{n-2}}, \tag{3}$$

instead of the Yamabe equation which would replace Δ in (3) by Δ_g, to determine the asymptotic profile of $\{u_k\}$ near blow up points. Note that Δ is the flat Laplacian in the z-coordinates.

The Taylor expansion of $R(z)$ in conformal normal coordinates is, for $\bar{l} \geqslant 2$,

$$R(z) = \sum_{l=2}^{\bar{l}} \sum_{|\alpha|=l} \frac{\partial_\alpha R}{\alpha!} z^\alpha + O\big(|z|^{\bar{l}+1}\big). \tag{4}$$

For convenience, we write, with $M_k := u_k(0)$,

$$v_k(y) := M_k^{-1} u_k\big(M_k^{-\frac{2}{n-2}} y\big).$$

Then (3) becomes

$$\Delta v_k(y) - \bar{c}(y) v_k(y) + n(n-2) v_k(y)^{\frac{n+2}{n-2}} = 0, \tag{5}$$

where

$$\bar{c}(y) := c(n) R_g\big(M_k^{-\frac{2}{n-2}} y\big) M_k^{-\frac{4}{n-2}} = c(n) \sum_{l=2}^{\bar{l}} M_k^{-\frac{4+2l}{n-2}} \sum_{|\alpha|=l} \frac{\partial_\alpha R}{\alpha!} y^\alpha$$

and v_k converges to

$$U(y) := \left(\frac{1}{1+|y|^2}\right)^{\frac{n-2}{2}} \quad \text{in } C^3_{loc}\big(\mathbb{R}^n\big).$$

For dimensions $n = 10, 11$, we only need to consider, in the formal expansion of v_k,

$$\tilde{v}_k = v^{(1)} + M_k^{-\frac{8}{n-2}} v^{(2)} + M_k^{-\frac{10}{n-2}} v^{(3)}.$$

The equations satisfied by $v^{(1)}$, $v^{(2)}$ and $v^{(3)}$ are, determined by (5),

$$\Delta v^{(1)} + n(n-2)\big[v^{(1)}\big]^{\frac{n+2}{n-2}} = 0,$$

$$\Delta v^{(2)} - c(n)\Bigg[\sum_{|\alpha|=2} \frac{\partial_\alpha R}{\alpha!} y^\alpha\Bigg] v^{(1)} + n(n+2)\big[v^{(1)}\big]^{\frac{4}{n-2}} v^{(2)} = 0,$$

and

$$\Delta v^{(3)} - c(n)\Bigg[\sum_{|\alpha|=3} \frac{\partial_\alpha R}{\alpha!} y^\alpha\Bigg] v^{(1)} + n(n+2)\big[v^{(1)}\big]^{\frac{4}{n-2}} v^{(3)} = 0.$$

Let

$$\bar{R}^{(l)} := \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\theta \in \mathbb{S}^{n-1}} \sum_{|\alpha|=l} \frac{\partial_\alpha R}{\alpha!} \theta^\alpha, \tag{6}$$

and

$$\tilde{R}^{(l)}(\theta) := -\bar{R}^{(l)} + \sum_{|\alpha|=l} \frac{\partial_\alpha R}{\alpha!}\theta^\alpha, \quad \theta \in \mathbb{S}^{n-1}. \tag{7}$$

We have, in polar coordinates,

$$v^{(1)} := U, \qquad v^{(2)}(r,\theta) = -c(n)\tilde{R}^{(2)}(\theta) f_2(r), \qquad v^{(3)}(r,\theta) = -c(n)\tilde{R}^{(3)}(\theta) f_3(r), \tag{8}$$

where f_2 and f_3 are, respectively, the solutions of (73) and (76).

Thus the asymptotic profile of u_k near the blow up point $\bar{P}_k$ should be

$$\begin{aligned}
\tilde{u}_k(x) &= M_k \tilde{v}_k\big(M_k^{\frac{2}{n-2}} x\big) \\
&= u_k(0)\big[v^{(1)}\big(u_k(0)^{\frac{2}{n-2}}x\big) + u_k(0)^{-\frac{8}{n-2}} v^{(2)}\big(u_k(0)^{\frac{2}{n-2}}x\big) + u_k(0)^{-\frac{10}{n-2}} v^{(3)}\big(u_k(0)^{\frac{2}{n-2}}x\big)\big] \\
&= u_k(0) U\big(u_k(0)^{\frac{2}{n-2}}x\big) - u_k(0)^{\frac{n-10}{n-2}}\big[c(n)\tilde{R}^{(2)}(\theta) f_2\big(u_k(0)^{\frac{2}{n-2}}|x|\big)\big] \\
&\quad - u_k(0)^{-\frac{12-n}{n-2}}\big[c(n)\tilde{R}^{(3)}(\theta) f_3\big(u_k(0)^{\frac{2}{n-2}}|x|\big)\big].
\end{aligned}$$

In the following, we give the previously mentioned sharp estimates in dimensions $n = 10, 11$. The asymptotic profile of blow up solutions is exactly the one described above.

For $Q \in M$ and $\mu > 0$, let

$$\xi_{Q,\mu}(P) := \left(\frac{\mu}{1 + \mu^2 \operatorname{dist}_g(P,Q)^2}\right)^{\frac{n-2}{2}}, \quad P \in M,$$

and, in polar coordinates,

$$\begin{aligned}
\tilde{\xi}_{Q,\mu}(P) = \xi_{Q,\mu}(P) &- c(n)\tilde{R}^{(2)}(\theta) f_2\big(\mu \cdot \operatorname{dist}_g(P,Q)\big)\mu^{\frac{n-10}{2}} \\
&- c(n)\tilde{R}^{(3)}(\theta) f_3\big(\mu \cdot \operatorname{dist}_g(P,Q)\big)\mu^{\frac{n-12}{2}}.
\end{aligned}$$

We use W_g to denote the Weyl tensor of the metric g.

Theorem 1.1. *Let (M^n, g) be a compact, smooth, connected Riemannian manifold of dimension $n = 10, 11$, and let u be a smooth solution of (1) with $1 < 1 + \epsilon \leqslant p \leqslant \frac{n+2}{n-2}$. Then for some positive constant C and some positive integer m which depend only on (M, g), there exist some local maximum points of u, denoted as $\mathcal{S} := \{P_1, \ldots, P_m\}$, such that*

$$\operatorname{dist}_g(P_i, P_j) \geqslant \frac{1}{C}, \qquad \frac{1}{C}u(P_i) \leqslant u(P_j) \leqslant Cu(P_i), \quad \forall i \neq j,$$

$$\big|W_g(P_i)\big|_g \leqslant Cu(P_i)^{-\frac{n-6}{n-2}}, \qquad \big|\nabla W_g(P_i)\big|_g \leqslant Cu(P_i)^{-\frac{n-8}{n-2}},$$

$$\big|\nabla_g^2 W_g(P_i)\big|_g \leqslant \begin{cases} \frac{C}{\sqrt{\log u(P_i)}}, & \text{if } n = 10, \\ Cu(P_i)^{-\frac{n-10}{n-2}}, & \text{if } n = 11, \end{cases} \qquad \forall i,$$

$$\frac{1}{C}\sum_{l=1}^{m}\xi_{P_l,u(P_l)^{\frac{2}{n-2}}} \leqslant u \leqslant C\sum_{l=1}^{m}\xi_{P_l,u(P_l)^{\frac{2}{n-2}}} \quad \text{on } M.$$

Moreover, for each l and modulo a conformal factor which makes g in conformal normal coordinates at P_l,

$$\big|\nabla_g^\alpha(u-\tilde{\xi}_{P_l,u(P_l)^{\frac{2}{n-2}}})(P)\big| \leqslant Cu(\bar{P})^{\frac{n-14+2|\alpha|}{n-2}}\big(1+u(\bar{P})^{\frac{2}{n-2}}\operatorname{dist}_g(P,P_l)^{8-n-|\alpha|}\big),$$
$$\forall \operatorname{dist}_g(P,P_l)<\frac{1}{C},\ |\alpha|=0,1,2.$$

A consequence of Theorem 1.1 is

Corollary 1.1. *Let (M^n,g), $n=10,11$, be a compact, smooth, connected Riemannian manifold which is not locally conformally flat, and let $1<1+\epsilon\leqslant p\leqslant\frac{n+2}{n-2}$. Then*

$$\|u\|_{H^1(M,g)}\leqslant C, \quad \forall u\in\mathcal{M}_p,$$

where C is some constant depending only on (M^n,g) and ϵ.

Remark 1.1. If the Positive Mass Theorem held in dimensions $n=10$ and 11, Theorem 1.1 would yield (2) in these dimensions.

In the following we give a result which is more local in nature. Let $B_1\subset\mathbb{R}^n$ be the unit ball centered at the origin, and let $(a_{ij}(x))$ be a smooth, $n\times n$ symmetric positive definite matrix function, defined on B_1, satisfying

$$\frac{1}{2}|\xi|^2\leqslant a_{ij}(x)\xi^i\xi^j\leqslant 2|\xi|^2, \quad \forall x\in B_1,\ \xi\in\mathbb{R}^n, \tag{9}$$

and, for some $\bar{a}>0$,

$$\|a_{ij}\|_{C^5(B_1)}\leqslant\bar{a}. \tag{10}$$

Consider

$$-L_g u=n(n-2)u^p, \qquad u>0 \quad \text{on } B_1, \tag{11}$$

where

$$g:=a_{ij}(x)\,dx^i\,dx^j. \tag{12}$$

If $\{x^1,\dots,x^n\}$ are conformal normal coordinates for g, let

$$v:=v^{(1)}+u(0)^{-\frac{8}{n-2}}v^{(2)}+u(0)^{-\frac{10}{n-2}}v^{(3)}, \tag{13}$$

where $v^{(1)}$, $v^{(2)}$ and $v^{(3)}$ are defined in (8).

Theorem 1.2. *Let (B_1, g) be as above and let u be a solution of* (11), *with $1 < 1+\epsilon \leqslant p \leqslant \frac{n+2}{n-2}$ and $n = 10, 11$. Assume, for some constant $\bar{b} \geqslant 1$,*

$$\nabla u(0) = 0, \qquad 1 \leqslant \sup_{B_1} u \leqslant \bar{b} u(0). \tag{14}$$

Then there exist some positive constants δ and C, depending only on $\bar{b}$, ϵ and $\bar{a}$, such that

$$u(0)u(x)|x|^{n-2} \leqslant C, \quad \forall 0 < |x| \leqslant \delta, \tag{15}$$

$$\begin{aligned} &\left|W_g(0)\right|_g^2 u(0)^{-\frac{8}{n-2}} + \left|\nabla_g W_g(0)\right|_g^2 u(0)^{-\frac{12}{n-2}} + \left|\nabla_g^2 W_g(0)\right|_g^2 u(0)^{-\frac{16}{n-2}} \log u(0) \\ &\quad \leqslant C u(0)^{-2}, \quad n = 10, \end{aligned} \tag{16}$$

$$\begin{aligned} &\left|W_g(0)\right|_g^2 u(0)^{-\frac{8}{n-2}} + \left|\nabla_g W_g(0)\right|_g^2 u(0)^{-\frac{12}{n-2}} + \left|\nabla_g^2 W_g(0)\right|_g^2 u(0)^{-\frac{16}{n-2}} \\ &\quad \leqslant C u(0)^{-2}, \quad n = 11, \end{aligned} \tag{17}$$

$$\frac{1}{C} u(0) U\big(u(0)^{\frac{2}{n-2}} x\big) \leqslant u(x) \leqslant C u(0) U\big(u(0)^{\frac{2}{n-2}} x\big), \quad \forall 0 < |x| \leqslant \delta, \tag{18}$$

and, if $\{x^1, \ldots, x^n\}$ are conformal normal coordinates for g, we have, with v given in (13),

$$\begin{aligned} &\left|\nabla^\alpha \big(u - u(0) v\big(u(0)^{\frac{2}{n-2}} \cdot\big)\big)\right| \leqslant C u(0)^{\frac{n-14+2|\alpha|}{n-2}} \big(1 + u(0)^{\frac{2}{n-2}} |x|\big)^{8-n-|\alpha|}, \\ &\quad \forall 0 < |x| \leqslant \delta, \ |\alpha| = 0, 1, 2. \end{aligned} \tag{19}$$

It is not difficult to see that Theorem 1.1 follows from Theorem 1.2. Our proof of Theorem 1.2 follows closely the arguments in [13]. In particular the sharp estimates on blow up solutions and on the decay rates of the Weyl tensor and its derivatives are obtained iteratively with improved estimates after each iteration. The main difference between the arguments in this paper and those of [13] is that some Riemannian tensor inequalities in conformal normal coordinates which we used for dimension $n = 8, 9$ are not sufficient for higher dimensions. Our proof of Theorem 1.2 requires an estimate from below of some integral quantity associated with $v^{(2)}$.

We will only prove Theorem 1.2 for $p = \frac{n+2}{n-2}$ since modifications of the arguments yield the result for $1 + \epsilon \leqslant p \leqslant \frac{n+2}{n-2}$, see [13, Section 5]. We will always assume that $n = 10, 11$ unless otherwise stated. In Section 2 we prove Theorem 1.2. In the appendices we establish some facts which we use for the proof.

2. The proof of Theorem 1.2

In this section we prove Theorem 1.2. In the first four subsections we establish (15) using the method of moving spheres. In the last section we derive (16)–(19) using the Pohozaev type identity.

2.1. The set up for proving (15)

Suppose the contrary of (15), then for some $\bar{a} > 0$, $\bar{b} \geqslant 1$, there exists a sequence of Riemannian metrics $\{\tilde{g}_k\}$ of the form (12) that satisfy (9) and (10), and some solutions u_k of (11), with $p = \frac{n+2}{n-2}$ and with g replaced by $\tilde{g}_k$, satisfying (14), such that

$$\max_{|x|<1/k} \big(u_k(0)u_k(x)|x|^{n-2}\big) \geqslant k. \tag{20}$$

We will simply use g to denote $\tilde{g}_k$, and we assume that $g_{ij}(z)\,dz^i\,dz^j$ is already in conformal normal coordinates centered at the origin—as in the proof of Theorem 2.1 in [13]. As in [13],

$$M_k := u_k(0) \to \infty.$$

Write

$$(g_k)_{ij}(y) = g_{ij}\big(M_k^{-\frac{2}{n-2}} y\big)\,dy^i\,dy^j,$$

$$v_k(y) := M_k^{-1} u_k\big(M_k^{-\frac{2}{n-2}} y\big),$$

$$c(x) = c(n)R_g(x) \quad \text{and} \quad \bar{c}(y) = c(n)R_g\big(M_k^{-\frac{2}{n-2}} y\big)M_k^{-\frac{4}{n-2}}.$$

Then

$$\begin{cases} \Delta_{g_k} v_k(y) - \bar{c} v_k(y) + n(n-2) v_k(y)^{\frac{n+2}{n-2}} = 0, & |y| \leqslant \frac{1}{2} M_k^{\frac{2}{n-2}}, \\ 1 = v_k(0) \geqslant (\bar{b}^{-1} + o(1)) v_k(y), \quad |y| \leqslant \frac{1}{2} M_k^{\frac{2}{n-2}}, & \nabla v_k(0) = 0. \end{cases} \tag{21}$$

By the Liouville type theorem of Caffarelli, Gidas and Spruck [6], together with some standard elliptic estimates, v_k converges to

$$U(y) := \left(\frac{1}{1+|y|^2}\right)^{\frac{n-2}{2}} \quad \text{in } C^3_{loc}\big(\mathbb{R}^n\big).$$

In local coordinates,

$$g_{pq}(x) = \delta_{pq} + \frac{1}{3} R_{pijq} x^i x^j + \frac{1}{6} R_{pijq,k} x^i x^j x^k + \left(\frac{1}{20} R_{pijq,kl} + \frac{2}{45} R_{pijm} R_{qklm}\right) x^i x^j x^k x^l + O\big(r^5\big).$$

In conformal normal coordinates, write

$$\Delta_g = \frac{1}{\sqrt{g}} \partial_i\big(\sqrt{g}\, g^{ij} \partial_j\big) = \Delta + b_i \partial_i + d_{ij} \partial_{ij},$$

where (g^{ij}) denotes the inverse matrix of (g_{ij}), $\partial_i = \frac{\partial}{\partial z^i}$, $\partial_{ij} = \frac{\partial^2}{\partial z^i \partial z^j}$, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial z^i \partial z^i}$,

$$\begin{aligned} b_i(x) &= \partial_j g^{ij}(x) \\ &= -\frac{1}{6}R_{ia,b}x^a x^b - \frac{1}{6}R_{iabp,p}x^a x^b - \left(\frac{1}{20}R_{ia,bc} - \frac{1}{15}R_{ipad}R_{pbcd} \right. \\ &\quad \left. - \frac{1}{15}R_{iapd}R_{pbcd} + \frac{1}{10}R_{iabp,pc}\right)x^a x^b x^c + O\big(r^4\big), \end{aligned}$$

and

$$\begin{aligned} d_{ij}(x) &= g^{ij} - \delta_{ij} \\ &= -\frac{1}{3}R_{ipqj}x^p x^q - \frac{1}{6}R_{ipqj,k}x^p x^q x^k \\ &\quad - \left(\frac{1}{20}R_{ipqj,kl} - \frac{1}{15}R_{ipqm}R_{jklm}\right)x^p x^q x^k x^l + O\big(r^5\big). \end{aligned}$$

Thus

$$\Delta_{g_k} = \Delta + \bar{b}_i \partial_i + \bar{d}_{ij}\partial_{ij},$$

where

$$\bar{b}_i(y) = M_k^{-\frac{2}{n-2}} b_i\big(M_k^{-\frac{2}{n-2}}y\big), \qquad \bar{d}_{ij}(y) = d_{ij}\big(M_k^{-\frac{2}{n-2}}y\big).$$

For $\lambda > 0$ and for any function v, let, as in [13],

$$v^\lambda(y) := \left(\frac{\lambda}{|y|}\right)^{n-2} v\big(y^\lambda\big), \quad y^\lambda := \frac{\lambda^2 y}{|y|^2},$$

denote the Kelvin transformation of v, and

$$\Sigma_\lambda := B\left(0, \frac{1}{\sqrt{k}}M_k^{\frac{2}{n-2}}\right) \Big\backslash \overline{B(0,\lambda)} = \left\{ y \;\Big|\; \lambda < |y| < \frac{1}{\sqrt{k}}M_k^{\frac{2}{n-2}} \right\},$$

$$w_\lambda(y) := v_k(y) - v_k^\lambda(y), \quad y \in \Sigma_\lambda.$$

As in [13, (33)–(35)], the equation for w_λ is

$$\Delta w_\lambda + \bar{b}_i \partial_i w_\lambda + \bar{d}_{ij}\partial_{ij} w_\lambda - \bar{c} w_\lambda + n(n+2)\xi^{\frac{4}{n-2}} w_\lambda = E_\lambda \quad \text{in } \Sigma_\lambda,$$

where $\xi^{\frac{4}{n-2}} = \int_0^1 (t v_k + (1-t)v_k^\lambda)^{\frac{4}{n-2}}\,dt$ and

$$\begin{aligned} E_\lambda &= \left(\bar{c}(y)v_k^\lambda(y) - \left(\frac{\lambda}{|y|}\right)^{n+2} \bar{c}\big(y^\lambda\big) v_k\big(y^\lambda\big)\right) - \big(\bar{b}_i \partial_i v_k^\lambda + \bar{d}_{ij}\partial_{ij} v_k^\lambda\big) \\ &\quad + \left(\frac{\lambda}{|y|}\right)^{n+2} \big(\bar{b}_i\big(y^\lambda\big)\partial_i v_k\big(y^\lambda\big) + \bar{d}_{ij}\big(y^\lambda\big)\partial_{ij} v_k\big(y^\lambda\big)\big). \end{aligned} \tag{22}$$

As in [13] we apply the method of moving spheres to $w_\lambda + h_\lambda$, an appropriate perturbation of w_λ, for λ in a fixed neighborhood of 1, to reach a contradiction. We recall that the perturbation h_λ is required to satisfy the following properties:

1. $h_\lambda = 0$ on ∂B_λ, $h_\lambda = o(1)|y|^{2-n}$ in Σ_λ.
2. For $O_\lambda := \{y \in \Sigma_\lambda \mid v_k(y) < 2v_k^\lambda(y)\}$,

$$\big(\Delta + \bar{b}_i\partial_i + \bar{d}_{ij}\partial_{ij} - \bar{c} + n(n+2)\xi^{\frac{4}{n-2}}\big)h_\lambda + E_\lambda \leqslant 0 \quad \text{in } O_\lambda. \tag{23}$$

Note that by the first property, $w_\lambda + h_\lambda > 0$ in $\Sigma_\lambda \setminus \bar{O}_\lambda$, so we do not need (23) to hold outside O_λ.

2.2. *Estimate of v_k in $|y| \leqslant M_k^{\frac{16-\epsilon}{(n-2)^2}}$*

In [13] we established estimates on $v_k - U$ with an error term of the order $M_k^{-\frac{8}{n-2}}$. Now, for dimension $n = 10, 11$, we need to work with terms in the formal expansion of v_k as described in the introduction which are of order $M_k^{-\frac{8}{n-2}}$ and $M_k^{-\frac{10}{n-2}}$. The main result of this subsection is the following estimate of v_k with an error term of the order $M_k^{-\frac{12}{n-2}}$ in the region $|y| \leqslant M_k^{\frac{16-\epsilon}{(n-2)^2}}$ for any $\epsilon > 0$.

Proposition 2.1. *For $n \geqslant 10$, and for any $\epsilon > 0$, there exists some positive constant $C(\epsilon)$ such that*

$$\begin{aligned}&\big|\nabla^l\big(v_k - \big(v^{(1)} + M_k^{-\frac{8}{n-2}}v^{(2)} + M_k^{-\frac{10}{n-2}}v^{(3)}\big)\big)\big|\\ &\quad\leqslant C(\epsilon)M_k^{-\frac{12}{n-2}}(1+r)^{8-n+\bar{a}-l}, \quad 0 < r \leqslant M_k^{\frac{16-\epsilon}{(n-2)^2}},\ l = 0,1,2,\end{aligned} \tag{24}$$

where $\bar{a} = \frac{3}{4}(n - 10 + \sqrt{\epsilon})$, $v^{(1)}$, $v^{(2)}$ and $v^{(3)}$ are defined in (8).

We first recall some notations in [13]. For $\bar{l} \geqslant 2$, write the Taylor expansion of $R(x)$ at 0 as (4). Let $\bar{R}^{(l)}$ and $\tilde{R}^{(l)}(\theta)$ be defined as in (6) and (7). We know (see [13, (44)]), with W denoting the Weyl tensor, that

$$\bar{R}^{(2)} = \frac{1}{2n}\Delta R = -\frac{1}{12n}|W|^2 \quad \text{and} \quad \bar{R}^{(3)} = 0. \tag{25}$$

We write

$$\tilde{R}^{(l)}(\theta) = \sum_{p\geqslant 1}\tilde{R}_{lp}e_p(\theta), \quad 2 \leqslant l \leqslant \bar{l}, \tag{26}$$

where e_p's, depending only on n, are non-constant eigenfunctions of $-\Delta_{\mathbb{S}^{n-1}}$. The following lemma, whose proof can be found in Appendix A, is used in our arguments.

Lemma 2.1.

$$\tilde{R}^{(3)} = \sum_{p=1}^{l_3} \tilde{R}_p^{(3)} e_{3p}(\theta) + O\big(|W|\big), \tag{27}$$

where $\{e_{3p}(\theta)\}_{1\leqslant p\leqslant l_3}$ *is a set of eigenfunctions of* $-\Delta_{\mathbb{S}^{n-1}}$ *associated with the eigenvalue* $3(n+1)$.

Let f_2 and f_3 be defined as in Appendix C, set

$$F^{(2)} := -c(n)\tilde{R}^{(2)}(\theta) f_2(r) M_k^{-\frac{8}{n-2}} = v^{(2)} M_k^{-\frac{8}{n-2}}$$

and

$$\begin{aligned} F^{(3)} &:= F^{(2)} - c(n)\sum_{p=1}^{l_3} \tilde{R}_p^{(3)} e_{3p}(\theta) f_3(r) M_k^{-\frac{10}{n-2}} \\ &= v^{(2)} M_k^{-\frac{8}{n-2}} + v^{(3)} M_k^{-\frac{10}{n-2}} + O\big(|W|\big) f_3(r) M_k^{-\frac{10}{n-2}}. \end{aligned}$$

By (73) and (76),

$$\big(\Delta + n(n+2)U^{\frac{4}{n-2}}\big)F^{(2)} = c(n)\tilde{R}^{(2)}(\theta) r^2 U M_k^{-\frac{8}{n-2}}$$

and

$$\begin{aligned} \big(\Delta + n(n+2)U^{\frac{4}{n-2}}\big)F^{(3)} &= c(n)\sum_{l=2}^{3} \tilde{R}^{(l)}(\theta) r^l U M_k^{-\frac{4+2l}{n-2}} \\ &\quad + O\big(|W|\big) M_k^{-\frac{10}{n-2}} (1+r)^{5-n}. \end{aligned} \tag{28}$$

Proof of Proposition 2.1. We claim that

$$(\Delta_{g_k} - \bar{c})\big(U + F^{(3)}\big) + n(n-2)\big(U + F^{(3)}\big)^{\frac{n+2}{n-2}} = O\big(M_k^{-\frac{12}{n-2}}\big)(1+r)^{6-n}. \tag{29}$$

To see this, we first recall some known facts. We know from [13, (21), (44) and (123)] that

$$v_k(y) \leqslant CU(y), \qquad |y| \leqslant M_k^{\frac{16-\epsilon}{(n-2)^2}}, \quad n \geqslant 10, \tag{30}$$

$$\big|\nabla^l R_{abcd}\big| = O\big(M_k^{-\frac{2(2-l)}{n-2}+\epsilon}\big), \quad l = 0, 1,\ n \geqslant 10, \tag{31}$$

$$\bar{R}^{(2s+2)} = O\big(M_k^{-\frac{4(2-s)}{n-2}+\epsilon}\big), \quad s = 0, 1,\ n \geqslant 10.$$

These lead to

$$\bar{b}_i(y) = O\big(M_k^{-\frac{8-\epsilon}{n-2}}\big)r^2 + O\big(M_k^{-\frac{8}{n-2}}\big)r^3,$$
$$\bar{d}_{ij}(y) = O\big(M_k^{-\frac{8-\epsilon}{n-2}}\big)(1+r)^3 + O\big(M_k^{-\frac{8}{n-2}}\big)r^4. \tag{32}$$

To derive (29) we use (28) and (31) to obtain

$$\begin{aligned}
&\Delta\big(U+F^{(3)}\big) + n(n-2)\big(U+F^{(3)}\big)^{\frac{n+2}{n-2}} \\
&= \Delta\big(U+F^{(3)}\big) + n(n-2)U^{\frac{n+2}{n-2}}\left(1+\frac{F^{(3)}}{U}\right)^{\frac{n+2}{n-2}} \\
&= \Delta U + n(n-2)U^{\frac{n+2}{n-2}} + \Delta F^{(3)} + n(n+2)U^{\frac{4}{n-2}}F^{(3)} + O\left(\frac{F^{(3)}}{U}\right)^2 U^{\frac{n+2}{n-2}} \\
&= c(n)\sum_{l=2}^{3}\tilde{R}^{(l)}r^l U M_k^{-\frac{4+2l}{n-2}} + O\big(|W|\big)M_k^{-\frac{10}{n-2}}(1+r)^{5-n} + O\big(M_k^{-\frac{16}{n-2}}\big)(1+r)^{6-n} \\
&= c(n)\sum_{l=2}^{3}\tilde{R}^{(l)}r^l U M_k^{-\frac{4+2l}{n-2}} + O\big(M_k^{-\frac{14-\epsilon}{n-2}}\big)(1+r)^{5-n}.
\end{aligned}$$

Note that we used the estimates of f_2 and f_3 in Appendix C. To estimate $(\Delta_{g_k} - \Delta)(U + F^{(3)})$, we observe that for any smooth functions $a(\theta)$ and $b(r)$, by the definition of the conformal normal coordinates, $(\Delta_{g_k} - \Delta)b(r) = 0$, consequently

$$(\Delta_{g_k} - \Delta)\big(a(\theta)b(r)\big) = \big((\Delta_{g_k} - \Delta)a(\theta)\big)b(r). \tag{33}$$

It follows, using the estimates of $\bar{b}_i$ and $\bar{d}_{ij}$ in (32), that

$$\begin{aligned}
(\bar{b}_i\partial_i + \bar{d}_{ij}\partial_{ij})\big(U+F^{(3)}\big) &= (\bar{b}_i\partial_i + \bar{d}_{ij}\partial_{ij})F^{(3)} \\
&= O\big(M_k^{-\frac{16-\epsilon}{n-2}}\big)(1+r)^{7-n} + O\big(M_k^{-\frac{16}{n-2}}\big)(1+r)^{8-n}.
\end{aligned}$$

Also, by $\bar{R}^{(2)} = O(M_k^{-\frac{8-\epsilon}{n-2}})$ and the estimates of f_2 and f_3,

$$\bar{c}\big(U+F^{(3)}\big) = c(n)\sum_{l=2}^{3}\tilde{R}^{(l)}r^l U M_k^{-\frac{4+2l}{n-2}} + O\big(M_k^{-\frac{12}{n-2}}\big)(1+r)^{6-n}.$$

Then (29) is the consequence of the above.

By (29) and the equation for v_k, we have

$$\begin{aligned}
&(\Delta_{g_k} - \bar{c})\big(v_k - U - F^{(3)}\big) + n(n-2)\big(v_k^{\frac{n+2}{n-2}} - \big(U+F^{(3)}\big)^{\frac{n+2}{n-2}}\big) \\
&= O\big(M_k^{-\frac{12}{n-2}}\big)(1+r)^{6-n}, \quad |y| \leqslant M_k^{\frac{16-\epsilon}{(n-2)^2}}.
\end{aligned} \tag{34}$$

Since $f_2'(0) = f_3'(0) = 0$, $\nabla(U + F^{(3)})(0) = 0$. By these facts and (30) we can prove

$$\Lambda_k := \max_{|y| \leqslant M_k^{\frac{16-\epsilon}{(n-2)^2}}} \left|\left(v_k - U - F^{(3)}\right)(y)\right| \leqslant C M_k^{-\frac{12}{n-2}}. \tag{35}$$

Indeed, let

$$w_k := \Lambda_k^{-1}\left(v_k - U - F^{(3)}\right).$$

Then we see from (34) and (31) that, for some $\bar{\epsilon} > 0$ independent of k,

$$\begin{aligned}
&\left(\Delta + \frac{o(1)\partial_{ij}}{(1+|y|)^{\bar{\epsilon}}} + \frac{o(1)\partial_i}{(1+|y|)^{1+\bar{\epsilon}}} + \frac{o(1)}{(1+|y|)^{2+\bar{\epsilon}}}\right) w_k(y) \\
&\quad = O(1)\Lambda_k^{-1} M_k^{-\frac{12}{n-2}}\left(1+|y|\right)^{6-n} + O(1)\left(1+|y|\right)^{-4} w_k \\
&\quad = O(1)\Lambda_k^{-1} M_k^{-\frac{12}{n-2}}\left(1+|y|\right)^{-2-\bar{\epsilon}} + O(1)\left(1+|y|\right)^{-2-\bar{\epsilon}}, \quad |y| \leqslant M_k^{\frac{16-\epsilon}{(n-2)^2}}.
\end{aligned}$$

If (35) did not hold, then $\Lambda_k^{-1} M_k^{-\frac{12}{n-2}} = o(1)$ along a subsequence, and the argument below (101) in [13] (with δR_k replaced by $M_k^{\frac{16-\epsilon}{(n-2)^2}}$) yields a contradiction. See also [7, Lemma 3.3] for a similar argument. (24) is proved for $l = 0$ and $|y| < R$ for R being a fixed large constant. Next we use (35) to compare $(v_k - U - F^{(3)})(y)$ with $Q M_k^{-\frac{12}{n-2}} r^{8-n+\bar{a}}$ for some large Q over $R < |y| < M_k^{\frac{16-\epsilon}{(n-2)^2}}$. By the maximum principle,

$$\left|v_k - \left(U + F^{(3)}\right)\right| \leqslant Q M_k^{-\frac{12}{n-2}} r^{8-n+\bar{a}}.$$

The estimates for the first and the second derivatives of $v_k - (U + F^{(3)})$ follow from this and the equation for $v_k - (U + F^{(3)})$ by elliptic estimates. Proposition 2.1 is established. □

2.3. Estimate of E_λ

In this and the next subsections, we assume $\lambda \in (\frac{1}{2}, 2)$ and we assume $\lambda \leqslant |y| \leqslant \frac{1}{2} M_k^{\frac{2}{n-2}}$ unless otherwise stated. We use $E_1, \ldots, E_4$ to denote the following terms:

$$\begin{aligned}
E_1 &= c(n) U^\lambda \sum_{s=0}^{2} \bar{R}^{(2s+2)} M_k^{-\frac{8+4s}{n-2}} r^{2s+2}\left(1 - \left(\frac{\lambda}{r}\right)^{4s+8}\right) \\
&\quad - \frac{c(n)^2}{2n(n+2)}\left(\sum_{i<j} 2(\partial_{ij} R)^2 + \sum_i (\partial_{ii} R)^2\right)\left(1 - \left(\frac{\lambda}{r}\right)^8\right) r^2 f_2^\lambda M_k^{-\frac{16}{n-2}},
\end{aligned}$$

$$E_2 = \begin{cases} c(n) U^\lambda \sum_{l=2}^{6} \tilde{R}^{(l)} M_k^{-\frac{4+2l}{n-2}} r^l \left(1 - (\frac{\lambda}{r})^{2l+4}\right), & n = 10, \\ c(n) U^\lambda \sum_{l=2}^{7} \tilde{R}^{(l)} M_k^{-\frac{4+2l}{n-2}} r^l \left(1 - (\frac{\lambda}{r})^{2l+4}\right), & n \geqslant 11, \end{cases}$$

$$E_3 = \sum_{s=1}^{J} \bar{a}_{s,k}(r) e_s,$$

$$E_4 = \begin{cases} O(M_k^{-\frac{18-\epsilon}{n-2}} r^{9-\frac{\epsilon}{2}-n}), & n = 10, \\ O(M_k^{-\frac{20-\epsilon}{n-2}} r^{10-\frac{\epsilon}{2}-n}), & n \geqslant 11, \end{cases} \tag{36}$$

where $e_s = e_s(\theta)$, independent of k, is a homogeneous spherical harmonic of degree s, J is a positive integer, $\bar{a}_{s,k}$ satisfies

$$\left|\bar{a}_{s,k}(r)\right| = O\big(M_k^{-\frac{16-\epsilon}{n-2}} r^{3-n}\big) + O\big(M_k^{-\frac{16}{n-2}} r^{4-n}\big).$$

From now on we say a term is E_3 or E_4 if it is of the form in (36). The main result in this subsection is

Proposition 2.2. *For $n \geqslant 10$, $\frac{1}{2} \leqslant \lambda \leqslant 2$,*

$$E_\lambda = E_1 + E_2 + E_3 + E_4, \quad \lambda \leqslant |y| \leqslant \frac{1}{2} M_k^{\frac{2}{n-2}}.$$

Proof. First by (24) we have

$$\left|\nabla^l\big[v_k^\lambda - \big(U^\lambda + F_\lambda^{(3)}\big)\big]\right| \leqslant C M_k^{-\frac{12}{n-2}} |y|^{2-n}, \quad l = 0, 1, 2, \tag{37}$$

where $F_\lambda^{(3)}$ is the Kelvin transformation of $F^{(3)}$. Note that (37) holds over the whole Σ_λ. Similarly we can define $F_\lambda^{(2)}$ as the Kelvin transformation of $F^{(2)}$ and we shall use this expression:

$$F_\lambda^{(2)} = -c(n) \tilde{R}^{(2)}(\theta) f_2^\lambda(r) M_k^{-\frac{8}{n-2}},$$

where

$$\tilde{R}^{(2)}(\theta) = \sum_{i<j} \partial_{ij} R \theta_i \theta_j + \sum_i \frac{\partial_{ii} R}{2}\left(\theta_i^2 - \frac{1}{n}\right) = \sum_{i<j} \partial_{ij} R \theta_i \theta_j + \frac{1}{2}\sum_i (\partial_{ii} R)\theta_i^2 + O\big(|W|^2\big),$$

and $f_2^\lambda(r) = (\frac{\lambda}{r})^{n-2} f_2(\lambda^2/r)$ is the Kelvin transformation of f_2. f_3^λ is understood similarly. Since $0 \leqslant f_2(r), f_3(r) \leqslant Cr$ for $0 \leqslant r \leqslant 1$, we have

$$\left|f_2^\lambda(r)\right| + \left|f_3^\lambda(r)\right| \leqslant C r^{1-n}, \qquad F_\lambda^{(3)} = O\big(M_k^{-\frac{8}{n-2}}\big) r^{1-n}. \tag{38}$$

Now we consider $(\bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij}) v_k^\lambda$. Since U^λ is radially symmetric, we have, using (33), (32), (37),

$$\begin{aligned}
(\bar{b}_i\partial_i+\bar{d}_{ij}\partial_{ij})v_k^\lambda &= (\bar{b}_i\partial_i+\bar{d}_{ij}\partial_{ij})F_\lambda^{(3)}+E_4\\
&= -c(n)\big\{(\bar{b}_i\partial_i+\bar{d}_{ij}\partial_{ij})\tilde{R}^{(2)}(\theta)\big\}f_2^\lambda(r)M_k^{-\frac{8}{n-2}}\\
&\quad - c(n)(\bar{b}_i\partial_i+\bar{d}_{ij}\partial_{ij})\sum_{p=1}^{l_3}\tilde{R}_p^{(3)}e_{3p}f_3^\lambda M_k^{-\frac{10}{n-2}}+E_4.
\end{aligned}$$

For any smooth function $a(\theta)$,

$$\int\limits_{S^{n-1}}(\bar{b}_i\partial_i+\bar{d}_{ij}\partial_{ij})a(\theta)=\int\limits_{S^{n-1}}(\Delta_{g_k}-\Delta)a(\theta)=0. \tag{39}$$

Expanding $\bar{b}_i(y)$ and $\bar{d}_{ij}(y)$ to the fourth and the fifth order, respectively, and using (31), we have

$$(\bar{b}_i\partial_i+\bar{d}_{ij}\partial_{ij})\tilde{R}^{(2)}(\theta)=\sum_{l=1}^{7}a_{l,k}(r)P_l(\theta)+O\big(M_k^{-\frac{12}{n-2}}r^4\big),$$

$$(\bar{b}_i\partial_i+\bar{d}_{ij}\partial_{ij})\sum_{p=1}^{l_3}\tilde{R}_p^{(3)}e_{3p}=\sum_{l=1}^{5}b_{l,k}(r)P_l(\theta)+O\big(M_k^{-\frac{12}{n-2}}r^4\big),$$

where $a_{l,k}(r)$ and $b_{l,k}(r)$ are radial functions satisfying

$$\big|a_{l,k}(r)\big|+\big|b_{l,k}(r)\big|=O\big(M_k^{-\frac{8-\epsilon}{n-2}}r\big)+O\big(M_k^{-\frac{8}{n-2}}r^2\big), \tag{40}$$

while $P_l(\theta)$ is a homogeneous polynomial in θ of degree l and is also independent of k. Consequently, using (39), we have

$$(\bar{b}_i\partial_i+\bar{d}_{ij}\partial_{ij})\tilde{R}^{(2)}(\theta)=\sum_{l=1}^{7}a_{l,k}(r)e_l(\theta)+O\big(M_k^{-\frac{12}{n-2}}r^4\big),$$

$$(\bar{b}_i\partial_i+\bar{d}_{ij}\partial_{ij})\sum_{p=1}^{l_3}\tilde{R}_p^{(3)}e_{3p}=\sum_{l=1}^{7}b_{l,k}(r)e_l(\theta)+O\big(M_k^{-\frac{12}{n-2}}r^4\big),$$

where $e_l(\theta)$, independent of k, is a homogeneous spherical harmonic of degree l, and $a_{l,k}(r)$ and $b_{l,k}(r)$, independent of θ, satisfy (40). Consequently, using also (38),

$$(\bar{b}_i\partial_i+\bar{d}_{ij}\partial_{ij})v_k^\lambda=(\bar{b}_i\partial_i+\bar{d}_{ij}\partial_{ij})F_\lambda^{(3)}+E_4=E_3+E_4.$$

Similarly we can show that

$$\left(\frac{\lambda}{|y|}\right)^{n+2}\big(\bar{b}_i\big(y^\lambda\big)\partial_i v_k\big(y^\lambda\big)+\bar{d}_{ij}\big(y^\lambda\big)\partial_{ij}v_k\big(y^\lambda\big)\big)=E_3+E_4.$$

We have discussed the minor terms in E_λ, by (22), the main term in E_λ is

$$\bar{c}(y)v_k^\lambda - \left(\frac{\lambda}{|y|}\right)^{n+2}\bar{c}\big(y^\lambda\big)v_k\big(y^\lambda\big).$$

We shall use the following two expansions of $\bar{c}$ according to circumstances.
First we know

$$\bar{c} = c(n)\sum_{l=2}^{7}\tilde{R}^{(l)}(\theta)r^l M_k^{-\frac{4+2l}{n-2}} + c(n)\sum_{s=0}^{2}\bar{R}^{(2s+2)}r^{2s+2}M_k^{-\frac{8+4s}{n-2}} + O\big(M_k^{-\frac{20}{n-2}}r^8\big), \quad \lambda < r < \frac{1}{2}M_k^{\frac{2}{n-2}}. \tag{41}$$

On the other hand, using (25), (27) and the rate of $|W|$,

$$\bar{c} = c^{(3)} + O\big(M_k^{-\frac{12}{n-2}}\big)r^4 + O\big(M_k^{-\frac{14-\epsilon}{n-2}}r^3\big), \quad \lambda < r < \frac{1}{2}M_k^{\frac{2}{n-2}}, \tag{42}$$

where

$$c^{(3)} = c^{(2)} + c(n)\sum_{p=1}^{l_3}\tilde{R}_p^{(3)}e_{3p}r^3 M_k^{-\frac{10}{n-2}}, \tag{43}$$

$$c^{(2)} = c(n)\left(\sum_{i<j}\partial_{ij}R\theta_i\theta_j + \frac{1}{2}\sum_i \partial_{ii}R\theta_i^2\right)r^2 M_k^{-\frac{8}{n-2}}.$$

Using (37), we have

$$\bar{c}v_k^\lambda - \left(\frac{\lambda}{r}\right)^{n+2}\bar{c}\big(y^\lambda\big)v_k\big(y^\lambda\big) = \bar{c}U^\lambda - \left(\frac{\lambda}{r}\right)^{n+2}\bar{c}\big(y^\lambda\big)U\big(y^\lambda\big) + c^{(3)}F_\lambda^{(3)} - \left(\frac{\lambda}{r}\right)^{n+2}c^{(3)}\big(y^\lambda\big)F^{(3)}\big(y^\lambda\big) + E_4. \tag{44}$$

To estimate the first two terms we use the expression (41) of $\bar{c}$.

$$\begin{aligned}
&\bar{c}U^\lambda - \left(\frac{\lambda}{r}\right)^{n+2}\bar{c}\big(y^\lambda\big)U\big(y^\lambda\big)\\
&= c(n)U^\lambda\left(\sum_{l=2}^{7}\tilde{R}^{(l)}r^l M_k^{-\frac{4+2l}{n-2}} + \sum_{s=0}^{2}\bar{R}^{(2s+2)}M_k^{-\frac{8+4s}{n-2}}r^{2s+2}\right)\\
&\quad - c(n)\left(\frac{\lambda}{r}\right)^{n+2}U\big(y^\lambda\big)\left(\sum_{l=2}^{7}\tilde{R}^{(l)}\left(\frac{\lambda^2}{r}\right)^l M_k^{-\frac{4+2l}{n-2}} + \sum_{s=0}^{2}\bar{R}^{(2s+2)}M_k^{-\frac{8+4s}{n-2}}\left(\frac{\lambda^2}{r}\right)^{2s+2}\right) + E_4
\end{aligned}$$

$$= c(n)U^\lambda \sum_{s=0}^{2} \bar{R}^{(2s+2)} M_k^{-\frac{8+4s}{n-2}} r^{2s+2}\left(1-\left(\frac{\lambda}{r}\right)^{4s+8}\right) + E_2 + E_4.$$

For the third and the fourth terms of (44) we use (42), (43) and (38) to obtain

$$\begin{aligned}
&c^{(3)} F_\lambda^{(3)} - \left(\frac{\lambda}{r}\right)^{n+2} c^{(3)}\big(y^\lambda\big) F^{(3)}\big(y^\lambda\big) \\
&\quad = \left(c^{(3)} - \left(\frac{\lambda}{r}\right)^4 c^{(3)}\big(y^\lambda\big)\right) F_\lambda^{(3)} \\
&\quad = \left(c^{(2)} - \left(\frac{\lambda}{r}\right)^4 c^{(2)}\big(y^\lambda\big)\right) F_\lambda^{(2)} + E_3 \\
&\quad = -c(n)^2 \left(\sum_{i<j} \partial_{ij} R\theta_i\theta_j + \frac{1}{2}\sum_i \partial_{ii} R\theta_i^2\right)^2 r^2 \left(1-\left(\frac{\lambda}{r}\right)^8\right) f_2^\lambda M_k^{-\frac{16}{n-2}} \\
&\qquad + E_3 + E_4.
\end{aligned}$$

In the expansion of the product in the last equality, we use the fact that homogeneous polynomials of degree 2 and 3 are orthogonal to each other—when a term has average 0 on $\mathbb{S}^{n-1}$ it contributes a term E_3.

The term that needs to be evaluated is

$$\left(\sum_{i<j} \partial_{ij} R\theta_i\theta_j + \frac{1}{2}\sum_i \partial_{ii} R\theta_i^2\right)^2.$$

It is elementary to verify the following identities:

$$\frac{1}{|S^{n-1}|}\int_{S^{n-1}} \theta_i^2\theta_j^2 = \frac{1}{n(n+2)}, \quad i \neq j, \qquad \frac{1}{|S^{n-1}|}\int_{S^{n-1}} (\theta_i)^4 = \frac{3}{n(n+2)}.$$

In the following we often write a polynomial $P(\theta)$ of degree less or equal to 7 as the sum of $\frac{1}{|S^{n-1}|}\int_{S^{n-1}} P(\theta)$ and $\sum_{p=1}^7 C_p e_p(\theta)$ where $e_p(\theta)$ are homogeneous spherical harmonics of degree p.

By the above, we write

$$\begin{aligned}
&\left(\sum_{i<j} \partial_{ij} R\theta_i\theta_j + \frac{1}{2}\sum_i \partial_{ii} R\theta_i^2\right)^2 \\
&\quad = \sum_{i<j} (\partial_{ij} R)^2 \theta_i^2\theta_j^2 + \frac{1}{4}\left(\sum_i \partial_{ii} R\theta_i^2\right)^2 + \sum_{p=1}^7 C_p e_p(\theta) \\
&\quad = \sum_{i<j} (\partial_{ij} R)^2 \theta_i^2\theta_j^2 + \frac{1}{4}\sum_{i\neq j} \partial_{ii} R \partial_{jj} R\theta_i^2\theta_j^2 + \frac{1}{4}\sum_i (\partial_{ii} R)^2\theta_i^4 + \sum_{p=1}^7 C_p e_p(\theta)
\end{aligned}$$

$$= \frac{1}{n(n+2)} \sum_{i<j} (\partial_{ij} R)^2 + \frac{1}{4n(n+2)} \sum_{i \neq j} \partial_{ii} R \partial_{jj} R + \frac{3}{4n(n+2)} \sum_i (\partial_{ii} R)^2 + \sum_{p=1}^{7} C_p e_p.$$

Note that

$$\sum_{i \neq j} \partial_{ii} R \partial_{jj} R + \sum_{i=1}^{n} (\partial_{ii} R)^2 = \left(\sum_i \partial_{ii} R \right)^2 = O\big(|W|^4\big) = O\big(M_k^{-\frac{16-\epsilon}{n-2}}\big).$$

We have

$$\left(\sum_{i<j} \partial_{ij} R \theta_i \theta_j + \frac{1}{2} \sum_i \partial_{ii} R \theta_i^2 \right)^2$$
$$= \frac{1}{2n(n+2)} \left[\sum_{i<j} 2(\partial_{ij} R)^2 + \sum_i (\partial_{ii} R)^2 \right] + O\big(M_k^{-\frac{16-\epsilon}{n-2}}\big) + \sum_{p=1}^{7} C_p e_p. \tag{45}$$

Thus

$$c^{(3)} F_\lambda^{(3)} - \left(\frac{\lambda}{r}\right)^{n+2} c^{(3)}\big(y^\lambda\big) F^{(3)}\big(y^\lambda\big)$$
$$= -\frac{c(n)^2}{2n(n+2)} \left[\sum_{i<j} 2(\partial_{ij} R)^2 + \sum_i (\partial_{ii} R)^2 \right] r^2 \left(1 - \left(\frac{\lambda}{r}\right)^8 \right) f_2^\lambda M_k^{-\frac{16}{n-2}} + E_3 + E_4.$$

Proposition 2.2 follows from the above. □

2.4. Construction of auxiliary functions and proof of (15)

The goal of this subsection is to finish the proof of (15) by finding a contradiction to (20). Before we construct the auxiliary functions, we discuss two relatively minor terms in E_1. Recall that $\bar{R}^{(2)} = -\frac{|W|^2}{12n}$. The following properties of conformal normal coordinates are established in [11]: If $W = 0$, then $R_{abcd} = 0$ and, for some constant $c_1(n) > 0$, $\bar{R}^{(4)} = -c_1(n)|R_{abcd,e}|^2$; if $W = 0$ and $\nabla W = 0$, then $R_{abcd,e} = 0$. Examining the proofs there, we arrive at

$$|R_{abcd}| = O(1)|W|, \qquad |R_{abcd,e}| = O\big(|W|\big) + O\big(|\nabla_g W|\big),$$

and

$$\bar{R}^{(4)} = -c_1(n)|\nabla R_{abcd}|^2 + O\big(|W|\big) O\big(|\nabla R_{abcd}|\big) + O\big(|W|^2\big).$$

Thus

$$\bar{R}^{(4)} \leqslant -\frac{1}{2} c_1(n) |\nabla R_{abcd}|^2 + O\big(|W|^2\big) = -\frac{1}{2} c_1(n) |\nabla R_{abcd}|^2 + O\big(M_k^{-\frac{8-2\epsilon}{n-2}}\big). \tag{46}$$

So

$$E_1 \leqslant c(n)U^\lambda \bar{R}^{(6)} r^6 M_k^{-\frac{16}{n-2}}\left(1-\left(\frac{\lambda}{r}\right)^{16}\right)$$
$$-\frac{c(n)^2}{2n(n+2)}\left(\sum_{i<j}2(\partial_{ij}R)^2+\sum_i(\partial_{ii}R)^2\right)\left(1-\left(\frac{\lambda}{r}\right)^8\right)r^2 f_2^\lambda M_k^{-\frac{16}{n-2}}. \tag{47}$$

Recall that the most important requirement for the test function h_λ is (23), for which we construct h_λ as the sum of four test functions $h_1,\dots,h_4$. Each of the first three functions is constructed with respect to Δ and V_λ (a radial function, to be defined later), rather than $\Delta_{g_k}-\bar{c}$ and ξ. So even though each of them cancels a major part of E_λ, they also create some minor extra errors because of the difference between Δ, V_λ and $\Delta_{g_k}-\bar{c},\xi$. Eventually all these new error terms will be put together and be controlled by h_4.

For the convenience of our discussion, we define

$$\bar{l}=6 \quad \text{if } n=10, \qquad \bar{l}=7 \quad \text{if } n=11.$$

To cancel the term

$$c(n)U^\lambda\sum_{l=2}^{\bar{l}}\tilde{R}^{(l)}M_k^{-\frac{4+2l}{n-2}}r^l\left(1-\left(\frac{\lambda}{r}\right)^{2l+4}\right) \quad \text{in } E_\lambda$$

we use (26). The $f_{2,\lambda}$ defined in Appendix C is to deal with $\tilde{R}^{(2)}$.

Let

$$V_\lambda(r):=n(n+2)\int_0^1\left(tU+(1-t)U^\lambda\right)^{\frac{4}{n-2}}dt.$$

For $3\leqslant l\leqslant\bar{l}$, let $\lambda_p>0$ be the eigenvalue corresponding to e_p, we consider

$$\begin{cases} f''_{p\lambda l}(r)+\dfrac{n-1}{r}f'_{p\lambda l}(r)+\left(V_\lambda-\dfrac{\lambda_p}{r^2}\right)f_{p\lambda l}(r)=-r^lU^\lambda(r)\left(1-\left(\dfrac{\lambda}{r}\right)^{2l+4}\right),\\ \lambda<r<2M_k^{\frac{2}{n-2}},\ 3\leqslant l\leqslant\bar{l},\\ f_{p\lambda l}(\lambda)=f_{p\lambda l}\big(2M_k^{\frac{2}{n-2}}\big)=0. \end{cases} \tag{48}$$

By Proposition 6.1 in [13, Appendix A], there exists some small $\epsilon_4=\epsilon_4(n)>0$ such that for $\lambda\in[1-\epsilon_4,1+\epsilon_4]$, Eq. (48) has a unique classical solution satisfying

$$0\leqslant f_{p\lambda l}(r)\leqslant Cr^{l+4-n},\quad 3\leqslant l\leqslant\bar{l},\ \lambda\leqslant r\leqslant 2M_k^{\frac{2}{n-2}}.$$

Let

$$h_1=c(n)\tilde{R}^{(2)}(\theta)f_{2,\lambda}M_k^{-\frac{8}{n-2}}.$$

This is a major part of h_λ. Let

$$h_2 = c(n) \sum_{l=3}^{\bar{l}} \sum_{p=1}^{I_l} \tilde{R}_{lp} e_p(\theta) f_{p\lambda l}(r) M_k^{-\frac{4+2l}{n-2}},$$

where $\tilde{R}_{lp}$ and $e_p(\theta)$ are the ones in (26). By the definitions of h_1 and h_2, we have

$$\Delta(h_1 + h_2) + V_\lambda(h_1 + h_2) = -E_2.$$

The extra error terms created by h_1 are

$$\big(\bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij} - \bar{c} + \big(n(n+2)\xi^{\frac{4}{n-2}} - V_\lambda\big)\big) h_1.$$

We need to estimate the above in O_λ, note that by definition $h_1, h_2 = o(1)|y|^{2-n}$ in Σ_λ. For $(\bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij}) h_1$, we just analyze it the same way as analyzing $(\bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij}) F_\lambda^{(3)}$ to obtain

$$(\bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij}) h_1 = \tilde{E}_3 + E_4.$$

Here and in the following, $\tilde{E}_3$ denotes a term of the form

$$\tilde{E}_3 = \sum_{s=1}^{J} \bar{c}_{s,k}(r) e_s$$

with $\bar{c}_{s,k}(r)$ depending only on r and satisfying

$$\bar{c}_{s,k}(r) = O\big(M_k^{-\frac{16-\epsilon}{n-2}} r^{7-n}\big) + O\big(M_k^{-\frac{16}{n-2}} r^{8-n}\big). \tag{49}$$

Next we consider $-\bar{c}h_1$. By the definition of h_1, (42) and (43) we have

$$-\bar{c}h_1 = -c^{(3)} h_1 + O\big(M_k^{-\frac{20}{n-2}}\big) r^{10-n}.$$

The major part to contribute to E_1 is, using (45),

$$\begin{aligned} -c^{(2)} h_1 &= -c(n)^2 \bigg(\sum_{i<j} \partial_{ij} R \theta_i \theta_j + \frac{1}{2} \sum_i \partial_{ii} R \theta_i^2\bigg)^2 r^2 f_{2,\lambda} M_k^{-\frac{16}{n-2}} + E_4 \\ &= -\frac{c(n)^2}{2n(n+2)} \bigg(\sum_{i<j} 2(\partial_{ij} R)^2 + \sum_i (\partial_{ii} R)^2\bigg) r^2 f_{2,\lambda} M_k^{-\frac{16}{n-2}} + \tilde{E}_3 + E_4. \end{aligned}$$

Since $\tilde{R}^{(2)}(\theta)$ is orthogonal to $e_{3p}(\theta)$,

$$\big(c^{(3)} - c^{(2)}\big) h_1 = \tilde{E}_3. \tag{50}$$

For $-\bar{c}h_1 + (n(n+2)\xi^{\frac{4}{n-2}} - V_\lambda)h_1$ we use (24) over $|y| \leqslant M_k^{\frac{16-\epsilon}{(n-2)^2}}$ and

$$n(n+2)\xi^{\frac{4}{n-2}} - V_\lambda = O\big(r^{-4}\big), \quad r \in \left(M_k^{\frac{16-\epsilon}{(n-2)^2}}, \frac{1}{\sqrt{k}} M_k^{\frac{2}{n-2}}\right).$$

First for $r \leqslant M_k^{\frac{16-\epsilon}{(n-2)^2}}$, using (24) and (37),

$$\begin{aligned} n(n+2)\xi^{\frac{4}{n-2}} &= n(n+2)\int_0^1 \big(t(U+a) + (1-t)\big(U^\lambda + b\big)\big)^{\frac{4}{n-2}}\,dt \\ &= V_\lambda + \frac{4n(n+2)}{n-2}\int_0^1 \big(tF^{(3)} + (1-t)F_\lambda^{(3)}\big)\big(tU + (1-t)U^\lambda\big)^{\frac{6-n}{n-2}}\,dt \\ &\quad + O\big(M_k^{-\frac{12}{n-2}} r^{2+\bar{a}}\big), \quad \lambda \leqslant r \leqslant M_k^{\frac{16-\epsilon}{(n-2)^2}}, \end{aligned} \tag{51}$$

where $a := v_k - U = F^{(3)} + O(M_k^{-\frac{12}{n-2}} r^{8-n+\bar{a}})$ and $b := v_k^\lambda - U^\lambda = F_\lambda^{(3)} + O(M_k^{-\frac{12}{n-2}} r^{2-n})$.

Since $\tilde{R}^{(2)}(\theta)$ is orthogonal to $e_{3p}(\theta)$

$$\left(\int_0^1 \big(tF^{(3)} + (1-t)F_\lambda^{(3)} - tF^{(2)} - (1-t)F_\lambda^{(2)}\big)\big(tU + (1-t)U^\lambda\big)^{\frac{6-n}{n-2}}\,dt\right)h_1 = \tilde{E}_3.$$

Clearly,

$$\big(tF^{(2)} + (1-t)F_\lambda^{(2)}\big)\big(tU + (1-t)U^\lambda\big)^{\frac{6-n}{n-2}} h_1 \leqslant 0.$$

So

$$\big(n(n+2)\xi^{\frac{4}{n-2}} - V_\lambda\big)h_1 \leqslant \tilde{E}_3 + E_4, \quad \lambda \leqslant r \leqslant M_k^{\frac{16-\epsilon}{(n-2)^2}}.$$

We use

$$\big|\big(n(n+2)\xi^{\frac{4}{n-2}} - V_\lambda\big)h_1\big| \leqslant O\big(M_k^{-\frac{8}{n-2}} r^{2-n}\big) = E_4, \quad M_k^{\frac{16-\epsilon}{(n-2)^2}} \leqslant r \leqslant \frac{1}{\sqrt{k}} M_k^{\frac{2}{n-2}}.$$

Now we estimate

$$\big(\bar{b}_i\partial_i + \bar{d}_{ij}\partial_{ij} - \bar{c} + n(n+2)\xi^{\frac{4}{n-2}} - V_\lambda\big)h_2.$$

As usual,

$$(\bar{b}_i\partial_i + \bar{d}_{ij}\partial_{ij})h_2 = \tilde{E}_3,$$

$$-\bar{c}h_2 = -c^{(2)}h_2 + E_4 = -c^{(2)}c(n)\left(\sum_{p=1}^{I_l} \tilde{R}_{lp}e_p(\theta) f_{p\lambda l}(r) M_k^{-\frac{4+2l}{n-2}}\right)\bigg|_{l=3} + E_4.$$

Compare (27) and (26), we deduce from above, using the orthogonality of $\tilde{R}^{(2)}$ and $e_{3p}(\theta)$ and the decay of W,

$$-\bar{c}h_2 = \tilde{E}_3 + E_4. \tag{52}$$

By (51)

$$n(n+2)\xi^{\frac{4}{n-2}} - V_\lambda = \frac{4n(n+2)}{n-2}\int_0^1 \left(tF^{(2)} + (1-t)F_\lambda^{(2)}\right)\left(tU + (1-t)U^\lambda\right)^{\frac{6-n}{n-2}} dt$$
$$+ O\left(M_k^{-\frac{10}{n-2}} r\right) + O\left(M_k^{-\frac{12}{n-2}} r^{2+\bar{a}}\right), \quad \lambda < r < M_k^{\frac{16-\epsilon}{(n-2)^2}}.$$

Therefore, as in the derivation of (52),

$$\begin{aligned}
&\left(n(n+2)\xi^{\frac{4}{n-2}} - V_\lambda\right)h_2 \\
&= \frac{4n(n+2)}{n-2} h_2 \int_0^1 \left(tF^{(2)} + (1-t)F_\lambda^{(2)}\right)\left(tU + (1-t)U^\lambda\right)^{\frac{6-n}{n-2}} dt + E_4 \\
&= \frac{4n(n+2)}{n-2}\left(c(n)\sum_{p=1}^{I_l} \tilde{R}_{lp}e_p(\theta) f_{p\lambda l}(r) M_k^{-\frac{4+2l}{n-2}}\right)\bigg|_{l=3} \\
&\quad \times \int_0^1 \left(tF^{(2)} + (1-t)F_\lambda^{(2)}\right)\left(tU + (1-t)U^\lambda\right)^{\frac{6-n}{n-2}} dt + E_4 \\
&= \tilde{E}_3 + E_4, \quad \lambda < r < M_k^{\frac{16-\epsilon}{(n-2)^2}}.
\end{aligned}$$

The first term on the right-hand side is of the form $\sum_{s=1}^{J} \bar{c}_{s,k}(r)e_s$ and $\bar{c}_{s,k}(r) = O(M_k^{-\frac{18}{n-2}} r^{7-n})$. So after this term is extended to $r \leqslant \frac{1}{\sqrt{k}} M_k^{\frac{2}{n-2}}$, it can be combined with E_3. The extended part has a good decay.

For $M_k^{\frac{16-\epsilon}{(n-2)^2}} \leqslant r \leqslant \frac{1}{\sqrt{k}} M_k^{\frac{2}{n-2}}$, we have

$$\left|\left(n(n+2)\xi^{\frac{4}{n-2}} - V_\lambda\right)h_2\right| \leqslant O\left(M_k^{-\frac{10}{n-2}} r^{3-n}\right) = E_4.$$

Recall that our purpose is to obtain (23). By putting h_1 and h_2 together and using (47), we have

$$\begin{aligned}&\big(\Delta+\bar b_i\partial_i+\bar d_{ij}\partial_{ij}-\bar c+n(n+2)\xi^{\frac{4}{n-2}}\big)(h_1+h_2)+E_1+E_2\\&\quad\leqslant c(n)U^\lambda\bar R^{(6)}M_k^{-\frac{16}{n-2}}r^6\left(1-\left(\frac{\lambda}{r}\right)^{16}\right)\\&\qquad-\frac{c(n)^2}{2n(n+2)}\sum_{i,j}(\partial_{ij}R)^2r^2\left(\left(1-\left(\frac{\lambda}{r}\right)^8\right)f_2^\lambda+f_{2,\lambda}\right)M_k^{-\frac{16}{n-2}}+\tilde E_3+E_4.\end{aligned}\tag{53}$$

For $\bar R^{(6)}$ we use (63), we also know the lower bounds for f_2 and $f_{2,\lambda}$, respectively (see (74) and (75)). These three estimates are in the appendix and are sufficient for h_1+h_2 to cancel the major part of E_λ. In fact, first by (74) and (75)

$$\begin{aligned}&\frac{c(n)}{2n(n+2)}\left(\left(1-\left(\frac{\lambda}{r}\right)^8\right)f_2^\lambda+f_{2,\lambda}\right)\\&\quad\geqslant U^\lambda r^4\left(1-\left(\frac{\lambda}{r}\right)^{16}\right)\frac{1}{8(n+4)(n+2)n}\left(\frac{n-8}{n-2}-\frac{49}{20n^2}+\epsilon\right),\end{aligned}$$

where we have used the following inequality that holds only for $n=10,11$.

$$\frac{1}{8(n+4)(n+2)n}\left(\frac{n-8}{n-2}-\frac{49}{20n^2}+\epsilon\right)\leqslant\frac{c(n)}{2n(n+2)}\frac{1}{6(n-4)}.$$

Thus, by using (63), we deduce from (53) that

$$\begin{aligned}&\big(\Delta+\bar b_i\partial_i+\bar d_{ij}\partial_{ij}-\bar c+n(n+2)\xi^{\frac{4}{n-2}}\big)(h_1+h_2)\\&\quad\leqslant -E_1-E_2+\tilde E_3+E_4\quad\text{in }\Sigma_\lambda.\end{aligned}$$

Next we construct test functions to control $\tilde E_3$ and E_4. The h_3 to be constructed later will create much minor error terms than before. Then eventually all the minors terms will be controlled by h_4. Let $\tilde f_{s\lambda}$ be the solution of

$$\begin{cases}\tilde f''_{s\lambda}(r)+\frac{n-1}{r}\tilde f'_{s\lambda}(r)+(V_\lambda-\frac{\lambda_s}{r^2})\tilde f_{s\lambda}(r)=-\bar c_{s,k}(r), & \lambda<r<M_k^{\frac{2}{n-2}},\\ \tilde f_{s\lambda}(\lambda)=\tilde f_{s\lambda}(M_k^{\frac{2}{n-2}})=0,\end{cases}$$

and let

$$h_3:=\sum_{s=1}^J\tilde f_{s\lambda}(r)e_s.$$

By (49), $\bar c_{s,k}(r)=O(M_k^{-\frac{14}{n-2}}r^{7-n})$. Consequently $|\tilde f_{s\lambda}(r)|\leqslant CM_k^{-\frac{14}{n-2}}r^{9-n}$. Therefore

$$(\Delta+V_\lambda)h_3=-\tilde E_3.$$

By the estimates of $\bar{b}_i$, $\bar{d}_{ij}$ and h_3, etc. we obtain

$$\big(\bar{b}_i\partial_i + \bar{d}_{ij}\partial_{ij} - \bar{c} + n(n+2)\xi^{\frac{4}{n-2}} - V_\lambda\big)h_3 = E_4.$$

Finally we define, for $Q \gg 1$ that

$$h_4(r) = \begin{cases} QM_k^{-\frac{18-2\sqrt{\epsilon}}{n-2}} f_{n,n-9+\sqrt{\epsilon}}(\frac{r}{\lambda}), & n = 10, \\ QM_k^{-\frac{20-2\sqrt{\epsilon}}{n-2}} f_{n,n-10+\sqrt{\epsilon}}(\frac{r}{\lambda}), & n = 11, \end{cases}$$

where $f_{n,\alpha}$ is defined in [13]. Let $h_\lambda := h_1 + h_2 + h_3 + h_4$, then (23) is obtained. This h_λ satisfies all the requirements for the test function to make the method of moving spheres work. Then the standard moving sphere argument leads to the following conclusion:

$$\min_{|y|\leqslant r} v_k \leqslant (1+\epsilon)U(r), \quad 0 \leqslant r \leqslant \frac{1}{\sqrt{k}} M_k^{\frac{2}{n-2}},$$

where ϵ is an arbitrary small positive constant. Then following the argument in [13] one gets a contradiction to (20). (15) is established.

2.5. *Vanishing rates of the Weyl tensor and the completion of the proof of Theorem 1.2*

In this subsection we use (15) to prove (16) and (17), the vanishing rates of the Weyl tensor and its covariant derivatives at the blow up point. By (15),

$$v_k(y) \leqslant CU(y), \quad |y| \leqslant \delta M_k^{\frac{2}{n-2}}. \tag{54}$$

This estimate leads to an improved estimate of v_k than that in Proposition 2.2.

Proposition 2.3. *There exists $\delta' > 0$, independent of k, such that*

$$\begin{aligned} &\big|\nabla^l\big(v_k - \big(v^{(1)} + M_k^{-\frac{8}{n-2}}v^{(2)} + M_k^{-\frac{10}{n-2}}v^{(3)}\big)\big)\big| \\ &\quad = O\big(M_k^{-\frac{12}{n-2}}\big)\big(1+|y|\big)^{8-n-l}, \quad |y| \leqslant \delta' M_k^{\frac{2}{n-2}},\ l = 0,1,2. \end{aligned}$$

Proof. Write

$$E^{(3)} := v_k - \big(U + F^{(3)}\big).$$

We only need to prove that

$$\big|\nabla^l E^{(3)}\big| = O\big(M_k^{-\frac{12}{n-2}}\big)\big(1+|y|\big)^{8-n-l}, \quad |y| \leqslant \delta' M_k^{\frac{2}{n-2}},\ l = 0,1,2. \tag{55}$$

It follows from (21) and (29) that

$$(\Delta_{g_k} - \bar{c})E^{(3)} + n(n+2)\bar{\xi}^{\frac{4}{n-2}}E^{(3)} = O\big(M_k^{-\frac{12}{n-2}}\big)(1+r)^{6-n}, \quad 0 < r < M_k^{\frac{2}{n-2}}, \tag{56}$$

where

$$\bar{\xi}^{\frac{4}{n-2}}(y) = \int_0^1 \big(tv_k + (1-t)\big(U + F^{(3)}\big)\big)^{\frac{4}{n-2}}\,dt.$$

Arguing as in [13, p. 212], we see that the operator $\Delta_{g_k} - \bar{c} + n(n+2)\bar{\xi}^{\frac{4}{n-2}}$ satisfies the maximum principle over $R_1 < |y| < \delta' M_k^{\frac{2}{n-2}}$ for some constants $R_1, \delta' > 0$ which are independent of k. For C_{10} large, but independent of k, we see, using (56), (24) and (54), that

$$\big(\Delta_{g_k} - \bar{c} + n(n+2)\bar{\xi}^{\frac{4}{n-2}}\big)\big(E^{(3)} - f\big) \geqslant 0, \quad R_1 < |y| < \delta' M_k^{\frac{2}{n-2}},$$

$$\big(E^{(3)} - f\big)(y) < 0 \quad \text{on } \{r = R_1\} \cup \big\{r = \delta' M_k^{\frac{2}{n-2}}\big\},$$

where $f(r) := C_{10} M_k^{-\frac{12}{n-2}} r^{8-n}$. Thus, in view of (24), estimate (55) for $l = 0$ follows from the maximum principle. The estimate for $l = 1, 2$ can then be deduced from the equation satisfied by $E^{(3)}$ using elliptic estimates. □

Recall the Pohozaev type identity (102) in [13], with $R_k' = \delta' M_k^{\frac{2}{n-2}}$,

$$I_1[v_k] + I_2[v_k] + I_3[v_k] + I_4[v_k] = I_5[v_k], \tag{57}$$

$$I_1[v_k] = \int_{|y| \leqslant R_k'} (-\bar{b}_i \partial_i v_k - \bar{d}_{ij} \partial_{ij} v_k)\left(\nabla v_k \cdot y + \frac{n-2}{2} v_k\right),$$

$$I_2[v_k] = -\frac{c(n)}{2} M_k^{-\frac{4}{n-2}} \int_{|y| \leqslant R_k'} \big\{\big(M_k^{-\frac{2}{n-2}} y\big) \cdot \nabla R\big(M_k^{-\frac{2}{n-2}} y\big) + 2R\big(M_k^{-\frac{2}{n-2}} y\big)\big\} v_k^2(y),$$

$$I_3[v_k] = \frac{c(n)}{2} M_k^{-\frac{4}{n-2}} R_k' \int_{|y| = R_k'} R\big(M_k^{-\frac{2}{n-2}} y\big) v_k^2(y),$$

$$I_4[v_k] = -\frac{(n-2)^2}{2} R_k' \int_{|y| = R_k'} v_k(y)^{\frac{2n}{n-2}},$$

$$I_5[v_k] = \int_{|y| = R_k'} \left\{\left(\left|\frac{\partial v_k}{\partial \nu}\right|^2 - \frac{1}{2}|\nabla v_k|^2\right) R_k' + \frac{n-2}{2} v_k \frac{\partial v_k}{\partial \nu}\right\} = O\big(M_k^{-2}\big).$$

Write

$$\nabla v_k \cdot y + \frac{n-2}{2} v_k = \tilde{U} + \tilde{F}^{(3)} + \tilde{E}^{(3)},$$

where

$$\tilde{U} = \nabla U \cdot y + \frac{n-2}{2}U, \qquad \tilde{F}^{(3)} = \nabla F^{(3)} \cdot y + \frac{n-2}{2}F^{(3)}.$$

Clearly,

$$\left|\nabla^l \tilde{F}^{(3)}\right| = O\big(M_k^{-\frac{8}{n-2}}\big)(1+r)^{6-n-l}, \quad l = 0, 1, 2, \tag{58}$$

and

$$\left|\nabla^l \tilde{E}^{(3)}\right| = O\big(M_k^{-\frac{12}{n-2}}|y|^{8-n-l}\big), \quad l = 0, 1, 2. \tag{59}$$

With these and (32), (55), (58), (59), we have

$$\begin{aligned} I_1[v_k] &= \int\limits_{B(0,M_k^{\frac{2}{n-2}})} (-\bar{b}_i\partial_i - \bar{d}_{ij}\partial_{ij})(v_k - U)\left(\nabla v_k \cdot y + \frac{n-2}{2}v_k\right)dy \\ &= \int\limits_{B(0,M_k^{\frac{2}{n-2}})} (\Delta - \Delta_{g_k})\big(F^{(3)} + E^{(3)}\big) \cdot \big(\tilde{U} + \tilde{F}^{(3)} + \tilde{E}^{(3)}\big) \\ &= \int\limits_{B(0,M_k^{\frac{2}{n-2}})} (\Delta - \Delta_{g_k})F^{(3)}\tilde{U} + O\big(M_k^{-2}\big). \end{aligned}$$

Since $\tilde{U}$ is radially symmetric, $\int_{B(0,M_k^{\frac{2}{n-2}})} (\Delta - \Delta_{g_k})F^{(3)}\tilde{U} = 0$. Thus $I_1[v_k] = O(M_k^{-2})$, and with notation in (26),

$$\begin{aligned} I_2[v_k] &= -\frac{c(n)}{2}\sum_{l=2}^{7}\sum_{|\alpha|=l}\int\limits_{|y|\leqslant R_k'}\left\{\left(\frac{l+2}{\alpha!}\right)\partial_\alpha R y^\alpha M_k^{-\frac{4+2l}{n-2}}\right\}v_k^2 + O\big(M_k^{-2}\big) \\ &= -\frac{c(n)}{2}\int\limits_{|y|\leqslant R_k'}\left\{\sum_{l=2}^{7}\sum_{p=1}^{I_l}(l+2)\tilde{R}_{lp}e_p r^l M_k^{-\frac{4+2l}{n-2}} + \sum_{s=0}^{2}(2s+4)\bar{R}^{(2s+2)}M_k^{-\frac{8+4s}{n-2}}r^{2s+2}\right\} \\ &\quad \times \big(U^2 + 2U\big(F^{(3)} + E^{(3)}\big) + \big(F^{(3)} + E^{(3)}\big)^2\big) + O\big(M_k^{-2}\big). \end{aligned}$$

Using Lemma A.2 and (46), and for small δ', we have

$$-\frac{c(n)}{2}\int\limits_{|y|\leqslant R_k'}\left\{\sum_{s=0}^{2}(2s+4)\bar{R}^{(2s+2)}M_k^{-\frac{8+4s}{n-2}}r^{2s+2}\right\}\big(U^2 + 2U\big(F^{(3)} + E^{(3)}\big) + \big(F^{(3)} + E^{(3)}\big)^2\big)$$

$$= -\frac{c(n)}{2}\int_{|y|\leqslant R_k'}\sum_{s=0}^{2}(2s+4)\bar{R}^{(2s+2)}r^{2s+2}U^2\big[1+O(\delta')+o(1)\big]M_k^{-\frac{8+4s}{n-2}}$$

$$\geqslant c_1(n)|W|^2M_k^{-\frac{8}{n-2}}+c_2(n)|\nabla R_{abcd}|^2M_k^{-\frac{12}{n-2}}$$

$$+4c(n)\bigg(\epsilon\big|\nabla^2R_{abcd}\big|^2-\frac{1}{8(n+4)(n+2)n}\bigg(\frac{n-8}{n-2}-\frac{49}{20n^2}+\epsilon'\bigg)\big|\nabla^2R\big|^2\bigg)$$

$$\times\bigg(\int_{|y|\leqslant R_k'}r^6U^2\,dy\,M_k^{-\frac{16}{n-2}}\bigg)+o(1)\big(|W|^2M_k^{-\frac{8}{n-2}}+|\nabla R_{abcd}|^2M_k^{-\frac{12}{n-2}}\big), \tag{60}$$

where $c_1(n)$, $c_2(n)$, ϵ and ϵ' are some positive constants depending only on n. ϵ and ϵ' are sufficiently small. Also we observe that

$$-\frac{c(n)}{2}\int_{|y|\leqslant R_k'}\Bigg(\sum_{l=2}^{7}\sum_{p=1}^{I_l}(l+2)\tilde{R}_{lp}e_pr^lM_k^{-\frac{4+2l}{n-2}}\Bigg)U^2=0.$$

We only need to deal with

$$-\frac{c(n)}{2}\int_{|y|\leqslant R_k'}\Bigg(\sum_{l=2}^{7}\sum_{p=1}^{I_l}(l+2)\tilde{R}_{lp}e_pr^lM_k^{-\frac{4+2l}{n-2}}+\sum_{s=0}^{2}(2s+4)\bar{R}^{(2s+2)}M_k^{-\frac{8+4s}{n-2}}r^{2s+2}\Bigg)$$

$$\times\big(2U\big(F^{(3)}+E^{(3)}\big)+\big(F^{(3)}+E^{(3)}\big)^2\big).$$

By previous estimates

$$\int_{|y|\leqslant R_k'}\Bigg(\sum_{l=3}^{7}\sum_{p=1}^{I_l}(l+2)\tilde{R}_{lp}e_pr^lM_k^{-\frac{4+2l}{n-2}}\Bigg)\big(2U\big(F^{(3)}+E^{(3)}\big)+\big(F^{(3)}+E^{(3)}\big)^2\big)=O\big(M_k^{-2}\big),$$

$$\int_{|y|\leqslant R_k'}\Bigg(\sum_{p=1}^{I_2}\tilde{R}_{2p}e_pr^2M_k^{-\frac{8}{n-2}}\Bigg)\big(2UE^{(3)}+\big(F^{(3)}+E^{(3)}\big)^2\big)=O\big(M_k^{-2}\big).$$

Finally $-\frac{c(n)}{2}\int_{|y|\leqslant R_k'}\sum_{l=2}^{7}\sum_{p=1}^{I_l}(l+2)\tilde{R}_{lp}e_pr^lM_k^{-\frac{4+2l}{n-2}}2UF^{(3)}$ contributes another important term:

$$-\frac{c(n)}{2}\int_{|y|\leqslant R_k'}\sum_{l=2}^{7}\sum_{p=1}^{I_l}(l+2)\tilde{R}_{lp}e_pr^lM_k^{-\frac{4+2l}{n-2}}2UF^{(3)}$$

$$=-c(n)\int_{|y|\leqslant R_k'}\Bigg(\sum_{p=1}^{I_2}4\tilde{R}_{2p}e_pr^2M_k^{-\frac{8}{n-2}}+\sum_{l=3}^{7}\sum_{p=1}^{I_l}(l+2)\tilde{R}_{lp}e_pr^lM_k^{-\frac{4+2l}{n-2}}\Bigg)$$

$$\times U\big(F^{(2)}+F^{(3)}-F^{(2)}\big).$$

Using the fact the eigenfunction corresponding to $l=2$ are orthogonal to those corresponding to $l=3$, we have

$$
\begin{aligned}
&-\frac{c(n)}{2}\int_{|y|\leqslant R_k'}\sum_{l=2}^{7}\sum_{p=1}^{I_l}(l+2)\tilde{R}_{lp}e_p r^l M_k^{-\frac{4+2l}{n-2}}2UF^{(3)}\\
&\quad=-4c(n)\int_{|y|\leqslant R_k'}\tilde{R}^{(2)}(\theta)r^2UF^{(2)}M_k^{-\frac{8}{n-2}}+O\big(M_k^{-2}\big)+o(1)M_k^{-\frac{8}{n-2}}|W|^2\\
&\quad=\frac{2c(n)^2}{n(n+2)}\Big(\sum_{i<j}2(\partial_{ij}R)^2+\sum_i(\partial_{ii}R)^2\Big)\int_{|y|\leqslant R_k'}r^2Uf_2\,dy\,M_k^{-\frac{16}{n-2}}+O\big(M_k^{-2}\big)\\
&\qquad+o(1)M_k^{-\frac{8}{n-2}}|W|^2\\
&\quad\geqslant\frac{2c(n)^2}{n(n+2)}\big|\nabla^2R\big|^2\int_{|y|\leqslant R_k'}r^2Uf_2\,dy\,M_k^{-\frac{16}{n-2}}+O\big(M_k^{-2}\big)+o(1)M_k^{-\frac{8}{n-2}}|W|^2. \qquad (61)
\end{aligned}
$$

The Pohozaev type identity (57) yields, in view of (60), (61) and the lower bound of f_2 in (74), that

$$
\begin{aligned}
&|W|^2M_k^{-\frac{8}{n-2}}+|\nabla R_{abcd}|^2M_k^{-\frac{12}{n-2}}+\big|\nabla^2R_{abcd}\big|^2M_k^{-\frac{16}{n-2}}\log M_k=O\big(M_k^{-2}\big),\quad n=10,\\
&|W|^2M_k^{-\frac{8}{n-2}}+|\nabla R_{abcd}|^2M_k^{-\frac{12}{n-2}}+\big|\nabla^2R_{abcd}\big|^2M_k^{-\frac{16}{n-2}}=O\big(M_k^{-2}\big),\quad n=11.
\end{aligned}
$$

Thus we have proved (16) and (17). Estimates (18) and (19) follow from (55). Theorem 1.2 is established.

Appendix A. Some curvature inequalities in conformal normal coordinates

A.1. *The inequality for $\bar{R}^{(6)}$*

In this appendix we prove the following two lemmas.

Lemma A.1. *If $|W(0)|=|\nabla W(0)|=0$, then we have, in conformal normal coordinates centered at 0,*

$$
\begin{aligned}
\bar{R}^{(6)}=&-\frac{R_{p_1p_2p_3p_4,p_5p_6}R_{p_1p_2p_3p_4,p_5p_6}}{40(n+4)(n+2)n}-\frac{R_{p_1p_2,p_3p_4}(R_{p_1p_2,p_3p_4}+R_{p_3p_4,p_1p_2})}{8(n+4)(n+2)n}\\
&+\frac{\sum_{p_1p_2}(\partial_{p_1p_2}R)^2}{8(n+4)(n+2)n}\quad \text{at } 0, \qquad (62)
\end{aligned}
$$

where repeated indices mean summation, and $R_{ijkl,pq}$ denotes covariant derivatives of R_{ijkl}.

Lemma A.2. *For some small* $\epsilon = \epsilon(n) > 0$, *we have, in conformal normal coordinates centered at* 0,

$$\bar{R}^{(6)} < -\epsilon R_{p_1p_2p_3p_4,p_5p_6} R_{p_1p_2p_3p_4,p_5p_6} + \frac{1}{8(n+4)(n+2)n} \times \left(\frac{n-8}{n-2} - \frac{49}{20n^2} + \epsilon\right) \sum_{p_1p_2} (\partial_{p_1p_2} R)^2 + O\big(|\nabla R_{abcd}|^2 + |W|^2\big). \tag{63}$$

We first assume Lemma A.1 and give the proof of Lemma A.2.

Proof of Lemma A.2. It was proved by Hebey and Vaugon in [11] that, if $|W(0)| = |\nabla W(0)| = 0$, then, in conformal normal coordinates centered at 0,

$$R_{p_1p_2,p_3p_4}(R_{p_1p_2,p_3p_4} + R_{p_3p_4,p_1p_2}) \geqslant \frac{6}{n-2} \sum_{p_1p_2} (\partial_{p_1p_2} R)^2 \quad \text{at } 0. \tag{64}$$

We also need the following inequality under the same assumption:

$$R_{p_1p_2p_3p_4,p_5p_6} R_{p_1p_2p_3p_4,p_5p_6} \geqslant \frac{49}{4n^2} \sum_{p_1p_2} (\partial_{p_1p_2} R)^2 \quad \text{at } 0. \tag{65}$$

Note that (65) with $\frac{49}{4n^2}$ replaced by $\frac{9}{n^2}$ was established in [11]. This weaker version leads to an inequality weaker than (A.2), which is nevertheless enough for applications in this paper. To prove (65), we consider

$$\|R_{ikmj,pq} - \alpha R_{,ij}\delta_{kp}\delta_{mq}\|^2 > 0.$$

Namely,

$$|\nabla_{pq} R_{ikmj}|^2 - 2\alpha R_{ikmj,km} R_{,ij} + \alpha^2 n^2 R_{,ij} R_{,ij} \geqslant 0.$$

By the second Bianchi identity, $R_{ikmj,k} = R_{im,j} - R_{ij,m}$. So

$$R_{ikmj,km} = R_{im,jm} - R_{ij,mm} = \frac{1}{2} R_{,ij} + 3R_{,ij} = \frac{7}{2} R_{,ij}.$$

It follows that

$$|\nabla_{pq} R_{ikmj}|^2 + \big(\alpha^2 n^2 - 7\alpha\big) R_{,ij} R_{,ij} \geqslant 0.$$

Inequality (65) follows from the above by taking $\alpha = \frac{7}{2n^2}$.

By (62), (64) and (65) we have

$$\bar{R}^{(6)} < \frac{1}{8(n+4)(n+2)n} \left(\frac{n-8}{n-2} - \frac{49}{20n^2}\right) \sum_{p_1p_2} (\partial_{p_1p_2} R)^2 \quad \text{at } 0.$$

Then (28) holds for some small $\epsilon(n) > 0$ under the assumption $|W(0)| = |\nabla W(0)| = 0$. In general if we do not assume $|W(0)| = |\nabla W(0)| = 0$, all the extra terms can be estimated by Cauchy's inequality, and we obtain (63). Lemma A.2 is established. □

Proof of Lemma A.1. It was proved in [11] that if $|W(0)| = |\nabla W(0)| = 0$, then, in conformal normal coordinates centered at 0,

$$C(2,2)\,\mathrm{Sym}_{p_1..p_6} R_{,p_1..p_6} + 864 R_{p_1p_2p_3p_4,p_5p_6} R_{p_1p_2p_3p_4,p_5p_6}$$
$$+ 4320 R_{p_1p_2,p_3p_4}(R_{p_1p_2,p_3p_4} + R_{p_3p_4,p_1p_2}) - 4320 \sum_{p_1p_2} (\partial_{p_1p_2} R)^2 = 0,$$

where $C(2,2)$ is the complete contraction:

$$C(2,2) = \sum_{p_1=p_2=1}^{n} \sum_{p_3=p_4=1}^{n} \sum_{p_5=p_6=1}^{n}.$$

Since we work in conformal normal coordinates and since $|W(0)| = |\nabla W(0)| = 0$,

$$\mathrm{Sym}_{p_1..p_6} R_{,p_1..p_6} = \mathrm{Sym}_{p_1..p_6} \partial_{p_1..p_6} R - \frac{36}{5} \mathrm{Sym}_{p_1..p_6} R_{,\nu p_1} R_{\nu p_2 p_3 p_4, p_5 p_6}$$
$$= \mathrm{Sym}_{p_1..p_6} \partial_{p_1..p_6} R,$$

where, for the second equality, we have used the skew-symmetry of R_{abcd}.

Thus we have

$$C(2,2)\,\mathrm{Sym}_{p_1..p_6} R_{,p_1..p_6} = 720 \Delta^3 R(0),$$

where Δ denotes the flat Laplacian.

Therefore

$$\Delta^3 R(0) + \frac{6}{5} R_{p_1p_2p_3p_4,p_5p_6} R_{p_1p_2p_3p_4,p_5p_6} + 6 R_{p_1p_2,p_3p_4}(R_{p_1p_2,p_3p_4} + R_{p_3p_4,p_1p_2})$$
$$- 6 \sum_{p_1p_2} (\partial_{p_1p_2} R)^2 = 0. \tag{66}$$

By some standard computations,

$$\bar{R}^{(6)} = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \sum_{|\alpha|=6} \frac{\partial_\alpha R}{\alpha!} \theta^\alpha = \frac{\Delta^3 R(0)}{48(n+4)(n+2)n}. \tag{67}$$

It follows from (66) and (67) that

$$\bar{R}^{(6)} = -\frac{R_{p_1p_2p_3p_4,p_5p_6} R_{p_1p_2p_3p_4,p_5p_6}}{40(n+4)(n+2)n} - \frac{R_{p_1p_2,p_3p_4}(R_{p_1p_2,p_3p_4} + R_{p_3p_4,p_1p_2})}{8(n+4)(n+2)n}$$
$$+ \frac{\sum_{p_1p_2} (\partial_{p_1p_2} R)^2}{8(n+4)(n+2)n}$$

at 0 where $|W(0)| = |\nabla W(0)| = 0$ is assumed. Lemma A.1 is established. □

A.2. Proof of Lemma 2.1

The following fact is elementary. Let $k \geqslant 1$ be an integer, $j \in [1, n]$ be a fixed integer, then

$$\int_{S^{n-1}} \partial_{p_1 .. p_{2k+1}} R(0) x^{p_1} \cdots x^{p_{2k+1}} \cdot x^j \, dx = C(n,k) \partial_j \big(\Delta^k R\big)(0), \tag{68}$$

where

$$C(n,k) = \frac{(2k+1)! |S^{n-1}|}{(2k+n) 2^k k! \prod_{i=0}^{k-1} (n+2i)}.$$

Proof of Lemma 2.1. In conformal normal coordinates,

$$\mathrm{Sym}_{p_1 \cdots p_{2k+3}} R_{p_1 p_2, p_3, \ldots, p_{2k+3}} = 0, \qquad \omega + 2 \leqslant 2k+3 \leqslant 2\omega + 3,$$

if $|\nabla^i R_{abcd}(0)| = 0$ for $0 \leqslant i \leqslant \omega - 1$. See [11]. After contraction this implies

$$\partial_j \big(\Delta^k\big) R(0) = 0, \quad j = 1, \ldots, n, \quad \text{if } \big|\nabla^i R_{abcd}(0)\big| = 0 \quad \text{for } 0 \leqslant i \leqslant \omega - 1.$$

For $n = 10, 11$, we only need to discuss $k = 1$, i.e., we have $\partial_j(\Delta R)(0) = 0$ if $|W(0)| = 0$. In general we have

$$\partial_j (\Delta R)(0) = O\big(|W|\big).$$

This and (68) imply

$$\int_{\mathbb{S}^{n-1}} \tilde{R}^{(3)}(\theta) \theta^j = O\big(|W|\big), \quad 1 \leqslant j \leqslant n.$$

On the other hand, it is clear that

$$\int_{\mathbb{S}^{n-1}} \tilde{R}^{(3)}(\theta) = 0, \qquad \int_{\mathbb{S}^{n-1}} \tilde{R}^{(3)}(\theta) \theta^i \theta^j = 0, \quad 1 \leqslant i, j \leqslant n.$$

Lemma 2.1 follows from the above. □

Appendix B. Some estimates on an ODE

Proposition B.1. *Let $n \geqslant 3$ be an integer, $\delta_0 \geqslant n$ be a constant, let $\hat{H}(r) \in C^0(0, \infty)$ satisfy, for some positive constants C, β and $\alpha > 2$, $\delta_0 + (\alpha - 2)(n - \alpha) > 0$,*

$$0 \leqslant \hat{H}(r) \leqslant C r^{\beta} (1+r)^{-\beta-\alpha}, \quad 0 < r < \infty.$$

Then for any constant p satisfying

$$0 < p \leqslant \beta + 2, \qquad p(p+n-2) < \delta_0,$$

there exists a unique $a(r) \in C^2(0,\infty)$ verifying

$$\begin{cases} Ta(r) := a''(r) + \frac{n-1}{r}a'(r) + (n(n+2)U(r)^{\frac{4}{n-2}} - \frac{\delta_0}{r^2})a(r) = -\hat{H}(r), & 0 < r < \infty, \\ \lim_{r\to 0} a(r) = \lim_{r\to\infty} a(r) = 0. \end{cases} \tag{69}$$

Moreover, for some positive constant C_0 depending only on n, δ_0, α, β, p and C,

$$0 \leqslant a(r) \leqslant C_0 r^p (1+r)^{-p+2-\alpha}, \quad 0 < r < \infty.$$

Lemma B.2. *Let $n \geqslant 3$ be an integer, $\delta_0 \geqslant n$ be a constant, and let $\hat{H}(r)$ be a non-negative function in $C^0(0,\infty)$. Then for any $0 < \epsilon < R$, there exists a unique solution $a_{\epsilon,R} \in C^2[\epsilon, R]$ to*

$$\begin{cases} a''_{\epsilon,R}(r) + \frac{n-1}{r}a'_{\epsilon,R}(r) + (n(n+2)U(r)^{\frac{4}{n-2}} - \frac{\delta_0}{r^2})a_{\epsilon,R}(r) = -\hat{H}(r), & \epsilon < r < R, \\ a_{\epsilon,R}(\epsilon) = a_{\epsilon,R}(R) = 0. \end{cases} \tag{70}$$

Moreover, $a_{\epsilon,R} \geqslant 0$ on $[\epsilon, R]$.

Proof. Let (S^n, g_0) be the standard sphere. It is known that in the stereographic projection coordinates

$$g_0 = \sum_{i=1}^{n+1} dx_i^2 = u_1(y)^{\frac{4}{n-2}} dy^2,$$

where

$$u_1(y) = \left(\frac{2}{1+|y|^2}\right)^{\frac{n-2}{2}} = 2^{\frac{n-2}{2}} U.$$

Also we know that

$$L_{g_0}(\phi) = \left(\Delta_{g_0} - \frac{n(n-2)}{4}\right)\phi = u_1^{-\frac{n+2}{n-2}} \Delta(u_1\phi).$$

If we let $\phi = a_{\epsilon,R}/u_1$, we can rewrite (70) as

$$\left(\Delta_{g_0}\phi - \frac{n(n-2)}{4}\phi\right)u_1^{\frac{n+2}{n-2}} = -\left(n(n+2)U^{\frac{4}{n-2}} - \frac{\delta_0}{r^2}\right)a_{\epsilon,R} - \hat{H}(r).$$

After simplification, we have

$$\Delta_{g_0}\phi + \left(n - \frac{\delta_0(1+r^2)^2}{4r^2}\right)\phi = -u_1^{-\frac{n+2}{n-2}} \hat{H}(r).$$

Since $\delta_0 \geqslant n$

$$n < \frac{\delta_0(1+r^2)^2}{4r^2} \quad \text{for } \epsilon \leqslant r \leqslant R,\ r \neq 1. \tag{71}$$

So the existence and the uniqueness of ϕ as well as $a_{\epsilon,R}$ are proved. □

Proof of Proposition B.1. Clearly for some $R_1 > 1$,

$$\frac{\delta_0}{r^2} \geqslant \frac{8n}{r^4} \quad \text{for } r \geqslant R_1/2, \quad \text{and} \quad \frac{\delta_0}{r^2} \geqslant 8n \quad \text{for } 0 < r \leqslant \frac{2}{R_1}.$$

Fix a $W \in C^0(0, \infty)$ satisfying:

$$W(r) = \frac{\delta_0}{r^2}, \quad \frac{2}{R_1} < r < \frac{R_1}{2},$$
$$\frac{\delta_0}{r^2} \geqslant W(r) \geqslant 4n, \quad \frac{1}{R_1} < r < \frac{2}{R_1},$$
$$W(r) = 4n, \quad 0 < r \leqslant \frac{1}{R_1},$$
$$\frac{\delta_0}{r^2} \geqslant W(r) \geqslant \frac{4n}{r^4}, \quad \frac{R_1}{2} < r \leqslant R_1,$$
$$W(r) = \frac{4n}{r^4}, \quad r > R_1.$$

Clearly $W(r) \leqslant \delta_0/r^2$ and

$$\frac{(1+r^2)^2}{4} W(r) > n, \quad \forall r > 0,\ r \neq 1.$$

With this fact, the first eigenvalue of $-\Delta_{g_0} + (\frac{(1+r^2)^2}{4} W(r) - n)$ is positive on S^n and the potential $(\frac{(1+r^2)^2}{4} W(r) - n)$ is in $C^0(S^n)$. Since $\alpha > 2$, $u_1^{-\frac{n+2}{n-2}} \hat{H}(r) \in L^q(S^n)$ for some $q > 1$. Let $\phi_1 \in W^{2,q}(S^n)$ be the solution of

$$\Delta_{g_0}\phi_1 + \left(n - \frac{(1+r^2)^2}{4} W(r)\right)\phi_1 = -u_1^{-\frac{n+2}{n-2}} \hat{H}(r).$$

By the symmetry of the data, the uniqueness of the solution, ϕ_1 depends only on r. Since both W and $\hat{H}$ are continuous for $0 < r < \infty$, ϕ_1 is C^2 in $0 < r < \infty$. By the maximum principle, $\phi_1 \geqslant 0$. Let

$$a_1(r) = \phi_1(r) u_1(r), \quad 0 < r < \infty.$$

Then $a_1 \in C^2(0, \infty)$, $a_1(r) \geqslant 0$, and

$$\begin{aligned} Ta_1(r) &:= a_1''(r) + \frac{n-1}{r} a_1'(r) + \left[n(n+2)U(r)^{\frac{4}{n-2}} - \frac{\delta_0}{r^2}\right] a_1(r) \\ &\leqslant a_1''(r) + \frac{n-1}{r} a_1'(r) + \left[n(n+2)U(r)^{\frac{4}{n-2}} - W(r)\right] a_1(r) \\ &= \left(\Delta_{g_0}\phi_1 - \frac{n(n-2)}{4}\phi_1\right) u_1^{\frac{n+2}{n-2}} - \left[n(n+2)U(r)^{\frac{4}{n-2}} - W(r)\right]\phi_1 u_1 \end{aligned}$$

$$= \left(\Delta_{g_0}\phi_1 + \left(n - \frac{(1+r^2)^2}{4}W(r)\right)\phi_1\right)u_1^{\frac{n+2}{n-2}} = -\hat{H}_1(r).$$

So $a_1(r)$ is a supersolution. A calculation gives

$$T\left(r^p\right) = -\left(\delta_0 - p(p+n-2) + O(r)\right)r^{p-2} \quad \text{as } r \to 0,$$
$$T\left(r^{2-\alpha}\right) = -\left(\delta_0 + (\alpha-2)(n-\alpha) + O(1/r)\right)r^{-\alpha} \quad \text{as } r \to \infty.$$

Since both $\delta_0 - p(p+n-2)$ and $\delta_0 + (\alpha-2)(n-\alpha)$ are positive, there exists $R_2 > 1$ such that

$$T\left(\gamma r^p\right) \leqslant -\hat{H}(r) \quad \text{for } 0 < r \leqslant \frac{1}{R_2},$$
$$T\left(\gamma r^{2-\alpha}\right) \leqslant -\hat{H}(r) \quad \text{for } r \geqslant R_2,$$

for some $\gamma > 1$. Choose γ larger if necessary such that

$$\gamma\left(\frac{1}{R_2}\right)^p > a_1\left(\frac{1}{R_2}\right), \qquad \gamma(R_2)^{2-\alpha} > a_1(R_2).$$

Define

$$\bar{a}(r) = \begin{cases} \min\{\gamma r^p, a_1(r)\}, & 0 < r < \frac{1}{R_2}, \\ a_1(r), & \frac{1}{R_2} \leqslant r \leqslant R_2, \\ \min\{\gamma r^{2-\alpha}, a_1(r)\}, & r > R_2. \end{cases}$$

Then $\bar{a}(r)$ is a continuous supersolution to $T\bar{a}(r) = -\hat{H}(r)$ in $(0,\infty)$. Therefore for any $0 < \epsilon < R < \infty$, the solution of (70) satisfies

$$0 \leqslant a_{\epsilon,R}(r) \leqslant \bar{a}(r), \quad \forall 0 < r < \infty.$$

Let $\epsilon \to 0$ and $R \to \infty$ along a subsequence, $a_{\epsilon,R}(r)$ tends to $a(r)$ in $C^{1,\lambda}_{loc}(0,\infty)$ for $0 < \lambda < 1$, which satisfies (69) in the weak sense. Since $\hat{H} \in C^0(0,\infty)$, we know that $a \in C^2(0,\infty)$.

Now we prove the uniqueness of the solution of (69). Let $a(r)$ and $b(r)$ be two solutions of (69), then their difference verifies the homogeneous equation

$$T(a-b) \equiv 0 \quad \text{in } (0,\infty).$$

By [8, Theorem 8.1], the homogeneous equation has two linearly independent solutions $a_+(r)$ and $a_-(r)$ with the asymptotic behavior

$$\lim_{r\to\infty}\frac{a_+(r)}{r^{\lambda_1}} = \lim_{r\to\infty}\frac{a_-(r)}{r^{\lambda_2}} = 1,$$

where λ_1 and λ_2 are the two solutions of $\lambda^2 + (n-2)\lambda - \delta_0 = 0$ such that $\lambda_1 > 0$ and $\lambda_2 < 2-n$. Since $(a-b)(r) = C_1 a_+ + C_2 a_-$ for some constants C_1 and C_2, and since $(a-b)(r) \to 0$ as $r \to \infty$, we must have $C_1 = 0$ and therefore

$$\lim_{r\to\infty} r^{n-2}(a-b)(r) = 0. \tag{72}$$

Since $T(a-b) = 0$ corresponds to

$$\left(\Delta_{g_0} + \left[n - \frac{\delta_0(1+r^2)^2}{4r^2}\right]\right)\frac{(a-b)(r)}{u_1} = 0 \quad \text{on } S^n \setminus \{P, N\},$$

where N, P are the south pole and the north pole, respectively. We know from (72) that $\frac{(a-b)}{u_1}(p) \to 0$ as $p \to \{N, P\}$, and in view of (71), we can apply the maximum principle to conclude $a - b \equiv 0$. Proposition B.1 is established. □

Appendix C. Two useful lower bounds

Proposition C.1. *For $n \geqslant 10$, there exists a unique $f_2 \in C^\infty((0,\infty))$ satisfying*

$$\begin{cases} f_2''(r) + \dfrac{n-1}{r} f_2'(r) + \left(n(n+2)U^{\frac{4}{n-2}} - \dfrac{2n}{r^2}\right) f_2 = -r^2 U, \quad 0 < r < \infty, \\ \lim\limits_{r\to 0} f_2(r) = \lim\limits_{r\to\infty} f_2(r) = 0. \end{cases} \tag{73}$$

Moreover, for some universal positive constant C,

$$\frac{U}{6(n-4)}\left(r^4 + \frac{3n-4}{n-2} r^2\right) \leqslant f_2(r) \leqslant C r^{\frac{3}{2}}(1+r)^{\frac{9}{2}-n}, \quad 0 < r < \infty. \tag{74}$$

Proposition C.2. *Let $V_\lambda = n(n+2)\int_0^1 (tU + (1-t)U^\lambda)^{\frac{4}{n-2}}\,dt$. Then there exists a unique $f_{2,\lambda} \in C^\infty(0,\infty)$ satisfying*

$$\begin{cases} f_{2,\lambda}''(r) + \dfrac{n-1}{r} f_{2,\lambda}'(r) + \left(V_\lambda - \dfrac{2n}{r^2}\right) f_{2,\lambda}(r) = -r^2 U^\lambda(r)\left(1 - \left(\dfrac{\lambda}{r}\right)^8\right), \quad r \in (\lambda, \infty), \\ f_{2,\lambda}(\lambda) = 0, \quad \lim\limits_{r\to\infty} f_{2,\lambda}(r) = 0. \end{cases}$$

Moreover, for any $\epsilon > 0$, there exist $\delta(\epsilon)$ satisfying $\delta(\epsilon) \to 0$ as $\epsilon \to 0$, and a universal constant C such that for $|\lambda - 1| \leqslant \delta(\epsilon)$,

$$\frac{1-\epsilon}{6(n-4)} U^\lambda\left(r^4\left(1 - \left(\frac{\lambda}{r}\right)^8\right) + \frac{3n-4}{n-2} r^2\left(1 - \left(\frac{\lambda}{r}\right)^4\right)\right) \leqslant f_{2,\lambda}(r) \leqslant C r^{6-n} \tag{75}$$

for $\lambda < r < \infty$.

Proposition C.3. *For $n \geqslant 8$, there exists a unique $f_3 \in C^\infty(0,\infty)$ satisfying*

$$\begin{cases} f_3''(r) + \frac{n-1}{r} f_3'(r) + \left(n(n+2)U^{\frac{4}{n-2}} - \frac{3(n+1)}{r^2} \right) f_3 = -r^3 U, \quad 0<r<\infty, \\ \lim_{r\to 0} f_3(r) = \lim_{r\to\infty} f_3(r) = 0 \end{cases} \tag{76}$$

and, for a universal constant C

$$0 \leqslant f_3(r) \leqslant C r^{\frac{5}{2}} (1+r)^{\frac{9}{2}-n}, \quad r>0.$$

The existence, uniqueness, and the upper bounds of f_2, $f_{2,\lambda}$ and f_3 follow from Proposition B.1. So we only prove the lower bound of f_2 and $f_{2,\lambda}$ in this section.

Let $\phi_1(r) = r^4 U/(6(n-4))$. Then by elementary computation

$$\Delta\phi_1 + \left(n(n+2)U^{\frac{4}{n-2}} - \frac{2n}{r^2} \right)\phi_1 > -r^2 U.$$

Consequently

$$\begin{cases} T(f_2-\phi_1) := \Delta(f_2-\phi_1) + \left(n(n+2)U^{\frac{4}{n-2}} - \frac{2n}{r^2} \right)(f_2-\phi_1) = -g \leqslant 0, \quad 0<r<\infty, \\ \lim_{r\searrow 0}(f_2-\phi_1)(r) = \lim_{r\to\infty}(f_2-\phi_1)(r) = 0, \end{cases}$$

where

$$g(r) = r^2 U \left(\frac{4(n-2)}{3(n-4)} \frac{1}{1+r^2} + \frac{2n}{3(n-4)} r^2 U^{\frac{4}{n-2}} \right).$$

By Proposition B.1, there exists a positive solution $a_1(r)$ of

$$\begin{cases} Ta_1(r) = -g(r), \quad 0<r<\infty, \\ \lim_{r\to 0} a_1(r) = \lim_{r\to\infty} a_1(r) = 0. \end{cases}$$

Since

$$\begin{cases} T(f_2-\phi_1-a_1) = 0, \quad 0<r<\infty, \\ \lim_{r\to 0}(f_2-\phi_1-a_1)(r) = \lim_{r\to\infty}(f_2-\phi_1-a_1)(r) = 0, \end{cases}$$

we know from the proof of Proposition B.1 that $f_2 - \phi_1 - a_1 \equiv 0$. So we only need to obtain a lower bound for a_1. Let

$$\phi_2(r) = \frac{3n-4}{6(n-4)(n-2)} r^2 U,$$

then direct computation gives

$$\Delta\phi_2 + \left(n(n+2)U^{\frac{4}{n-2}} - \frac{2n}{r^2} \right)\phi_2 = -\frac{2(3n-4)}{3(n-4)} U \left(\frac{r^2}{1+r^2} - \frac{n}{n-2} r^2 U^{\frac{4}{n-2}} \right).$$

Then one verifies immediately

$$g(r) > \frac{2(3n-4)}{3(n-4)} U\left(\frac{r^2}{1+r^2} - \frac{n}{n-2} r^2 U^{\frac{4}{n-2}}\right).$$

This means

$$f_2(r) - \frac{1}{6(n-4)} r^4 U > \frac{3n-4}{6(n-4)(n-2)} r^2 U.$$

(74) is established.

To prove (75), we still use maximum principle as before, but instead of comparing $f_{2,\lambda}$ directly with the right-hand side in (75), we compare $f_{2,\lambda}$ with $(1-2\epsilon)(\phi_3+\phi_4)$ where

$$\phi_3 := \phi_1 - \phi_1^\lambda = \frac{1}{6(n-4)}\left(r^4 U^\lambda \left(1 - \left(\frac{\lambda}{r}\right)\right)^8 + r^4\left(U - U^\lambda\right)\right),$$

$$\phi_4 := \phi_2 - \phi_2^\lambda = \frac{3n-4}{6(n-4)(n-2)}\left(r^2 U^\lambda \left(1 - \left(\frac{\lambda}{r}\right)\right)^4 + r^2\left(U - U^\lambda\right)\right).$$

Our purpose is to show

$$f_{2,\lambda} \geqslant (1-2\epsilon)(\phi_3+\phi_4), \tag{77}$$

where ϵ is any fixed small positive constant and λ is close to 1 depending on ϵ. Once we have (77), (75) follows from (77) and the following well-known fact:

$$U(r) - U^\lambda(r) = (1-\lambda)\left(1 - \frac{\lambda}{r}\right) O\left(r^{2-n}\right).$$

Let $T := \Delta + V_\lambda - \frac{2n}{r^2}$, then by elementary computation,

$$\begin{aligned} T\phi_3 = {} & -r^2 U^\lambda \left(1 - \left(\frac{\lambda}{r}\right)^8\right) + \frac{4(n-2)}{3(n-4)} U^\lambda \left(\frac{r^2}{1+r^2} - \frac{\lambda^8}{r^6(1+\lambda^4/r^2)}\right) \\ & + \frac{2n}{3(n-4)} r^4 \left(U^\lambda\right)^{\frac{n+2}{n-2}} \left(1 - \left(\frac{\lambda}{r}\right)^8\right) + \delta(\lambda)\left(1 - \frac{\lambda}{r}\right) r^{4-n}, \end{aligned}$$

$$\begin{aligned} T\phi_4 = {} & -\frac{2(3n-4)}{3(n-4)} U^\lambda \left(1 - \left(\frac{\lambda}{r}\right)^4\right) + \frac{2(3n-4)}{3(n-4)} U^\lambda \left(\frac{1}{1+r^2} - \left(\frac{\lambda}{r}\right)^4 \frac{1}{1+\lambda^4/r^2}\right) \\ & + \frac{2n(3n-4)}{3(n-4)(n-2)} r^2 \left(U^\lambda\right)^{\frac{n+2}{n-2}} \left(1 - \left(\frac{\lambda}{r}\right)^4\right) + \delta(\lambda)\left(1 - \frac{\lambda}{r}\right) r^{2-n}, \end{aligned}$$

where we use $\delta(\lambda)$ to indicate a function of λ which tends to 0 as $\lambda \to 1$.

Then one verifies that

$$T\left(f_{2,\lambda} - (1-2\epsilon)(\phi_3+\phi_4)\right) < 0, \quad \lambda < r < \infty,$$

if λ is close to 1 enough. Since we also have $f_{2,\lambda}(\lambda) = \phi_3(\lambda) = \phi_4(\lambda) = 0$ and $\lim_{r\to\infty}(f_{2,\lambda} - (1-2\epsilon)(\phi_3 + \phi_4)) = 0$, we have proved (77) by maximum principle. Proposition C.1 is established.

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