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Absolute algebra III—The saturated spectrum

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ABSTRACT

We investigate the algebraic and topological preliminaries to a geometry in characteristic one.

1. Introduction

The theory of *characteristic one semirings* (*i.e.* semirings with $1 + 1 = 1$) originated in many different contexts: pure algebra (see *e.g.* LaGrassa's Ph.D. thesis [8]), idempotent analysis and the study of $\mathbf{R}_+^{max}$ [1,3], and Zhu's theory [12], itself inspired by considerations of Hopf algebras (see [11]). Its main motivation is now the Riemann Hypothesis, via adeles and the theory of hyperrings (cf. [2–4], notably Section 6 from [4]).

For example, it has by now become clear (see [4], Theorem 3.11) that the classification of finite hyperfield extensions of the Krasner hyperring K is one of the main problems of the theory. If H denotes a hyperring extension of K, B_1 the smallest characteristic one semifield and S the sign hyperring, then there are canonical mappings $B_1 \to S \to K \to H$, whence mappings

$$Spec(H) \to Spec(K) \to Spec(S) \to Spec(B_1),$$

thus $Spec(H)$ "lies over" $Spec(B_1)$ (see [4], Section 6, notably diagram (43), where B_1 is denoted by $\mathbf{B}$).

The ultimate goal of our investigation is to provide a proper algebraic geometry in characteristic one. The natural procedure is to construct "affine B_1-schemes" and endow them with an appropriate topology and a sheaf of semirings; a suitable glueing procedure will then produce general "B_1-schemes". This program is not yet completed; in this paper, we deal with a natural first step: the extension to B_1-algebras of the notions of spectrum and Zariski topology, and the fundamental topological properties of these objects. In order to construct a structure sheaf over the spectrum of a B_1-algebra, Castella's localization procedure [1] will probably be useful.

As in our two previous papers, we work in the context of B_1-algebras, *i.e.* characteristic one semirings. For such an A, one may define prime ideals by analogy to the classical commutative algebra. In order to define the spectrum of a B_1-algebra A, two candidates readily suggest themselves: the set $Spec(A)$ of prime (in a suitable sense) congruences, and the set $Pr(A)$ of prime ideals; in contrast to the classical situation, these two approaches are not equivalent. In fact both sets may be equipped with a natural topology of Zariski type (see [10], Theorem 2.4 and Proposition 3.15), but they do not in general correspond bijectively to one another; nevertheless, the subset $Pr_s(A) \subseteq Pr(A)$ of *saturated* prime ideals is in natural bijection with the set of *excellent* prime congruences (see below) on A.

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It turns out (Section 3) that there is another, far less obvious, bijection between $Pr_s(A)$ and the *maximal spectrum* $MaxSpec(A) \subseteq Spec(A)$ of A. This mapping is actually a homeomorphism for the natural (Zariski-type) topologies mentioned above. As a by-product, we find a new point of view on the description of the maximal spectrum of the polynomial algebra $B_1[x_1, \ldots, x_n]$ found in [9,12]. The homeomorphism in question is actually functorial in A (Section 4).

In Section 5, we show that the theory of the nilradical and of the root of an ideal carry over, with some precautions, to our setting; the situation is even better when one restricts oneself to *saturated* ideals. This allows us, in Section 6, to establish some nice topological properties of

$$MaxSpec(A) \simeq Pr_s(A);$$

namely, T_0 and quasi-compact (Theorem 6.1), and the open quasi-compact sets constitute a basis stable under finite intersections. Furthermore this space is *sober* , *i.e.* each irreducible closed set has a (necessarily unique) generic point. In other words, $Pr_s(A)$ satisfies the usual properties of a ring spectrum that are used in the algebraic geometry (see *e.g.* the canonical reference [6]): $Pr_s(A)$ is a *spectral space* in the sense of Hochster [7].

In the last paragraph, we discuss the particular case of a *monogenic* B_1-algebra, that is, a quotient of the polynomial algebra $B_1[x]$; in [9], we had listed the smallest finite such algebras.

In a subsequent work I shall investigate how higher concepts and methods of the commutative algebra (minimal prime ideals, zero divisors, primary decomposition) carry over to characteristic one semirings.

2. Definitions and notation

We shall review some of the definitions and notation of our previous two papers [9,10].

$B_1 = \{0, 1\}$ denotes the smallest *characteristic one semifield* ; the operations of addition and multiplication are the obvious ones, with the slight change that

$$1 + 1 = 1.$$

A B_1-module M is a nonempty set equipped with an action

$$B_1 \times M \to M$$

satisfying the usual axioms (see [9], Definition 2.3); as first seen in [12], Proposition 1 (see also [9], Theorem 2.5), B_1-modules can be canonically identified with ordered sets having a smallest element (0) and in which any two elements a and b have a least upper bound ($a + b$). In particular, one may identify finite B_1-modules and nonempty finite lattices.

A (commutative) B_1-algebra is a B_1-module equipped with an associative multiplication that has a neutral element and satisfies the usual axioms relative to addition (see [9], Definition 4.1). In the sequel, except when otherwise indicated, A will denote a B_1-algebra.

An *ideal* I of A is by definition a subset containing 0, stable under addition, and having the property that

$$\forall x \in A \, \forall y \in I \quad xy \in I;$$

I is termed prime if $I \neq A$ and

$$ab \in I \Longrightarrow a \in I \quad \text{or} \quad b \in I.$$

By a *congruence* on A, we mean an equivalence relation on A compatible with the operations of addition and multiplication. The trivial congruence $\mathcal{C}_0(A)$ is characterized by the fact that any two elements of A are equivalent under it; the congruences are naturally ordered by inclusion, and

$$MaxSpec(A)$$

will denote the set of maximal nontrivial congruences on A.

For $\mathcal{R}$ a congruence on A, we set

$$I(\mathcal{R}) := \{x \in A | x \, \mathcal{R} \, 0\};$$

it is an ideal of A.

A nontrivial congruence $\mathcal{R}$ is termed *prime* if

$$ab \, \mathcal{R} \, 0 \Longrightarrow a \, \mathcal{R} \, 0 \quad \text{or} \quad b \, \mathcal{R} \, 0;$$

the set of prime congruences on A is denoted by $Spec(A)$. It turns out that (see [10], Proposition 2.3)

$$MaxSpec(A) \subseteq Spec(A).$$

For J an ideal of A, there is a unique smallest congruence $\mathcal{R}_J$ such that $J \subseteq I(\mathcal{R})$; it is denoted by $\mathcal{R}_J$. Such congruences are termed *excellent* .

An ideal J of A is termed *saturated* if it is of the form $I(\mathcal{R})$ for some congruence $\mathcal{R}$; this is the case if and only if $J = \bar{J}$, where

$$\bar{J} := I(\mathcal{R}_J).$$

We shall denote the set of prime ideals of A by $Pr(A)$, and the set of saturated prime ideals by $Pr_s(A)$.

For $S \subseteq A$, let us set

$$W(S) := \{\mathcal{P} \in Pr(A) | S \subseteq \mathcal{P}\},$$

and

$$V(S) := \{\mathcal{R} \in Spec(A) | S \subseteq I(\mathcal{R})\}.$$

As seen in [10], Theorem 2.4 and Proposition 3.4, the family $(W(S))_{S\subseteq A}$ is the family of closed sets for a topology on $Pr(A)$, and the family $(V(S))_{S\subseteq A}$ is the family of closed sets for a topology on $Spec(A)$. We shall always consider $Spec(A)$ and $Pr(A)$ as equipped with these topologies, and their subsets with the induced topologies.

For M a commutative monoid, we define the *Deitmar spectrum* $Spec_{\mathcal{D}}(M)$ as the set of prime ideals (including $\emptyset$) of M (in [5], this is denoted by $Spec\,\mathbf{F}_M$). We define $\mathcal{F}(M) = B_1[M]$ as the "monoid algebra of M over B_1"; the functor $\mathcal{F}$ is adjoint to the forgetful functor from the category of B_1-algebras to the category of monoids (for the details, see [9], Section 5). Furthermore, there is an explicit canonical bijection between $Spec_{\mathcal{D}}(M)$ and a certain subset of $Spec(\mathcal{F}(M))$ (see [10], Theorem 4.2).

For S a subset of A, let $\langle S\rangle$ denote the intersection of all the ideals of A containing S (there is always at least one such ideal: A itself). It is clear that $\langle S\rangle$ is an ideal of A, and therefore is the smallest ideal of A containing S. As in ring theory, one may see that

$$\langle S\rangle = \left\{\sum_{j=1}^{n} a_j s_j | n \in \mathbf{N}, (a_1, \ldots, a_n) \in A^n, (s_1, \ldots, s_n) \in S^n\right\}.$$

We shall denote by $\mathcal{SP}$ the category whose objects are spectra of B_1-algebras and whose morphisms are the continuous maps between them.

3. A new description of maximal congruences

Let A denote a B_1-algebra.

For $\mathcal{P}$ a saturated prime ideal of A, let us define a relation $\mathcal{S}_{\mathcal{P}}$ on A by:

$$x\mathcal{S}_{\mathcal{P}}y \equiv (x \in \mathcal{P} \text{ and } y \in \mathcal{P}) \quad \text{or} \quad (x \notin \mathcal{P} \text{ and } y \notin \mathcal{P}).$$

Then $\mathcal{S}_{\mathcal{P}}$ is a congruence on A: if $x\mathcal{S}_{\mathcal{P}}y$ and $x'\mathcal{S}_{\mathcal{P}}y'$, then one and only one of the following holds:

(i) $x \in \mathcal{P}, y \in \mathcal{P}, x' \in \mathcal{P}$ and $y' \in \mathcal{P}$,
(ii) $x \in \mathcal{P}, y \in \mathcal{P}, x' \notin \mathcal{P}$ and $y' \notin \mathcal{P}$,
(iii) $x \notin \mathcal{P}, y \notin \mathcal{P}, x' \in \mathcal{P}$ and $y' \in \mathcal{P}$,
(iv) $x \notin \mathcal{P}, y \notin \mathcal{P}, x' \notin \mathcal{P}$ and $y' \notin \mathcal{P}$.

In case (i), $x + x' \in \mathcal{P}$ and $y + y' \in \mathcal{P}$, whence $x + x'\mathcal{S}_{\mathcal{P}}y + y'$; in cases (ii) and (iv), $x + x' \notin \mathcal{P}$ and $y + y' \notin \mathcal{P}$ (as $\mathcal{P}$ is saturated), whence $x + x'\mathcal{S}_{\mathcal{P}}y + y'$. Case (iii) is symmetrical relatively to case (ii), therefore, in all cases, $x + x'\mathcal{S}_{\mathcal{P}}y + y'$: $\mathcal{S}_{\mathcal{P}}$ is compatible with addition.

In cases (i), (ii) and (iii), $xx' \in \mathcal{P}$ and $yy' \in \mathcal{P}$, whence $xx'\mathcal{S}_{\mathcal{P}}yy'$; in case (iv) $xx' \notin \mathcal{P}$ and $yy' \notin \mathcal{P}$ (as $\mathcal{P}$ is prime), whence also $xx'\mathcal{S}_{\mathcal{P}}yy'$: $\mathcal{S}_{\mathcal{P}}$ is compatible with multiplication, hence is a congruence on A.

As $0 \in \mathcal{P}$ and $1 \notin \mathcal{P}$, $0 \not\mathcal{S}_{\mathcal{P}} 1$, therefore $\mathcal{S}_{\mathcal{P}}$ is nontrivial; but each $x \in A$ is either in $\mathcal{P}$ (whence $x\mathcal{S}_{\mathcal{P}}0$) or not (whence $x\mathcal{S}_{\mathcal{P}}1$). It follows that

$$\frac{A}{\mathcal{S}_{\mathcal{P}}} = \{\bar{0}, \bar{1}\} \simeq B_1;$$

in particular, $\mathcal{S}_{\mathcal{P}}$ is maximal: $\mathcal{S}_{\mathcal{P}} \in MaxSpec(A)$.

Obviously, $I(\mathcal{S}_{\mathcal{P}}) = \mathcal{P}$.

Furthermore, let $(x, y) \in A^2$ be such that $x\mathcal{R}_{\mathcal{P}}y$; then there is $z \in \mathcal{P}$ such that $x+z = y+z$. If $x \in \mathcal{P}$ then $y+z = x+z \in \mathcal{P}$, whence $y \in \mathcal{P}$ (as $y + (y + z) = y + z$ and $\mathcal{P}$ is saturated); symmetrically, $y \in \mathcal{P}$ implies $x \in \mathcal{P}$, whence the assertions $(x \in \mathcal{P})$ and $(y \in \mathcal{P})$ are equivalent, and $x\mathcal{S}_{\mathcal{P}}y$. We have shown that

$$\mathcal{R}_{\mathcal{P}} \leq \mathcal{S}_{\mathcal{P}}.$$

We shall denote by α_A the mapping

$$\begin{aligned}\alpha_A\ &: Pr_s(A) \to MaxSpec(A)\\ &\mathcal{P} \mapsto \mathcal{S}_{\mathcal{P}}.\end{aligned}$$

Let $\mathcal{R} \in MaxSpec(A)$; then $\mathcal{R} \in Spec(A)$, whence $I(\mathcal{R})$ is prime; by Theorem 3.8 of [10], it is saturated, *i.e.* $I(\mathcal{R}) \in Pr_s(A)$. Let us set

$$\beta_A(\mathcal{R}) := I(\mathcal{R}).$$

Theorem 3.1. *The mappings*

$$\alpha_A : Pr_s(A) \mapsto MaxSpec(A)$$

and

$$\beta_A : MaxSpec(A) \mapsto Pr_s(A)$$

are bijections, inverse of one another. They are continuous for the topologies on $Pr_s(A)$ and $MaxSpec(A)$ induced by the topologies on $Pr(A)$ and $Spec(A)$ mentioned above, whence $Pr_s(A)$ and $MaxSpec(A)$ are homeomorphic.

Proof. Let $\mathcal{R} \in MaxSpec(A)$; then

$$\alpha_A(\beta_A(\mathcal{R})) = \alpha_A(I(\mathcal{R})) = \mathcal{S}_{I(\mathcal{R})}.$$

Let us assume $x\mathcal{R}y$; then, if $x \in I(\mathcal{R})$ one has $x\mathcal{R}0$, whence $y\mathcal{R}0$ and $y \in I(\mathcal{R})$; by symmetry, $y \in I(\mathcal{R})$ implies $x \in I(\mathcal{R})$, thus $(x \in I(\mathcal{R}))$ and $(y \in I(\mathcal{R}))$ are equivalent, *i.e.* $x\mathcal{S}_{I(\mathcal{R})}y$. We have proved that $\mathcal{R} \leq \mathcal{S}_{I(\mathcal{R})}$. As $\mathcal{R}$ is maximal, we have $\mathcal{R} = \mathcal{S}_{I(\mathcal{R})}$, whence

$$\alpha_A(\beta_A(\mathcal{R})) = \mathcal{S}_{I(\mathcal{R})} = \mathcal{R},$$

and

$$\alpha_A \circ \beta_A = Id_{MaxSpec(A)}.$$

Let now $\mathcal{P} \in Pr_s(A)$; then

$$\begin{aligned}(\beta_A \circ \alpha_A)(\mathcal{P}) &= \beta_A(\alpha_A(\mathcal{P}))\\ &= \beta_A(\mathcal{S}_{\mathcal{P}})\\ &= I(\mathcal{S}_{\mathcal{P}})\\ &= \mathcal{P},\end{aligned}$$

whence

$$\beta_A \circ \alpha_A = Id_{Pr_s(A)},$$

and the first statement follows.

Let now F denote a closed subset of $Pr_s(A)$; then $F = G \cap Pr_s(A)$ for G a closed subset of $Pr(A)$ and $G = W(S) := \{\mathcal{P} \in Pr(A) | S \subseteq \mathcal{P}\}$ for some subset S of A. But then, for $\mathcal{R} \in MaxSpec(A)$, $\mathcal{R} \in \beta_A^{-1}(F)$ if and only if $\beta_A(\mathcal{R}) \in F$, *i.e.* $I(\mathcal{R}) \in G \cap Pr_s(A)$, that is $I(\mathcal{R}) \in G$, or $S \subseteq I(\mathcal{R})$, which means $\mathcal{R} \in V(S)$. Thus

$$\beta_A^{-1}(F) = V(S) \cap MaxSpec(A)$$

is closed in $MaxSpec(A)$. We have shown the continuity of β_A.

Let now $H \subseteq MaxSpec(A)$ be closed; then $H = MaxSpec(A) \cap L$ for some closed subset L of $Spec(A)$, and $L = V(T)$ for some subset T of A. Then a saturated prime ideal $\mathcal{P}$ of A belongs to $\alpha_A^{-1}(H)$ if and only if $\alpha_A(\mathcal{P}) \in H$, that is

$$\mathcal{S}_{\mathcal{P}} \in MaxSpec(A) \cap L,$$

i.e.

$$\mathcal{S}_{\mathcal{P}} \in V(T)$$

or $T \subseteq I(\mathcal{S}_{\mathcal{P}})$. But $I(\mathcal{S}_{\mathcal{P}}) = \mathcal{P}$ whence $\mathcal{P}$ belongs to $\alpha_A^{-1}(H)$ if and only if $T \subseteq \mathcal{P}$, that is

$$\alpha_A^{-1}(H) = W(T) \cap Pr_s(A),$$

which is closed in $Pr_s(A)$. □

Let us consider the special case in which A is in the image of $\mathcal{F}$: $A = \mathcal{F}(M)$, for M a commutative monoid. Let P be a prime ideal of M; as seen in [10], Theorem 4.2, $\tilde{P}$ is a saturated prime ideal in A, and one obtains in this way a bijection between $Spec_{\mathcal{D}}(M)$ and $Pr_s(A)$. The following is now obvious.

Theorem 3.2. *The mapping*

$$\begin{aligned}\psi_M\ &: Spec_{\mathcal{D}}(M) \to MaxSpec(\mathcal{F}(M))\\ &P \mapsto \alpha_{\mathcal{F}(M)}(\tilde{P})\end{aligned}$$

is a bijection.

The following two particular cases are of special interest.

1. M is a group; then $Spec_{\mathcal{D}}(M) = \{\emptyset\}$, whence $MaxSpec(\mathcal{F}(G))$ has exactly one element.
2. $M = C_n := \langle x_1, \ldots, x_n\rangle$ is the free monoid on n variables $x_1, \ldots, x_n$. Then the elements of $Spec_{\mathcal{D}}(M)$ are the $(P_J)_{J\subseteq\{1,\ldots,n\}}$, where

$$P_J := \bigcup_{j\in J} x_j C_n$$

(a fact that was already used in [10], Example 4.3). Then

$$\psi_M(P_J) = \alpha_{\mathcal{F}(M)}(\tilde{P_J}) = \mathcal{S}_{\tilde{P_J}}$$

whence $x\psi_M(P_J)y$ if and only if either ($x \in \tilde{P_J}$ and $y \in \tilde{P_J}$) or ($x \notin \tilde{P_J}$ and $y \notin \tilde{P_J}$). But we have seen in [9], Theorem 4.5, that

$$\mathcal{F}(M) = B_1[x_1, \ldots, x_n]$$

could be identified with the set of finite formal sums of elements of M. Obviously, an element x of $\mathcal{F}(M)$ belongs to $\tilde{P_J}$ if and only if at least one of its components involves at least one factor $x_j (j \in J)$. It is now clear that, using the notation of [9], Definition 4.6 and Theorem 4.7,

$$\psi_M(P_J) = \widetilde{\ }_J.$$

We hereby recover the description of $MaxSpec(B_1[x_1, \ldots, x_n])$ given in [9] (Theorems 4.7, 4.8 and 4.10).

The following result will be useful.

Theorem 3.3. *Any proper saturated ideal of a B_1-algebra A is contained in a saturated prime ideal of A.*

Proof. Let J be a proper saturated ideal of A; as $I(\mathcal{R}_J) = \bar{J} = J \neq A$, $\mathcal{R}_J \neq \mathcal{C}_0(A)$. By Zorn's Lemma, one has $\mathcal{R}_J \leq \mathcal{R}$ for some $\mathcal{R} \in MaxSpec(A)$. According to Theorem 2.1, $\mathcal{R} = \alpha_A(\mathcal{P}) = \mathcal{S}_{\mathcal{P}}$ for a saturated prime ideal $\mathcal{P}$ of A, therefore $\mathcal{R}_{\mathcal{J}} \leq \mathcal{S}_{\mathcal{P}}$ and

$$J = \bar{J} = I(\mathcal{R}_J) \subseteq I(\mathcal{S}_{\mathcal{P}}) = \mathcal{P}. \quad \Box$$

4. Functorial properties of spectra

Let $\varphi : A \to C$ denote a morphism of B_1-algebras, and let $\mathcal{R} \in Spec(C)$. We define a binary relation $\tilde{\varphi}(\mathcal{R})$ on A by:

$$\forall (a, a') \in A^2 \quad a\tilde{\varphi}(\mathcal{R})a' \equiv \varphi(a)\mathcal{R}\varphi(a').$$

It is clear that $\tilde{\varphi}(\mathcal{R})$ is a congruence on A, and that

$$I(\tilde{\varphi}(\mathcal{R})) = \varphi^{-1}(I(\mathcal{R})).$$

In particular $I(\tilde{\varphi}(\mathcal{R}))$ is a prime ideal of A, hence $\tilde{\varphi}(\mathcal{R}) \in Spec(A)$: $\tilde{\varphi}$ maps $Spec(C)$ into $Spec(A)$. Let $F := V(S)$ be a closed subset of $Spec(A)$, and let $\mathcal{R} \in Spec(C)$; then $\mathcal{R} \in \tilde{\varphi}^{-1}(F)$ if and only if $\tilde{\varphi}(\mathcal{R}) \in F$, that is $S \subseteq I(\tilde{\varphi}(\mathcal{R}))$, or $S \subseteq \varphi^{-1}(I(\mathcal{R}))$, *i.e.* $\varphi(S) \subseteq I(\mathcal{R})$, or $\mathcal{R} \in V(\varphi(S))$. Therefore $\tilde{\varphi}^{-1}(F) = V(\varphi(S))$ is closed in $Spec(C)$: $\tilde{\varphi}$ is continuous.

Furthermore, for $\varphi : A \to C$ and $\psi : C \to D$ one has

$$\widetilde{\psi \circ \varphi} = \tilde{\varphi} \circ \tilde{\psi} : Spec(D) \to Spec(A).$$

It follows that the equations $\mathcal{H}(A) = Spec(A)$ and $\mathcal{H}(\varphi) = \tilde{\varphi}$ define a contravariant functor $\mathcal{H}$ from $\mathcal{Z}_a$ to $\mathcal{SP}$.

Let J denote an ideal in C, and let us assume $a\mathcal{R}_{\varphi^{-1}(J)}a'$; then there is an $x \in \varphi^{-1}(J)$ with $a + x = a' + x$. Now $\varphi(x) \in J$ and

$$\begin{aligned}\varphi(a) + \varphi(x) &= \varphi(a + x)\\ &= \varphi(a' + x)\\ &= \varphi(a') + \varphi(x),\end{aligned}$$

whence $\varphi(a)\mathcal{R}_J\varphi(a')$ and $a\tilde{\varphi}(\mathcal{R}_J)a'$. We have established.

Proposition 4.1. *Let A and C denote B_1-algebras, $\varphi : A \to C$ a morphism and J an ideal of C: then*

$$\mathcal{R}_{\varphi^{-1}(J)} \leq \tilde{\varphi}(\mathcal{R}_J).$$

Theorem 4.2. *Let A and C denote two B_1-algebras, and $\varphi : A \to C$ a morphism. Then $\tilde{\varphi} : Spec(C) \to Spec(A)$ maps $MaxSpec(C)$ into $MaxSpec(A)$, and the diagram*

$$\begin{array}{ccc} Pr_s(C) & \stackrel{\varphi^{-1}}{\to} & Pr_s(A) \\ \downarrow\alpha_C & & \downarrow\alpha_A \\ MaxSpec(C) & \stackrel{\tilde{\varphi}}{\to} & MaxSpec(A) \end{array}$$

commutes.

Proof. Let $\mathcal{P} \in Pr_s(C)$, then, for all $(a, a') \in A^2$

$$\begin{aligned} a\tilde{\varphi}(\mathcal{S}_{\mathcal{P}})a' &\Longleftrightarrow \varphi(a)\mathcal{S}_{\mathcal{P}}\varphi(a') \\ &\Longleftrightarrow (\varphi(a) \in \mathcal{P} \quad \text{and} \quad \varphi(a') \in \mathcal{P}) \\ \text{or} \quad (\varphi(a) \notin \mathcal{P} \text{ and } \varphi(a') \notin \mathcal{P}) \\ &\Longleftrightarrow (a \in \varphi^{-1}(\mathcal{P}) \quad \text{and} \quad a' \in \varphi^{-1}(\mathcal{P})) \\ \text{or} \quad (a \notin \varphi^{-1}(\mathcal{P}) \text{ and } a' \notin \varphi^{-1}(\mathcal{P})) \\ &\Longleftrightarrow a\mathcal{S}_{\varphi^{-1}(\mathcal{P})}a'. \end{aligned}$$

Therefore

$$\begin{aligned} (\tilde{\varphi} \circ \alpha_C)(\mathcal{P}) &= \tilde{\varphi}(\alpha_C(\mathcal{P})) \\ &= \tilde{\varphi}(\mathcal{S}_{\mathcal{P}}) \\ &= \mathcal{S}_{\varphi^{-1}(\mathcal{P})} \\ &= \alpha_A(\varphi^{-1}(\mathcal{P})) \\ &= (\alpha_A \circ \varphi^{-1})(\mathcal{P}) \end{aligned}$$

whence $\tilde{\varphi} \circ \alpha_C = \alpha_A \circ \varphi^{-1}$.

Incidentally we have proved that $\tilde{\varphi}$ maps $MaxSpec(C) = \alpha_C(Pr_s(C))$ into $\alpha_A(Pr_s(A)) = MaxSpec(A)$, *i.e.* the first assertion. □

5. Nilpotent radicals and prime ideals

The usual theory generalizes without major problem to B_1-algebras.

Theorem 5.1. *In the B_1-algebra A,let us define*

$$Nil(A) := \{x \in A | (\exists n \geq 1) x^n = 0\}.$$

Then Nil(A) is a saturated ideal of A, and one has

$$\bigcap_{\mathcal{P} \in Pr(A)} \mathcal{P} = \bigcap_{\mathcal{P} \in Pr_s(A)} \mathcal{P} = Nil(A).$$

Proof. Let $M := \bigcap_{\mathcal{P} \in Pr(A)} \mathcal{P}$ and $N = \bigcap_{\mathcal{P} \in Pr_s(A)} \mathcal{P}$. If $x \in Nil(A)$ and $\mathcal{P} \in Pr(A)$, then, for some $n \geq 1$, $x^n = 0 \in \mathcal{P}$, whence (as $\mathcal{P}$ is prime) $x \in \mathcal{P}$: $Nil(A) \subseteq M$.

As $Pr_s(A) \subseteq Pr(A)$, we have $M \subseteq N$.

Let now $x \notin Nil(A)$; then

$$(\forall n \in \mathbf{N}) \quad x^n \neq 0.$$

Define

$$\mathcal{E} := \{J \in Id_s(A) | (\forall n \geq 0) x^n \notin J\}.$$

This set is nonempty ($\{0\} \in \mathcal{E}$) and inductive for $\subseteq$, therefore, by Zorn's Lemma, there exists a maximal element $\mathcal{P}$ of $\mathcal{E}$. As $1 = x^0 \notin \mathcal{P}$, $\mathcal{P} \neq A$.

Let us assume $ab \in \mathcal{P}$, $a \notin \mathcal{P}$ and $b \notin \mathcal{P}$; then $\overline{\mathcal{P} + Aa}$ and $\overline{\mathcal{P} + Ab}$ are saturated ideals of A strictly containing $\mathcal{P}$, whence there exists two integers m and n with $x^m \in \overline{\mathcal{P} + Aa}$ and $x^n \in \overline{\mathcal{P} + Ab}$. By definition of the closure of an ideal, there are $u = p_1 + \lambda a \in \mathcal{P} + Aa$ and $v = p_2 + \mu b \in \mathcal{P} + Ab$ such that $x^m + u = u$ and $x^n + v = v$. Then

$$ub = p_1 b + \lambda(ab) \in \mathcal{P}$$

and

$$x^m b + ub = (x^m + u)b = ub,$$

whence, as $\mathcal{P}$ is saturated, $x^m b \in \mathcal{P}$.

Then

$$x^m v = x^m p_2 + \mu x^m b \in \mathcal{P};$$

as

$$\begin{aligned} x^{m+n} + x^m v &= x^m(x^n + v) \\ &= x^m v, \end{aligned}$$

we obtain $x^{m+n} \in \mathcal{P}$, a contradiction.

Therefore $\mathcal{P}$ is prime and saturated and $x = x^1 \notin \mathcal{P}$, whence $x \notin N$. We have proved that $N \subseteq Nil(A)$, whence $M = N = Nil(A)$. □

Corollary 5.2.

$$Nil(A) = \bigcap_{\mathcal{P} \in Pr(A)} \overline{\mathcal{P}}.$$

Proof.

$$\begin{aligned} Nil(A) &= \bigcap_{\mathcal{P} \in Pr(A)} \mathcal{P} \quad \text{(by Theorem 5.1)} \\ &\subseteq \bigcap_{\mathcal{P} \in Pr(A)} \overline{\mathcal{P}} \\ &\subseteq \bigcap_{\mathcal{P} \in Pr_S(A)} \overline{\mathcal{P}} \\ &= \bigcap_{\mathcal{P} \in Pr_S(A)} \mathcal{P} \\ &= Nil(A) \quad \text{(also by Theorem 5.1)}. \quad \square \end{aligned}$$

Definition 5.3. *For I an ideal of A, we define the root $r(I)$ of I* by

$$r(I) := \{x \in A | (\exists n \geq 1) x^n \in I\}.$$

Lemma 5.4. (i) *$r(I)$ is an ideal of A.*
(ii) *$\overline{r(I)} \subseteq r(\overline{I})$; in particular, if I is saturated then so is $r(I)$.*
(iii) *$r(\{0\}) = Nil(A)$.*

Proof. (i) Obviously, $0 \in r(I)$.

If $x \in r(I)$ and $y \in r(I)$, then $x^m \in I$ for some $m \geq 1$ and $y^n \in I$ for some $n \geq 1$, whence

$$\begin{aligned} (x+y)^{m+n-1} &= \sum_{j=0}^{m+n-1} \binom{m+n-1}{j} x^j y^{m+n-1-j} \\ &= \left(\sum_{j=0}^{m+n-1} x^j y^{m+n-1-j} \right) \\ &\in I, \end{aligned}$$

as $x^j \in I$ for $j \geq m$ and $y^{m+n-1-j} \in I$ for $j \leq m-1$ (as, then, $m+n-1-j \geq n$). Thus $x+y \in r(I)$.

For $a \in A$, $(ax)^m = a^m x^m \in I$, whence $ax \in r(I)$. Therefore $r(I)$ is an ideal of A.

(ii) Let $x \in \overline{r(I)}$ then there is $u \in r(I)$ such that $x + u = u$, and there is $n \geq 1$ such that $u^n \in I$. Let us show by induction on $j \in \{0, \ldots, n\}$ that $u^{n-j}x^j \in \overline{I}$. This is clear for $j = 0$. Let then $j \in \{0, \ldots, n-1\}$, and assume that $u^{n-j}x^j \in \overline{I}$; then

$$\begin{aligned} u^{n-j-1}x^{j+1} + u^{n-j}x^j &= u^{n-j-1}x^j(x+u) \\ &= u^{n-j-1}x^j u \\ &= u^{n-j}x^j, \end{aligned}$$

whence $u^{n-j-1}x^{j+1} \in \overline{\overline{I}} = \overline{I}$. Thus, for $j = n$, we obtain

$$x^n = u^{n-n}x^n \in \overline{I},$$

whence $x \in r(\overline{I})$.

If now I is saturated, then

$$\begin{aligned} r(I) &\subseteq \overline{r(I)} \\ &\subseteq r(\bar{I}) \quad \text{(by the above)} \\ &= r(I), \end{aligned}$$

whence $r(I) = \overline{r(I)}$ is saturated.

(iii) That assertion is obvious. □

Proposition 5.5. *For each saturated ideal I of the B_1-algebra A , one has*

$$r(I) = \bigcap_{\mathcal{P} \in Pr_s(A); I \subseteq \mathcal{P}} \mathcal{P}.$$

Remark 5.6. *For $I = \{0\}$, this is part of Theorem 5.1.*

Proof. Let $x \in r(I)$, and let $\mathcal{P} \in Pr_s(A)$ with $I \subseteq \mathcal{P}$; then, for some $n \geq 1$ $x^n \in I$, whence $x^n \in \mathcal{P}$ and $x \in \mathcal{P}$:

$$r(I) \subseteq \bigcap_{\mathcal{P} \in Pr_s(A); I \subseteq \mathcal{P}} \mathcal{P}.$$

Let now $y \in A$, $y \notin r(I)$, and denote by π the canonical projection

$$\pi : A \twoheadrightarrow A_0 := \frac{A}{\mathcal{R}_I}.$$

As I is saturated, one has

$$\forall n \geq 1 \quad y^n \notin \bar{I},$$

whence

$$\forall n \geq 1 y^n \not{\mathcal{R}}_I 0,$$

or

$$\forall n \geq 1 \quad \pi(y)^n = \pi(y^n) \neq \bar{0}.$$

Therefore $\pi(y) \notin Nil(A_0)$, whence, according to Theorem 5.1, there exists a saturated prime ideal $\mathcal{P}_0$ of A_0 such that $\pi(y) \notin \mathcal{P}_0$. But then $\mathcal{P} := \pi^{-1}(\mathcal{P}_0)$ is a saturated prime ideal of A containing I with $y \notin \mathcal{P}$, whence

$$y \notin \bigcap_{\mathcal{P} \in Pr_s(A); I \subseteq \mathcal{P}} \mathcal{P}. \quad \square$$

6. Topology of spectra

We can now establish the basic topological properties of the spectra $Pr_s(A)$ (analogous, in our setting, to Corollary 1.1.8 and Proposition 1.1.10(ii) of [6]).

Theorem 6.1. *$Pr_s(A)$ and $MaxSpec(A)$ are T_0 and quasi-compact.*

Proof. According to Theorem 3.1, $Pr_s(A)$ and $MaxSpec(A)$ are homeomorphic, therefore it is enough to establish the result for $Pr_s(A)$.

Let $\mathcal{P}$ and $\mathcal{Q}$ denote two different points of $Pr_s(A)$; then either $\mathcal{P} \nsubseteq \mathcal{Q}$ or $\mathcal{Q} \nsubseteq \mathcal{P}$. Let us for instance assume that $\mathcal{P} \nsubseteq \mathcal{Q}$; then $\mathcal{Q} \notin W(\mathcal{P})$; set

$$O := Pr_s(A) \cap (Pr(A) \setminus W(\mathcal{P})).$$

Then O is an open set in $Pr_s(A)$, $\mathcal{Q} \in O$ and, obviously, $\mathcal{P} \notin O$. Therefore $Pr_s(A)$ is T_0.

Let $(U_i)_{i \in I}$ denote an open cover of $Pr_s(A)$:

$$Pr_s(A) = \bigcup_{i \in I} U_i;$$

each $Pr_s(A) \setminus U_i$ is closed, whence $Pr_s(A) \setminus U_i = Pr_s(A) \cap W(S_i)$ for some subset S_i of A. Therefore $Pr_s(A) \cap (\bigcap_{i \in I} W(S_i)) = \emptyset$, *i.e.* $Pr_s(A) \cap W(\bigcup_{i \in I} S_i) = \emptyset$. Therefore $Pr_s(A) \cap W(\overline{\langle \bigcup_{i \in I} S_i \rangle}) = \emptyset$, whence, according to Theorem 3.3, $\overline{\langle \bigcup_{i \in I} S_i \rangle} = A$. Let $J = \langle \bigcup_{i \in I} S_i \rangle$; then $1 \in \bar{J}$, hence there is $x \in J$ such that $1 + x = x$. Furthermore, there exist $n \in \mathbf{N}$, $(i_1, \ldots, i_n) \in I^n$, $x_{i_k} \in S_{i_k}$ and $(a_1, \ldots, a_n) \in A^n$ such that $x = a_1 x_{i_1} + \cdots + a_n x_{i_n}$. But then

$$1 + a_1 x_{i_1} + \cdots + a_n x_{i_n} = a_1 x_{i_1} + \cdots + a_n x_{i_n}$$

whence

$$1 \in \overline{\langle\{x_{i_1}, \ldots, x_{i_n}\}\rangle} \subseteq \overline{\bigcup_{j=1}^{n} S_{i_j}}$$

and

$$\overline{\bigcup_{j=1}^{n} S_{i_j}} = A.$$

It follows that

$$Pr_s(A) \cap W\left(\bigcup_{j=1}^{n} S_{i_j}\right) = \emptyset,$$

that is

$$Pr_s(A) \cap \bigcap_{j=1}^{n} W(S_{i_j}) = \emptyset,$$

or

$$Pr_s(A) = \bigcup_{j=1}^{n} U_{i_j} :$$

$Pr_s(A)$ is quasi-compact. □

For $f \in A$, let

$$\begin{aligned} D(f) &:= Pr_s(A) \setminus (Pr_s(A) \cap W(\{f\})) \\ &= \{\mathcal{P} \in Pr_s(A) | f \notin \mathcal{P}\}. \end{aligned}$$

Proposition 6.2. 1. *Each $D(f)(f \in A)$ is open and quasi-compact in $Pr_s(A)$ (see [6], Proposition 1.1.10(ii)).*
2. *The family $(D(f))_{f \in A}$ is an open basis for $Pr_s(A)$ (see [6], Proposition 1.1.10(i)); in particular, the open quasi-compact sets constitute an open basis.*
3. *A subset O of $Pr_s(A)$ is open and quasi-compact if and only if it is of the form $Pr_s(A) \cap W(I)$ for I an ideal of finite type in A.*
4. *The family of open quasi-compact subsets of $Pr_s(A)$ is stable under finite intersections.*
5. *Each irreducible closed set in $Pr_s(A)$ has a unique generic point (see [6], Corollary 1.1.14(ii)).*

Proof. 1. The openness of $D(f)$ is obvious.

Let us assume $D(f) = \bigcup_{i \in I} U_i$, where the U_i's are open sets in $D(f)$. Each U_i can be written as

$$U_i = D(f) \cap V_i,$$

for V_i an open set in $Pr_s(A)$, *i.e.* $Pr_s(A) \setminus V_i = W(S_i)$ for S_i a subset of A. Then

$$D(f) \subseteq \bigcup_{i \in I} V_i = Pr_s(A) \Big\backslash \left(\bigcap_{i \in I} W(S_i)\right),$$

whence

$$Pr_s(A) \cap W\left(\bigcup_{i \in I} S_i\right) \subseteq W(\{f\}),$$

that is, setting

$$S := \bigcup_{i \in I} S_i,$$

$$f \in \bigcap_{\mathcal{P} \in W(S) \cap Pr_s(A)} \mathcal{P} = \bigcap_{\mathcal{P} \in Pr_s(A); S \subseteq \mathcal{P}} \mathcal{P}.$$

Therefore, by Proposition 5.5, $f \in r(\overline{\langle S\rangle})$: there is $n \geq 1$ such that $f^n \in \overline{\langle S\rangle}$. Thus, there is $g \in \langle S\rangle$ such that $f^n + g = g$; one has $g = \sum_{j=1}^{m} a_j s_j$ for $a_j \in A$, $s_j \in S$; for each $j \in \{1, \ldots, m\}$, $s_j \in S_{i_j}$ for some $i_j \in I$. Let $S_0 = \{s_1, \ldots, s_m\}$; then $g \in \langle\bigcup_{j=1}^{n} S_{i_j}\rangle$, whence $f^n \in \overline{\langle\bigcup_{j=1}^{m} S_{i_j}\rangle}$, and reading the above argument in reverse order with S replaced by $\bigcup_{j=1}^{m} S_{i_j}$ yields that

$$D(f) = \bigcup_{j=1}^{m} U_{i_j},$$

whence the quasi-compactness of $D(f)$.

2. Let U be an open set in $Pr_s(A)$, and $\mathcal{P} \in U$. We have $Pr_s(A) \setminus U = Pr_s(A) \cap W(S)$ for some subset S of A. As $\mathcal{P} \notin W(S)$, $S \nsubseteq \mathcal{P}$, whence there is an $s \in S$ with $s \notin \mathcal{P}$. It is now clear that $\mathcal{P} \in D(s)$ and
$$D(s) \subseteq Pr_s(A) \setminus W(S) = U.$$
3. Let $O \subseteq Pr_s(A)$ be open and quasi-compact; according to (2), one may write $O = \bigcup_{j \in J} D(f_j)$ with $f_j \in A$. But then, there is a finite subset J_0 of J such that $O = \bigcup_{j \in J_0} D(f_j)$. Now
$$\begin{aligned} Pr_s(A) \setminus O &= \bigcap_{j \in J_0} D(f_j) \\ &= Pr_s(A) \cap W(\langle f_j | j \in J_0 \rangle) \end{aligned}$$
is of the required type.
Conversely, if $Pr_s(A) \setminus O = Pr_s(A) \cap W(I)$ with $I = \langle g_1, \ldots, g_n \rangle$, it is clear that $O = \bigcup_{i=1}^{n} D(g_i)$; as a finite union of quasi-compact subspaces of $Pr_s(A)$, O is therefore quasi-compact.
4. Let $O_1, \ldots, O_n$ denote quasi-compact open subsets of $Pr_s(A)$; then, according to (iii), we may write
$$Pr_s(A) \setminus O_j = Pr_s(A) \cap W(I_j)$$
for some finitely generated ideal I_j of A. Thus
$$\begin{aligned} Pr_s(A) \setminus (O_1 \cap \cdots \cap O_m) &= \bigcup_{j=1}^{m} (Pr_s(A) \setminus O_j) \\ &= \bigcup_{j=1}^{m} (Pr_s(A) \cap W(I_j)) \\ &= Pr_s(A) \cap \bigcup_{j=1}^{m} W(I_j) \\ &= Pr_s(A) \cap W\left(\prod_{j=1}^{m} I_j\right) \\ &= Pr_s(A) \cap W(I_1 \ldots I_m), \end{aligned}$$
whence, according to (iii), $O_1 \cap \cdots \cap O_m$ is quasi-compact, as $I_1 \ldots I_m$ is finitely generated.
5. Let F denote an irreducible closed set in $Pr_s(A)$; then $F = Pr_s(A) \cap W(S)$ for S a subset of A. We have seen above that, setting $I := \overline{\langle S \rangle}$, one has $F = Pr_s(A) \cap W(I)$. As F is not empty, $I \neq A$. Let us assume $ab \in I$; then, for each $\mathcal{P} \in F$, one has $ab \in I \subseteq \mathcal{P}$, whence $a \in \mathcal{P}$ or $b \in \mathcal{P}$, *i.e.* $\mathcal{P} \in F \cap W(\{a\})$ or $\mathcal{P} \in F \cap W(\{b\})$:
$$F = (F \cap W(\{a\})) \cup (F \cap W(\{b\})).$$
As F is irreducible, it follows that either $F = F \cap W(\{a\})$ or $F = F \cap W(\{b\})$. In the first case we get $F \subseteq W(\{a\})$, *i.e.*
$$a \in \bigcap_{\mathcal{P} \in Pr_s(A); I \subseteq \mathcal{P}} \mathcal{P} = I \text{(Proposition 5.5)};$$
similarly, in the second case, $b \in I$: I is prime. But then
$$\begin{aligned} \overline{\{I\}} &= Pr_s(A) \cap W(I) \\ &= F \end{aligned}$$
and I is a generic point for F.
It is unique as, in a T_0-space, an (irreducible) closed set admits **at most one** generic point (see [6], (0.2.1.3)). □

Corollary 6.3. *$Pr_s(A)$ and $MaxSpec(A)$ are spectral spaces in the sense of Hochster ([7], p. 43).*

Theorem 6.4 (*Cf.* [6], Corollary 1.1.14)**.** *Let $F = Pr_s(A) \cap W(S)$ be a nonempty closed set in $Pr_s(A)$; then F is homeomorphic to $Pr_s(B)$, where $B := \frac{A}{\mathcal{R}_I}$ with $I := \overline{\langle S \rangle}$.*

Proof. As seen above, one has $F = Pr_s(A) \cap W(I)$, whence, as $F \neq \emptyset$, $I \neq A$. Let $A_0 := \frac{A}{\mathcal{R}_I}$, and let $\pi : A \to A_0$ denote the canonical projection.
Let us now define
$$\begin{aligned} \psi &: Pr_s(A_0) \to F \\ &\mathcal{Q} \mapsto \pi^{-1}(\mathcal{Q}). \end{aligned}$$

Then ψ is well-defined (as $\pi^{-1}(\mathcal{Q})$ is a saturated prime ideal of A that contains I), and injective (as, for each $\mathcal{Q} \in Pr_s(A_0)$, $\pi(\psi(\mathcal{Q})) = \mathcal{Q}$).

Let $\mathcal{P} \in F$; then $\pi(\mathcal{P})$ is an ideal of A_0. Let us assume $\pi(v) \in \overline{\pi(\mathcal{P})}$; then

$$\pi(v) + \pi(a) = \pi(a)$$

for some $a \in \mathcal{P}$, that is

$$\pi(a + v) = \pi(a).$$

But then

$$a + v + i = a + i$$

for some $i \in I$, whence

$$v + (a + i) = a + i.$$

As $a + i \in \mathcal{P}$ and $\mathcal{P}$ is saturated, it follows that $v \in \mathcal{P}$: $\pi(\mathcal{P})$ is saturated.

Furthermore , if $\pi(1) \in \pi(\mathcal{P})$, one has $\pi(1) + \pi(v) = \pi(v)$ for some $v \in \mathcal{P}$, whence there is $w \in I$ such that $1 + v + w = v + w$, whence $1 + v + w \in \mathcal{P}$ and (as $\mathcal{P}$ is saturated) $1 \in \mathcal{P}$ and $\mathcal{P} = A$, a contradiction. Therefore $\pi(\mathcal{P}) \neq A_0$.

Let us assume $\pi(x)\pi(y) \in \pi(\mathcal{P})$: then $xy + i = q + i$ for some $i \in I$, whence

$$(x + i)(y + i) = xy + xi + iy + i^2 \in \mathcal{P},$$

and $x + i \in \mathcal{P}$ or $y + i \in \mathcal{P}$; as $\mathcal{P}$ is saturated, it follows that $x \in \mathcal{P}$ or $y \in \mathcal{P}$, whence $\pi(x) \in \pi(\mathcal{P})$ or $\pi(y) \in \pi(\mathcal{P})$: $\pi(\mathcal{P})$ is prime.

As $\mathcal{P}$ is saturated, one sees in the same way that $\psi(\pi(\mathcal{P})) = \pi^{-1}(\pi(\mathcal{P})) = \mathcal{P}$, whence ψ is surjective.

Let $G := F \cap W(S_0)$ be closed in F; then $\mathcal{P} \in \psi^{-1}(G)$ if and only if $\psi(\mathcal{P}) \in F \cap W(S_0)$, that is $S \subseteq \pi^{-1}(\mathcal{P})$ and $S_0 \subseteq \pi^{-1}(\mathcal{P})$, *i.e.* $\pi(S \cup S_0) \subseteq \mathcal{P}$:

$$\psi^{-1}(G) = Pr_s(A_0) \cap W(\pi(S \cup S_0))$$

is closed in F, and ψ is continuous.

Let now $H := Pr_s(A_0) \cap W(\bar{G})$ be closed in $Pr_s(A_0)$, and let $\mathcal{Q} \in Pr_s(A_0)$; as π is surjective, $\bar{G} \subseteq \mathcal{Q}$ if and only if $\pi^{-1}(\bar{G}) \subseteq \pi^{-1}(\mathcal{Q}) = \psi(\mathcal{Q})$, and it follows that

$$\psi(H) = F \cap W(\pi^{-1}(\bar{G}))$$

is closed in F. Therefore ψ is a homeomorphism. □

7. Remarks on the one-generator case

Let us now consider the case of a nontrivial monogenic B_1-algebra containing strictly B_1, *i.e.* $A = \frac{B_1[x]}{\sim}$ is a quotient of the free algebra $B_1[x]$ with $x \nsim 0, x \nsim 1$. Denote by α the image of x in A; then $\alpha \notin \{0, 1\}$, and α generates A as a B_1-algebra.

Let us suppose that, for some $(u, v) \in A^2$, $\alpha u = 1 + \alpha v$; then α is not nilpotent, as from $\alpha^n = 0$ would follow

$$0 = \alpha^n v = \alpha^{n-1}(\alpha v) = \alpha^{n-1}(1 + \alpha u) = \alpha^{n-1} + \alpha^n u = \alpha^{n-1},$$

whence $\alpha^{n-1} = 0$ and, by induction on n, $1 = \alpha^0 = 0$, a contradiction.

Therefore the following three cases may appear.

(i) α is nilpotent.
(ii) α is not nilpotent and there does not exist $(u, v) \in A^2$ such that $\alpha u = 1 + \alpha v$.
(iii) (α is not nilpotent) and there exists $(u, v) \in A^2$ such that $\alpha u = 1 + \alpha v$.

In case (i), any prime ideal of A must contain α, hence contain αA; the ideal αA is, according to the above remark, saturated, and is not contained in a strictly bigger saturated ideal other than A itself (in both cases, as any element of A not in αA is of the shape $1 + \alpha x$). Therefore $Pr_s(A) = \{\alpha A\}$, whence $Nil(A) = \alpha A$. In this case we see that

$$\frac{A}{\mathcal{R}_{Nil(A)}} \simeq B_1.$$

In cases (ii) and (iii), no power of α belongs to $Nil(A)$; as $Nil(A)$ is saturated, it follows that $Nil(A) = \{0\}$. In fact, A is integral, whence $\{0\} \in Pr_s(A)$. If $\mathcal{P} \in Pr_s(A)$ and $\mathcal{P} \neq \{0\}$, then $\mathcal{P}$ contains some power of α, hence contains α, hence contains αA. As above we see that $\mathcal{P} = \alpha A$; but, in case (iii), αA is not saturated. In case (ii) it is easy to see that αA is prime and saturated. Therefore:

1. in case (ii), $Pr_s(A) = \{\{0\}, \alpha A\}$; $\{0\}$ is a generic point, that is

$$\overline{\{\{0\}\}} = Pr_s(A),$$

and αA a "closed point" ($\{\alpha A\}$ is closed);
2. in case (iii), $Pr_s(A) = \{\{0\}\}$.

One may remark that $B_1[x]$ itself falls into case (ii).

In [9], pp. 75–79, we have enumerated (up to isomorphism) monogenic B_1-algebras of cardinality ≤ 5. It is easy to see where these algebras fall in the above classification; we keep the numbering used in [9]. Let then $3 \leq |A| \leq 5$. We have the following repartition.

Case (i): (6), (8), (12), (15), (18), (24)
Case (ii): (7), (10), (11), (16), (19), (25), (26)
Case (iii): (5), (9), (13), (14), (17), (20), (21), (22), (23), (27), (28).

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