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An Application of the Amalgam Method: The 2-Local Structure of N-Groups of Characteristic 2 Type

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In [9] Thompson classified the finite (simple) groups H satisfying:

(N) Every p-local subgroup of H is solvable, for every prime p.

Later it turned out that his proof could be used as a pattern for the classification of the finite simple groups in general. In the course of this later classification Thompson's result was generalized in [5], [6], and [7] to finite (simple) groups H satisfying:

(N_2) Every 2-local subgroup of H is solvable.

The proofs of both of the above results are subdivided as to whether the invariant $e(H)$ is small or large. Here $e(H)$ denotes the rank of a largest elementary abelian p-subgroup contained in a 2-local subgroup of H, where p ranges over all odd primes. It is characteristic for those proofs that the general approach used fails if $e(H)$ is small.

In the last 10 years a new method in group theory—the amalgam method—has been developed which seems to be well suited to deal with groups (of characteristic 2 type) where $e(H)$ is small. In contrast to other methods in group theory it is a completely local method. That is, it focuses on the structure of finite (local) subgroups of a group rather than the structure of the group itself. In fact, usually neither simplicity nor finiteness of the group is required.

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This paper can be seen as an attempt to explore the reaches of the amalgam method in general classification problems [with arbitrary $e(H)$] and how global properties of the group in question (finiteness, simplicity, etc.) can be used to facilitate the amalgam method. We thought that N_2-groups, i.e., finite groups satisfying (N_2), would be an interesting class of groups for studying these aspects.

The following two theorems will be proven in this paper.

THEOREM 1. *Let H be a finite group, $S_0 \in \mathrm{Syl}_2(H)$, and $B = C_{S_0}(\Omega_1(Z(J(S_0))))$. Suppose that*

(i) *every 2-local subgroup of H containing B is solvable and of characteristic 2 type and*

(ii) *there exist (at least) two maximal 2-local subgroups of H containing S_0.*

Then H is type $L_3(2)$, $Sp_4(2)$, $G_2(2)'$, ${}^2F_4(2)'$, M_{12}, $\Omega_6^+(2)$, $\Omega_6^-(3)$, or $\Omega_8^+(3)$. In particular $|S_0| \le 2^{15}$. Moreover, if H is of type M_{12}, $\Omega_6^+(2)$, $\Omega_6^-(3)$, or $\Omega_8^+(3)$, then there exists a 2-local subgroup of H which is not solvable.

In Theorem 1, "H is of type X" means that H contains a pair of subgroups P_1 and P_2 such that $O_2(\langle P_1, P_2\rangle) = 1$ and $S_0 \le P_1 \cap P_2$, and the structure of P_1 and P_2 is like that of a pair of maximal 2-local subgroups of X_0, where $X \le X_0 \le \mathrm{Aut}(X)$. The precise definition for "H is of type X" is given in Section 8 [for $X = L_3(2)$, $Sp_4(2)$, $G_2(2)'$, $\Omega_6^-(3)$, and $\Omega_8^+(3)$], Section 9 [for $X = \Omega_6^+(2)$], and Section 10 [for $X = M_{12}$ and ${}^2F_4(2)'$].

THEOREM 2. *Suppose that H is an N_2-group of even order and $S_0 \in \mathrm{Syl}_2(H)$. Then one of the following holds:*

(a) *H is of type $L_3(2)$, $Sp_4(2)$, $G_2(2)'$, or ${}^2F_4(2)'$.*

(b) *S_0 is a dihedral or semidihedral group.*

(c) *$|S_0| = 2^5$ and there exists a maximal 2-local subgroup isomorphic to $C_2 \times \Sigma_4$ in H.*

(d) *H contains a strongly embedded subgroup.*

(e) *There exists a 2-local subgroup U of H such that $O_{2'}(U) \ne 1$.*

Some remarks about the proof of Theorem 1. The amalgam method works with a pair of subgroups P_1 and P_2 having the following property:

(1) $S_0 \le P_1 \cap P_2$, $O_2(P_i) \ne 1$, $i = 1, 2$, and $O_2(\langle P_1, P_2\rangle) = 1$.

By a nice argument of Gomi [see (4.4)], such a pair of subgroups exists with the additional property:

(2) S_0 is contained in a unique maximal subgroup of P_i, $i = 1, 2$.

Property (2) yields a particularly transparent structure for $P_i/O_2(P_i)$; see (3.3). A refinement of Gomi's argument gives a pair of subgroups which, apart from (1) and (2), satisfies [see (4.7)]:

(3) Either $O^2(P_2)$ is subnormal in $C_H(\Omega_1(Z(S_0)))$ or $\Omega_1(Z(S_0))$ is neither normal in P_1 nor in P_2.

In the proof of (4.7) a pushing-up theorem for $SL_2(2)$ is used. This is a special case of Baumann's pushing-up theorem [1]. The subgroup B used in hypothesis (i) of Theorem 1—the Baumann subgroup—was first used in that paper. However, we decided to quote [8] rather than [1] [see (2.4) and (2.5)] because the proof in [8] is short, elementary and, more important, stays within the framework of this paper since its proof also uses the amalgam method.

It should be pointed out that, apart from [8] and textbook material the reader should be familiar with, the proof has been made self-contained.

The importance of (3) for the proof can only be appreciated by reading the proofs of Sections 7–10, but some evidence can be given here.

If $\Omega_1(Z(S_0))$ is neither normal in P_1 nor in P_2, then the normal subgroup $Z_i = \langle \Omega_1(Z(S_0))^{P_i} \rangle$ is a noncentral $GF(2)$ P_i-module with $C_{S_0}(Z_i) = O_2(P_i)$, $i = 1, 2$. Hence, the structure of $P_i/O_2(P_i)$ can be investigated by its action on Z_i. That this, together with the amalgam method, is a very effective procedure can be seen in (8.2), where this entire case is treated.

If $\Omega_1(Z(S_0))$ is normal in P_2, then P_2 acts trivially on Z_2, and the action gives no further information about $P_2/O_2(P_2)$. This lack of information from the action is compensated by the subnormality of $O^2(P_2)$ in $C_H(\Omega_1(Z(S_0)))$. Indeed, it is mainly here where the global structure of H facilitates the application of the amalgam method.

There is another aspect of the proof which should be mentioned. There is no subdivision of the proof according to the size of $e(H)$, but, of course, a priori, there is no bound on $e(P_i)$, $i = 1, 2$. The easiest way to see how the proof treats larger values of $e(P_i)$ may be by referring to an example.

Let $H = (E_1 \times E_2)\langle t \rangle$, where $E_i \cong L_3(2)$, $E_1^t = E_2$, and $t^2 = 1$. Then $H \cong L_3(2) \wr C_2$ and $e(H) = 2$ [larger values of $e(H)$ can be produced by substituting $\langle t \rangle$ by a larger 2-group]. Let $S_0 \in \mathrm{Syl}_2(H)$, $t \in S_0$, and $T = S_0 \cap F^*(H)$, and let $L_1, L_2 \leq E_1T$ such that $T = L_1 \cap L_2$ and $O^2(L_i) \cong A_4$, $i = 1, 2$. Then $P_1 = \langle L_1, t \rangle$ and $P_2 = \langle L_2, t \rangle$ are the only maximal 2-local subgroups of H containing S_0. They both are solvable and satisfy (1)–(3).

We now forget about the global structure of H and only use the structure of P_1 and P_2. More precisely, we only use the following properties:

$$(*)\quad [\Omega_1(Z(S_0)), O^2(L_i), O^2(L_i^t)] = 1 \text{ for } i = 1, 2.$$

$$(**)\quad C_{Z(O_2(P_1))}(O^2(L_1^t)) \cap C_{Z(O_2(P_2))}(O^2(L_2^t)) \cap Z(T) \neq 1.$$

Then $\langle L_1^t, L_2^t \rangle \leq C_H(x)$ for suitable $1 \neq x \in Z(T)$, and it is fairly easy to see that $C_H(x)$ is not solvable [if we assume the global information, we get that $E_2 \leq C_H(x)$]. Hence, H is not an N_2-group.

The crucial observation is that the Bauman subgroup B (of S_0) is contained in L_1 and L_2; in fact $B = T$ in our example. In general, property $(*)$ can be established if $B \leq L_1 \cap L_2$ and B is neither normal in L_1 nor L_2; see (3.10). This then allows one to use the above argument in fairly general situations. Usually the element x can be found in

$$[\Omega_1(Z(S_0)), O^2(L_1)] \cap [\Omega_1(Z(S_0)), O^2(L_2)] \cap Z(T),$$

but often the argument also works if $[\Omega_1(Z(S_0)), O^2(L_i)] = 1$ for some i.

Most of the notation used in this paper is standard or will be defined in the section where it occurs for the first time. Modules will be written multiplicatively since they usually arise as normal subgroups or sections of groups.

Let Y be a group and V be a $GF(2)$ Y-module. Then Y operates quadratically on V if $[V, Y, Y] = 1$. An element t of Y induces a transvection on V if $|V/C_V(t)| = 2$, and Y induces transvections on V if $[V, Y] \neq 1$ and every element of $Y \setminus C_Y(V)$ induces a transvection on V.

1.

In this section G is a finite solvable group of even order and V is a finite $GF(2)$ G-module such that $C_G(V) = O_2(G) = 1$.

We will use the following

Notation. $S \in \mathrm{Syl}_2(G)$, $W = O_{2'}(G)$, and $m(Y) = |V|(|C_V(Y)|\,|Y|)^{-1}$ for $Y \leq S$.

$$\mathscr{A}(V, S) = \{A \leq S \mid A \text{ is elementary abelian and } m(A) \leq 1\}.$$

$$J(V, S) = \langle A \mid A \in \mathscr{A}(V, S) \rangle.$$

$$B = C_S(C_V(J(V, S))).$$

$$E = \langle J(V, S)^G \rangle.$$

$\mathscr{G}(S)$ is the set of all subgroups A of S such that $m(A) \leq m(S)$, $C_S([C_W(A), N_S(A)]) = A$, and $C_S(C_V(A)) = A$.

$\Omega(W)$ is the set of all subgroups D in W such that $|D| = 3$ and $\|[V, D]\| = 4$.

The first lemma is well known and will be used in this paper without any further reference.

(1.1) *The following hold*:

(a) $W = [W, S]C_W(S)$.

(b) $W = \langle C_W(a) \mid a \in S^{\#} \rangle$ *if S is noncyclic and abelian.*

(c) $W = \langle C_W(S_0) \mid |S/S_0| = 2 \rangle$ *if S is elementary abelian.*

(d) *S is elementary abelian if* $[V, S, S] = 1$.

(1.2) *Let $G = WS$. Suppose that S is quadratic on V and $V = \langle C_V(S)^G \rangle$. Then*

$$V = \langle C_V(S_0) \mid |S/S_0| = 2 \rangle.$$

Proof. We have

$$G = C_G(S)\big\langle [C_W(S_0), S] \mid |S/S_0| = 2 \big\rangle.$$

Let H be a fixed subgroup of index 2 in S, $N = C_W(H)$ and $V_0 = \langle C_V(S_0) \mid |S/S_0| = 2 \rangle$. It suffices to show that $[N, S]$ normalizes V_0.

Let S_0 be any subgroup of index 2 in S. Then the quadratic action of S on V gives

$$[S, C_V(S_0), N] \leq C_V(H) \quad \text{and} \quad [C_V(S_0), N, S] \leq C_V(H).$$

Hence the 3-subgroup lemma implies $[N, S, C_V(S_0)] \leq C_V(H) \leq V_0$.

(1.3) *Let x be an involution of G and $F = [W, x]$. Suppose that F is a p-group and $|V/C_V(x)| \leq 4$. Then one of the following holds*:

(a) $\|[V, F]\| = 4$ *and* $F \cong C_3$.

(b) $\|[V, F]\| = 2^4$, $|V/C_V(x)| = 4$, *and* $F \cong C_3$, C_5, *or* $C_3 \times C_3$.

(c) $\|[V, F]\| = 2^6$, $|V/C_V(x)| = 4$, $[Z(F), x] = 1$, *and F is extra special of order* 3^3.

Proof. We proceed by induction on $|F||V|$. Then $G = F\langle x \rangle$ and $V = [V, F]$; in particular $C_V(F) = 1$. Note that $|V/C_V(x)| = \|[V, x]\|$.

Assume first that there exists $1 \neq a \in Z(F)$ such that $a^x = a^{-1}$ and $\|[V, a]\| = 2^4$. Then $[V, x] \leq [V, a]$, and $F = [F, x]$ implies $V = [V, a]$. Hence, F is a subgroup of $GL_4(2)$ ($\cong A_8$), and (a) or (b) follows.

Assume next that there exists $1 \neq a \in Z(F)$ such that $a^x = a^{-1}$ and $\|[V,a]\| = 4$. Note that $C_F(V/[V,a]) = \langle a \rangle$ since $C_G(V) = 1$. Set $\overline{V} = V/[V,a]$ and $\overline{G} = G/\langle a \rangle$. By induction either $F = \langle a \rangle$ or $\overline{F} \cong C_3$ and $F = Z(F)$ since $|\overline{V}/C_{\overline{V}}(x)| \leq 2$. The first case gives (a). In the second case there exists $b \in Z(F)$ such that $b^x = b^{-1}$ and $\|[V,b]\| = 2^4$, and the above argument applies.

We may assume now that $[Z(F), x] = 1$. Let $z \in Z(F)$. If $[V, x, z] = 1$, then $[V, F, x] \leq C_V(z)$ and $[V, x, F] \leq C_V(z)$. Hence also

$$[F, x, V] = [F, V] = V \leq C_V(z)$$

and $z = 1$. This shows that $\|[V, x]\| = 4$, $Z(F) \cong C_3$, and $V = [V, Z(F)]$. In particular, V can be regarded as a $GF(4)$ G-module via the action of $Z(F)$.

Assume next that $F = \langle a, b \rangle$, where $a^x = a^{-1}$ and $b^x = b^{-1}$. Then $V = [V, a][V, b]$ and $|V| \leq 2^6$ since $\|[V, a]\| = \|[V, b]\| = 2^4$ and $[V, x] \leq [V, a] \cap [V, b]$. Hence, G is a subgroup of $SL_3(4)$ and (c) follows.

Assume finally that there exist $a, b \in F$ so that $a^x = a^{-1}$, $b^x = b^{-1}$, $[a, b] \neq 1$, and $F \neq \langle a, b \rangle$. Set $F_0 = \langle a, b \rangle$ and $V_0 = [V, F_0]$. By induction $|V_0| = 2^6$ and F_0 is extra special of order 3^3. Note that the structure of $SL_3(4)$ gives $V_0 \neq V$ and $C_F(Z(F_0)) \neq F$. Let $F_1 = C_F(Z(F_0))$ and $F_2 = N_F(F_1)$. Again by induction $F_1 = C_{F_1}(x)F_0$ and $[Z(F_1), x] = 1$. It follows that $[F_2, x] \neq F_0$ and so $[F_2, x, Z(F_1)] \neq 1$. On the other hand, $[x, Z(F_1), F_2] = [Z(F_1), F_2, x] = 1$ which contradicts the 3-subgroup lemma.

(1.4) *Let* $\Omega(W) = \{F_1, \ldots, F_r\}$ *and* $W_0 = \langle F_1, \ldots, F_r \rangle \leq O_3(G)$. *Then*

(a) $W_0 = F_1 \times \cdots \times F_r$ *and*

(b) $V = V_0 \times V_1 \times \cdots \times V_r$, *where* $V_0 = C_V(W_0)$ *and* $V_i = [V, F_i]$.

Proof. Note that $\|[V, \langle F_1, F_2 \rangle]\| \leq 2^4$. Hence, the structure of $GL_4(2)$ gives (a) and (b).

(1.5) *Let* U *be a subgroup of* S. *Then*

(a) $|C_V(U)/C_V(S)| = m(S)m(U)^{-1}|S/U|$.

Moreover, if S *is elementary abelian, then*

(b) $S = \langle A \mid A \in \mathscr{G}(S), |A| = 2 \rangle$,

(c) $W = \langle C_W(A) \mid A \in \mathscr{G}(S), |S/A| = 2 \rangle$,

(d) $W = \langle C_W(A) \mid A \in \mathscr{G}(S), |A| = 2 \rangle$ *if* $|S| \geq 4$,

(e) $m(S) \geq 1$.

Proof. Claim (a) is obvious. Let S be elementary abelian and A be any subgroup of index 2 in S. From (a) we get

$$|C_V(A)/C_V(S)| = 2m(S)m(A)^{-1}.$$

Hence either $C_V(A) = C_V(S)$ or $m(A) \leq m(S)$.

Let $Y = [C_W(A), S]$. Assume that $C_V(A) = C_V(S)$. Then Y centralizes $C_V(S)$ and $V = C_V(Y) \times [V, Y]$. Since $[V, Y]$ is S-invariant and $C_V(S) \leq C_V(Y)$ we get that $[V, Y] = 1$ and $C_W(A) = C_W(S)$. Note further that $C_S(Y) \neq A$ also implies $C_W(A) = C_W(S)$. Hence (c) follows.

From (c) we get that $S = \langle A \mid A \in \mathscr{G}(S), |S/A| = 2\rangle$. If $\mathscr{G}(A) \subseteq \mathscr{G}(S)$ for every $A \in \mathscr{G}(S)$ with $|S/A| = 2$, then (b) and (d) follow by induction on $|S|$. Hence, to prove (b) and (d), it suffices to prove the above inclusion $\mathscr{G}(A) \subseteq \mathscr{G}(S)$.

Let $A \in \mathscr{G}(S)$ with $|S/A| = 2$ and $A_0 \in \mathscr{G}(A)$. Set $Y_0 = [C_W(A_0), S]$ and $V_0 = C_V(A_0)$. Since $Y \leq Y_0$ and $C_V(A) \leq V_0$ we get that $C_S(Y_0) \leq C_S(Y) = A$ and $C_S(V_0) \leq C_S(C_V(A)) = A$; i.e., $C_S(Y_0) = C_A(Y_0)$ and $C_S(V_0) = C_A(V_0)$. Now $A_0 \in \mathscr{G}(A)$ implies that $A_0 \in \mathscr{G}(S)$.

To prove (e), let $A \in \mathscr{G}(S)$ and $|A| = 2$. Then (a) implies that $m(A) \geq 1$ and thus $m(S) \geq 1$.

1.6 *Let $V^* = [V, W]$. Suppose that S is elementary abelian, $W = [W, S]$, and $m(S) \leq m(Y)$ for every subgroup $Y \neq 1$ of S. Then one of the following holds*:

(a) $|S| = 2$ *and* $m(S) > 1$.

(b) $m(S) = 2$, $G \cong SL_2(2) \times SL_2(2)$, $|V^*| = 2^4$, *and* $[V^*, S, S] \neq 1$.

(c) $W = F_1 \times \cdots \times F_r$, *where* $F_i \in \Omega(W)$, $[V, S, S] = 1$, *and* $|V^*/C_{V^*}(S)| = 2|S|$.

(d) $WS = E_1 \times \cdots \times E_r$, *where* $E_i \cong SL_2(2)$ *and* $E_i' \in \Omega(W)$, $[V, S, S] = 1$, *and* $|V^*/C_{V^*}(S)| = |S|$.

Proof. If $|S| = 2$, then (a) resp. (d) follows easily. Hence, we may assume that $|S| \geq 4$.

Let $\mathscr{G}_2(S) = \{A \in \mathscr{G}(S) \mid |A| = 2\}$ and $A \in \mathscr{G}_2(S)$, and let $V_A = C_V(A)$ and $W_A = [C_W(A), S]$. Note that $m(A) = m(S)$ and that $C_{W_A}(V_A) = 1$ by the $P \times Q$ lemma. From (1.5)(a) we get

$$|V_A/C_{V_A}(S)| = |S/A| = |S/C_S(V_A)|.$$

Hence, by induction on $|S|$ we may assume that $W_AS/A = \bar{E}_1 \times \cdots \times \bar{E}_s$ and $V_A = U_0 \times U_1 \times \cdots \times U_s$, where $\bar{E}_i \cong SL_2(2)$, $U_0 = C_{V_A}(W_AS)$, $U_i = [V_A, E_i]$, and $|U_i| = 4$. In particular, W_A is a 3-group. Since W is solvable and $[W, S] = W$, (1.5)(d) implies that W is a 3-group.

Suppose that there exists E_i such that $[V, E_i'] \not\leq V_A$. Let $A_i = S \cap E_i$, $U = [V, E_i]$, and $S_0 = C_S(U)$. Then $[U, A_i, A_i] \neq 1$ and $|A_i| = 4$. Moreover, $|V_A/C_{V_A}(A_i)| = 2$ and so for $a \in A_i^\#$ either $C_V(a) \leq V_A$ or $|C_V(a)/C_{V_A}(a)| = 2$ since $m(\langle a \rangle) \geq m(S) = m(A)$. Since $U \not\leq V_A$ we get

$$|C_V(a)/C_{V_A}(a)| = |[V, a]/[V_A, a]| = 2 \qquad \text{for } a \in A_i \setminus A.$$

It follows that $|U| = 2^4$ and $|[U, a]| = 4$ for $a \in S \setminus S_0$. In particular, $[U, a, s] = [U, a, A]$ for $a \in A_i \setminus A$ and $s \in C_S(E_i) \setminus S_0$. This gives $[U, s] = [U, A]$ by the action of E_i, and $C_S(E_i) = A$. Hence $S = A_i S_0$.

Assume that $S_0 \neq 1$. Then by (1.5)(a),

$$|C_V(S_0)/C_V(S)| = 4m(S)m(S_0)^{-1},$$

while $|U/C_U(S)| = 8$, a contradiction. Hence $S_0 = 1$ and $|S| = 4$.

Let $W_1 = C_W(U)$. Then $W_1 \leq C_W(S)$ since $[V, S] \leq U$. As shown above $[C_W(X), S] \cong C_3$ for $X \in \mathscr{G}_2(S)$. Hence $[C_W(X), S, W_1] = 1$ and $W_1 \leq Z(W)$ by (1.5)(d). It follows that $[V, W_1, W] = 1$ since $[V, W_1, S] = 1$ and $W = [W, S]$. This implies that $W_1 = 1$ and $W \cong C_3 \times C_3$. Now (b) is easy to verify.

Suppose now that for every $A \in \mathscr{G}_2(S)$, $[V, E_i'] \leq V_A$ for $i = 1, \dots, s$. Together with (1.4) and (1.5)(d) this shows that $W = F_1 \times \cdots \times F_r$, $F_i \in \Omega(W)$, and $S \leq N_G(F_i)$. Hence $[V^*, S, S] = 1$. We now choose $A \in \mathscr{G}_2(S)$ with the additional property the s is maximal; i.e., $|[W, A]|$ is minimal.

As above, for $i = 1, \dots, s$ there exists $A \leq A_i \leq S$ such that $|A_i| = 4$, $|V_A/C_{V_A}(A_i)| = 2$, and $[W_A, A_i] = F_i$. Let i be fixed and $A = \langle a \rangle$. Among all subgroups of order 2 in A_i we choose B_i such that $B_i \neq A$ and $|[W, B_i]|$ is minimal with that property. Let $B_i = \langle x \rangle$. Since a inverts $[W, A]$ the minimality of $[W, x]$ gives $C_{[W, A]}(x) \neq 1$. Hence, there exists $F_k \in \Omega(W)$ such that $[F_k, x] = 1$ and $[F_k, A] = F_k$. It follows that $[V, F_k] \leq C_V(x) \not\leq V_A$ and $m(B_i) \leq m(A)$; in particular $B_i \in \mathscr{G}_2(S)$. Thus, the minimality of $[W, A]$ gives $|[W, A]| = |[W, B]| = 3$ or 9. Let $B_0 = A$ and $V_0 = C_V(W)$, and note that $S = \langle B_0, \dots, B_s \rangle$.

Assume that $|[W, A]| = 3$. Then $WS \simeq SL_2(2) \times \cdots \times SL_2(2)$, $|S| = 2^r$, $|V^*/C_{V^*}(S)| = |S|$, and $|V^*/C_{V^*}(B_i)| = 2$. It follows that

$$|V/C_V(S)| = |S|\,|V_0/C_{V_0}(S)| = |S|m(S)$$

and

$$|V/C_V(B_i)| = 2|V_0/C_{V_0}(B_i)| = 2m(B_i).$$

Hence $C_{V_0}(S) = C_{V_0}(B_i)$ and $[V, S, S] = 1$ since $m(B_i) = m(S)$, and (d) follows.

Assume now that $\|[W, A]\| = 9$. Then $|S| = 2^{r-1}$ since $W_A S/A \simeq SL_2(2) \times \cdots \times SL_2(2)$ and $|V^*/C_{V^*}(B_i)| = 4$. Now as above $C_{V_0}(S) = C_{V_0}(B_i)$ and $[V, S, S] = 1$, and (c) follows.

(1.7) *Suppose that* $J(V, S) \neq 1$. *Then the following hold*:

(a) $W = E' \times C_W(E)$.

(b) $B = J(V, S)$ *and* $J(V, S) \in \mathscr{A}(V, S)$.

(c) $E = E_1 \times \cdots \times E_n$ *and* $V = V_0 \times V_1 \cdots \times V_n$, *where* $V_0 = C_V(E)$, $E_i \cong \mathrm{SL}_2(2)$, $V_i = [V, E_i]$, *and* $|V_i| = 4$ *for* $i = 1, \ldots, n$.

Proof. Let $A \in \mathscr{A}(V, S)$. Then $m(A) \leq 1$, and (1.5)(e) gives $m(A) = 1$. Now (1.6)(d) and again (1.5)(e) imply that $[W, A]A = E_1 \times \cdots \times E_s$ and $V = C_V([W, A]A) \times V_1 \times \cdots \times V_s$, where $E_i \cong SL_2(2)$, $V_i = [V, E_i]$, and $|V_i| = 4$. Since $E_i = [W, E_i \cap A](E_i \cap A)$ each of the subgroups E_i is normal in WA.

Let Ω^* be the set of all subgroups F of G such that $F \cong SL_2(2)$, $\|[V, F]\| = 4$, and F is normal in WF, and let $E_0 = \langle F \mid F \in \Omega^* \rangle$. Then $E_0 = E_1 \times \cdots \times E_r$, where $\Omega^* = \{E_1, \ldots, E_r\}$. Note that E_0 is normal in G.

Let $S_0 = S \cap E_0$. As we have seen above $J(V, S) \leq S_0$. On the other hand, $|V/C_V(S_0)| \leq |S_0|$ and thus $S_0 \in \mathscr{A}(V, S)$; in particular $S_0 = J(V, S)$ and $E_0 = E$.

It remains to prove $B = S_0$. Clearly, B normalizes every $E_i \in \Omega^*$. Hence $B = S_0 C_B(E)$. However, $C_B(E)$ centralizes V and thus $B = S_0$.

2.

In this section G is a finite solvable group of even order and $C_G(O_2(G)) \leq O_2(G)$.

Notation. $S \in \mathrm{Syl}_2(G)$, $Z = \Omega_1(Z(S))$, $V = \langle Z^G \rangle$, and $\overline{G} = G/C_G(V)$. $\mathscr{A}(S)$ is the set of all elementary abelian subgroups of maximal order of S, $J(S) = \langle A \mid A \in \mathscr{A}(S) \rangle$, $\tilde{Z} = \Omega_1(Z(J(S)))$, $B = C_S(\tilde{Z})$, and $L = \langle B^G \rangle$.

(2.1) $O_2(\overline{G}) = 1$.

Proof. This follows directly from the definition of V.

(2.2) *Let* $\overline{E} = [O_{2'}(\overline{G}), \overline{J(S)}]\overline{J(S)}$. *Suppose that* $\overline{J(S)} \neq 1$. *Then the following hold*:

(a) $\overline{E}$ *is a normal subgroup of* $\overline{G}$.

(b) $\overline{J(S)} = \overline{B}$.

(c) $\bar{E} = \bar{E}_1 \times \cdots \times \bar{E}_n$, *where* $\bar{E}_i \cong SL_2(2)$.

(d) $V = V_0 \times \cdots \times V_n$, *where* $V_0 = C_V(\bar{E})$, $V_i = [V, E_i]$, *and* $|V_i| = 4$ *for* $i = 1, \ldots, n$.

Proof. Let $A \in \mathscr{A}(S)$. Then the maximality of $|A|$ yields

$$|VC_A(V)| = |V|\,|C_A(V)|\,|V \cap A|^{-1} \le |A|.$$

It follows that $|V/C_V(A)| \le |\bar{A}|$ and $\bar{A} \in \mathscr{A}(V, \bar{S})$ (for definition see Section 1); in particular $\overline{J(S)} \le J(V, \bar{S})$. Now (1.6)(d) and (1.7) applied to $\bar{G}$ and V give (c), (d), and, together with an easy Frattini argument, also (a).

Note that $\bar{B}$ normalizes $\bar{E}_i$ for $i = 1, \ldots, n$ and centralizes V_0 since $V_0 \le \tilde{Z}$. It follows that $C_{\bar{B}}(\bar{E}') = C_{\bar{B}}(V) = 1$. Hence $\overline{J(S)} \le \bar{B}$ implies that $\overline{J(S)} = \bar{B}$.

(2.3) *Suppose that* $O_2(G) = C_S(V)$. *Then* $B \in \mathrm{Syl}_2(L)$.

Proof. Assume first that $[V, J(S)] = 1$. Then $V \le \tilde{Z}$ and thus $B \le C_S(V) = O_2(G)$. Hence, B is normal in G.

Assume now that $[V, J(S)] \ne 1$. Let $E = \langle J(S)^G \rangle$. We apply (2.2). Then $\overline{J(S)} = \bar{B}$ and $L = EB$. Hence, it suffices to show that $O_2(E) \le B$. Let $\tilde{Z}_0 = \Omega_1(Z(J(O_2(E))))$ and $A \in \mathscr{A}(S)$. Note that by (1.5)(e), $C_A(V)V \in \mathscr{A}(S)$. Hence $\tilde{Z} \le \tilde{Z}_0$ and $\tilde{Z}_0 \in C_{\tilde{Z}_0}(A)V$. It follows that $[\tilde{Z}_0, E] \le V$; in particular, $\tilde{Z}V$ is normal in E. Now the structure of $\bar{E}$ and its operation on V shows that there exists $x \in E$ so that $E = \langle J(S), J(S)^x \rangle C_E(V)$ and $\tilde{Z}V = Z^*V$, where $Z^* = \tilde{Z} \cap \tilde{Z}^x$ and $\tilde{Z} = Z^*(\tilde{Z} \cap V)$. Moreover, $C_E(V)$ centralizes $\tilde{Z}V/V$ and so $C_E(V) = C_E(Z^*V)O_2(E)$. Since $O_2(E)$ normalizes Z^* we conclude that Z^* is normal in E. However, now $[Z^*, J(S)] = 1$ implies $[Z^*, E] = 1$, and $O_2(E) \le C_S(\tilde{Z}) = B$.

(2.4) *Suppose that the following hold*:

(i) *No nontrivial characteristic subgroup of S is normal in G.*

(ii) *S is contained in a unique maximal subgroup of G.*

Then $[O_2(G), O^2(G)] \le V$.

Proof. By (i) neither Z nor $J(S)$ is normal in G; i.e., $[V, J(S)] \ne 1$. Moreover, (ii) implies that $C_S(V) = O_2(G)$. We apply (2.2) and (2.3).

Let $B \le L_1 \le L$ so that $\bar{L}_1/O_2(\bar{L}_1) \cong SL_2(2)$ and $|L_1|$ is minimal with that property. Then $B \in \mathrm{Syl}_2(L_1)$ and $L_1/O_2(L_1) \cong D_{2\cdot 3^n}$. Moreover, by (ii), $\langle L_1, S \rangle = G$ and thus no nontrivial characteristic subgroup of B is normal in L_1. Hence, [8] implies that $[O_2(L_1), O^2(L_1)] \le V$. Let T be a Hall $2'$-subgroup of $C_L(O_2(L)/V)$. Then $L_1 \le TS$ and $TS = G$. It follows that $[O_2(G), O^2(G)] \le V$.

(2.5) *Suppose that G satisfies the hypothesis of* (2.4) *and* $\overline{G} \cong SL_2(2)$. *Let T be a subgroup of* $\mathrm{Aut}(S)$ *of odd order. Then* $\langle V\tau \mid \tau \in T \rangle$ *is a normal subgroup of G in $O_2(G)$.*

Proof. This is [8, 3.5].

3.

In this section G is a finite group of even order, S is a nontrivial 2-subgroup of G, and U is a subgroup of G containing S.

Notation. $\mathscr{L}(U, S) = \{E \le U \mid S \in \mathrm{Syl}_2(E),\ O_2(E) \ne 1,$ and $S \ne O_2(E)\}$; $\mathscr{P}(U, S) = \{E \in \mathscr{L}(U, S) \mid S$ is contained in a unique maximal subgroup of $E\}$; $\mathscr{P}^*(U, S) = \{E \in \mathscr{P}(U, S) \mid O^2(E)$ is subnormal in L for some maximal element $L \in \mathscr{L}(U, S)\}$. $\Phi_2(U)$ is the inverse image of $\Phi(U/O_2(U))$ in U. If $U = G$, we also write $\mathscr{L}(S)$, $\mathscr{P}(S)$, and $\mathscr{P}^*(S)$ instead of $\mathscr{L}(G, S)$, $\mathscr{P}(G, S)$, and $\mathscr{P}^*(G, S)$, respectively.

(3.1) *Let* $S \in \mathrm{Syl}_2(G)$ *and $\mathscr{S}$ be the set of all subgroups U of G such that $S \le U$ and*

(∗) *$S \ne O_2(U)$ and S is contained in a unique maximal subgroup of U.*

Then either $\mathscr{S} = \emptyset$ and $S = O_2(G)$, or $O^{2'}(G) = \langle P \mid P \in \mathscr{S} \rangle$.

Proof. Let G be a minimal counterexample and $G_0 = \langle P \mid P \in \mathscr{S} \rangle$, and let M be any maximal subgroup of G containing S. Then either $M = N_G(S)$ or, by induction, $M = N_M(S)(M \cap G_0)$. Since $N_G(S)$ normalizes G_0 we conclude that $G_0 N_G(S)$ is the unique maximal subgroup of G containing S, but now $G \in \mathscr{S}$. Since $G = O^{2'}(G)N_G(S) = O^{2'}(G)M$ it follows that $G = O^{2'}(G)$, and G is not a counterexample.

(3.2) *Let $L \in \mathscr{L}(S)$. Then* $O^{2'}(L) = \langle P \mid P \in \mathscr{P}(L, S) \rangle$.

Proof. This is a direct consequence of (3.1).

(3.3) *Let $P \in \mathscr{P}(S)$ and B be the maximal subgroup of P containing S, and let P_0 be the largest normal subgroup of P contained in B. Suppose that P is solvable. Then*

(a) *$O^2(P/O_2(P))$ is a p-group for some odd prime p,*

(b) *$O^2(P/P_0)$ is an irreducible S-module,*

(c) $P_0/O_2(P) = \Phi(O^2(P/O_2(P)))$.

Proof. Let $\overline{P} = P/O_2(P)$. The existence of Hall subgroups in $\overline{P}$ shows that $\overline{P}$ is a $\{2, p\}$-group, where p is some odd prime, and the Frattini argument shows that $\overline{P}_0$ is a p-group.

Clearly, $\Phi(O^2(\overline{P})) \leq \overline{P}_0$, and Maschke's theorem gives $\overline{P}_0 = \Phi(O^2(\overline{P}))$ and (b).

(3.4) *Let $P \in \mathscr{P}(S)$ and T be a normal subgroup of S. Suppose that P is solvable. Then either $T \leq O_2(P)$ or $[O^2(P), T] = O^2(P)$.*

Proof. Since $[O^2(P), T]$ is normal in P, the claim follows from (3.3).

(3.5) *Let $P \in \mathscr{P}(S)$ and N be a normal subgroup of P in $O_2(P)$. Suppose that P is solvable and $[N, O_2(P) \cap O^2(P)] = 1$. Then either $[Z(S), O^2(P)] \neq 1$ or $[N, O^2(P)] = 1$.*

Proof. Note that $[N, O^2(P)] \leq Z(O_2(O^2(P)))$ and that $[N, O^2(P), O^2(P)] = 1$ implies $[N, O^2(P)] = 1$. Hence we may assume that $N \leq O^2(P)$ and N is abelian. By Maschke's theorem $N = C_N(O^2(P)) \times [N, O^2(P)]$. Since $[N, O^2(P)]$ is S-invariant, the claim follows.

(3.6) *Let $P \in \mathscr{P}(S)$, $\overline{P} = P/O_2(P)$, and Z and T be two normal subgroups of S, and let A be a subgroup of S satisfying $\Phi(A) \leq O_2(P)$ and $a \in A \setminus O_2(P)$. Suppose that P is solvable, $Z \leq O_2(P)$, and $T \not\leq O_2(P)$. Then there exists $x \in P$ so that for $L = \langle A, A^x \rangle$ the following hold:*

(a) $\overline{L} = \overline{E} \times \overline{A}_0$, *where $\overline{E} \cong D_{2p^n}$ and $A = \langle a \rangle A_0$.*

(b) $O^2(L) \not\leq \Phi_2(O^2(P))$.

(c) *Any two elements in Z^L are interchanged by an involution of L.*

(d) $O^2(L) \leq [O^2(L), T]$.

Proof. Let $F = [O^2(P), a]$. If $F \leq \Phi_2(O^2(P))$, then $a \in O_2(P)$ which is not the case. Hence $F \not\leq \Phi_2(O^2(P))$. Let $F_0 \leq F$ such that

(∗) F_0 is A-invariant and $F_0 \not\leq \Phi_2(O^2(P))$.

Among all such F_0 satisfying (∗) we choose F_0 such that first $|F_0[F_0, T]|$ and then $|F_0|$ is minimal. Since $\overline{F} = \langle C_{\overline{F}}(\overline{A}_0) \mid |\overline{A}/\overline{A}_0| = 2 \rangle$ there exists $A_0 \leq A$ such that $|\overline{A}/\overline{A}_0| = 2$ and $\overline{F}_0 \leq C_{\overline{F}}(\overline{A}_0)$ by the minimal choice of $\overline{F}_0$. Note that $\overline{F}_0 \not\leq \Phi(\overline{F})$ and so $\bar{a} \notin \overline{A}_0$.

Choose $\bar{e} \in \overline{F}_0 \setminus \Phi(O^2(\overline{P}))$ such that $\bar{e}^{\bar{a}} = \bar{e}^{-1}$. The minimality of $\overline{F}_0$ gives $\overline{F}_0 = \langle \bar{e} \rangle$. Set $L = \langle A, A^e \rangle$. Then (a) and (b) hold. Since $O_2(L)$ normalizes every element in Z^L, the structure of $L/O_2(L)$ also gives (c).

By (3.4), $\overline{F}_0 \leq [\overline{F}_0, \overline{T}]\Phi(O^2(\overline{P}))$. Hence $F_1 := [F_0, T, a] \not\leq \Phi_2(O^2(P))$. Now the minimality of $F_0[F_0, T]$ gives $F_0 \leq [F_0, T]$, and (d) follows.

(3.7) *Let $L \in \mathscr{L}(S)$ and $\overline{L} = L/O_2(L)$. Suppose that L is solvable. Then*

$$\left[F(\overline{L}), \overline{S}\right] = \langle O^2(\overline{P}) \mid P \in \mathscr{P}^*(L, S) \rangle.$$

Proof. Let F be the inverse image of $[F(\bar{L}), \bar{S}]$ in L. By (3.2), $FS = \langle P \mid P \in \mathscr{P}(FS, S)\rangle$. Moreover, $O^2(P)$ is subnormal in L for every $P \in \mathscr{P}(FS, S)$; i.e., $P \in \mathscr{P}^*(L, S)$.

Now let $P \in \mathscr{P}^*(L, S)$. Then $O^2(P)$ is subnormal in L since L is the unique maximal element of $\mathscr{L}(L, S)$. It follows that $O^2(\bar{P}) \le F(\bar{L})$ and so $P \in \mathscr{P}(FS, S)$ and $\mathscr{P}^*(L, S) = \mathscr{P}(FS, S)$.

(3.8) *Let $P_1, P_2 \in \mathscr{P}(S)$ and $H = \langle P_1, P_2\rangle$, and let N be a normal subgroup of H which is maximal (with respect to inclusion) such that $O^2(P_i) \not\le N$ for $i = 1, 2$. Suppose that P_1 and P_2 are solvable. Then there exists a normal series $Q \le N \le H_0 \le H_1 \le H$ such that*

(a) *$Q = S \cap N$ and Q is the largest subgroup of S which is normal in H,*

(b) *H_0/N is a minimal normal subgroup of H/N and $O^2(P_j) \le H_0$ for some $j \in \{1, 2\}$,*

(c) *$H_1 = H_0$ or $H_1 = H_0 O^2(P_k)$, where $O^2(P_k) \not\le H_0$.*

(d) *$H = SH_1$.*

Proof. Let $Q^* = S \cap O_2(H)$. Clearly, Q^* is the largest normal subgroup of H in S. Hence $Q^* \le N$ by the maximality of N; i.e., $Q^* \le S \cap N = Q$. On the other hand, by (3.4), $Q = O_2(P_i) \cap N$ since $O^2(P_i) \not\le N$. Hence $Q = Q^*$.

Let H_0 be the inverse image of a minimal normal subgroup of H/N. Then there exists $j \in \{1, 2\}$ such that $O^2(P_j) \le H_0$. Let $\{1, 2\} = \{j, k\}$. If $O^2(P_k) \le H_0$, then $H_0 S = H$. If $O^2(P_k) \not\le H_0$, then $H_0 O^2(P_k) S = H$.

(3.9) *Let P_1, P_2 and H be as in* (3.8), *and let $S \le T \in \mathrm{Syl}_2(H)$ and $Q = S \cap O_2(H)$. Suppose that H is solvable, $\Omega_1(Z(S)) \le Q$, and $J(S) = J(T)$. Then one of the following holds*:

(a) $[\Omega_1(Z(S)), O^2(P_i), O^2(P_j)] = 1$ *for $i \ne j$, or*

(b) $C_S(\Omega_1(Z(J(S)))) \le O_2(P_i)$ *for some $i \in \{1, 2\}$, or*

(c) $[\Omega_1(Z(S)), O^2(P_1)] = [\Omega_1(Z(S)), O^2(P_2)]$.

Proof. Let $Z = \Omega_1(Z(S))$, $V = \langle Z^H\rangle$, $\bar{H} = H/C_H(V)$, and $B = C_S(\Omega_1(Z(J(S))))$. We may assume that $B \not\le O_2(P)$, and $O^2(P_i) \not\le C_H(V)$ for $i = 1, 2$. We apply (3.8) with $C_H(V) \le N$.

Assume that $J(S) \le C_H(V)$. Then $J(S) \le Q$ and $V \le Z(J(S))$. Hence $B \le C_H(V)$ and so $B \le Q \le O_2(P_i)$, $i = 1, 2$, a contradiction.

Assume now that $J(S) \not\le C_H(V)$. Then there exists $i \in \{1, 2\}$ such that $J(S) \not\le O_2(P_i)$; i.e., $O^2(P_i) \le \langle J(S)^{P_i}\rangle$ by (3.4). Let $\bar{F} = O_{2'}(F(\bar{H})) \cdot O^2(\bar{P}_i) J(\bar{S})$.

Since $J(S) = J(T)$ and $\overline{S} \cap O_2(\overline{H}) = 1$ we get $[\overline{J(S)}, O_2(\overline{H})] = 1$ and thus $[\overline{F}, O_2(\overline{H})] = 1$. It follows that

$$O_2(\overline{F}) \le C_{\overline{S}}\big(F(\overline{H})\big) = \overline{S} \cap O_2(\overline{H}) = 1.$$

Hence, (1.7) applied to $\overline{F}$ and V gives $O^2(\overline{P}_i) \le O_3(\overline{H})$; in particular, $\overline{H} = O_3(\overline{H})\overline{P}_j$, where $\{1, 2\} = \{i, j\}$, and $\overline{S} \in \mathrm{Syl}_2(\overline{H})$.

Assume that $J(S) \not\le O_2(P_j)$. Then by the same argument, $O^2(\overline{P}_j) \le O_3(\overline{H})$ and $\overline{H} = O_3(\overline{H})\overline{S}$. If $\overline{P}_1 \ne \overline{P}_2$, then again (1.7) gives (a). In the other case, (c) follows.

Assume that $J(S) \le O_2(P_j)$. Since $B \not\le O_2(P_j)$ we get that $[\Omega_1(Z(J(S))), O^2(P_j)] = 1$. However, now (1.7) applied to $\overline{H}$ and V shows that $[V, O^2(P_i), O^2(P_j)] = 1$, and again (a) holds.

4.

In this section G is a finite group of even order, $S \in \mathrm{Syl}_2(G)$, and the following hold:

(a) Every 2-local subgroup of G containing S is solvable and of characteristic 2 type.

(b) S is contained in two different maximal 2-local subgroups of G.

Notation. $Z = \Omega_1(Z(S))$, $C = C_G(Z)$, $D = \bigcap_{P \in \mathscr{P}(S)} O_2(P)$, $M = N_G(D)$, $B = C_S(\Omega_1(Z(J(S))))$,

$$\mathscr{P}_0(S) = \mathscr{P}^*(C, S) \cup \big(\mathscr{P}^*(S) \setminus \mathscr{P}(C, S)\big),$$

$$\mathscr{P}_1(S) = \mathscr{P}^*(M, S) \cup \big(\mathscr{P}^*(S) \setminus \mathscr{P}(M, S)\big),$$

$$\Lambda = \big\{(P_1, P_2) \mid P_i \in \mathscr{P}(S),\ O_2(\langle P_1, P_2\rangle) = 1\big\}.$$

Remark. A subgroup similar to D was used in [3].

(4.1) *The following hold*:

(a) $\mathscr{L}(S) \ne \emptyset \ne \mathscr{P}(S)$.

(b) $O_2(M) \ne 1$.

(c) $N_G(S) \le M$.

Proof. $N_G(S)$ is not the only 2-local subgroup of G containing S. Hence (3.2) implies (a).

Claim (b) holds since $Z \le D \le O_2(M)$, and (c) holds since $N_G(S)$ operates on the elements of $\mathscr{P}(S)$ by conjugation.

(4.2) *Let* $L \in \mathscr{L}(S)$, $D_L = \bigcap_{P \in \mathscr{P}(L,S)} O_2(P)$, *and* $D_L^* = \bigcap_{P \in \mathscr{P}^*(L,S)} O_2(P)$. *Then* $D_L = D_L^* = O_2(L)$ *and* $D = \bigcap_{L \in \mathscr{M}\mathscr{L}(S)} O_2(L)$, *where* $\mathscr{M}\mathscr{L}(S)$ *is the set of maximal elements of* $\mathscr{L}(S)$.

Proof. Let F_0 be the inverse image of $F(L/O_2(L))$ in L and $F = [F_0, S]$. By (3.7), $FS = \langle P \mid P \in \mathscr{P}^*(L,S)\rangle$. It follows that $[F_0, D_L^*] \leq F$ and $[F, D_L^*] \leq O_2(L)$. We conclude that

$$O_2(L) \leq D_L \leq D_L^* \leq O_2(L).$$

and the claim follows.

(4.3) $D = \bigcap_{P \in \mathscr{P}_i(S)} O_2(P)$ *for* $i = 0, 1$.

Proof. For $\varnothing \neq \mathscr{X} \subseteq \mathscr{L}(S)$ define $O_2(\mathscr{X}) = \bigcap_{P \in \mathscr{X}} O_2(P)$. Let $M_0 = C$ and $M_1 = M$. By (4.2) it suffices to show that $O_2 \mathscr{P}_i(S)) \leq O_2(L)$ for every maximal element $L \in \mathscr{L}(S)$.

We have $\mathscr{P}^*(L,S) \subseteq \mathscr{P}(M_i, S) \cup (\mathscr{P}^*(S) \setminus \mathscr{P}(M_i,S))$, and by (4.2),

$$O_2\big(\mathscr{P}^*(M_i,S)\big) = O_2\big(\mathscr{P}(M_i,S)\big) = O_2(M_i).$$

Hence

$$\begin{aligned} O_2\big(\mathscr{P}_i(S)\big) &= O_2\big(\mathscr{P}(M_i,S)\big) \cap O_2\big(\mathscr{P}^*(S) \setminus \mathscr{P}(M_i,S)\big) \\ &\leq O_2\big(\mathscr{P}^*(L,S)\big) = O_2(L). \end{aligned}$$

(4.4) *Let* $P \in \mathscr{P}(S) \setminus \mathscr{P}(M,S)$ *and* $k \in \{0,1\}$. *Then there exists* $P^* \in \mathscr{P}_k(S)$ *such that* $(P, P^*) \in \Lambda$.

Proof. Let $\mathscr{P}_k(S) = \{P_1, \ldots, P_n\}$ and $H_i = \langle P, P_i \rangle$, $i = 1, \ldots, n$. We may assume that $O_2(H_i) \neq 1$ for $i = 1, \ldots, n$. Hence by (4.2) and (4.3),

$$D \leq \bigcap_{i=1}^{n} O_2(H_i) \leq \bigcap_{i=1}^{n} O_2(P_i) = D$$

and $D = \bigcap_{i=1}^{n} O_2(H_i)$. However, now $P \leq M$, a contradiction.

(4.5) *Let* $\mathscr{P}(S) = \mathscr{P}(M,S) \cup \mathscr{P}(C,S)$ *and* $P \in \mathscr{P}(S) \setminus \mathscr{P}(M,S)$. *Then there exists* $P^* \in \mathscr{P}^*(M,S)$ *so that* $(P, P^*) \in \Lambda$. *If in addition* $\mathscr{P}^*(C,S) \subseteq \mathscr{P}(M,S)$, *then* $O_2(M) \leq O_2(C)$ *and* $O_2(C) \in \mathrm{Syl}_2(O^2(P^*)O_2(C))$.

Proof. By (4.4) there exists $P^* \in \mathscr{P}_1(S)$ such that $(P, P^*) \in \Lambda$. Since $P \leq C$, we have $P^* \nleq C$ and thus $P^* \in \mathscr{P}^*(M,S)$.

Suppose that $\mathscr{P}^*(C,S) \subseteq \mathscr{P}(M,S)$. Then (4.2) implies that $O_2(M) \leq O_2(C)$. Since $O_2(O^2(P^*)) \leq O_2(M)$, we conclude that $O_2(C) \in \mathrm{Syl}_2(O^2(P^*)O_2(C))$.

(4.6) *Suppose that* $\mathscr{P}(S) = \mathscr{P}(C,S) \cup \mathscr{P}(M,S)$. *Then* $\mathscr{P}^*(C,S) \not\subseteq \mathscr{P}(M,S)$.

Proof. Assume that $\mathscr{P}^*(C,S) \subseteq \mathscr{P}(M,S)$. By (4.5) there exist $P \in \mathscr{P}(C,S)$ and $P^* \in \mathscr{P}^*(M,S)$ such that $(P, P^*) \in \Lambda$ and $O_2(C) \in \mathrm{Syl}_2(O^2(P^*)O_2(C))$.

Let $B_0 = C_{O_2(C)}(\Omega_1(Z(J(O_2(C)))))$, $L = \langle B_0^{P^*} \rangle$, $V = \langle Z^{P^*} \rangle$, and $\overline{L} = L/C_L(V)$. Note that B_0 is normal in P and thus not normal in P^*. Hence by (3.4), $O^2(P^*) \le L$, and by (3.3), $O_2(L) \in \mathrm{Syl}_2(C_L(V))$ since $P^* \not\le C$. Thus (2.3) gives $B_0 \in \mathrm{Syl}_2(L)$. Moreover, since $(P, P^*) \in \Lambda$ there is no nontrivial characteristic subgroup of B_0 which is normal in L. Hence $LS = P^*$ and (2.2) imply

(i) $\overline{L} = \overline{E}_1 \times \cdots \times \overline{E}_n$,

(ii) $V = V_0 \times \cdots \times V_n$,

where the notation is as in (2.2). Since $V_0 \cap Z$ is normalized by P and P^*, we get $V_0 \cap Z = 1$ and thus $V_0 = 1$.

Let T be a Sylow 3-subgroup of P^*. Then by (2.4), $[O_2(P^*), T] = V$ and $O_2(P^*) = VC_{O_2(P^*)}(T)$. The Frattini argument shows that $C_{O_2(P^*)}(T)$ is normal in S, and $V_0 = 1$ gives $C_{O_2(P^*)}(T) = 1$ and $O_2(P^*) = V_1 \times \cdots \times V_n$; in particular, $O_2(P^*) = O_2(M)$.

Let $R_i = [V_i, B_0]$ and $\Omega = \{R_i \mid i = 1, \ldots, n\}$. Note that $\Omega = \{R_1^s \mid s \in S\}$ since $P^* \in \mathscr{P}(S)$. Choose $\alpha \in \mathrm{Aut}(B_0)$. If $[V_i\alpha, V_j] \ne 1$, then $R_j = [V_i\alpha, V_j] = [V_i\alpha, B_0] = R_i\alpha$. If $V_i\alpha \le V$, then there exists $\overline{E}_j$ so that $[V_i\alpha, \overline{E}_j \cap \overline{B}_0] = R_j = R_i\alpha$. We conclude that $\mathrm{Aut}(B_0)$ operates on Ω. Since $C \le N_G(B_0)$ we get that $C = SC_0$, where $C_0 = N_C(R_1)$.

Let U be a Hall $2'$-subgroup of C_0 and $u \in U$. By (2.5), $[V_1, V_1^u] = 1$, and $[V_i, V_1^u] \le R_1 \cap R_i = 1$ for $2 \le i$. It follows that $\langle V_1^U \rangle \le O_2(P^*)$. Hence also

$$\langle \langle V_1^U \rangle^S \rangle = \langle V_1^{US} \rangle \le O_2(P^*).$$

However, $US = C$ and $\langle V_1^S \rangle = O_2(P^*)$. Now $O_2(P^*) = O_2(M)$ implies that $C \le M$, and M is the unique maximal 2-local subgroup containing S. a contradiction.

(4.7) *There exists* $(P, P^*) \in \Lambda$ *such that* $P \not\le C$ *and* $P^* \in \mathscr{P}_0(S)$.

Proof. Assume that $\mathscr{P}(S) \ne \mathscr{P}(C,S) \cup \mathscr{P}(M,S)$. Then (4.4) yields the assertion. Assume that $\mathscr{P}(S) = \mathscr{P}(C,S) \cup \mathscr{P}(M,S)$. Then (4.6) implies that $\mathscr{P}^*(C,S) \not\subseteq \mathscr{P}(M,S)$. Now (4.7) follows from (4.5) for $P^* \in \mathscr{P}^*(C,S) \setminus \mathscr{P}(M,S)$.

5.

In this section we assume

HYPOTHESIS 1. *H is a finite group of even order, $S_0 \in \mathrm{Syl}_2(H)$, and the following hold*:

(i) *Every 2-local subgroup of H containing S_0 is solvable and of characteristic 2 type.*

(ii) *$O_2(H) = 1$ and H does not contain a strongly embedded subgroup.*

(5.1) *There exists a nontrivial subgroup S of S_0 and $P_1, P_2 \in \mathscr{P}(S)$ such that $O_2(\langle P_1, P_2 \rangle) = 1$ and one of the following holds*:

(a) *$S = S_0$ and $\Omega_1(Z(S))$ is neither normal in P_1 nor in P_2.*

(b) *$S = S_0$ and $P_2 \in \mathscr{P}^*(C_H(\Omega_1(Z(S))), S)$.*

(c) *$S \neq S_0$, S_0 is contained in a unique maximal 2-local subgroup M of H, and*

(c_1) *$\Omega_1(Z(S))$ and $J(S)$ are neither normal in P_1 nor in P_2,*

(c_2) *$P_i \not\leq M$ and $S \in \mathrm{Syl}_2(N_H(O_2(P_i)))$ for $i = 1, 2$, and*

(c_3) *if $J(S) \leq T \leq S_0$, $P_1^*, P_2^* \in \mathscr{P}(T)$, and $O_2(\langle P_1^*, P_2^* \rangle) \neq 1$, then either $\langle P_1^*, P_2^* \rangle \leq M$ or $J(S) = J(S_1)$ for $T \leq S_1 \in \mathrm{Syl}_2(\langle P_1^*, P_2^* \rangle)$.*

Proof. Assume first that S_0 is contained in two different maximal 2-local subgroups of H. Then (a) or (b) follows from (4.7).

Assume now that there exists a unique maximal 2-local subgroup M of H containing S_0. Since M is not strongly embedded in H there exists $1 \neq Q \leq S_0$ such that $N_H(Q) \not\leq M$. Among all 2-local subgroups which are not in M we choose N such that for $S \in \mathrm{Syl}_2(N \cap M)$ consecutively

(i) $|J(S)|$ is maximal,

(ii) $|S|$ is maximal.

After conjugation in M we may assume that $S \leq S_0$. Since $S \neq S_0$ we have that

$$(*) \qquad N_H(J(S)) \leq M \text{ and } C_H\big(\Omega_1(Z(S))\big) \leq M;$$

in particular $S \in \mathrm{Syl}_2(N)$. Hence by (3.2), $\mathscr{P}(N, S) \not\subseteq \mathscr{P}(M, S)$.

Let $P \in \mathscr{P}(N, S) \setminus \mathscr{P}(M, S)$ and let $x \in N_{S_0}(S) \setminus S$ with $x^2 \in S$. Set $P_1 = P$, $P_2 = P_1^x$, and $H_0 = \langle P_1, P_2 \rangle$. Since $x \in N_H(O_2(H_0))$ and $H_0 \not\leq M$ the maximality of S implies that $O_2(H_0) = 1$; and similarly $S \in \mathrm{Syl}_2(N_H(O_2(P_i)))$. Together with $(*)$, (c_1) and (c_2) follow.

Now let P_1^* and P_2^* be as in (c_3) and let $H^* = \langle P_1^*, P_2^* \rangle$ and $T \leq S_1 \in \mathrm{Syl}_2(H^*)$. Assume that $J(S_1) \neq J(S)$. Then the maximality of $J(S)$ gives $H^* \leq M$. Hence (c_3) holds.

(5.2) *Let* $Z_0 = \Omega_1(Z(S_0))$, $B_0 = C_{S_0}(\Omega_1(Z(J(S_0))))$, $C = C_H(Z_0)$, *and* $P \in \mathscr{P}^*(C, S_0)$, *and let K be a subgroup of P. Suppose that the following hold*:

(i) *Every* 2-*local subgroup of H containing* B_0 *is solvable and of characteristic* 2 *type, and*

(ii) $K = [K, B_0]$.

Then K is subnormal in every 2-*local subgroup of H containing* $B_0 K$.

Proof. Let D be a 2-subgroup of H which is normalized by KB_0. Note that B_0 is normal in DB_0 and by (ii), $K = O^2(K)$. Hence $[D, B_0] \le D \cap B_0$ and

$$[D, KB_0] = \left[D, \langle B_0^K \rangle\right] \le \langle (D \cap B_0)^K \rangle \le KB_0.$$

It follows that $D \in N_H(KB_0)$, and $K = O^2(KB_0)$ yields $D \le N_H(K)$. We have shown:

(1) Let D be a 2-subgroup of H and $KB_0 \le N_H(D)$. Then $D \le N_H(K)$.

From (1) we get that K is normal in $KO_2(P)$ and from (3.3) that $KO_2(P)$ is subnormal in P. Hence

(2) K is subnormal in $O^2(P)$.

Assume now that K is a counterexample such that $|K|$ is maximal. Set $N = N_H(K)$. By (1) there exists $T \in \mathrm{Syl}_2(N)$ such that $O_2(C)B_0 \le T$. Let $g \in H$ such that $T \le S_0^g$. Then $g \in N_H(B_0)$ and $Z_0^g \le Z(O_2(C))$. On the other hand, the subnormality of $O^2(P)$ in C gives $O_2(O^2(P)) \le O_2(C)$ and by (3.5), $[\Omega_1(Z(O_2(C))), O^2(P)] = 1$. Hence $O^2(P) \le C^g$.

If $K \ne O^2(P)$, then the maximality of K implies that $O^2(P)$ and thus by (2) also K is subnormal in C^g. If $K = O^2(P)$, then $S_0 \in \mathrm{Syl}_2(N)$ and by (2), K is subnormal in C. Hence, we have shown:

(3) There exists $h \in N_H(B_0)$ such that $S_0^h \cap N \in \mathrm{Syl}_2(N)$ and K is subnormal in C^h.

We set $T_0 = S_0^h$ and $\hat{Z}_0 = Z_0^h$, where h and S_0 are as in (3). Since K is a counterexample there exists a subgroup M such that the following hold:

(4) $C_H(O_2(M)) \le O_2(M)$ and $KB_0 \le M$.

(5) K is not subnormal in M.

(6) K is subnormal in every proper subgroup of M containing $KB_0O_2(M)$.

By (1), $O_2(M) \le N$ and thus by (3), $O_2(M)B_0 \le T_0^g$ for some $g \in N$. Then $B_0^g = B_0$ and we may assume that $T_0 = T_0^g$. Now (4) implies that $\hat{Z}_0 \le Z(O_2(M))$.

Let $V = \langle \hat{Z}_0^M \rangle$ and $C_0 = C_M(V)$. If $K \leq C_0$, then by (3), K is subnormal in C_0 since $C_0 \leq C^h$, and thus K is subnormal in M which contradicts (5).

Suppose that $O_2(K) \leq C_0$. Then $[V, K] \leq Z(O_2(K))$ by (1) and thus $[Z(O_2(K)), K] \neq 1$ since $K \not\leq C_0$. This contradicts (3.5). We have shown:

(7) $O_2(K) \not\leq C_0$.

Let $\overline{M} = M/C_0$ and $\overline{W} = O_2(\overline{M})$. Then as above in the proof of (1), $[\overline{W}, \overline{B}_0] \leq \overline{W} \cap \overline{B}_0$ and $\overline{W} \leq N_{\overline{M}}(\overline{K})$, i.e., $W \leq N_M(KC_0)$. On the other hand, K is subnormal in KC_0 and thus $O_2(K) \leq O_2(KC_0)$. Hence

$$O_2(K) \cap W \leq O_2(W) \leq O_2(M) \leq C_0.$$

Since $K/O_2(K)$ is a p-subgroup we get that $[\overline{W}, \overline{K}] = O_2(\overline{K}) \cap \overline{W} = 1$. In particular,

$$\left[O_{2'}\left(F(\overline{M})\right), O_2(\overline{K})\right] \neq 1$$

by (7). Now (6) implies that $\overline{M} = O_{2'}(F(\overline{M}))\overline{KB_0}$ and $\overline{W} \leq \overline{T_0 \cap M}$. The definition of V implies that $\overline{W} = 1$. Since by (7), $V \not\leq Z(B_0)$ we get $J(T_0) \not\leq C_0$. Hence (2.2) gives $\overline{K} \leq O_3(\overline{M})$ and $O_2(\overline{K}) = 1$, a contradiction to (7).

We now have set the stage for the amalgam method, which will deal with a triple (P_1, P_2, S) as in (5.1). In case (5.1)(c), P_1 and P_2 need not be solvable or of characteristic 2 type. To get these properties we will make a further hypothesis which will be used in most of the following sections.

HYPOTHESIS 2. *Hypothesis* 1 *holds, S, P_1, and P_2 are as in* (5.1), *and* $B = C_S(\Omega_1(Z(J(S))))$. *In addition*:

(iii) *Every* 2-*local subgroup of H containing B is solvable and of characteristic* 2 *type*.

(5.3) *Assume Hypothesis* 2. *Then P_1 and P_2 are solvable and of characteristic* 2 *type*.

Proof. Let $N = N_H(O_2(P_i))$. By (5.1), $S \in \mathrm{Syl}_2(N)$, and by Hypothesis 2, N is solvable and of characteristic 2 type.

(5.4) *Assume Hypothesis* 2. *Let $B \leq T \leq S$, $F_1, F_2 \in \mathscr{P}(T)$, and $H_0 = \langle F_1, F_2 \rangle$, and let M be a maximal* 2-*local subgroup of H containing S_0. Suppose that $O_2(H_0) \neq 1$ and either $S = S_0$ or $H_0 \not\leq M$. Then $\Omega_1(Z(S)) \leq O_2(H_0)$.*

Proof. Let $T_0 = T \cap O_2(H_0)$, $N = N_H(T_0)$, and $T \leq S_1 \in \mathrm{Syl}_2(H_0)$. If $S = S_0$, then obviously $J(S) = J(S_1)$; and if $S \neq S_0$, then (5.1)(c) shows that $J(S) = J(S_1)$. Hence $[O_2(H_0), J(S)] \leq J(S) \cap O_2(H_0) \leq T_0$ and $T_0 \neq 1$ since $\Omega_1(C_{S_1}(J(S))) \leq J(S)$.

We have shown that N is a 2-local subgroup of H. By (3.8), $H_0 \leq N$. Thus, N is of characteristic 2 type. Let $Z = \Omega_1(Z(S))$ and $W = \langle Z^N \rangle$. As above $J(S) = J(S_2)$ for $T \leq S_2 \in \mathrm{Syl}_2(N)$. Since $Z \leq J(S)$ we get that $[O_2(N), W] \leq T_0$ and $[T_0, W] = 1$. It follows that $[O^2(W), O_2(N)] = 1$ and thus $Z \leq W \leq O_2(N)$. We conclude that $Z \leq O_2(H_0)$.

6.

In this section we assume Hypothesis 2.

Notation. $Q_i = O_2(P_i)$, $L_i = \langle B^{P_i} \rangle$, $Z_i = \langle \Omega_1(Z(B))^{P_i} \rangle$, $V = \langle \Omega_1(Z(S))^{P_1} \rangle$, $\overline{P}_1 = P_1/C_{P_1}(V)$, and $J(V, \overline{S})$ is defined as in Section 1.

(6.1) $B \not\leq Q_2$.

Proof. Assume first that $J(S) \leq Q_1$. Then $J(S) \not\leq Q_2$ since $O_2(\langle P_1, P_2 \rangle) = 1$. Hence $B \not\leq Q_2$.

Assume now that $J(S) \not\leq Q_1$. By (5.3) and (3.3), $C_S(V) = Q_1$ and by (2.3), $B \in \mathrm{Syl}_2(L_1)$. Assume that $B \leq Q_2$. Then $J(S)$ is normal in P_2. In particular (5.1)(a) or (b) holds and $S = S_0$ According to (2.2) there exists $E_1 \in \mathscr{P}(L_1, B)$ such that $E_1/C_{E_1}([V, E_1]) \cong SL_2(2)$ and $\langle E_1, S \rangle = P_1$. Since $\langle E_1, P_2 \rangle = \langle P_1, P_2 \rangle$ there is no nontrivial characteristic subgroup of B which is normal in E_1. Hence (2.4) gives $\|[O_2(E_1), O^2(E_1)]\| = 4$.

Let $W_1 = [O_2(E_1), O^2(E_1)]$ and $W = \langle W_1^u \mid u \in U \rangle$, where U is a Hall $2'$-subgroup of P_2. Then by (2.5), W is normal in $\langle E_1, U \rangle$. Let $u \in U$ and $H_0 = \langle E_1, E_1^u \rangle$. Clearly $J(B) = J(S_1)$ for $B \leq S_1 \in \mathrm{Syl}_2(H_0)$ since $S = S_0$. Moreover, by (5.4), $\Omega_1(Z(S)) \leq B \cap O_2(H_0)$. Hence (3.9) gives $O^2(E_1^u) \leq N_H(W_1)$ and $O^2(E_1) \leq N_H(W_1^u)$; i.e., by (2.2), $\langle L_1, E_1^u \mid u \in U \rangle \leq N_H(W_1)$. In particular, $O_2(\langle E_1^s, E_1^u \rangle) \neq 1$ for $s \in S$. Now the same argument with E_1^s, $s \in S$, in place of E_1 gives $O^2(E_1^s) \leq N_H(W_1^u)$. It follows that $\langle L_1, U \rangle \leq N_H(W)$.

Set $W^* = \langle W^S \rangle$. Then W^* is a normal subgroup of P_1 in Q_1. Since $P_2 = SU = US$ we get that $U \leq N_H(W^*)$ and thus $\langle P_1, U \rangle = \langle P_1, P_2 \rangle \leq N_H(W^*)$, a contradiction.

(6.2) *Suppose that* $[P_2, \Omega_1(Z(S))] \neq 1$. *Then* $J(S) \not\leq Q_i$, $i = 1, 2$.

Proof. The hypothesis is symmetric in P_1 and P_2 since also $[P_1, \Omega_1(Z(S))] \neq 1$. Hence (6.1) implies $B \not\leq Q_i$, $i = 1, 2$. Now (3.4) gives $[O^2(P_i), B] = O^2(P_i)$. Assume that $J(S) \leq Q_i$. Then $\Omega_1(Z(Q_i)) \leq Z(B)$ and $[\Omega_1(Z(Q_i)), O^2(P_i)] = 1$. Thus (5.3) yields $[P_i, \Omega_1(Z(S))] = 1$, a contradiction.

(6.3) *Suppose that* $[P_2, \Omega_1(Z(S))] \neq 1$. *Then* $P_i/Q_i \cong D_{2\cdot 3^{n_i}}$, $i = 1, 2$.

Proof. From (6.2) we get that $J(S) \nleq Q_i$, $i = 1, 2$. Hence $B \in \mathrm{Syl}_2(L_i)$ by (3.3) and (2.3). We apply (2.2).

There exists $E_i \in \mathscr{P}(L_i, B)$ such that

$$\langle E_i, S\rangle = P_i, \qquad \big|[Z_i, O^2(E_i)]\big| = 4, \quad \text{and}$$
$$E_i/C_{E_i}\big([Z_i, O^2(E_i)]\big) \cong SL_2(2).$$

Assume first that $O^2(E_2)$ normalizes $[Z_1, O^2(E_1^s)]$ for every $s \in S$. Then E_2 normalizes $[Z_1, O^2(L_1)]$ and $\langle P_1, P_2\rangle \leq N_H([Z_1, L_1])$, a contradiction.

We may assume now that E_1 and E_2 are chosen such that $O^2(E_2) \nleq N_H([Z_1, O^2(E_1)])$. Let $H_0 = \langle E_1, E_2\rangle$. Suppose that $O_2(H_0) \neq 1$. If (5.1)(c) holds, then $H_0 \nleq M$ [M as in (5.1)(c)] since $P_i \nleq M$. Thus $J(S) = J(S_1)$ for $B \leq S_1 \in \mathrm{Syl}_2(H_0)$ in any case. Moreover, (5.4) yields $\Omega_1(Z(S)) \leq O_2(H_0)$, and H_0 satisfies the hypothesis of (3.9). Hence $O^2(E_2) \leq N_H([Z_1, O^2(E_1)])$, a contradiction.

Suppose now that $O_2(H_0) = 1$. Let $Z_0 = C_{\Omega_1(Z(B))}(E_1)$. Then $C_{Z_0}(E_2) = 1$. Since by (2.2),

$$|\Omega_1(Z(B))/\Omega_1(Z(B)) \cap C_H(E_i)| = 2,$$

we conclude that $|Z_0| \leq 2$ and $|\Omega_1(Z(B))| \leq 4$; in particular $[\Omega_1(Z(B)), S] \leq \Omega_1(Z(S))$.

Assume first that $L_1 = E_1 C_{L_1}(Z_1)$. Then $S = BQ_1$ and $Z_0 \leq Z(L_1)$. Since L_1 is normal in P_1 and $[\Omega_1(Z(S)), P_1] \neq 1$ we get that $[Z_0, Q_1] = 1$ and $\Omega_1(Z(B)) = \Omega_1(Z(S))$. Hence also $L_2 = E_2 C_{L_2}(Z_2)$, and the assertion follows with (3.3).

Assume now that $L_1/C_{L_1}(Z_1) \cong SL_2(2) \times SL_2(2)$ and by symmetry also $L_2/C_{L_2}(Z_2) \cong SL_2(2) \times SL_2(2)$. Then $\Omega_1(Z(B)) = Z_0 \times Z_0^s$ and $[Z_0, E_2^s] = 1$, where $s \in S \setminus B$. The above argument applied to $\langle E_1, E_2^s\rangle$ shows that $E_2^s \leq N_H([Z_1, O^2(E_1)])$. Since $L_1 \leq N_H([Z_1, O^2(E_1)])$ we conclude that $O_2(\langle E_1^s, E_2^s\rangle) = O_2(H_0^s) \neq 1$. This contradicts $O_2(H_0) = 1$.

(6.4) *Suppose that* $[P_2, \Omega_1(Z(S))] = 1$ *and* $J(V, \bar{S}) \neq 1$. *Then* $P_2 \neq \langle C_{P_2}(w), S\rangle$ *for every* $w \in C_V(J(V, \bar{S})) \setminus \Omega_1(Z(S))$.

Proof. Let $Z_0 = C_V(J(V, \bar{S}))$ and $\bar{E} = O^2(\bar{P}_1)J(V, \bar{S})$. We apply (2.2). Then

(1) $\bar{B} \leq J(V, \bar{S})$,

(2) $\bar{E} = \bar{E}_1 \times \cdots \times \bar{E}_r$, $\bar{E}_i \cong SL_2(2)$, and

(3) $V = V_0 \times V_1 \times \cdots \times V_r$, $V_0 = C_V(\bar{E})$, $V_i = [V, \bar{E}_i]$, and $|V_i| = 4$ for $i = 1, \ldots, r$.

Note that $[P_2, \Omega_1(Z(S))] = 1$ and (5.3) imply that $\Omega_1(Z(P_1)) = 1$. If $r = 1$, then $Z_0 = \Omega_1(Z(S))$ and there is nothing to prove.

Let $r > 1$, $w \in Z_0 \setminus \Omega_1(Z(S))$, and $T = C_S(w)$. By (1), $B \leq T$, and by (2) and (3) there exists $j \in \{1, \ldots, r\}$ such that $[w, \overline{E}_j] = 1$. Let $F_1 \in \mathscr{P}(ET, T)$ such that $[w, F_1] = 1$ and $\overline{E}_j \leq \overline{F}_1$.

Assume that $\langle C_{P_2}(w), S\rangle = P_2$. By (6.1) and (3.4), T is not normal in $O^2(C_{P_2}(w))T$. Hence, there exists $F_2 \in \mathscr{P}(C_{P_2}(w), T)$ such that $\langle F_2, S\rangle = P_2$. Let $L = \langle F_1, F_2\rangle$ and $Q = T \cap O_2(L)$. Note that $w \in O_2(L)$ and thus $O_2(L) \neq 1$.

Again (6.1) and (3.4) imply that $[O^2(F_2), B] = O^2(F_2)$. Let $C = C_H(\Omega_1(Z(S)))$. Then $O^2(P_2)$ is subnormal in C. Since $O^2(F_2)$ is subnormal in $O^2(P_2)$ we get

(4) $O^2(F_2)$ is subnormal in C and $[O^2(F_2), B] = O^2(F_2)$.

By (5.2), $O^2(F_2)$ is subnormal in L and by (5.4), $\Omega_1(Z(S)) \leq Z(Q)$. Moreover, by (3.8), Q is normal in L. Let $W = O_2(O^2(F_2))$ and $U = [\Omega_1(Z(S)), O^2(F_1)]$. Then $W \leq Q$ and $U \leq Z(Q)$; in particular, $[U, W] = 1$. It follows that $[U, O^2(F_2)] \leq Z(W)$.

Suppose that $[U, O^2(F_2)] \neq 1$. Then (4) and the $P \times Q$ lemma yield $[Z(O_2(C)), O^2(F_2)] \neq 1$ and thus $[Z(O_2(C)), O^2(P_2)] \neq 1$. Now (3.5) gives $[\Omega_1(Z(S)), O^2(P_2)] \neq 1$, a contradiction.

We have shown that $[U, O^2(F_2)] = 1$ and thus $[V_j, O^2(F_2)] = 1$ since $V_j \leq U$. Now

$$\langle O^2(P_1)B, F_2B\rangle \leq N_H(V_j),$$

and the above argument with $V = [\Omega_1(Z(S)), O^2(P_1)]$ in place of U yields $[V, O^2(F_2)] = 1$ and $\langle P_1, F_2\rangle = \langle P_1, P_2\rangle \leq N_H(V)$, a contradiction.

7.

In this section G is a finite group and S a is nontrivial 2-subgroup of G such that the following hold:

(i) There exist $P_1, P_2 \in \mathscr{P}(S)$ with $G = \langle P_1, P_2\rangle$.

(ii) P_1 and P_2 are solvable and of characteristic 2 type.

(iii) $O_2(G) = 1$.

DEFINITION. Let $\Gamma = \Gamma(G, P_1, P_2)$ be the graph whose vertices are the right cosets of P_1 and P_2 in G and whose edges are the unordered pairs $\{P_1 g, P_2 h\}$ with $P_1 g \cap P_2 h \neq \emptyset$.

Γ is called the *coset graph* of G with respect to P_1 and P_2, and G operates on Γ by right multiplication. We identify Γ with its set of vertices.

For $\delta \in \Gamma$ we define:

$$G_\delta \text{ is the stabilizer of } \delta \text{ in } G,$$

$$d(\ ,\ ) \text{ is the usual distance metric on } \Gamma,$$

$$\Delta(\delta) = \{\lambda \in \Gamma \mid d(\lambda, \delta) = 1\},$$

$$Q_\delta = O_2(G_\delta),$$

$$E_\delta = O^2(G_\delta),$$

$$Z_\delta = \langle \Omega_1(Z(T)) \mid T \in \mathrm{Syl}_2(G_\delta)\rangle,$$

$$V_\delta = \langle Z_\lambda \mid \lambda \in \Delta(\delta)\rangle.$$

The properties collected in (7.1) are elementary and independent from the structure of P_1 and P_2. They will be used without reference.

(7.1) *The following hold*:

(a) Γ *is connected.*

(b) G *operates edge- but not vertex-transitively on* Γ.

(c) *There exists an edge* $\{\alpha, \beta\}$ *such that that* $G_\alpha = P_1$ *and* $G_\beta = P_2$; *i.e., the vertex stabilizers are conjugate to* P_1 *or* P_2, *and the edge stabilizers are conjugate to* $P_1 \cap P_2$.

(d) G_δ *operates transitively on* $\Delta(\delta)$.

Remark. According to (7.1)(b) and (c) any statement about G_δ, $\delta \in \Gamma$, is, after conjugation, also a statement about P_1 and P_2, respectively. This fact will be used freely. One particular application used frequently is the following:

Let $\{\delta, \lambda\}$ be an edge of Γ and let N be a subgroup of Q_δ which is normal in G_δ and G_λ. Then a suitable conjugate of N in S is normal in P_1 and P_2 and thus contained in $O_2(\langle P_1, P_2\rangle) = O_2(G) = 1$. Hence $N = 1$.

(7.2) *The kernel of the operation of* G *on* Γ *is trivial*.

Proof. Let K be the kernel of the operation of G on Γ. Then K is a normal subgroup of G contained in P_1. Since P_1 is of characteristic 2 type, either $K = 1$ or $O_2(K) \neq 1$. The latter case contradicts $O_2(G) = 1$.

(7.3) *Let* $\delta \in \Gamma$, $\lambda \in \Delta(\delta)$, *and* $T \in \mathrm{Syl}_2(G_\delta \cap G_\lambda)$. *Then the following hold*:

(a) $T \in \mathrm{Syl}_2(G_\delta) \cap \mathrm{Syl}_2(G_\lambda)$; *in particular*, $Q_\delta \leq G_\lambda$.

(b) $Z_\delta \leq \Omega_1(Z(Q_\delta))$.

(c) *Either* $C_T(Z_\delta) = Q_\delta$ *or* $Z_\delta = \Omega_1(Z(G_\delta)) = \Omega_1(Z(T))$.

(d) *If* $Z_\delta = \Omega_1(Z(G_\delta))$, *then* $Z(G_\lambda) = 1$.

Proof. We may assume that $\{G_\delta, G_\lambda\} = \{P_1, P_2\}$ and $T = S$. Now (a) is obvious and (b) follows since P_1 and P_2 are of characteristic 2 type.

Let $S_0 = C_S(Z_\delta)$. Then $Q_\delta \leq S_0$ by (b). Assume that $S_0 \not\leq Q_\delta$. Then $E_\delta \leq \langle S_0^{G_\delta} \rangle$ by (3.4); i.e., $G_\delta = G_{G_\delta}(Z_\delta)S$. Now the definition of Z_δ gives (c).

Suppose that $Z_\delta = \Omega_1(Z(G_\delta))$. Note that $\Omega_1(Z(G_\lambda)) \leq \Omega_1(Z(S)) = Z_\delta$ and so $\Omega_1(Z(G_\lambda))$ is central in $\langle G_\delta, G_\lambda \rangle = G$. Now $O_2(G) = 1$ yields $Z(G_\lambda) = 1$.

DEFINITION.

$$b = \min\{d(\delta, \delta') \mid \delta, \delta' \in \Gamma,\ Z_\delta \not\leq Q_{\delta'}\}.$$

According to (7.2) and (7.3)(b), b is a well-defined integer larger than zero. A pair (α, α') is called a *critical pair* if $d(\alpha, \alpha') = b$ and $Z_\alpha \not\leq Q_{\alpha'}$. In the following let (α, α') be a critical pair and γ be a path of length b from α to α'. We denote γ by $(\alpha, \alpha + 1, \dots, \alpha + b) = (\alpha' - b, \dots, \alpha')$; i.e., $\alpha + b = \alpha'$ and $\alpha' - b = \alpha$. Without loss of generality we may assume that $S \leq G_\alpha \cap G_{\alpha+1}$ and $\{G_\alpha, G_{\alpha+1}\} = \{P_1, P_2\}$.

(7.4) *The following hold*:

(a) $Z_\alpha \leq V_{\alpha+1} \leq G_{\alpha'}$.

(b) $Z_{\alpha'} \leq G_\alpha$ *and* $V_{\alpha'} \leq G_{\alpha+1}$.

(c) $C_S(Z_\alpha) = Q_\alpha$.

(d) *If* $[Z_\alpha, Z_{\alpha'}] \neq 1$, *then* $C_T(Z_{\alpha'}) = Q_{\alpha'}$ *for* $T \in \mathrm{Syl}_2(G_{\alpha'})$ *and* (α', α) *is also a critical pair*.

(e) Z_α *is quadratic on* $Z_{\alpha'}$ *and vice versa*.

Proof. (a) and (b) follow from (7.3)(a) and the minimality of b.

To (c). Suppose that $C_S(Z_\alpha) \neq Q_\alpha$. Then by (7.3)(c), $Z_\alpha = \Omega_1(Z(S))$. Hence $Z_\alpha \leq Z_{\alpha+1} \not\leq Q_{\alpha'}$, which contradicts the minimality of b.

To (d). Suppose that $[Z_\alpha, Z_{\alpha'}] \neq 1$. By (b), $Z_{\alpha'} \leq G_\alpha$ and by (c), $Z_{\alpha'} \not\leq Q_\alpha$. Hence, (α', α) is also a critical pair and (d) follows from (c).

To (e). Z_α and $Z_{\alpha'}$ normalize each other by (a) and (b), and they are abelian by (7.3)(b). Hence (e) follows.

(7.5) *Suppose that* $[Z_\alpha, Z_{\alpha'}] = 1$. *Then the following hold*:

(a) b *is odd*; *i.e.*, $\alpha' \in (\alpha + 1)^G$.

(b) $Z_{\alpha+1} = \Omega_1(Z(S)) = \Omega_1(Z(G_{\alpha+1}))$.

(c) $C_{Q_\alpha}(E_\alpha) = 1$; *in particular*, $Z(G_\alpha) = 1$.

(d) *If* $b > 1$, *then* $V_{\alpha+1}$ *is elementary abelian and* $V_{\alpha+1}$ *is quadratic on* $V_{\alpha'}$, *and vice versa*.

Proof. By (7.3)(c), $Z_{\alpha'} = \Omega_1(Z(G_{\alpha'}))$, and (7.4)(c) implies that $\alpha' \notin \alpha^G$, i.e., $\alpha' \in (\alpha + 1)^G$ and b is odd. Hence (a) and (b) follow.

To (c). $Z(S) \cap C_{Q_\alpha}(E_\alpha) \leq Z(G_\alpha)$. Hence, (7.3)(d) implies (c).

To (d). By (7.4)(a) and (b), $V_{\alpha+1}$ and $V_{\alpha'}$ normalizes each other, and if $b > 1$, then $b \geq 3$ by (a), and both subgroups are abelian. Now (d) follows.

Remark. Assume Hypothesis 2. Then (7.4)(c) and (5.1) lead to the following two cases:

Case I. $\Omega_1(Z(S))$ is neither normal in P_1 nor in P_2. Hence $Z_{\alpha+1} \nleq Z(G_{\alpha+1})$ and $[Z_\alpha, Z_{\alpha'}] \neq 1$ by (7.5)(b). This case will be treated in (8.2).

Case II. $\Omega_1(Z(S))$ is normal in P_2; i.e., (5.1)(b) holds. It follows that $P_1 = G_\alpha$ and $P_2 = G_{\alpha+1}$; in particular, $Z_{\alpha+1} \leq Z(G_{\alpha+1})$ and most importantly,

$$E_{\alpha+1} \text{ is subnormal in } C_H(Z_{\alpha+1}).$$

This case will be treated in (8.6) and in Sections 9 and 10.

(7.6) *The following hold*:

(a) $Q_\alpha \cap Q_{\alpha+1}$ *is not normal in* $G_{\alpha+1}$.

(b) $O_2(E_{\alpha+1}) \nleq Q_\alpha$; *in particular*, $E_\alpha \leq \langle O_2(E_{\alpha+1})^{G_\alpha} \rangle$.

(c) $Q_\alpha \cap Q_{\alpha+1} \nleq Q_\mu$ *for every* $\mu \in \Delta(\alpha + 1)$ *with* $\langle Q_\mu, G_\alpha \cap G_{\alpha+1} \rangle = G_{\alpha+1}$.

(d) $\langle C_{G_{\alpha+1}}(Z_\alpha), G_\alpha \cap G_{\alpha+1} \rangle \neq G_{\alpha+1}$.

Proof. Suppose that $Q_\alpha \cap Q_{\alpha+1}$ is normal in $G_{\alpha+1}$. Then $Q_\alpha \cap Q_{\alpha+1} = Q_{\alpha+1} \cap Q_\mu$ for every $\mu \in \Delta(\alpha + 1)$.

Assume that $b = 1$. Then $E_{\alpha+1} \leq \langle Z_\alpha^{G_{\alpha+1}} \rangle$ by (3.4). However, $[Q_{\alpha+1}, Z_\alpha] \leq Q_\alpha \cap Q_{\alpha+1}$ and $[Q_\alpha \cap Q_{\alpha+1}, Z_\alpha] = 1$. Hence $[Q_{\alpha+1}, E_{\alpha+1}] = 1$ and $G_{\alpha+1}$ is not of characteristic 2 type, a contradiction.

Assume that $b > 1$ and choose $\mu = \alpha + 2$. Then $Z_{\alpha'} \leq Q_{\alpha+2} \cap Q_{\alpha+1} \leq Q_\alpha$ and $[Z_\alpha, Z_{\alpha'}] = 1$. Hence by (7.5), $\alpha' \in (\alpha + 1)^G$, $\alpha' - 1 \in \alpha^G$, and $Z_{\alpha+1} = \Omega_1(Z(G_{\alpha+1}))$. Since $Z_\alpha \leq Q_{\alpha'-1}$ but $Z_\alpha \nleq Q_{\alpha'}$ we conclude that $Q_\alpha \nleq Q_{\alpha+1}$. Thus (3.4) gives $O_2(E_{\alpha+1}) \leq Q_\alpha \cap Q_{\alpha+1}$. Now (3.5) yields $[Z_\alpha, E_{\alpha+1}] = 1$ and $Z_\alpha \leq O_2(G) = 1$, a contradiction. This shows (a).

Suppose that $O_2(E_{\alpha+1}) \leq Q_\alpha$. Then $Q_\alpha \cap Q_{\alpha+1}$ is normal in $G_{\alpha+1}$ which contradicts (a). Hence $O_2(E_{\alpha+1}) \nleq Q_\alpha$ and (3.4) gives (b).

Suppose that $Q_\alpha \cap Q_{\alpha+1} \leq Q_\mu$ for some $\mu \in \Delta(\alpha + 1)$ with $\langle Q_\mu, G_\alpha \cap G_{\alpha+1} \rangle = G_{\alpha+1}$. Then $Q_\alpha \cap Q_{\alpha+1} = Q_\mu \cap Q_{\alpha+1}$ since $\mu \in \alpha^{G_{\alpha+1}}$. Hence $Q_\alpha \cap Q_{\alpha+1}$ is normal in $G_{\alpha+1}$ which contradicts (a).

Note that $Q_\alpha \cap Q_{\alpha+1} = C_{Q_{\alpha+1}}(Z_\alpha)$ by (7.4)(c). Hence (a) implies (d).

(7.7) *Let* $C = C_G(\Omega_1(Z(S)))$. *Suppose that* $E_{\alpha+1}$ *is subnormal in* C, $O_2(C) \le S$, *and* C *is of characteristic* 2 *type. Then the following hold*:

(a) $[C_G(Z_\alpha), E_\alpha O_2(E_{\alpha+1})] \le Q_\alpha$.

(b) $C_G(V_{\alpha+1})$ *is a* 2-*group*.

(c) $E_{\alpha'} \not\le C$ *if* $b > 1$; *in particular*, $Z_{\alpha'} \ne Z_{\alpha+1}$ *if* $b > 1$.

Proof. Note that $Z_{\alpha+1} = \Omega_1(Z(S))$ since $E_{\alpha+1} \le C$. Moreover, $O_2(E_{\alpha+1}) \le O_2(C)$ by the subnormality of $E_{\alpha+1}$. This gives $[C_G(Z_\alpha), O_2(E_{\alpha+1})] \le O_2(C) \cap C_G(Z_\alpha) \le C_S(Z_\alpha)$. Hence (7.4)(c) gives $[C_G(Z_\alpha), O_2(E_{\alpha+1})] \le Q_\alpha$, and (7.6)(b) and (3.4) yield (a).

Let $E = O^2(C_G(V_{\alpha+1}))$. By (a), $[E, E_\alpha] \le EQ_\alpha$. Since $E = O^2(EQ_\alpha)$ we conclude that E is normal in $\langle E_\alpha, G_{\alpha+1}\rangle = G$, and $O_2(E) = 1$. On the other hand, $C_G(V_{\alpha+1})$ is of characteristic 2 type since $C_G(V_{\alpha+1}) \le C$. Hence $E = 1$ and $C_G(V_{\alpha+1})$ is a 2-group.

Suppose that $b > 1$ and $E_{\alpha'} \le C$. Since $E_{\alpha+1}$ is transitive on $\Delta(\alpha+1)$ we get that $Z_\alpha \le Z_{\alpha+2} O_2(E_{\alpha+1})$. Moreover $O_2(E_{\alpha+1}) \le O_2(C)$ since $E_{\alpha+1}$ is subnormal in C, and $Z_{\alpha+2} \le Q_{\alpha'}$ by the minimality of b. Hence $Z_\alpha \le (C \cap Q_{\alpha'})O_2(C)$ and $[Z_\alpha, E_{\alpha'}] \le O_2(E_{\alpha'})$. Now (3.3) gives $Z_\alpha \le Q_{\alpha'}$, a contradiction.

(7.8) *Let* $\delta \in \Gamma$, $\lambda \in \Delta(\delta)$, *and* $\overline{G}_\delta = G_\delta/Q_\delta$, *and let* A *be a subgroup of* Q_λ *with* $A \not\le Q_\delta$ *and* $\Phi(A) \le Q_\delta$. *Then there exists* $x \in G_\delta$ *and* $A_0 \le A$ *such that for* $L = \langle A, A^x\rangle$ *the following hold*:

(a) $|A/A_0| = 2$, $x \in L$, $x^2 \in Q_\delta$, *and* $A_0 = A \cap O_2(L)$.

(b) $\langle L, G_\lambda \cap G_\delta\rangle = G_\delta$ *and* $\overline{L} \cong D_{2p^n} \times \overline{A}_0$.

(c) *Any two elements in* Z_λ^L *are interchanged by an involution of* L.

(d) $L = \langle a, A^x\rangle$ *for every* $a \in A \setminus A_0$.

(e) *For* $T \in \mathrm{Syl}_2(G_\delta \cap G_\lambda)$ *either* $B(T) \le Q_\delta$ *or* $O^2(L) \le [O^2(L), B(T)]$.

Proof. Apply (3.6).

8.

In this section we assume Hypothesis 2 and set $G = \langle P_1, P_2\rangle$. We use the notation concerning $\Gamma = \Gamma(G, P_1, P_2)$ as it was introduced in Section 7; in particular, $S \le G_\alpha \cap G_{\alpha+1}$. The additional hypothesis for this section is $[Z_\alpha, Z_{\alpha'}] \ne 1$.

Notation. $R = [Z_\alpha, Z_{\alpha'}]$, $\overline{G}_\delta = G_\delta / C_{G_\delta}(Z_\delta)$, and $J(Z_\alpha, \overline{S})$ is defined as in Section 1.

(8.1) *The following hold*:

(a) $|Z_\alpha / Z_\alpha \cap Q_{\alpha'}| = |Z_{\alpha'} / Z_{\alpha'} \cap Q_\alpha|$.

(b) $\overline{E}_\alpha J(Z_\alpha, \overline{S}) = E_1 \times \cdots \times E_r$ *and* $Z_\alpha = V_0 \times \cdots \times V_r$, *where* $V_0 = C_{Z_\alpha}(\overline{E}_\alpha J(Z_\alpha, \overline{S}))$, $E_i \cong SL_2(2)$, $V_i = [Z_\alpha, E_i]$, *and* $|V_i| = 4$, *for* $i \geq 1$.

(c) $R = (R \cap V_1) \times \cdots \times (R \cap V_r)$.

Proof. We apply (7.4)(c) and (d). Since the configuration is symmetric in α and α' we may assume that

$$(*) \qquad |Z_{\alpha'} / Z_{\alpha'} \cap Q_\alpha| \geq |Z_\alpha / Z_\alpha \cap Q_{\alpha'}|.$$

Hence $\overline{Z}_{\alpha'} \leq J(Z_\alpha, \overline{S})$ and (1.5)(e) implies equality in $(*)$. Thus (a) holds.

Claim (b) is a consequence of (1.7), and (b) implies (c).

(8.2) *Suppose that* $Z_{\alpha+1} \not\leq Z(G_{\alpha+1})$. *Then* $G_\delta \cong \Sigma_4$ *or* $G_\delta \cong C_2 \times \Sigma_4$ *for every* $\delta \in \Gamma$.

Proof. By (7.3)(c), $Z_\delta \not\leq Z(G_\delta)$ for every $\delta \in \Gamma$. Hence $[P_i, \Omega_1(Z(S))] \neq 1$ for $i = 1, 2$. Now (6.3) yields $G_\delta / Q_\delta \cong D_{2 \cdot 3^{n_\delta}}$ and $\overline{G}_\delta \cong SL_2(2)$.

Pick $\alpha - 1 \in \Delta(\alpha)$ such that $\langle G_{\alpha-1} \cap G_\alpha, Z_{\alpha'} \rangle = G_\alpha$. Assume first that $Z_{\alpha-1} \leq G_{\alpha'}$. Then (7.1)(b) and (7.4) give $b > 1$; in particular, $V_{\alpha+1} \leq Q_{\alpha+1}$.

Since $Z_{\alpha-1} \leq Z_\alpha Q_{\alpha'}$ we get that $[Z_{\alpha-1}, Z_{\alpha'}] \leq R$, and $Z_{\alpha-1} Z_\alpha$ is normal in G_α. By (7.3)(c), $C_S(Z_{\alpha-1} Z_\alpha) = Q_\alpha \cap Q_{\alpha-1}$ and thus also $Q_\alpha \cap Q_{\alpha-1}$ is normal in G_α. It follows that $Q_{\alpha+1} \in \mathrm{Syl}_2(E_\alpha Q_{\alpha+1})$. Since $O_2(\langle G_\alpha, G_{\alpha+1} \rangle) = 1$ no nontrivial characteristic subgroup of $Q_{\alpha+1}$ is normal in $E_\alpha Q_{\alpha+1}$. Let U be a Hall $2'$-subgroup of $G_{\alpha+1}$. Then $V_{\alpha+1} = \langle Z_\alpha^U \rangle$, and (2.5) implies that $V_{\alpha+1}$ is normal in G_α, which contradicts $O_2(\langle G_\alpha, G_{\alpha+1} \rangle) = 1$.

Assume now that $Z_{\alpha-1} \not\leq G_{\alpha'}$. Then $(\alpha - 1, \alpha' - 1)$ is also a critical pair. With the same argument as above there exists $\alpha - 2 \in \Delta(\alpha - 1)$ such that $\langle G_{\alpha-2} \cap G_{\alpha-1}, Z_{\alpha'-1} \rangle = G_{\alpha-1}$ and $(\alpha - 2, \alpha' - 2)$ is a critical pair (here $\alpha' - 2 = \alpha - 1$ if $b = 1$).

Set $R_i = [Z_{\alpha-i}, Z_{\alpha'-i}]$, $i = 1, 2$, and assume that $b > 1$. Then $R_2 \leq Z(G_{\alpha-1})$ since R_2 centralizes $Z_{\alpha'-1}$ and $G_{\alpha-2} \cap G_{\alpha-1}$, and similarly $R_1 \leq Z(G_\alpha)$. If $b > 2$, then $R_2 \leq Z_{\alpha'-2} \leq Q_{\alpha'}$, and so R_2 is centralized by G_α. Hence $R_2 \leq O_2(\langle G_\alpha, G_{\alpha-1} \rangle)$, a contradiction.

Assume $b = 2$. Let $V_0 = V_\alpha \cap Q_{\alpha-1} \cap Q_{\alpha+1}$. Then $V_\alpha = Z_{\alpha-1} Z_{\alpha+1} V_0$ and $|V_\alpha / V_0| = 4$. In addition, $[V_0, Z_{\alpha+2}] = R \leq Z_\alpha \leq V_0$ and $[V_0, Z_{\alpha-2}] \leq R_2 \leq Z_\alpha \leq V_0$. Hence, V_0 is normal in G_α and $V_0 \leq Z(V_\alpha)$.

If V_0 is elementary abelian, then $|V_\alpha / V_0| = 4$ and the action of G_α on V_α / V_0 imply that $V_\alpha' = 1$. However, this contradicts $[Z_{\alpha-1}, Z_{\alpha+1}] \neq 1$. Hence $\Phi(V_0) \neq 1$. Note that $\Phi(V_0) \leq Q_\delta$ for every $\delta \in \Delta(\alpha - 1)$; i.e.,

$[\Phi(V_0), V_{\alpha-1}] = 1$. On the other hand, with the above argument there exists $\alpha - 3 \in \Delta(\alpha - 2)$ such that $(\alpha - 3, \alpha - 1)$ is a critical pair and $\langle G_{\alpha-1} \cap G_{\alpha-2}, Z_{\alpha+1}\rangle = G_{\alpha-1}$. Thus $V_{\alpha-2} \not\leq Q_{\alpha-1}$ and $\langle V_{\alpha-2}, Z_{\alpha+1}\rangle Q_{\alpha-1} = G_{\alpha-1}$. Note that $V_0 = Z_\alpha(V_0 \cap Q_\kappa)$ for $\kappa \in \Delta(\alpha - 1)$ and thus $\Phi(V_0) = \Phi(V_0 \cap Q_\kappa) \leq Q_\nu$ for $\nu \in \Delta(\kappa)$. It follows that $[\Phi(V_0), V_{\alpha-2}] = 1$ and $\Phi(V_0)$ is normal in $\langle V_{\alpha-2}, G_\alpha \cap G_{\alpha-1}\rangle = G_{\alpha-1}$ and G_α, a contradiction.

Assume finally that $b = 1$. Let $\delta \in \{\alpha - 1, \alpha\}$. Then $Q_\delta = (Q_\alpha \cap Q_{\alpha-1})Z_\delta$ and so $\Phi(Q_\delta) = \Phi(Q_\alpha \cap Q_{\alpha-1})$. This gives $\Phi(Q_\delta) = 1$ and $Q_\delta = [Z_\delta, E_\delta] \times Z(G_\delta)$. By (8.1), $\|[Z_\delta, E_\delta]\| = 4$ and so $G_\delta/Q_\delta \cong SL_2(2)$. Since $Z(G_{\alpha-1}) \leq Q_\alpha$, but $Z(G_\alpha) \cap Z(G_{\alpha-1}) = 1$ we also get $|Z(G_{\alpha-1})| \leq 2$ and with the same argument $|Z(G_\alpha)| \leq 2$. Now (8.2) follows.

Definition. Let H be a finite group. Then H is of *type* $L_3(2)$ resp. $Sp_4(2)$ provided H contains two subgroups G_α and $G_{\alpha+1}$ such that $G_\alpha \cap G_{\alpha+1} = S \in \mathrm{Syl}_2(H)$, $O_2(\langle G_\alpha, G_{\alpha+1}\rangle) = 1$, and $G_\alpha \cong G_{\alpha+1} \cong \Sigma_4$ resp. $C_2 \times \Sigma_4$.

Note that $L_3(2)$ and $Sp_4(2)$ are examples for groups of type $L_3(2)$ and $Sp_4(2)$, respectively.

(8.3) *Suppose that $Z_{\alpha+1} \leq Z(G_{\alpha+1})$. Then $O_2(E_\alpha) \not\leq Q_{\alpha+1}$.*

Proof. Assume that $O_2(E_\alpha) \leq Q_{\alpha+1}$. Then $Q_{\alpha+1} \in \mathrm{Syl}_2(E_\alpha Q_{\alpha+1})$. By (3.2) and (3.3) there exists $F \in \mathscr{P}(E_\alpha Q_{\alpha+1}, Q_{\alpha+1})$ such that $\langle F, G_\alpha \cap G_{\alpha+1}\rangle = G_\alpha$. Hence, no nontrivial characteristic subgroup of $Q_{\alpha+1}$ is normal in F. Now (2.4) gives $[O_2(E_\alpha), O^2(F)] \leq Z_\alpha$. This implies that $O_2(E_\alpha) \leq Z_\alpha$ and then $O_2(E_\alpha) = Z_\alpha$ since by (7.3)(d), $Z(G_\alpha) = 1$. However, now again $Z(G_\alpha) = 1$ yields $Q_\alpha = Z_\alpha = O_2(E_\alpha)$.

An application of (2.2) and (8.1) to Z_α and $\overline{E}_\alpha \overline{J(Q_{\alpha+1})}$ resp. $\overline{E}_\alpha \overline{J(S)}$ shows that $J(S) = J(Q_{\alpha+1}) = B$, but now $B \leq Q_{\alpha+1}$, which contradicts (6.1).

(8.4) *Let $Z = C_{Z_\alpha}(J(Z_\alpha, \overline{S}))$. Suppose that $Z_{\alpha+1} \leq Z(G_{\alpha+1})$. Then Z is normal in $G_{\alpha+1}$.*

Proof. Let $S_1 = \langle Z_{\alpha'}^{G_{\alpha+1}}\rangle$ and $V^*_{\alpha+1} = \langle Z^{G_{\alpha+1}}\rangle$. We may assume that $V^*_{\alpha+1} \neq Z$. Note that $Z(G_\alpha) = 1$ by (7.3)(d) and $\alpha' \in \alpha^G$ by (7.4); in particular, b is even. We now apply (8.1). Then $\overline{S}_1 = J(Z_\alpha, \overline{S})$, and with the notation given in (8.1):

(1) $\overline{E}_\alpha \overline{S}_1 = \overline{E}_1 \times \cdots \times \overline{E}_r$, $\overline{E}_i \cong SL_2(2)$,

(2) $Z_\alpha = V_1 \times \cdots \times V_r$, $V_i = [V, E_i]$, and $|V_i| = 4$,

(3) $C_{Z_\alpha}(S_1) = Z$.

Since S_1 is a normal subgroup of $G_{\alpha+1}$ in $Q_{\alpha+1}$ we get from (2) and (3) that $[V_{\alpha+1}, S_1] = V^*_{\alpha+1}$ and $[V^*_{\alpha+1}, S_1] = 1$; in particular,

(4) $V^*_{\alpha+1} \leq Q_{\alpha'}$ and $V^*_{\alpha+1} \cap Z_\alpha = Z$.

According to (1) and (2) we may assume that $[V_1, Z_{\alpha'}] \neq 1$; i.e., $V_1 \nleq Q_{\alpha'}$. We apply (7.8) with $\delta = \alpha'$, $\lambda = \alpha' - 1$, and $A = V_1$:

(5) There exists $x \in G_{\alpha'}$ and $L_0 \leq G_{\alpha'}$ such that for $\alpha' + 1 = (\alpha' - 1)^x$:

(i) $x \in L_0$ and $L_0 = \langle V_1, V_1^x \rangle$ and

(ii) $\langle L_0, G_{\alpha'} \cap G_{\alpha'+1} \rangle = G_{\alpha'}$.

Let $L = L_0 Q_{\alpha'}$ and $Q = O_2(O^2(L))$. Note that $\langle Q^h \mid h \in G_{\alpha'} \cap G_{\alpha'+1} \rangle = O_2(E_{\alpha'})$ and that by (8.3), $O_2(E_{\alpha'}) \nleq Q_{\alpha'+1}$. It follows that $Q \nleq Q_{\alpha'+1}$. We now apply (7.8) with $\delta = \alpha' + 1$ and $\lambda = \alpha'$:

(6) There exists $y \in G_{\alpha'+1}$ and $\tilde{L} \leq G_{\alpha'+1}$ such that for $\alpha' + 2 = \alpha'^y$:

(i) $y \in \tilde{L}$ and $\langle \tilde{L}, G_{\alpha'+1} \cap G_{\alpha'+2} \rangle = G_{\alpha'+1}$ and

(ii) either $\tilde{L} = \langle V^*_{\alpha+1}, Q_{\alpha'+2} \rangle$ and $|V^*_{\alpha+1}/V^*_{\alpha+1} \cap G_{\alpha'+2}| = 2$ or $\tilde{L} = \langle Q, Q_{\alpha'+2} \rangle$ and $V^*_{\alpha+1} \leq G_{\alpha'+2}$.

Define $Z_{\alpha,\alpha+1} = Z$ and $Z_{\delta,\lambda} = Z^x_{\alpha,\alpha+1}$, where $(\alpha, \alpha+1)^x = (\delta, \lambda)$ and $V^*_\delta = \langle Z_{\lambda,\delta} \mid \lambda \in \Delta(\delta) \rangle$ for $\delta \in (\alpha+1)^G$. Let $D \leq Z_{\alpha'+2,\alpha'+1}$ such that $[D, V_1] \leq Z_{\alpha'}$. Note that $D \leq V^*_{\alpha'+1}$ and so $[D, V_1^x] = 1$. It follows from (5) that $D \leq V^*_{\alpha'-1} Z_{\alpha'} \cap V^*_{\alpha'+1} Z_{\alpha'}$ and $[D, V^*_{\alpha+1}] = 1$. Let $D^* = \langle D^{Q_{\alpha'}} \rangle$. Then also $D^* \leq V^*_{\alpha'-1} Z_{\alpha'} \cap V^*_{\alpha'+1} Z_{\alpha'}$ and $[D^*, O^2(L)] \leq Z_{\alpha'}$. This implies that $[D^*, Q] \leq V^*_{\alpha'+1} \cap Z_{\alpha'} = Z_{\alpha',\alpha'+1}$ by (4). On the other hand, $D^* Z_{\alpha'}$ and thus also $[D^*, Q]$ is normal in L. Now $[D^*, Q] \leq V^*_{\alpha'+1}$ and $[V^*_{\alpha'+1}, V_1^x] = 1$ imply that $[D^*, Q, O^2(L)] = 1$, and the 3-subgroup lemma gives $[D^*, Q] = 1$. Hence by (4) and (6) $[D, \tilde{L}] = 1$, and (6.4) and (6) yield $D \leq Z_{\alpha'+1}$. We have shown:

(7) If $D \leq Z_{\alpha'+2,\alpha'+1}$ and $[D, V_1] \leq Z_{\alpha'}$, then $D \leq Z_{\alpha'+1}$.

Assume that $b = 2$. Then, according to (1) and (2), there exists $\mu \in \Delta(\alpha')$ such that $E_{\alpha'} \leq \langle V_{\alpha+1}, V_\mu \rangle$. Since by (4), $[V^*_{\alpha+1}, V^*_\mu] \leq Z(V_{\alpha+1}) \cap Z(V_\mu)$ we get that $[V^*_{\alpha+1}, V^*_\mu, E_{\alpha'}] = 1$, and $Z(G_{\alpha'}) = 1$ implies that $[V^*_{\alpha+1}, V^*_\mu] = 1$. It follows that $V^*_\mu \leq Q_{\alpha+1}$ since $V^*_{\alpha+1} \neq Z$, and thus by (1) and (2), $[V^*_\mu, V_1] \leq Z_{\alpha'}$. Hence, $V^*_\mu Z_{\alpha'}$ is normal in $G_{\alpha'}$. Now, as above for (7), $[V^*_\mu, O_2(E_{\alpha'})] = 1$ which contradicts (8.3). We have shown:

(8) $b > 2$.

In particular, (8) implies that $[Z_{\alpha'}, Z_{\alpha'+2}] = 1$ and thus $[V_1, Z_{\alpha'}, Z_{\alpha'+2}] = 1$. Hence, by (1) and (2), $Z_{\alpha'+2} \cap G_\alpha$ normalizes V_1. It follows that $[Z_{\alpha'+2,\,\alpha'+1} \cap G_\alpha, V_1] \leq Z_{\alpha'}$, and (7) yields

(9) $Z_{\alpha'+2,\,\alpha'+1} \cap G_\alpha = Z_{\alpha'+1}$; in particular, $Z_{\alpha'+2} \not\leq G_\alpha$.

Let $R_2 = [Z_{\alpha+2}, Z_{\alpha'+2}]$. Then $R_2 \leq Z_{\alpha'+2,\,\alpha'+1} \cap G_\alpha$ by (1), (2), (3), and (8). Hence, (9) gives $R_2 \leq Z_{\alpha'+1}$, and by (5), R_2 centralizes $\langle G_{\alpha'+1}, L\rangle = \langle G_{\alpha'+1}, G_{\alpha'}\rangle$. Thus $R_2 = 1$. We conclue that $Z_{\alpha'+2} \leq Q_{\alpha+2} \leq G_{\alpha+1}$. On the other hand, by (9), $Z_{\alpha'+2} \not\leq Q_{\alpha+1}$ and thus $[V^*_{\alpha+1}, Z_{\alpha'+2}] \neq 1$.

Let $R_1 = [V^*_{\alpha+1} \cap G_{\alpha'+2}, Z_{\alpha'+2}] \cap Z_{\alpha'+2,\,\alpha'+1}$. As for R_2 we get that $R_1 = 1$. Now (1), (2), and (3) imply that $[V^*_{\alpha+1} \cap G_{\alpha'+2}, Z_{\alpha'+2}] = 1$. Since $[V^*_{\alpha+1}, Z_{\alpha'+2}] \neq 1$ we conclude that $V^*_{\alpha+1} \not\leq G_{\alpha'+2}$ and by (6), $|V^*_{\alpha+1}/C_{V^*_{\alpha+1}}(Z_{\alpha'+2})| = 2$. Now (1.2) gives $|Z_{\alpha'+2}/Z_{\alpha'+2} \cap Q_{\alpha+1}| = 2$. Together with (9) we get $|Z_{\alpha'+2,\,\alpha'+1}| = 4$, and (1), (2), and (3) imply $|Z_\alpha| = 4^2$ and $\overline{G}_\alpha \cong SL_2(2) \wr C_2$.

Assume that $[V^*_{\alpha+1}, Z_{\alpha'+2} \cap Q_{\alpha+1}] = 1$. Then by (6), $Z_{\alpha'} \cap Z_{\alpha'+2} = Z_{\alpha'+2} \cap Q_{\alpha+1}$. It follows that $Z_{\alpha'+2,\,\alpha'+1} \leq [Z_{\alpha'+2}, Q_{\alpha'+1}] \leq Z_{\alpha'+2} \cap Q_{\alpha+1}$ since $Q_{\alpha'+1}$ normalizes $Z_{\alpha'} \cap Z_{\alpha'+2}$, but now (9) yields $Z_{\alpha'+2,\,\alpha'+1} = Z_{\alpha'+1} = V^*_{\alpha'+1}$, a contradiction.

Assume that $[V^*_{\alpha+1}, Z_{\alpha'+2} \cap Q_{\alpha+1}] \neq 1$. Then $S = Q_\alpha Q_{\alpha+1}$. Let $\lambda \in \Delta(\alpha+1)$ such that $Z_{\lambda,\,\alpha+1} \not\leq G_{\alpha'+2}$ and let $U = C_{Z_{\alpha'+2} \cap Q_{\alpha+1}}(Z_{\lambda,\,\alpha+1})$. Then $4 \leq |U|$ and by (6), $U \leq Z_{\alpha'} \cap Z_{\alpha'+2}$. However, now $Z_{\alpha'+2,\,\alpha'+1} \leq [Z_{\alpha'} \cap Z_{\alpha'+2}, Q_{\alpha'+1}]$ or $Z_{\alpha'} \cap Z_{\alpha'+2} = Z_{\alpha'+2,\,\alpha'+1}$. Hence, in both cases $Z_{\alpha'+2,\,\alpha'+1} \leq Z_{\alpha'} \cap Z_{\alpha'+2} \leq Z_{\alpha'+2} \cap Q_{\alpha+1}$ and $Z_{\alpha'+2,\,\alpha'+1} = Z_{\alpha'+1}$, a contradiction

(8.5) *Suppose that* $Z_{\alpha+1} \leq Z(G_{\alpha+1})$. *Then* $G_\alpha/Q_\alpha \cong SL_2(2)$ *and* $b = 2$.

Proof. Let $Z = C_{Z_\alpha}(J(Z_\alpha, \overline{S}))$. By (7.3)(d) and (8.1), $|Z_\alpha| = |Z|^2$, and (8.3) gives $Q_\alpha \not\leq Q_{\alpha+1}$. Hence, (8.4) yields $[Z, O^2(G_{\alpha+1})] = 1$, and (6.4) gives $Z = Z_{\alpha+1}$. From (8.1) we get that $|Z_\alpha| = 4$ and $\overline{G}_\alpha \cong SL_2(2)$; in particular, by (3.3), $G_\alpha/Q_\alpha \cong D_{2\cdot 3^n}$. Now (7.7)(a) shows that either $G_\alpha/Q_\alpha \cong SL_2(2)$ or $O_2(E_{\alpha+1}) \leq Q_\alpha$. In the second case $Q_\alpha \cap Q_{\alpha+1}$ is normal in $G_{\alpha+1}$ which contradicts (7.6)(a). Thus, we have $G_\alpha/Q_\alpha \cong SL_2(2)$.

Assume that $b > 2$. Since $\overline{G}_\alpha \cong SL_2(2)$ we get from (8.1) that $R = Z = Z_{\alpha+1}$. The same argument in $G_{\alpha'}$ gives $R = Z_{\alpha'-1}$. Hence $\langle G_{\alpha+1}, G_{\alpha'-1}\rangle \leq C_H(R)$, but $E_{\alpha+1}$ is subnormal in $C_H(R)$ and so $O_2(E_{\alpha+1}) \leq O_2(C_H(R))$.

Let $\alpha - 1 \in \Delta(\alpha) \setminus \{\alpha+1\}$. Note that $V_{\alpha-1} \leq Q_{\alpha+1}$ since $b > 2$. Hence, conjugation in $G_{\alpha+1}$ shows that

$$V_{\alpha-1} \leq V_\mu O_2(E_{\alpha+1}) \leq V_\mu O_2(C_H(R))$$

for suitable $\mu \in \Delta(\alpha + 2)$. Since $d(\mu, \alpha' - 1) \leq b - 1$ we have $V_\mu \leq G_{\alpha'}$, and since $O_2(C_H(R)) \leq Q_{\alpha' - 1}$ we get $V_{\alpha-1} \leq G_{\alpha'}$. Now $[Z_{\alpha'}, V_{\alpha-1}] = R$ and $V_{\alpha-1}$ is normal in $G_{\alpha-1}$ and $\langle G_{\alpha-1} \cap G_\alpha, Z_{\alpha'} \rangle = G_\alpha$, a contradiction.

(8.6) *Suppose that* $Z_{\alpha+1} \leq Z(G_{\alpha+1})$. *Let* $\alpha - 1 \in \Delta(\alpha) \setminus \{\alpha + 1\}$, $D = Q_{\alpha-1} \cap Q_{\alpha+1}$, $L = \langle Q_{\alpha-1}^{G_\alpha} \rangle$, $Q = O_2(L)$, *and* $T \in \mathrm{Syl}_3(G_\alpha)$. *Then* $[D, L] = Z_\alpha$, Q/D *and* D *are elementary abelian and one of the following holds*:

(a) $2^5 \leq |S| \leq 2^6$, $G_\delta/Q_\delta \cong SL_2(2)$ *for every* $\delta \in \Gamma$, *and*

(a_1) $Q = Q_\alpha$, $O_2(E_\alpha) \cong C_4 \times C_4$, *and* $Q = O_2(E_\alpha)\langle t \rangle$, *where either* $t = 1$ *or* t *inverts the elements in* $O_2(E_\alpha)$,

(a_2) $Q_{\alpha+1} \cong C_4 \curlyvee Q_8$ *or* $Q_8 \curlyvee Q_8$, *and* $Q_{\alpha+1} \cap E_{\alpha+1} \cong Q_8$.

(b) $2^8 \leq |S| \leq 2^{10}$, $G_\alpha/Q_\alpha \cong SL_2(2)$, *and* $G_{\alpha+1}/Q_{\alpha+1} \cong SL_2(2) \wr C_2$, *and*

(b_1) $|Q/O_2(E_\alpha)| \leq 4$, *and* $O_2(E_\alpha)$ *is special of order* 2^6 *with* $C_{O_2(E_\alpha)}(T) = 1$.

(b_2) $V_{\alpha+1} \cong Q_8 \curlyvee Q_8$ *and* $\Phi(Q_{\alpha+1}) = Z_{\alpha+1}$,

(b_3) *there exists an elementary abelian normal subgroup* W *of order* 2^4 *in* L *such that* $N_G(W)$ *is nonsolvable*.

(c) $2^{14} \leq |S| \leq 2^{15}$, $G_\alpha/Q_\alpha \cong SL_2(2)$, *and* $E_{\alpha+1}/O_2(E_{\alpha+1})$ *is elementary abelian of order* 3^4, *and*

(c_1) $|Q/D| = 2^6$, $|D| = 2^5$, $|Z_\alpha| = 4$, *and* $D = C_Q(T) \times Z_\alpha$,

(c_2) $Q_{\alpha+1}$ *is extra special of order* 2^9 *and* $Q/Q \cap Q_{\alpha+1}$ *is elementary abelian of order* 2^3.

(c_3) *Every* Q-*invariant subgroup of order* 3 *in* $E_{\alpha+1}/O_2(E_{\alpha+1})$ *operates fixed-point-freely on* $Q_{\alpha+1}/Z_{\alpha+1}$, *and every involution in* $QQ_{\alpha+1}/Q_{\alpha+1}$ *centralizes a subgroup of order* 2^5 *in* $Q_{\alpha+1}$;

(c_4) *there exists* $\lambda \in \Delta(\alpha + 1) \setminus \{\alpha\}$ *such that* $N_G(Z_\lambda Z_\alpha)/C_G(Z_\lambda Z_\alpha) \cong L_3(2)$.

Proof. We apply (8.1) and (8.5). Then $b = 2$, $G_\alpha/Q_\alpha \cong SL_2(2)$ and $|Z_\alpha| = 4$; in particular, $Z(G_\alpha) = 1$. We choose the following additional notation:

$$A = V_{\alpha-1} \cap Q_\alpha, \qquad \tilde{A} = V_{\alpha+1} \cap Q_\alpha, \qquad V = V_{\alpha+1}/Z_{\alpha+1}.$$

Note that $E_\alpha \leq L$ and that D is normal in G_α. Note further that $L \cap S = V_{\alpha+1}Q$ since $V_{\alpha+1} \not\leq Q_\alpha$ and that $[V_{\alpha+1}, Q_{\alpha+1}] = Z_{\alpha+1}$. It follows that

(1) $[D, O^2(L)] = Z_\alpha$, $Q = A\tilde{A}D$, $[Q, Q_{\alpha+1}]D = \tilde{A}D$, and $\Phi(Q) \leq D$.

Let $D_0 \le C_D(V_{\alpha+1})$ such that $[D_0, S] \le Z_{\alpha+1}$. Then by (1), $D_0 Z_\alpha$ is normal in G_α. On the other hand $[Q, D_0 Z_\alpha] \le Z_{\alpha+1}$ and so $D_0 \le Z(Q)$. Now $Z(G_\alpha) = 1$ yields $D_0 \le Z_\alpha$ and then $D_0 \le Z_{\alpha+1}$. Moreover, $|V_{\alpha+1}/\tilde{A}| = 2$ and $|[C_D(\tilde{A}), V_{\alpha+1}]| \le 2$ also gives $C_D(\tilde{A}) = Z_\alpha$. We have shown:

(2) $C_D(V_{\alpha+1}) = Z_{\alpha+1}$ and $C_D(\tilde{A}) = Z_\alpha = C_D(A)$.

Since $[Q_{\alpha+1}, V_{\alpha+1}] = Z_{\alpha+1}$ we have $[\Phi(D), V_{\alpha+1}] = 1$. Now (2) gives $\Phi(D) \le Z_{\alpha+1}$. Since $\Phi(D)$ is normal in G_α we get

(3) $\Phi(D) = 1$.

An easy consequence of (1) is

(4) If $x \in Q_{\alpha+1} \setminus Q$ and $U \le Q_{\alpha-1} \cap Q_\alpha$ such that $[U, x] \le D$, then $U \le D$.

Assume that $A \le D$. Then by (1), $O_2(E_\alpha) = Z_\alpha$ and $Z(G_\alpha) = 1$ implies $Q_\alpha = Z_\alpha$. Now $b = 2$ gives $Q_\alpha \le Q_{\alpha+1}$ and $Q_{\alpha+1} = S$, a contradiction.

Assume that there exists $A \cap D \le A_0 \le A$ such that $A_0 \not\le D$ and A_0 operates quadratically on V. Then by (1.2) there exists $x \in V_{\alpha+1} \setminus Q_\alpha$ and $A_1 \le A_0$ such that $[A_1, x] \le Z_{\alpha+1}$ and $|A_0/A_1| = 2$. Now (4) implies that $A_1 \le D$. We have shown:

(5) $A \not\le Q_{\alpha+1}$, and no noncyclic subgroup of $AQ_{\alpha+1}/Q_{\alpha+1}$ operates quadratically on V.

Suppose that $|A/A \cap D| = 2$. Then by (1), $|Q/D| = 4$. Moreover, since $[D, A] \le Z_{\alpha-1}$ and D is elementary abelian we get that $|D/D \cap Z(Q)| \le 2$. Now $Z(G_\alpha) = 1$ gives $D \cap Z(Q) = Z_\alpha$ and $|D/Z_\alpha| \le 2$.

Note that $S = Q_\alpha Q_{\alpha+1}$ and so $Q_\alpha/Q_\alpha \cap Q_{\alpha+1} \cong S/Q_{\alpha+1} = \bar{S}$. Since $|\bar{A}| = 2$ we get $\bar{A} \le Z(\bar{S})$. On the other hand,

$$Q_\alpha = C_{Q_\alpha}(T) A(Q_\alpha \cap Q_{\alpha+1}) \qquad \text{for } T \in \mathrm{Syl}_3(G_\alpha)$$

and $\overline{G_{Q_\alpha}(T)} \cap \bar{A} = 1$. By (3.4), $Z(\bar{S})$ is cyclic and thus $Q_\alpha = Q$, $|S/Q_{\alpha+1}| = 2$, and $|S| = 2^5$, if $D = Z_\alpha$, and $|S| = 2^6$, if $|D/Z_\alpha| = 2$. Assume that $|V_{\alpha+1}| \le 2^4$. Then $G_{\alpha+1}/Q_{\alpha+1} \cong SL_2(2)$ and $V_{\alpha+1} \cong C_4 \curlyvee Q_8$, and (a) is easy to check since $V_{\alpha+1} \cap Q_\alpha \cong C_2 \times C_4$.

Assume that $|V_{\alpha+1}| = 2^5$. Then $V_{\alpha+1} \cong Q_8 \curlyvee Q_8$ since D is elementary abelian of order 2^3 in $V_{\alpha+1}$. In particular 5 does not divide $|G_{\alpha+1}|$ and so $G_{\alpha+1}/Q_{\alpha+1} \cong SL_2(2)$. However, now there are only three conjugates of Z_α in $V_{\alpha+1}$ and $|V_{\alpha+1}| \le 2^4$, a contradiction. We have shown:

(6) If $|A/A \cap D| = 2$, then (a) holds.

From now on we assume

(7) $|A/A \cap D| \ge 4$.

Choose $a \in A \setminus Q_{\alpha+1}$ such that $\|[V, a]\|$ is minimal. Then by (7.8) and (3.6) there exists $y \in G_{\alpha+1}$ and $E = \langle A, A^y \rangle$ such that for $\lambda = \alpha^y$:

(i) $y \in E$,

(ii) $|A/A \cap G_\lambda| = 2$ and $a \notin A \cap G_\lambda$,

(iii) $\langle E, G_\alpha \cap G_{\alpha+1} \rangle = G_{\alpha+1}$,

(iv) $[A \cap G_\lambda, E] \leq Q_{\alpha+1}$.

Let $V_1 = \langle Z_\alpha^E \rangle$, $V_0 = C_{Q_{\alpha+1}}(O^2(E))$, and $Y = [Q_{\alpha+1}, O^2(E)]$. Note that by (1), $Q_{\alpha+1} = V_{\alpha+1} D$ and that $[D, A] \leq Z_{\alpha-1} \leq V_{\alpha+1}$. Hence (5) gives

(8) $Y \leq [Q_{\alpha+1}, E_{\alpha+1}] \leq V_{\alpha+1}$.

Suppose that $V_1 \not\leq Q_\alpha$. Then we may assume that $Z_\lambda \not\leq Q_\alpha$. Since $[A \cap G_\lambda, Z_\lambda] \leq Z_{\alpha+1} \leq D$ we get from (4) that $A \cap G_\lambda \leq D$. Hence $|A/A \cap D| = 2$, which contradicts (7). We have shown:

(9) $V_1 \leq Q_\alpha$.

Note that $[V_0 \cap D, A] \leq Z_{\alpha-1} \cap V_0$. Since by (iii), Z_α is not normal in E we get that $Z_{\alpha-1} \not\leq V_0$ and $[V_0 \cap D, A] = 1$. Now (1) and (3) yield $V_0 \cap D \leq Z(Q)$ since $(V_0 \cap D)Z_\alpha$ is normal in L, and $Z(G_\alpha) = 1$ gives $V_0 \cap D \leq Z_\alpha$. Hence, $Z_{\alpha-1} \not\leq V_0$ implies:

(10) $V_0 \cap D = Z_{\alpha+1}$.

Since $[V_0 \cap Q_\alpha, A] \leq V_0 \cap D$ we get from (10), $[V_0 \cap Q_\alpha, A] \leq Z_{\alpha+1}$, and by (iv), $[V_1, A \cap G_\lambda] \leq Z_{\alpha+1}$. Thus, we have

(11) $[V_1(V_0 \cap Q_\alpha), A \cap G_\lambda] \leq Z_{\alpha+1}$ and $[V_0 \cap Q_\alpha, A] \leq Z_{\alpha+1}$.

From (11) we get that A acts as a cyclic group of order 2 on $V_1/Z_{\alpha+1}$; i.e., $[V_1, A, A] \leq Z_{\alpha+1}$. On the other hand, by (9), $[V_1, A, A] \leq Z_{\alpha-1}$. We conclude that

$$[V_1, A, A] \leq Z_{\alpha-1} \cap Z_{\alpha+1} = 1.$$

Hence (2) yields $[V_1, A]Z_{\alpha+1} = Z_\alpha$. This gives

(12) $|V_1| = 2^3$ and $V_1 \leq Y$; in particular, $E/C_E(V_1) \cong SL_2(2)$.

Suppose that a induces a transvection on $V/C_V(O^2(E))$. Then $Y = V_1$ and $V_{\alpha+1} = V_1(V_0 \cap V_{\alpha+1})$; in particular, by (9), $V_0 \cap V_{\alpha+1} \not\leq Q_\alpha$. Hence (11) implies that $[V_{\alpha+1} \cap Q_\alpha,\ A \cap G_\lambda] \leq Z_{\alpha+1}$, and by (7),

$$|V/C_V(A \cap G_\lambda)| = 2.$$

Now the minimality of $[V, a]$ shows that $\|[V, a]\| = 2$. Since $[Y, a] \not\leq Z_{\alpha+1}$ we get $[V_0 \cap V_{\alpha+1}, a] \leq Z_{\alpha+1}$ which contradicts (4) since $a \notin D$. We have shown:

(13) a does not induce a transvection on $V/C_V(O^2(E))$.

Assume that $V_0 \not\leq Q_\alpha$. Then by (1), $[V_0, Q] \leq [V_0, A]D \leq V_0 D$ and thus $Q_{\alpha+1} = V_0 D$ and $[V_{\alpha+1}, A] \leq Z_{\alpha-1} V_0$. This contradicts (13). Hence we have:

(14) $V_0 \leq Q_\alpha$ and $Y \not\leq Q_\alpha$.

Suppose that $|V/C_V(a)| = 4$. From (12) and (13) we get

$$|Y/V_1| = |V_1/Z_{\alpha+1}| = 4;$$

in particular, $|Y| = 2^5$ and $E/O_2(E) \cong SL_2(2)$. Moreover, by (11) either $[Y, A \cap G_\lambda] \leq Z_{\alpha+1}$ or $[Y, A \cap G_\lambda]Z_{\alpha+1} = V_1$. The first case contradicts (4) and (7) since $Y \not\leq Q_\alpha$. Thus, we have $[Y, A \cap G_\lambda]Z_{\alpha+1} = V_1$. Now (11) and (14) show that $A \cap G_\lambda$ is quadratic on V, and (5) yields that $|A \cap G_\lambda/D| = 2$ and $|A/A \cap D| = 4$.

Again by (11) and (14), $|V/C_V(A)| = 8$ and $|V/C_V(x)| = 4$ for $x \in A \setminus Q_{\alpha+1}$. Hence (1.6) shows that $G_{\alpha+1}/Q_{\alpha+1} \cong SL_2(2) \wr C_2$ and $V_{\alpha+1} = Y$; in particular, $V_{\alpha+1} \cong Q_8 \curlyvee Q_8$.

Since $A\tilde{A}$ is normal in G_α we get that $A\tilde{A} = O_2(E_\alpha)$, $|O_2(E_\alpha)| = 2^6$, and $C_{A\tilde{A}}(T) = 1$ for $T \in \mathrm{Syl}_3(G_\alpha)$. By (14), $V_0 \leq Q_\alpha$, and by (1) and (10), $V_0 \cap Q_{\alpha-1} = Z_{\alpha+1}$ and $V_0/Z_{\alpha+1}$ is elementary abelian. Since $V_0/Z_{\alpha+1}$ is isomorphic to a subgroup of D_8 we conclude that $|V_0/Z_{\alpha+1}| \leq |A/A \cap D| = 4$. To prove (b) it remains to prove (b$_3$).

Let $W = [V_1, E_\alpha]$. Since $|V_1/Z_\alpha| = 2$ and $C_W(T) = 1$ we get that W is elementary abelian of order 2^4. In addition, $E_\alpha/C_{E_\alpha}(W) \cong \Sigma_4$.

Since $[V_1, W] = 1$ we have that $[W, E] \leq V_{\alpha+1}$ and $\langle W^E \rangle \cap V_{\alpha+1} \leq C_{V_{\alpha+1}}(V_1) = V_1$. Hence $E \leq N_G(W)$ and $E/C_E(W) \cong \Sigma_4$. Now E fixes $Z_{\alpha+1}$ in W while E_α operates fixed-point-freely on W, and (b$_3$) follows. We may assume now:

(15) $|V/C_V(x)| \geq 8$ for every $x \in A \setminus Q_{\alpha+1}$.

Assume that $V_1 \not\leq D$. Let $v \in V_1 \setminus Q_{\alpha-1}$. By (12), $[v, A] \leq Z_\alpha$ and thus $|[v, V_{\alpha-1}]Z_{\alpha-1}/Z_{\alpha-1}| \leq 4$. This contradicts (15). Hence we have:

(16) $V_1 \leq D$.

In particular, by (1), V_1 is normal in L. Now (c$_4$) is easy to check.

Let $V_2 = \langle (Y \cap D)^E \rangle$. Then $[V_2, A \cap G_\lambda] \leq V_1 \leq D$, and by (4) and (7), $V_2 \leq Q_\alpha$. Moreover $[Y \cap Q_\alpha, A] \leq Y \cap D \leq V_2$, and A induces transvections on Y/V_2. We conclude that $|Y/V_2| = |V_1/Z_{\alpha+1}| = 4$.

Let $\overline{Y} = YZ_{\alpha+1}/Z_{\alpha+1}$ and $b \in (A \cap G_\lambda) \setminus Q_{\alpha+1}$. Since $\overline{Y}/C_{\overline{Y}}(b)$ is E-invariant we get from (5) that $|\overline{Y}/C_{\overline{Y}}(b)| \geq 2^4$. On the other hand,

$\|[\bar{Y}, b]/[\bar{V}_2, b]\| \leq 4$ and $[\bar{V}_2, b] \leq \bar{V}_1$. It follows that $\|[\bar{Y}, b]\| \leq 2^4$. Now $|\bar{Y}/C_{\bar{Y}}(b)| = \|[\bar{Y}, b]\|$ implies

$$2^4 = |\bar{Y}/C_{\bar{Y}}(b)| = \|[\bar{Y}, b]\|$$

and $\|[\bar{V}_2, b]\| = 4$; in particular, $|\bar{Y}| \geq 2^8$.

From (11) we get that $[\bar{Y}, b] = [V, b]$ since by (14), $V_0 \leq Q_\alpha$ and by (10), $V_0 \cap D = Z_{\alpha+1}$. Hence, the minimality of $[V, a]$ and the fixed-point-free action of $O^2(E)$ on Y give

(17) $|Y| = 2^9$ and $\|[V, a]\| = \|[Y, a]\| = 2^4$.

By (10), $V_0 \cap D = Z_{\alpha+1}$. Hence $[A, V_0] \leq Z_{\alpha+1}$ and $\|[V_{\alpha-1}, w] Z_{\alpha-1}/Z_{\alpha-1}| \leq 4$ for every $w \in V_0$. Now (15), applied to $V_{\alpha-1}/Z_{\alpha-1}$, yields $V_0 \leq D$ and thus $V_0 = Z_{\alpha+1}$. We have shown:

(18) $Q_{\alpha+1} = V_{\alpha+1} = Y$.

Now (7.3)(b) shows that $V_{\alpha+1}$ is extra special. Since $[V_2, A] \leq D$ and $[D, A] \leq V_1$ the action of E also gives $|V_{\alpha+1} \cap D| = 2^5$ and $|V_{\alpha+1}/D| = 2^4$; in particular, $|\bar{A}| = 2^3$. Moreover, as seen before,

(19) $|V/C_V(x)| = |C_V(x)| = 2^4$ for every $x \in A \setminus Q_{\alpha+1}$.

Since $C_{\bar{E}_{\alpha+1}}(x)$ operates faithfully on $C_V(x)$, (19) and (12) imply

(20) $C_{\bar{E}_{\alpha+1}}(x) \cong U_x \leq C_3 \times C_3$ for every $x \in A \setminus Q_{\alpha+1}$.

Note that $|\bar{A}| = 2^3$. Let $\bar{A}_1, \ldots, \bar{A}_n$ be the subgroups of index 2 in $\bar{A}$ such that $\bar{E}_i = C_{\bar{E}_{\alpha+1}}(\bar{A}_i) \neq 1$. Then

$$\bar{E}_{\alpha+1} = \langle \bar{E}_i \mid i = 1, \ldots, n \rangle$$

and (20) implies that

(21) $\bar{E}_{\alpha+1}$ is elementary abelian and $|\bar{E}_i| = 3$.

In particular, there exists $z \in \bar{A}$ which inverts $\bar{E}_{\alpha+1}$. Since z is in 3 subgroups of index 2 in $\bar{A}$ we have $n \leq 4$. In addition, an easy argument shows that $\bar{E}_{\alpha+1} = \bar{E}_1 \times \cdots \times \bar{E}_n$. Hence $\bar{S}$ is transitive on $\{\bar{E}_1, \ldots, \bar{E}_n\}$ and $n = 2$ or 4. The first case contradicts $|\bar{A}| = 2^3$, and (c) is proven.

DEFINITION. Let H be a finite group. Then H is of *type* $G_2(2)'$, $\Omega_6^-(3)$, and $\Omega_8^+(3)$, respectively, if H contains two subgroups G_α and $G_{\alpha+1}$ such that $G_\alpha \cap G_{\alpha+1} = S \in \mathrm{Syl}_2(H)$ and $O_2(\langle G_\alpha, G_{\alpha+1} \rangle) = 1$, and G_α and $G_{\alpha+1}$ satisfy (8.6)(a), (b), and (c), respectively.

Note that $G_2(2)$, $\mathrm{Aut}(\Omega_6^-(3))$, and $\mathrm{Aut}(\Omega_8^+(3))$ provide examples for such groups H.

9.

In this section we assume Hypothesis 2 and $G = \langle P_1, P_2 \rangle$, and we use the notation concerning $\Gamma = \Gamma(G, P_1, P_2)$ as introduced in Section 7. The additional hypothesis for this section is $[Z_\alpha, Z_{\alpha'}] = 1$. According to (7.5) this gives $Z_{\alpha'} = \Omega_1(Z(T)) \le Z(G_{\alpha'})$ for $T \in \mathrm{Syl}_2(G_{\alpha'})$ and $Z(G_\alpha) = 1$. Hence, as mentioned in the remark after (7.5), it follows that $E_{\alpha'}$ is subnormal in $C_H(Z_{\alpha'})$.

(9.1) *Suppose that* $b = 1$. *Let* $V_{\alpha'}^* = \langle (Z_\alpha \cap Q_{\alpha'})^{G_{\alpha'}} \rangle$. *Then the following hold*:

(a) $G_\alpha / Q_\alpha \cong SL_2(2) \wr C_2$ *and* $G_{\alpha'}/Q_{\alpha'} \cong SL_2(2)$.

(b) $|S| = 2^7$, $Q_\alpha = Z_\alpha$, *and* $V_{\alpha'}^* \cong Q_8 \curlyvee Q_8$.

(c) *There exists* $U \le V_{\alpha'}^*$ *such that* $|U| = 2^3$ *and* $N_H(U)/U \cong L_3(2)$.

Proof. Since $b = 1$ we have $\alpha' = \alpha + 1$ and $Z_\alpha \not\le Q_{\alpha+1}$. We apply (7.8) with $\delta = \alpha + 1$, $\lambda = \alpha$, and $A = Z_\alpha$. Then there exists $x \in G_{\alpha+1}$ such that for $E = \langle Z_\alpha, Z_\alpha^x \rangle$ and $\alpha + 2 = \alpha^x$:

(i) $x \in E$,

(ii) $|Z_\alpha / Z_\alpha \cap G_{\alpha+2}| = 2$,

(iii) $\langle E, G_{\alpha+1} \cap G_\alpha \rangle = G_{\alpha+1}$,

(iv) $[Z_\alpha \cap G_{\alpha+2}, E] \le Q_{\alpha+1}$,

(v) $E = \langle a, Z_{\alpha+2} \rangle$ for every $a \in Z_\alpha \setminus G_{\alpha+2}$.

Let $V = (Z_\alpha \cap G_{\alpha+2})(Z_{\alpha+2} \cap G_\alpha)$ and $V_0 = Z_\alpha \cap Z_{\alpha+2}$. Then

(1) $V_0 \le Z(E)$ and $[VQ_{\alpha+1}, E] \le V$, and

(2) $V \cap Q_\alpha = V \cap Z_\alpha = Z_\alpha \cap G_{\alpha+2} = C_V(a)$ for $a \in Z_\alpha \setminus G_{\alpha+2}$.

By (1), $[Q_{\alpha+1}, O^2(E)] \le C_{Q_{\alpha+1}}(Z(V))$ and by (7.5)(b), $\Omega_1(C_{Z(V)}(Q_{\alpha+1})) = Z_{\alpha+1}$. Hence $[C_{Z(V)}(Q_{\alpha+1}), O^2(E)] = 1$ and the $P \times Q$ lemma gives

(3) $[Z(V), O^2(E)] = 1$.

On the other hand, $|V/V_0| \le 4$ shows that V is abelian since V is generated by involutions. Hence, (3) gives

(4) $|V/V_0| > 4$; in particular, $|V/V \cap Q_\alpha| \ge 4$.

Let $\overline{G}_\alpha = G_\alpha / C_{G_\alpha}(Z_\alpha)$. Assume that X is a subgroup of V such that $\overline{X} \ne 1$ and $[Z_\alpha, X, X] = 1$. By (1.2) there exists $X_1 \le X$ with $|\overline{X}/\overline{X}_1| = 2$ and $C_{Z_\alpha}(X_1) \not\le Z_\alpha \cap V$. Now (2) and (v) imply that $X_1 \le V_0 \le Z_\alpha$ and $\overline{X}_1 = 1$. We have shown:

(5) No noncyclic subgroup of $\overline{V}$ operates quadratically on Z_α.

For $\bar{X} \leq \bar{V}$ define as in Section 1,

$$m(\bar{X}) = |Z_\alpha / C_{Z_\alpha}(X)|\,|\bar{X}|^{-1}.$$

Note that $m(\bar{V}) = 2$ by (ii) and (2). Suppose that there exists $1 \neq X \leq V$ such that $\bar{X} \neq 1$ and $m(\bar{X}) < m(\bar{V})$. By (1.5)(b) we may assume that $|\bar{X}| = 2$ and thus $|Z_\alpha / C_{Z_\alpha}(X)| = 2$. However, now $[Z_\alpha, X, V] = 1$ and by (3), $[Z_\alpha, X] \leq V_0$. Thus (v) yields $X \leq V_0$ and $\bar{X}_1 = 1$, a contradiction. We have shown:

(6) $m(\bar{Y}) \geq m(\bar{V})$ for every $1 \neq \bar{Y} \leq \bar{V}$; in particular, no element of $\bar{V}^{\#}$ induces a transvection on Z_α.

Let $F = [E_\alpha, V]$, $W = [Z_\alpha, F]$, and $Y_0 = C_{Z_\alpha}(F)$. Then (5) and (6) together with (1.6) imply:

(7) $\overline{FV} \cong SL_2(2) \times SL_2(2)$ and $|W| = 2^4$; in particular, $|V/V_0| = 2^4$ and $|W/W \cap V| = 2$.

Thus we have for $x \in V \setminus Q_\alpha$,

$$|W/C_W(x)| = |Z_\alpha / C_{Z_\alpha}(x)| = 4 \quad \text{and} \quad [W, x] = [Z_\alpha, x].$$

In particular $V \cap Z_\alpha = [Z_\alpha, x]V_0$. This gives $[Y_0, x] = 1$ and $Y_0 \leq V_0$ by (3). Assume that $Y_0 \neq 1$. Let $Y = Y_0 \cap Z(B)$. If $J(S) \leq Q_\alpha$, then $Y = Y_0 \neq 1$. If $J(S) \nleq Q_\alpha$, then (2.2) shows that $Y \neq 1$.

Now (6.4) and (v) give $Y = Z_{\alpha+1}$. Hence $\langle F, G_\alpha \cap G_{\alpha+1} \rangle = G_\alpha \leq C_H(Z_{\alpha+1})$, a contradiction. We have shown that $Y_0 = 1$ and thus:

(8) $|Z_\alpha| = 2^4$, $\bar{G}_\alpha \cong SL_2(2) \wr C_2$, and $Z_\alpha \cap G_{\alpha+2} = Z_\alpha \cap Q_{\alpha+1}$.

By (5) and (8), $V_0 = Z_{\alpha+1}$. Let $V_1 \leq Q_{\alpha+1}$ be maximal with $[V_1, Q_{\alpha+1}] = Z_{\alpha+1}$ and $[V_1, E_{\alpha+1}] = V_1$. Then by (1), $|V_1 \cap V| \geq 8$ and $V_1 \nleq Q_\alpha$. Since $[\bar{V}_1, \bar{V}] = 1$ and V_1 is normal in S we get from (8) and (3.4) that

(9) $V_1 \leq VQ_\alpha$, $1 \neq |V_1 / C_{V_1}(Z_\alpha)| \leq 4$, and $[E_\alpha, V_1] = E_\alpha$.

Again by (3.4), $E_{\alpha+1} = [E_{\alpha+1}, Z_\alpha]$. Note that $[Q_\alpha, Z_\alpha \cap V_1] = 1$ and $|Z_\alpha \cap V_1 / Z_{\alpha+1}| \leq 4$. Hence (1.3) yields that $|V_1 / Z_{\alpha+1}| \leq 2^4$ and $|Q_\alpha Q_{\alpha+1} / Q_{\alpha+1} Z_\alpha| \leq 2$ or that $|V_1/Z_{\alpha+1}| = 2^6$ and $E_{\alpha+1}/C_{E_{\alpha+1}}(V_1)$ is extra special of order 3^3.

Assume first that there is a noncentral chief factor of G_α in Q_α / Z_α. Then $[Q_\alpha, V_1] \nleq Z_\alpha$ and thus $|V_1/Z_{\alpha+1}| = 2^6$ and $|Q_\alpha \cap V_1 / Z_\alpha \cap V_1| = 4$. Hence (8) and (7.7)(a) imply:

(10) $|Q_\alpha / C| = 2^4$, where C is maximal in Q_α with $[C, E_\alpha] = Z_\alpha$, $|\bar{V}_1| = 4$, and $[Q_\alpha, V_1, V_1] \leq Z_\alpha$.

Now (10), (9), and (5) imply that Q_α/C and Z_α are nonisomorphic G_α/Q_α modules. Since $Z(G_\alpha) = 1$ this shows that $C = Z_\alpha$ and $|Q_\alpha| = 2^8$; in particular, $E_\alpha/Q_\alpha \cong C_3 \times C_3$ and $E_{\alpha+1}/Q_{\alpha+1}$ is extra special of order 3^3.

Let $D \in \mathrm{Syl}_3(G_\alpha)$ and let D^* be a subgroup of order 3 in D with $V \leq N_{G_\alpha}(D^*Q_\alpha)$ and $Q = C_{Q_\alpha}(D^*)$. Since D^* operates fixed-point-freely on Z_α we get $|Q| = 4$ and $Q \cap Z_\alpha = 1$. Hence QZ_α is elementary abelian. On the other hand, QZ_α is normal in $Q_\alpha V_1$ and thus $[V_1, QZ_\alpha, QZ_\alpha] = 1$. It follows that QZ_α centralizes $Z(E_{\alpha+1}/Q_{\alpha+1})$. However, then $Q \leq Z_\alpha Q_{\alpha+1}$ and $[Q, V_1] \leq Z_\alpha$ which contradicts the action of D^*V on Q_α/Z_α.

We have shown that $[Q_\alpha, E_\alpha] = Z_\alpha$. Now as above $Z(G_\alpha) = 1$ yields

(11) $Q_\alpha = Z_\alpha$.

In particular, we conclude from (8) that $|S| = 2^7$ and $G_\alpha/Q_\alpha \cong SL_2(2) \wr C_2$. Now $|V_1 \cap V| = 8$ and $V \cong Q_8 \curlyvee Q_8$ follow. In particular $G_{\alpha+1}/Q_{\alpha+1} \cong SL_2(2)$ and $V = \langle (Z_\alpha \cap Q_{\alpha+1})^{G_{\alpha+1}} \rangle$. It remains to prove (c).

The elements of order 3 in $E_{\alpha+1}$ operate fixed-point-freely on $V/Z_{\alpha+1}$. Hence, there exists an elementary abelian normal subgroup U of order 8 in $E_{\alpha+1}$ different from $V \cap V_1$ such that $[U, Z_\alpha] \leq U$. Clearly $N_{G_{\alpha+1}}(U)/U \cong \Sigma_4$. Since $U \nleq V_1 Z_\alpha$ we also have that $[E_\alpha, U] \neq E_\alpha$ and $N_{G_\alpha}(U)/U \cong \Sigma_4$. It follows that $N_H(U)/C_G(U) \cong L_3(2)$. Since $C_H(U) \leq C_H(Z_{\alpha+1})$ and $C_H(Z_{\alpha+1})$ is of characteristic 2 type, it is easy to see that $C_H(U) = U$.

DEFINITION. Let H be a finite group. Then H is of *type* $\Omega_6^+(2)$ if H contains two subgroups G_α and $G_{\alpha'}$ such that $G_\alpha \cap G_{\alpha'} = S \in \mathrm{Syl}_2(H)$ and $O_2(\langle G_\alpha, G_{\alpha'} \rangle) = 1$, and G_α and $G_{\alpha'}$ satisfy (9.1)(a)–(c).

Note that $\Omega_6^+(2)$ $[\cong \Sigma_8 \cong \mathrm{Aut}(L_4(2))]$ is a group of type $\Omega_6^+(2)$.

(9.2) *Let* $\delta \in (\alpha')^G$ *and* $\lambda, \mu \in \Delta(\delta)$. *Suppose that* $|Z_\lambda/Z_\lambda \cap Z_\mu| = 2$ *and* $\langle Q_\mu, G_\lambda \cap G_\delta \rangle = G_\delta$. *Then* $G_\lambda/Q_\lambda \cong SL_2(2)$ *and* $|Z_\lambda| = 4$.

Proof. After conjugation we may assume that $\delta = \alpha + 1$ and $\lambda = \alpha$. Let $L = \langle Q_\alpha, Q_\mu \rangle$ and $Q = Q_{\alpha+1} \cap Q_\mu$. By (7.6)(c), $Q \nleq Q_\alpha$. Thus, Q induces transvections on Z_α. We apply (1.7).

Let $X = [Z_\alpha, Q]$. Then $X \leq Z_\alpha \cap Z_\mu \cap Z(B)$. Since $\langle Q_\alpha, G_{\alpha+1} \cap G_\mu \rangle = G_{\alpha+1}$ we get from (6.4) that $X \leq Z_{\alpha+1}$. Now (1.7) yields $|Z_\alpha| = 4$. Hence $G_\alpha/C_{G_\alpha}(Z_\alpha) \cong SL_2(2)$ and $E_\alpha/O_2(E_\alpha)$ is cyclic. Now (7.7)(a) yields $Q_\alpha = C_{G_\alpha}(Z_\alpha)$.

(9.3) *Suppose that* $b > 1$. *Then* $G_\lambda/Q_\lambda \cong SL_2(2)$ *and* $|Z_\lambda| = 4$ *for every* $\lambda \in \alpha^G$.

Proof. It suffices to prove the claim for α. Recall that $Z(G_\alpha) = 1$. According to (9.2) we may assume that

(1) $|Z_\alpha| > 4$.

We apply (7.8) with $\delta = \alpha'$, $\lambda = \alpha' - 1$, and $A = V_{\alpha+1}$. Then there exists $x \in G_{\alpha'}$ such that for $E = \langle V_{\alpha+1}, V_{\alpha+1}^x \rangle$ and $\mu = (\alpha' - 1)^x$:

(i) $x \in O^2(E)$.

(ii) $Z_\alpha \not\leq G_\mu$ and $|V_{\alpha+1}/V_{\alpha+1} \cap G_\mu| = 2$,

(iii) $\langle E, G_{\alpha'} \cap G_\mu \rangle = G_{\alpha'}$,

(iv) $E = \langle a, V_{\alpha+1}^x \rangle$ for every $a \in V_{\alpha+1} \setminus G_\mu$.

(v) $[E, V_{\alpha+1} \cap G_\mu] \leq Q_{\alpha'}$.

By (7.5)(d), $V_{\alpha+1}$ operates quadratically on $V_{\alpha'}$, and vice versa. Since Z_μ is not normal in $G_{\alpha'}$ and $[Z_\mu, V_{\alpha+1}^x] = 1$ we get from (ii), (iii), and (iv) that $Z_\mu \not\leq Q_\alpha$.

Note that by (7.5)(d), $[Z_\alpha, Z_\mu \cap G_\alpha, Z_\mu \cap G_\alpha] \leq [V_{\alpha'}, Z_\mu \cap G_\alpha] = 1$. Hence, by (1.2) there exists $W \leq Z_\mu \cap G_\alpha$ such that $|Z_\mu \cap G_\alpha/W| \leq 2$ and $C_{Z_\alpha}(W) \not\leq G_\mu$. Now (i) and (iv) imply that $W \leq Z_\mu \cap Z_{\alpha'-1}$ and $W \leq Q_\alpha$. Together with (1) and (9.2) this gives

(2) $Z_\mu \cap Q_\alpha = Z_\mu \cap Z_{\alpha'-1}$, $|Z_\mu \cap G_\alpha/Z_\mu \cap Q_\alpha| \leq 2$, and $Z_\mu \not\leq G_\alpha$; in particular, $V_{\alpha'} \not\leq Q_{\alpha+1}$.

We now apply (7.8) with $\delta = \alpha + 1$, $\lambda = \alpha + 2$, and $A = V_{\alpha'}$. Then there exist $y \in G_{\alpha+1}$, $\tilde{E} = \langle V_{\alpha'}, V_{\alpha'}^y \rangle$, and $\tilde{\mu} = (\alpha + 2)^y$ such that (i)–(v) hold for $(y, \tilde{E}, \alpha', \tilde{\mu}, \alpha + 1, \alpha + 2)$ in place of $(x, E, \alpha + 1, \mu, \alpha', \alpha' - 1)$. Moreover, by (7.8) and (6.1) we may also assume that

(3) $O^2(\tilde{E}) \leq [O^2(\tilde{E}), B(T)]$, where $T \in \mathrm{Syl}_2(G_{\alpha+1} \cap G_{\tilde{\mu}})$.

The same argument as above with $(\tilde{\mu}, \mu)$ in place of (μ, α) shows that $Z_{\tilde{\mu}} \not\leq G_\mu$ and $|Z_{\tilde{\mu}} \cap G_\mu/Z_{\tilde{\mu}} \cap Z_{\alpha+2}| \leq 2$. Hence, without loss of generality we may assume that $\tilde{\mu} = \alpha$. This gives together with (2) and (9.2):

(4) $|Z_\mu/Z_\mu \cap G_\alpha| = |Z_\mu \cap G_\alpha/Z_\mu \cap Z_{\alpha'-1}| = |Z_\alpha/Z_\alpha \cap G_\mu| = |Z_\alpha \cap G_\mu/Z_\alpha \cap Z_{\alpha+2}| = 2$ and $O^2(\tilde{E}) \leq [O^2(\tilde{E}), B]$.

Let $R_1 = [Z_\alpha \cap G_\mu, Z_\mu \cap G_\alpha] \cap Z(B)$ and $C = C_H(R_1)$. Assume first that $R_1 \neq 1$. Then C is of characteristic 2 type since $B \leq C$. Note further that $R_1 \leq Z_\alpha \cap Z_{\alpha+2} \cap Z_\mu \cap Z_{\alpha'-1}$ and thus $\langle E, \tilde{E} \rangle \leq C$.

Let $F = [O^2(\tilde{E}), B]$. Then $F = [F, B] \leq E_{\alpha+1}$. Hence, (5.2) shows that F is subnormal in C; in particular, $O^2(\tilde{E}) \leq F \leq O_{2,2'}(C)$. Since α and $\alpha + 2$ are conjugate by an element of $O^2(\tilde{E})$ we get

$$Z_\alpha \leq Z_{\alpha+2} O_2(C) \leq (Q_{\alpha'} \cap C) O_2(C),$$

a contradiction since $[Z_\alpha, E]$ is not a 2-group.

Assume now that $R_1 = 1$. Then $Z_\alpha \nleq Z(J(S))$ and thus $J(S) \nleq Q_\alpha$. From (6.4) we get that $Z_\alpha \cap Z_{\alpha+2} \cap Z(B) = Z_{\alpha+1}$ since $[Z_\alpha \cap Z_{\alpha+2}, \tilde{E}] = 1$. On the other hand by (4), $|Z_\alpha / Z_\alpha \cap Z_{\alpha+2}| = 4$. Thus, (1.7) yields

$$|Z_\alpha| = 2^4 \quad \text{and} \quad \overline{G}_\alpha \cong SL_2(2) \wr C_2.$$

Moreover, since $R_1 = 1$ we have $\overline{S} = \overline{B}(\overline{Z_\mu \cap G_\alpha})$.

Let $R_0 = [Z_\mu \cap G_\alpha, Z_\alpha \cap G_\mu]$ and $C_0 = C_H(R_0)$. Note that $\langle E, \tilde{E} \rangle \leq C_0$ since $R_0 \leq Z_\alpha \cap Z_{\alpha+2} \cap Z_\mu \cap Z_{\alpha'-1}$ and that $|R_0| = 2$ since $R_1 = 1$. Because $C_{\overline{G}_\alpha}(\overline{Z_\mu \cap G_\alpha})$ acts transitively on $[Z_\alpha, Z_\mu \cap G_\alpha]^\#$ there exists $S^g \leq G_\alpha$ such that $S^g \leq C_0$. Hence, C_0 is of characteristic 2 type.

If $O^2(E) \leq O_{2,2'}(C_0)$, then, as above for $\tilde{E}$ and C, $Z_\mu \leq Z_{\alpha'-1} O_2(C_0)$ and $[Z_\mu, O^2(\tilde{E})]$ is a 2-group, a contradiction. Assume that $O^2(E) \nleq O_{2,2'}(C_0)$. Then $[O_{2,2',2}(C_0), Z_\alpha] \nleq O_2(C_0)$ and thus $|Z_\alpha \cap O_2(C_0)/R_0| \leq 2$. Hence, Z_α induces transvections in $O_2(C_0)/\Phi(O_2(C_0))R_0$ since $O_2(C_0) \leq S^g \leq G_\alpha$. Now (1.7) gives $O^2(E) \leq O_{2,2'}(C_0)$, a contradiction.

Remark. In the following lemmata we will use (9.3) without reference. Note that (9.3) has the following easy consequences which will be used frequently.

Suppose that $b > 1$. Then $Z_\alpha = Z_{\alpha-1} \times Z_{\alpha+1}$, where $\alpha - 1 \in \Delta(\alpha) \setminus \{\alpha + 1\}$ and $|Z_{\alpha+1}| = 2$. By (7.6)(b), $S = Q_\alpha Q_{\alpha+1} = G_\alpha \cap G_{\alpha+1}$ and $[Z_\alpha, S] = Z_{\alpha+1}$. It follows that $[V_{\alpha+1}, Q_{\alpha+1}] = Z_{\alpha+1}$, and $V_{\alpha+1}/Z_{\alpha+1}$ is a $G_{\alpha+1}/Q_{\alpha+1}$ module. Moreover, by (7.7)(b) this module is faithful.

Note further that by (9.3), E_α is 2-transitive in $\Delta(\alpha)$. Hence, all paths of length 2 with initial vertex in $(\alpha + 1)^G$ are conjugate under G.

DEFINITION. Let $\delta \in \Gamma$. Then $W_\delta = \langle V_\lambda \mid \lambda \in \Delta(\delta) \rangle$ if $\delta \in \alpha^G$ and $W_\delta = \langle V_\lambda \mid d(\delta, \lambda) = 2 \rangle$ if $\delta \in (\alpha + 1)^G$.

(9.4) *Let $b > 1$ and $\rho \in \Gamma$ with $d(\rho, \alpha + 1) = 2$. Suppose that there exist $t \in C_{G_{\alpha+1}}(V_\rho)$, $x \in [E_{\alpha+1}, t]$, and $A \leq V_\rho^x$ such that*

(i) $[A, t] \leq V_{\alpha+1}$,

(ii) $\langle G_{\alpha+1} \cap G_\nu, t \rangle = G_{\alpha+1}$ *for* $\nu \in \Delta(\alpha + 1) \cap \Delta(\rho^x)$,

(iii) $|[V_{\alpha+1}, t] Z_{\alpha+1}/Z_{\alpha+1}| = 2$.

Then $A \leq V_{\alpha+1}$.

Proof. Possibly after conjugation in $G_{\alpha+1}$ we may assume that $\nu = \alpha$. Set $\rho^x = \alpha - 1$. Assume that $A \nleq V_{\alpha+1}$ and without loss $V_{\alpha-1} \cap V_{\alpha+1} \leq A$. We choose the following notation: $T = \langle (t^x)^{Q_\alpha} \rangle$, $F = \langle Q_\alpha \cap Q_{\alpha-1}, t \rangle$, $Q = O_2(O^2(F))$, $\overline{G}_{\alpha+1} = G_{\alpha+1}/Q_{\alpha+1}$, $\overline{V}_{\alpha+1} = V_{\alpha+1}/Z_{\alpha+1}$, and $V_a = [\langle a \rangle V_{\alpha+1}, Q] Z_{\alpha+1}$ for $a \in A$.

Note that $t \notin Q_{\alpha+1}$ by (ii) and that $[V_{\alpha-1}, T] = 1$, i.e., $T \le Q_{\alpha-1}$ by (7.4)(c) and (7.7)(b). Now (iii), (1.7), and (7.7)(b) imply:

(1) $\bar{E}_{\alpha+1}\bar{T} = \bar{E}_1 \times \cdots \times \bar{E}_r$, $\bar{E}_i \cong SL_2(2)$, and

(2) $[\bar{V}_{\alpha+1}, \bar{E}_{\alpha+1}] = \bar{V}_1 \times \cdots \times \bar{V}_r$, $\bar{V}_i = [\bar{V}_{\alpha+1}, \bar{E}_i]$, and $|\bar{V}_i| = 4$.

We may assume that $[\bar{E}_{\alpha+1}, t] \le \bar{E}_1$; i.e., $\bar{x} \in \bar{E}_1$ and $O^2(\bar{F}) = \langle O^2(\bar{E}_1)^y | y \in \overline{Q_\alpha \cap Q_{\alpha-1}}\rangle$. Note that V_a is F-invariant and $V_a \le [V_{\alpha+1}, O^2(F)]Z_{\alpha+1}$. Since $|Q/Q \cap Q_\alpha| \le 2$ we also get that

(3) $|V_a/V_a \cap V_{\alpha-1}| \le 2$.

Assume that $|V_{\alpha+1}/V_{\alpha+1} \cap V_{\alpha-1}| = 2$. Then $V_{\alpha-1} = A$ and $\langle Q_\alpha, t\rangle$ normalizes $V_{\alpha-1}V_{\alpha+1}$. Since $G_\alpha \cap G_{\alpha+1} = Q_\alpha Q_{\alpha+1}$ we get from (ii) that $E_{\alpha+1} \le \langle Q_\alpha, t\rangle$. Now (7.6)(b) shows that $W_\alpha = V_{\alpha-1}V_{\alpha+1}$. It follows that W_α is normal in G_α and $G_{\alpha+1}$, a contradiction. We have shown:

(4) $|V_{\alpha+1}/V_{\alpha-1} \cap V_{\alpha+1}| \ge 4$.

Suppose first that $V_a \ne Z_{\alpha+1}$. Then $[V_a, O^2(F)] \ne 1$ and thus $V_1 \le V_a$ since $O^2(\bar{F}) = \langle O^2(\bar{E}_1)^y | y \in \overline{Q_\alpha \cap Q_{\alpha-1}}\rangle$. Now (3) gives $V_a = V_1(V_a \cap V_{\alpha-1})$ and $[V_a, T] = [V_1, T] \le V_1$. It follows that $O^2(\bar{F}) = O^2(\bar{E}_1)$ and $F/O_2(F) \cong SL_2(2)$; in particular,

(5) $V_a = V_1$.

Since $\underline{Q_\alpha \cap Q_{\alpha-1}}$ is normal in Q_α and $\bar{S} = \bar{Q}_\alpha$ we conclude that $\overline{Q_\alpha \cap Q_{\alpha-1}}$ normalizes $\bar{E}_i$ for $i = 1, \ldots, r$ and $T \le Q_\alpha \cap Q_{\alpha-1} \le TQ_{\alpha+1}$. Now $[V_{\alpha-1}, T] = 1$ gives

$$[V_{\alpha-1} \cap V_{\alpha+1}, Q_\alpha \cap Q_{\alpha-1}] = [V_{\alpha-1} \cap V_{\alpha+1}, Q_\alpha \cap Q_{\alpha-1} \cap Q_{\alpha+1}] = 1$$

since $Z_{\alpha-1} \cap Z_{\alpha+1} = 1$.

On the other hand, $V_{\alpha-1} \cap V_{\alpha+1}$ is normal in G_α and $O_2(E_\alpha) \le \langle (Q_\alpha \cap Q_{\alpha-1})^{G_\alpha}\rangle$. It follows that $[V_{\alpha-1} \cap V_{\alpha+1}, O_2(E_\alpha)] = 1$, and (7.5)(c) yields

(6) $V_{\alpha-1} \cap V_{\alpha+1} = Z_\alpha$.

Thus, (3) and (5) show that $Z_\alpha \le V_a = V_1$, and $V_{\alpha+1} = V_1$ since $O^2(\bar{F})$ is normal in $\bar{E}_{\alpha+1}$; in particular, $\bar{G}_{\alpha+1} = \bar{F}$ and $|V_{\alpha+1}| = 8$. This contradicts (4).

Suppose now that $V_a = Z_{\alpha+1}$ for every $a \in A$ and thus

(7) $[A, Q] \le Z_{\alpha+1}$.

If $Q \not\le Q_\alpha$, then $A \le V_{\alpha+1}$, which contradicts the assumption on A. Hence $Q \le Q_\alpha$. Let $Q^* = \bigcap_{y \in F}(Q_\alpha \cap Q_{\alpha+1})^y$ and $\bar{Q}_{\alpha+1} = Q_{\alpha+1}/Q^*$. Then $Q_{\alpha-1} \cap Q_{\alpha+1} = Q_{\alpha-1} \cap Q_\alpha \cap Q_{\alpha+1} \le O_2(F)$ and so $Q_{\alpha-1} \cap Q_{\alpha+1} \le Q^*$. It follows that $[\overline{Q_{\alpha+1} \cap Q_\alpha}, Q_\alpha \cap Q_{\alpha-1}] = 1$. Hence $|Q_{\alpha+1}/Q_{\alpha+1}$

$\cap Q_\alpha| = 2$ implies that $F/C_F(\bar{Q}_{\alpha+1}) \cong SL_2(2)$. However, $C_F(\bar{Q}_{\alpha+1})$ normalizes $Q_{\alpha+1} \cap Q_\alpha$ and thus by (7.6)(c) and (1), $C_F(\bar{Q}_{\alpha+1})$ is a 2-group. Now $F/O_2(F) \cong SL_2(2)$ and $O^2(\bar{F}) = O^2(\bar{E}_1)$. As above we get that $Q_\alpha \cap Q_{\alpha-1} \leq TQ_{\alpha+1}$ and $[V_{\alpha-1} \cap V_{\alpha+1}, Q_\alpha \cap Q_{\alpha-1}] = 1$ and then $V_{\alpha-1} \cap V_{\alpha+1} = Z_\alpha$.

Since $Q_\alpha \cap Q_{\alpha-1} \leq TQ_{\alpha+1}$ we also get that

$$[A, Q_\alpha \cap Q_{\alpha-1}] = [A, Q_\alpha \cap Q_{\alpha-1} \cap Q_{\alpha+1}] \leq Z_{\alpha-1},$$

and by (7), $[A, Q_\alpha \cap Q_{\alpha-1} \cap Q_{\alpha+1}]Z_{\alpha+1}$ is F-invariant. Hence $[A, Q_\alpha \cap Q_{\alpha-1}] = 1$ since by (ii), Z_α is not normal in F. On the other hand, $|Q_{\alpha-1}/Q_\alpha \cap Q_{\alpha-1}| = 2$ and thus $|A/C_A(Q_{\alpha-1})| \leq 2$; in particular, $A = Z_\alpha C_A(Q_{\alpha-1})$ and $C_A(Q_{\alpha-1}) \neq Z_{\alpha-1}$ since $A \neq Z_\alpha$. However, $Z_\alpha \leq A \leq [V_{\alpha-1}, E_{\alpha-1}]Z_\alpha$, and by (1) and (2), $C_{[V_{\alpha-1}, E_{\alpha-1}]}(Q_{\alpha-1}) = Z_{\alpha-1}$. We conclude that

$$\begin{aligned} |C_{V_{\alpha-1}}(Q_{\alpha-1})/Z_{\alpha-1}| &= |C_{V_{\alpha-1}}(Q_{\alpha-1})[V_{\alpha-1}, E_{\alpha-1}]/[V_{\alpha-1}, E_{\alpha-1}]| \\ &= |C_A(Q_{\alpha-1})[V_{\alpha-1}, E_{\alpha-1}]/[V_{\alpha-1}, E_{\alpha-1}]| \\ &= |C_A(Q_{\alpha-1})/Z_{\alpha-1}|. \end{aligned}$$

Thus $C_A(Q_{\alpha-1}) = C_{V_{\alpha-1}}(Q_{\alpha-1})$ and $A = Z_\alpha C_{V_{\alpha-1}}(Q_{\alpha-1})$.

Now $G_\alpha \cap G_{\alpha-1} \leq N_G(A)$. Hence $\langle Q_\alpha, F\rangle$ and thus $E_{\alpha+1}$ normalizes $AV_{\alpha+1}$. Let $V_a^* = [\langle a\rangle V_{\alpha+1}, O_2(E_{\alpha+1})]Z_{\alpha+1}$ for $a \in A$. Then, as for V_a, $|V_a^*/V_a^* \cap V_{\alpha-1}| \leq 2$ and $V_a^* = Z_{\alpha+1}$ or $Z_\alpha \leq V_a^*$. If $Z_\alpha \leq V_a^*$, then $V_a^* = [V_{\alpha+1}, E_{\alpha+1}]$ and $|V_{\alpha+1}/V_{\alpha+1} \cap V_{\alpha-1}| = 2$, which contradicts (4). Thus, we have $[A, O_2(E_{\alpha+1})] \leq Z_{\alpha+1}$ and $A \leq V_{\alpha+1}$ since by (7.6)(b), $O_2(E_{\alpha+1}) \not\leq Q_\alpha$, a contradiction.

(9.5) *Suppose that $b > 1$. Let $t \in V_{\alpha+1} \setminus Q_{\alpha'}$ such that $|[V_{\alpha'}, t]Z_{\alpha'}/Z_{\alpha'}| = 2$ and $[V_{\alpha'}, t] \leq V_{\alpha'-2}$. Then either*

(a) $|V_{\alpha'}| = 2^3$ *and* $G_{\alpha'}/Q_{\alpha'} \cong SL_2(2)$, *or*

(b) $|V_{\alpha'}| = 2^5$, $G_{\alpha'}/Q_{\alpha'} \cong SL_2(2) \wr C_2$, *and* $|V_{\alpha'} \cap V_{\alpha'-2}| = 2^3$.

Proof. Let $\bar{V}_{\alpha'} = V_{\alpha'}/Z_{\alpha'}$, $R = [V_{\alpha'}, t]$, and $\bar{E}_{\alpha'} = E_{\alpha'}/Q_{\alpha'}$. Then t induces a transvection on $\bar{V}_{\alpha'}$. Hence (1.7) and (7.7)(b) imply:

(i) $\bar{E}_{\alpha'} = \bar{E}_1 \times \cdots \times \bar{E}_r$, $\bar{E}_i \cong C_3$ and

(ii) $\bar{V}_{\alpha'} = \bar{V}_0 \times \bar{V}_1 \times \cdots \times \bar{V}_r$, $V_i = [V_{\alpha'}, E_i]$, and $|\bar{V}_i| = 4$ for $i \geq 1$, and $V_0 = C_{V_{\alpha'}}(E_{\alpha'})$.

We may assume that $R \leq V_1$. Note that $RZ_{\alpha'-1}$ is normal in $\langle Q_{\alpha'-2}, Q_{\alpha'}\rangle$ and

$$[R, \langle Q_{\alpha'-2}, Q_{\alpha'}\rangle] \leq Z_{\alpha'-1}.$$

On the other hand, by (7.6)(b), $E_{\alpha'-1} \leq \langle Q_{\alpha'-2}, Q_{\alpha'} \rangle$, and by (7.5)(c), $R \nleq Z(O_2(E_{\alpha'-1}))$. Hence there exists $w \in O_2(E_{\alpha'-1})$ such that $[w, R]Z_{\alpha'} = Z_{\alpha'-1}$. It follows that $Z_{\alpha'-1} \leq V_1 V_1^w$. Since $V_1 V_1^w$ is normalized by $E_{\alpha'}$ we conclude that $V_{\alpha'} = V_1 V_1^w$. If $V_1 = V_1^w$, then (a) holds, and if $V_1 \neq V_1^w$, then (b) holds.

(9.6) *Suppose that* $b > 1$ *and* $G_{\alpha+1}/Q_{\alpha+1} \cong SL_2(2)$. *Then* $|V_{\alpha+1}/V_{\alpha+1} \cap V_{\alpha+3}| = 2$.

Proof. Let $\alpha - 1 \in \Delta(\alpha) \setminus \{\alpha + 1\}$ and let $R = [V_{\alpha+1}, V_{\alpha'}]$ and $R_0 = [V_{\alpha-1}, V_{\alpha'-2}]$. Since $|Z_\alpha| = 4$ and there are only three $G_{\alpha+1}$-conjugates of Z_α we get that $|V_{\alpha+1}| \leq 2^4$ and $|[V_{\alpha+1}, E_{\alpha+1}]| = 2^3$. Hence, we may assume:

$(*)$ $|V_{\alpha+1}| = 2^4$, $V_{\alpha-1} \cap V_{\alpha+1} = Z_\alpha$, and $Z_\alpha \nleq [V_{\alpha+1}, E_{\alpha+1}]$.

Suppose first that $R_0 = 1$. Then $V_{\alpha-1} \leq C_{G_{\alpha'-2}}(V_{\alpha'-2}) \leq Q_{\alpha'-1} \leq G_{\alpha'}$ by (7.7)(b); in particular, $[V_{\alpha-1}, R] = 1$. Hence $[V_{\alpha-1}, V_{\alpha'}] \leq RZ_{\alpha'}$. Since $|V_{\alpha'}/Z_{\alpha'-1}R| = 2$ and $[V_{\alpha-1}, Z_{\alpha'-1}R] = 1$ there exists $A \leq V_{\alpha-1}$ such that $|V_{\alpha-1}/A| = 2$ and $[V_{\alpha'}, E_{\alpha'}, A] = R$. If $[V_{\alpha'}, E_{\alpha'}] \leq Q_{\alpha+1}$, then $R = Z_{\alpha+1}$ and $E_\alpha \leq \langle Q_{\alpha-1}, [V_{\alpha'}, E_{\alpha'}] \rangle$. Thus $[A, E_\alpha] \leq Z_\alpha$ and $A \leq V_{\alpha-1} \cap V_{\alpha+1}$, which contradicts $(*)$. If $[V_{\alpha'}, E_{\alpha'}] \nleq Q_{\alpha+1}$, then (9.4) with $\rho = \alpha + 3$, $t \in [V_{\alpha'}, E_{\alpha'}] \setminus Q_{\alpha+1}$, and $(\alpha + 3)^x = \alpha - 1$ shows that $A \leq V_{\alpha+1}$, a contradiction as above. We have shown that $R_0 \neq 1$.

Suppose next that $b = 3$. Then $RZ_{\alpha+2} \leq V_{\alpha+1} \cap V_{\alpha+3}$. Now $(*)$ yields $R \leq Z_{\alpha+2}$ and $Z_{\alpha+2} \leq [V_{\alpha'}, E_{\alpha'}]$. This contradicts $(*)$ with α' in place of $\alpha + 1$.

Suppose finally that $b > 3$ and $R_0 \neq 1$; i.e., $b \geq 5$. Then $[R_0, V_{\alpha'}] = 1$ and, as above, with (9.4), $R_0 \leq V_{\alpha-1} \cap V_{\alpha+1} = Z_\alpha$. It follows that either $Z_\alpha \leq [V_{\alpha-1}, E_{\alpha-1}]$ or $V_{\alpha'-2} \leq Q_{\alpha-1}$ and $R_0 = Z_{\alpha-1}$. The first case contradicts $(*)$ with $\alpha - 1$ in place of $\alpha + 1$. The second case gives $[Z_\alpha, V_{\alpha'}] = 1$, which contradicts (7.7)(b).

(9.7) *Suppose that* $b > 1$ *and* $|V_{\alpha+1}/V_{\alpha+1} \cap V_{\alpha+3}| = 2$. *Then* $b = 3$, $|V_{\alpha+1}| = 8$, *and* $G_{\alpha+1}/Q_{\alpha+1} \cong SL_2(2)$.

Proof. Let $t \in V_{\alpha+1} \setminus Q_{\alpha'}$. Then $[V_{\alpha'}, t] \leq V_{\alpha'} \cap V_{\alpha'-2} = C_{V_{\alpha'}}(t)$ and $|[V_{\alpha'}, t]Z_{\alpha'}/Z_{\alpha'}| = 2$. Hence (9.5) implies that $|V_{\alpha'}| = 8$ and $G_{\alpha'}/Q_{\alpha'} \cong SL_2(2)$ since $C_{V_{\alpha'}}(t)$ is $Q_{\alpha'-1}$-invariant and $G_{\alpha'} = Q_{\alpha'-1}E_{\alpha'}Q_{\alpha'}$. It remains to prove that $b = 3$.

Assume that $b > 3$. Let $R = [V_{\alpha'}, Z_\alpha]$. Then $|R| = 2$ and $R \leq Z_{\alpha'-1}$ since $|Z_\alpha/C_{Z_\alpha}(V_{\alpha'})| = 2$ and $|V_{\alpha'}| = 8$. With the same argument $R \leq Z_{\alpha+2}$. Hence, there exist $\rho \in \Delta(\alpha + 2)$ and $\rho' \in \Delta(\alpha' - 1)$ such that $R = Z_\rho = Z_{\rho'}$.

Assume that $b = 7$. Then $R \leq V_{\alpha+3} \cap V_{\alpha+5} = Z_{\alpha+4}$ and thus $R = Z_{\alpha+3} = Z_{\alpha+5}$, which contradicts $Z_{\alpha+4} = Z_{\alpha+3} \times Z_{\alpha+5}$.

Assume that $b > 7$. Let $U = \langle W_\tau \mid \tau \in \Delta(\alpha)\rangle$. Then U is abelian and $U \le Q_\rho$. Let $C = O_2(C_G(R))$. Then $C \le Q_{\rho'} \le G_{\alpha'-1}$. Hence $W_{\alpha'-1}$ is C-invariant. Note that E_ρ is subnormal in $C_G(R)$ and thus $O_2(E_\rho) \le C$. If $\rho \notin \{\alpha+1, \alpha+3\}$, then by (7.6)(b) there exists $x \in C$ such that $(\alpha+1)^x = \alpha+3$. Hence $[V_{\alpha+1}, W_{\alpha'-1}][V_{\alpha+3}, W_{\alpha'-1}] = 1$ which contradicts $[V_{\alpha+1}, V_{\alpha'}] \ne 1$. Thus, we have $\rho \in \{\alpha+1, \alpha+3\}$. On the other hand, E_ρ is subnormal in $C_H(R)$, $W_\tau \le Q_\rho$ for $\tau \in \Delta(\alpha)$, and E_ρ is 2-transitive on $\Delta(\rho)$. Hence, there exists $\kappa \in \Gamma$ such that $W_\tau \le CW_\kappa$, $d(\kappa, \alpha') \le b-3$, and $d(\kappa, \alpha'-1) \le b-4$. We conclude that $U \le G_{\alpha'-1}$.

Assume that $U \le G_{\alpha'}$. Then $[U, V_{\alpha'}] \le RZ_{\alpha'}$ and either $[W_\alpha, V_{\alpha'}] = R$ or $[U, V_{\alpha'}] = [W_\alpha, V_{\alpha'}]$. Suppose that $V_{\alpha'} \le Q_{\alpha+1}$. Then $R = Z_{\alpha+1} \le Z_\alpha$ and the first case shows that V_τ is normal in G_α for $\tau \in \Delta(\alpha)$, while the second case shows that W_τ is normal in G_α for $\tau \in \Delta(\alpha)$, a contradiction in both cases. Suppose that $V_{\alpha'} \not\le Q_{\alpha+1}$. Then either W_α is normal in $G_{\alpha+1}$ or U is normal in $G_{\alpha+1}$, a similar contradiction.

Assume that $U \not\le G_{\alpha'}$. Then $Z_{\alpha'} \not\le [W_{\alpha+1} \cap G_{\alpha'}, V_{\alpha'}]$ since U is abelian and $Z_{\alpha'-1} = Z_{\alpha'} \times Z_{\alpha'-2}$. It follows that $[W_\alpha \cap G_{\alpha'}, V_{\alpha'}] = R$.

Suppose that $W_\alpha \le G_{\alpha'}$. Then $[W_\alpha, V_{\alpha'}] = R \le V_{\alpha+1} \le W_\alpha$ and $V_{\alpha'} \le Q_{\alpha+1}$ since W_α is not normal in $G_{\alpha+1}$. It follows that $R = Z_{\alpha+1}$ and $[W_\alpha, V_{\alpha'}] = Z_{\alpha+1}$. Hence V_τ is normal in G_α for $\tau \in \Delta(\alpha)$, a contradiction.

Suppose that $W_\alpha \not\le G_{\alpha'}$. Then $[W_\alpha, Z_{\alpha'-1}] = Z_{\alpha'-2} = R$. On the other hand, $Z_{\alpha'-1} \le Q_\alpha$ and so $[W_\alpha, Z_{\alpha'-1}] \le Z_\alpha \cap Q_{\alpha'} = Z_{\alpha+1}$. It follows that $R = Z_{\alpha+1} = Z_{\alpha'-2}$. In particular, $V_{\alpha'} \le Q_{\alpha+1}$ and $[U, V_{\alpha'}] = [W_\alpha, V_{\alpha'}]$ since $Z_{\alpha'} \not\le U$, but now W_τ is normal in G_α for $\tau \in \Delta(\alpha)$, a contradiction.

It remains to discuss the case $b = 5$. Then $R \le Z_{\alpha+2} \cap Z_{\alpha+4} = Z_{\alpha+3}$; i.e., $\rho = \rho' = \alpha+3$. Now the proof can be finished by a nice argument of Goldschmidt [2] which we will repeat here.

Note that $|\Delta(\delta)| = 3$ for every $\delta \in \Gamma$. Hence, a subgroup of $G_\delta \cap G_\tau$, $\tau \in \Delta(\delta)$, is transitive on $\Delta(\delta) \setminus \{\tau\}$ whenever it is not in Q_δ. Thus, $G_{\alpha+1} \cap G_{\alpha+2}$ is transitive on $\Delta(\alpha+2) \setminus \{\alpha+1\}$ and $Q_{\alpha+2}$ is transitive on $\Delta(\alpha+3) \setminus \{\alpha+2\}$ since $Q_{\alpha+2}Q_{\alpha+3} \in \mathrm{Syl}_3(G_{\alpha+3})$. Moreover, $Q_{\alpha+2} \cap Q_{\alpha+3} \ne Q_{\alpha+3} \cap Q_{\alpha+4}$ by (7.6)(c), and so $Q_{\alpha+2} \cap Q_{\alpha+3}$ is transitive on $\Delta(\alpha+4) \setminus \{\alpha+3\}$. We conclude that $G_{\alpha+1}$ is transitive on paths of length 4 with initial vertex $\alpha+1$, and so G is transitive on paths of length 4 with initial vertex in $(\alpha+1)^G$.

We now investigate a path $(\alpha+1, \dots, \alpha+5, \dots, \alpha+7)$ of length 6. Then $V_{\alpha+7} \not\le Q_{\alpha+3}$ and $[V_{\alpha+3}, V_{\alpha+7}] = Z_{\alpha+5}$. Hence, there exists $y \in V_{\alpha+7} \setminus Q_{\alpha+3}$ and $w \in V_{\alpha+1} \setminus Q_{\alpha+5}$ such that for $z = [y, y^w]$ and $z' = [w, w^y]$,

$$z \in \left[V_{\alpha+7}, V_{\alpha+7}^w\right] = Z_{\alpha+5} \quad \text{and} \quad z' \in \left[V_{\alpha+1}, V_{\alpha+1}^y\right] = Z_{\alpha+3}.$$

On the other hand, $\langle y, w\rangle$ is a dihedral group of order $2|\langle y, y^w\rangle| \le 16$. It follows that $(yw)^4 z = (wy)^4 = z'$ and $z = z' \in Z_{\alpha+3} \cap Z_{\alpha+5} = 1$. Since

$V_{\alpha+1} = Z_\alpha Z_{\alpha+2}$ and $[V_{\alpha+1}, Z^y_{\alpha+2}] = [Z_{\alpha+2}, V^y_{\alpha+1}] = 1$ we have shown that $[V_{\alpha+1}, V^y_{\alpha+1}] = 1$. However, the path $(\alpha+1, \dots, \alpha+3, \dots, (\alpha+1)^y)$ is a G-conjugate of $(\alpha+1, \dots, \alpha+5)$ and thus also $[V_{\alpha+1}, V_{\alpha+5}] = 1$, a contradiction.

(9.8) *Suppose that* $Z_{\alpha'} \le V_{\alpha+1}$. *Then* $b \le 3$.

Proof. Assume that $b > 3$; i.e., $b \ge 5$ by (7.5)(a). Let $\lambda \in \Delta(\alpha+1)$. Then W_λ is abelian and $W_\lambda \le G_{\alpha'-2}$; in particular, $[W_\lambda, V_{\alpha+1}] = 1$ and thus $[W_\lambda, Z_{\alpha'}] = 1$. Since $Z_{\alpha'-1} = Z_{\alpha'-2} \times Z_{\alpha'}$ we conclude that $W_\lambda \le C_G(Z_{\alpha'-1})$. Now (7.7)(a) gives $[W_\lambda, E_{\alpha'-1}] \le Q_{\alpha'-1}$. Since $E_{\alpha'-1}$ is transitive on $\Delta(\alpha'-1)$ we get that $W_\lambda \le G_{\alpha'}$.

According to (7.8) and (7.6)(d) there exists $\mu \in \Delta(\alpha')$ such that

$$|W_\alpha / W_\alpha \cap G_\mu| = 2, \qquad Z_\alpha \not\le G_\mu \quad \text{and} \quad \left[Z_\alpha, Z_\mu\right] \ne 1.$$

Assume first that $Z_\mu \le G_\alpha$. Then $[Z_\alpha, Z_\mu] = Z_{\alpha+1}$ and $[W_\alpha, Z_\mu] \le Z_{\alpha+1} Z_{\alpha'}$. Let $\alpha - 1 \in \Delta(\alpha) \setminus \{\alpha+1\}$. Then there exists a subgroup A of index 2 in $V_{\alpha-1}$ such that $[A, Z_\mu] = Z_{\alpha+1}$. Since $E_\alpha \le \langle Q_{\alpha-1}, Z_\mu \rangle$ we get that $A \le V_{\alpha+1}$, which contradicts (9.7).

Assume now that $Z_\mu \not\le G_\alpha$; in particular, $Z_\mu \not\le Q_{\alpha+1}$. Note that

$$\left[W_\alpha, Z_\mu\right] \le \left[Z_\alpha, Z_\mu\right] Z_{\alpha'} \le V_{\alpha+1} \le W_\alpha.$$

Hence, W_α is normal in $\langle G_\alpha \cap G_{\alpha+1}, Z_\mu \rangle$. Since W_α is not normal in $G_{\alpha+1}$ we conclude that $\langle G_\alpha \cap G_{\alpha+1}, Z_\mu \rangle \ne G_{\alpha+1}$. On the other hand, $Z_\mu \not\le G_\alpha$ and so by (3.3),

$(*)$ $E_{\alpha+1}/O_2(E_{\alpha+1})$ is not elementary abelian.

Assume that $Z_{\alpha+1} \not\le V_{\alpha'}$. According to (1.2) there exists $A \le V_{\alpha'}$ and $U \le V_{\alpha+1}$ such that $|V_{\alpha'}/A| = 2$, $U \not\le Q_{\alpha'}$, and $[U, A] \le Z_{\alpha+1}$. Since $Z_{\alpha+1} \not\le V_{\alpha'}$ we get that $[A, U] = 1$, and U induces transvections on $V_{\alpha'}/Z_{\alpha'}$. Now (1.7) and (7.7)(b) contradict $(*)$ with α' in place of $\alpha+1$.

Assume that $Z_{\alpha+1} \le V_{\alpha'}$. Then our hypothesis is symmetric in $\alpha+1$ and α'. Thus, with the above argument, $W_\tau \le G_{\alpha+1}$ for $\tau \in \Delta(\alpha')$. Again by (7.8) there exists $x \in G_{\alpha+1}$ such that for $E = \langle Z_\mu, Z^x_\mu \rangle$ and $\lambda = (\alpha+2)^x$,

$$\langle Z_\mu, G_\lambda \cap G_{\alpha+1} \rangle = G_{\alpha+1} \quad \text{and} \quad |W_\mu / W_\mu \cap G_\lambda| = 2.$$

Let $\rho = (\alpha+3)^x$. Note that $[Z_\mu, Z_\lambda] \ne 1$ since Z_λ is not normal in $G_{\alpha+1}$.

Suppose that $Z_\lambda \le G_\mu$. Then $E_\mu \le \langle Z_\lambda, Q_\kappa \rangle$ for $\kappa \in \Delta(\mu) \setminus \{\alpha'\}$. On the other hand, $[W_\mu, Z_\lambda] = [Z_\mu, Z_\lambda][W_\mu \cap G_\lambda, Z_\lambda] \le Z_{\alpha+1} Z_{\alpha'}$ and, as above for μ and α, $|V_\kappa / V_\kappa \cap V_{\alpha'}| = 2$, which contradicts (9.7).

Suppose that $Z_\lambda \nleq G_\mu$. Then $Z_\lambda \nleq Q_{\alpha'}$ and, as above for α, by (7.8) and (7.6)(d) there exists $\mu' \in \Delta(\alpha')$ such that

$$|W_\lambda / W_\lambda \cap G_{\mu'}| = 2, \qquad Z_\lambda \nleq G_{\mu'}, \quad \text{and} \quad \left[Z_\lambda, Z_{\mu'}\right] \neq 1.$$

Note that $[W_\lambda, Z_{\mu'}] \leq V_{\alpha+1} \leq W_\lambda$. If $Z_{\mu'} \leq G_\lambda$, then, as above for α and μ, $|V_\rho/V_\rho \cap V_{\alpha+1}| = 2$ for $\rho \in \Delta(\lambda)$, which contradicts (9.7). If $Z_{\mu'} \nleq G_\lambda$, then W_λ is normal in $\langle G_\lambda \cap G_{\alpha+1}, Z_{\mu'}\rangle = \langle G_\lambda \cap G_{\alpha+1}, Z_\mu\rangle = G_{\alpha+1}$, a contradiction.

(9.9) *Suppose that $Z_{\alpha+1} \leq V_{\alpha'}$. Then $b \leq 3$.*

Proof. Assume that $b > 3$. Then (9.8) with $\alpha + 1$ and α' interchanged yields $V_{\alpha'} \leq Q_{\alpha+1}$. In particular, $V_{\alpha'} \leq G_\alpha$ and $[Z_\alpha, V_{\alpha'}] = Z_{\alpha+1}$, and Z_α induces transvections on $V_{\alpha'}/Z_{\alpha'}$. Hence, according to (1.7), there exists $V_1 \leq V_{\alpha'}$ such that

(1) $[V_1, Z_\alpha] = Z_{\alpha+1}$ and $[V_1, C_{G_{\alpha'}}(Z_\alpha)] \leq Z_{\alpha+1}Z_{\alpha'}$.

Let $\alpha - 1 \in \Delta(\alpha)\setminus\{\alpha+1\}$ and A be maximal in $V_{\alpha-1}$ such that $[A, V_1] \leq Z_{\alpha+1}$. Then $A = V_{\alpha-1} \cap V_{\alpha+1}$ since $\langle Q_{\alpha-1}, V_1\rangle$ contains E_α. From (9.7) we get that

(2) $|V_{\alpha-1}/A| \geq 4$.

Hence, (1) implies that $V_{\alpha-1} \nleq G_{\alpha'}$.

Suppose that $V_{\alpha-1} \leq G_{\alpha'-1}$. Then $|V_{\alpha-1}/V_{\alpha-1} \cap G_{\alpha'}| = 2$ and $[V_{\alpha-1}, Z_{\alpha'}] \neq 1$ since $Z_{\alpha'-1} = Z_{\alpha'-2} \times Z_{\alpha'}$. We conclude that $Z_{\alpha'} \nleq W_\alpha$ since W_α is abelian and (1) yields $[V_{\alpha-1} \cap G_{\alpha'}, V_1] = Z_{\alpha+1}$. This contradicts (2).

We have shown that $V_{\alpha-1} \nleq Q_{\alpha'-2}$. Hence (9.8) implies that $Z_{\alpha'-2} \nleq V_{\alpha-1}$. Since $[Z_\alpha, V_1] = [Z_{\alpha-1}, V_1] \neq 1$ and $b \geq 5$ we also have $Z_{\alpha-1} \nleq V_{\alpha'-2}$ and thus $V_{\alpha'-2} \nleq Q_{\alpha-1}$. By (7.8) and (7.6)(d) there exists $\rho \in \Delta(\alpha'-2)$ such that $|V_{\alpha-1}/V_{\alpha-1} \cap G_\rho| = 2$, and $[V_{\alpha-1} \cap G_\rho, Z_\rho] = 1$ since $Z_{\alpha'-2} \nleq V_{\alpha-1}$. Hence, Z_ρ induces transvections on $V_{\alpha-1}/Z_{\alpha-1}$. Moreover, $[V_{\alpha-1}, Z_\rho, V_1] = 1$ since $b > 3$ and thus $[V_{\alpha-1}, Z_\rho] \leq V_{\alpha+1}$. From (9.5) we conclude that

(3) $G_{\alpha-1}/Q_{\alpha-1} \cong SL_2(2) \wr C_2$, $|V_{\alpha-1}| = 2^5$, and $|A| = 2^3$.

Conjugation to $G_{\alpha'}$ gives $Z_{\alpha+1} \leq V_{\alpha'} \cap V_{\alpha'-2}$. Hence, there exists $y \in C_{G_{\alpha'-1}}(Z_{\alpha+1})$ such that $\alpha'^y = \alpha' - 2$. Let $X = N_{G_{\alpha'}}(Z_{\alpha+1}Z_{\alpha'})$. By (3) applied to α' we get that $X/Q_{\alpha'} \cong SL_2(2) \times C_2$ and $X \cap G_{\alpha'-1} \in \mathrm{Syl}_2(X)$. Hence X^y has the same properties with α' replaced by $\alpha' - 2$. It follows that $|V_{\alpha-1}/V_{\alpha-1} \cap G_{\alpha'-1}| = 2$. On the other hand, $V_{\alpha-1} \cap G_{\alpha'-1} \leq Q_{\alpha'-1}$

since $Z_{\alpha'-2} \not\leq V_{\alpha-1}$, and $[V_{\alpha-1} \cap Q_{\alpha'-1}, V_1] \leq Z_{\alpha+1}$ since $[Z_{\alpha'}, V_{\alpha-1}] \neq 1$. However, now $|V_{\alpha-1}/A| = 2$, which contradicts (2).

(9.10) $b \leq 3$.

Proof. Assume that $b > 3$. By (9.8) and (9.9), $Z_{\alpha'} \not\leq V_{\alpha+1}$ and $Z_{\alpha+1} \not\leq V_{\alpha'}$; in particular, $V_{\alpha'} \not\leq Q_{\alpha+1}$. According to (7.8) and (7.6)(d) there exists $\mu \in \Delta(\alpha')$ such that $|V_{\alpha+1}/V_{\alpha+1} \cap G_\mu| = 2$, $\langle G_{\alpha'} \cap G_\mu, V_{\alpha+1}\rangle = G_{\alpha'}$, and $[Z_\mu, V_{\alpha+1}] \neq 1$. Since $Z_{\alpha'} \not\leq V_{\alpha+1}$ we get that $[V_{\alpha+1} \cap G_\mu, Z_\mu] = 1$, and since $Z_{\alpha+1} \not\leq V_{\alpha'}$ we get that $Z_\mu \not\leq Q_{\alpha+1}$.

Let $\Lambda = \{\lambda \in \Delta(\alpha+1) \mid \langle G_\lambda \cap G_{\alpha+1}, Z_\mu\rangle = G_{\alpha+1}\}$. Pick $\lambda \in \Lambda$. Then $[Z_\mu, Z_\lambda] \neq 1$ since Z_λ is not normal in $G_{\alpha+1}$. Hence $Z_\lambda \not\leq G_\mu$ and

(1) (λ, α') is a critical pair for every $\lambda \in \Lambda$.

Again by (7.8) and (7.6)(d) there exists $x \in G_{\alpha+1}$ such that for $\lambda = (\alpha+2)^x$ and $E = \langle Z_\mu, Z_\mu^x\rangle$,

(i) $x \in E$ and $[O^2(E), V_{\alpha'} \cap G_\lambda] \leq Q_{\alpha+1}$,

(ii) $\lambda \in \Lambda$ and $[Z_\lambda, Z_\mu] \neq 1$,

(iii) $|V_{\alpha'}/V_{\alpha'} \cap G_\lambda| = 2$.

Since $Z_{\alpha+1} \not\leq V_{\alpha'}$ we get that $[Z_\lambda, V_{\alpha'} \cap G_\lambda] = 1$, and since $Z_{\alpha'} \not\leq V_{\alpha+1}$ we get that $Z_\lambda \not\leq Q_{\alpha'}$. Hence, we may assume that $\lambda = \alpha$. We have shown:

(2) Z_α induces transvections on $V_{\alpha'}/Z_{\alpha'}$.

Let $\bar{V}_{\alpha'} = V_{\alpha'}/Z_{\alpha'}$ and $\bar{E}_{\alpha'} = E_{\alpha'}Q_{\alpha'}/Q_{\alpha'}$. Then (1), (1.7), and (7.7)(b) show:

(3) $\bar{E}_{\alpha'} = \bar{E}_1 \times \cdots \times \bar{E}_r$, $\bar{E}_i \cong C_3$, and

(4) $\bar{V}_{\alpha'} = \bar{V}_0 \times \bar{V}_1 \times \cdots \times \bar{V}_r$, $V_i = [V_{\alpha'}, E_i]$, and $|\bar{V}_i| = 4$ for $i \geq 1$, and $V_0 = C_{V_{\alpha'}}(E_{\alpha'})$.

Let $R = [Z_\alpha, V_{\alpha'}]$. Then we may assume that $R \leq V_1$ and $Z_\mu \leq V_1 Z_{\alpha'-1}$. Let $\alpha - 1 = (\alpha+3)^x$, x as in (i).

Assume first that $[V_{\alpha-1}, Z_{\alpha'-1}] = 1$. Then by (7.7)(b), $V_{\alpha-1} \leq G_{\alpha'}$ and

$$[V_{\alpha-1}, Z_\mu] = [V_{\alpha-1}, V_1] \leq RZ_{\alpha'}.$$

Hence, there exists a subgroup $A \leq V_{\alpha-1}$ such that $|V_{\alpha-1}/A| = 2$ and $[A, Z_\mu] \leq R$. Now (9.4) (with $\rho = \alpha + 3$ and $t \in Z_\mu \setminus Q_{\alpha+1}$) implies that $A \leq V_{\alpha+1}$, which contradicts (9.7).

We have shown that $[V_{\alpha-1}, Z_{\alpha'-1}] \neq 1$. Hence, either $Z_{\alpha'-1} \leq Q_{\alpha-1}$ or $(\alpha'-1, \alpha-1)$ is a critical pair. The first case gives $Z_{\alpha-1} \leq V_{\alpha'-2}$ and $[Z_{\alpha-1}, V_{\alpha'}] = 1$ since $b \geq 5$. However, $Z_\alpha = Z_{\alpha-1} \times Z_{\alpha+1}$. Hence $[Z_\alpha, Z_\mu] = 1$, which contradicts (ii). We have shown:

(5) $(\alpha'-1, \alpha-1)$ is a critical pair.

Set $R_0 = [V_{\alpha-1}, Z_{\alpha'-1}]$. By (9.4), $R_0 \le V_{\alpha+1} \cap V_{\alpha-1}$. Hence, possibly after substituting (α, α') by $(\alpha'-1, \alpha-1)$, we may assume that $R \le V_{\alpha'} \cap V_{\alpha'-2}$. Now (9.5), (9.6), and (9.7) give

(6) $G_{\alpha'}/Q_{\alpha'} \cong SL_2(2) \wr C_2$, $|V_{\alpha'}| = 2^5$, and $|V_{\alpha'} \cap V_{\alpha'-2}| = 2^3$.

Moreover, there exists $y \in C_{G_{\alpha'-1}}(R)$ such that $\alpha'^y = \alpha' - 2$. Let $X = N_{G_{\alpha'}}(RZ_{\alpha'})$. Then, as in the proof of (9.9), we get from (4) that $X/Q_{\alpha'} \cong SL_2(2) \times C_2$ and $X \cap G_{\alpha'-1} \in \mathrm{Syl}_2(X)$. Hence X^y has the same properties with α' replaced by $\alpha' - 2$. Since $b > 3$ and $W_{\alpha+1} \le G_{\alpha'-2}$ we get that $W_{\alpha+1} \le X^y$ and

(7) $|W_{\alpha+1}/W_{\alpha+1} \cap G_{\alpha'-1}| = 2$.

Assume that $b > 5$. Then $W_{\alpha+1}$ is abelian. Note that by (7), $|V_{\alpha-1}/V_{\alpha-1} \cap G_{\alpha'-1}| \le 2$ and that by (5) and (9.8), $Z_{\alpha'-2} \nleq R_0$; i.e., $V_{\alpha-1} \cap G_{\alpha'-1} \le Q_{\alpha'-1} \le G_{\alpha'}$. Hence

$$\left[V_{\alpha-1} \cap G_{\alpha'-1}, Z_\mu\right] = \left[V_{\alpha-1} \cap G_{\alpha'-1}, V_1\right] \le RZ_{\alpha'}.$$

Suppose that $Z_{\alpha'} \le [V_{\alpha-1} \cap G_{\alpha'-1}, Z_\mu]$. Then $Z_{\alpha'} \le W_{\alpha+1}$ and $[Z_{\alpha'}, V_{\alpha-1}] = 1$ since $W_{\alpha+1}$ is abelian. Thus, by (6), $V_{\alpha-1} \le C_{G_{\alpha'-2}}(Z_{\alpha'-1}) \le G_{\alpha'-1}$ and $V_{\alpha-1} \le G_{\alpha'}$. In particular, there exists a subgroup $A \le V_{\alpha-1}$ such that

$$(*) \qquad \left[A, Z_\mu\right] = R \quad \text{and} \quad |V_{\alpha-1}/A| = 2.$$

Suppose that $Z_{\alpha'} \nleq [V_{\alpha-1} \cap G_{\alpha'-1}, Z_\mu]$. Then, for $A = V_{\alpha-1} \cap G_{\alpha'-1}$, property $(*)$ is satisfied. Hence, a subgroup A with $(*)$ exists in both cases. Now (9.4) gives $A \le V_{\alpha+1}$ which contradicts (9.7). We have shown:

(8) $b = 5$.

According to (6) there exists $\lambda \in \Lambda$ such

$$(**) \quad V_\rho \cap V_{\alpha+1} \cap V_{\alpha+3} = Z_{\alpha+1} \qquad \text{for every } \rho \in \Delta(\lambda) \setminus \{\alpha + 1\}.$$

We fix λ with this property and pick $\rho \in \Delta(\lambda) \setminus \{\alpha + 1\}$. Then $[V_\rho, V_{\alpha+3}] \le V_\rho \cap V_{\alpha+1} \cap V_{\alpha+3} = Z_{\alpha+1}$, and (6) gives $[V_\rho, V_{\alpha+3}] = 1$. Hence $W_\lambda \le G_{\alpha'}$, and since $Z_\mu \le V_1 Z_{\alpha'-1}$ we get

(9) $[W_\lambda, Z_\mu] \le RZ_{\alpha'}$.

Let $Q = O_2(E_{\alpha+1})$ and $Q_0 = C_{Q_{\alpha+2}}(V_{\alpha+1} \cap V_{\alpha+3})$. Then $|Q_{\alpha+1}/Q_0| = 4$ and $Q_{\alpha+2} = [Q_{\alpha+2}, E_{\alpha+2}]Q_0$. Now (7.6)(b) gives $[Q, Q_{\alpha+2}] \nleq Q_0$; in particular, $[Q \cap Q_{\alpha+2}, V_{\alpha+1} \cap V_{\alpha+3}] \ne 1$. From $[W_{\alpha+1}, V_{\alpha+1} \cap V_{\alpha+3}] = 1$ and (6) we conclude

(10) $(Q \cap Q_{\alpha+2})Q_{\alpha+3} = G_{\alpha+2} \cap G_{\alpha+3}$.

Let C be the largest normal subgroup of $G_{\alpha+1}$ in $W_{\alpha+1}$ such that $[C, E_{\alpha+1}] \le V_{\alpha+1}$. Then $(C \cap V_{\alpha-1})V_{\alpha+1} = (C \cap V_{\alpha+3})V_{\alpha+1}$ by the action of $E_{\alpha+1}$. Hence $[(C \cap V_{\alpha-1})V_{\alpha+1}, Z_{\mu}] = R$ and (9.4) yields

(11) $C \cap V_{\alpha-1} \le V_{\alpha+1}$.

Set $W = [W_{\alpha+1}, Q]C$. Assume that $W \le C$. Then by (11), $[V_{\alpha+3}, Q \cap Q_{\alpha+2}] \le V_{\alpha+1} \cap V_{\alpha+3}$, but this contradicts (6) and (10). Thus, we have

(12) $W \cap V_{\alpha+3} \not\le V_{\alpha+1}$.

Assume that $Z_{\alpha'} \le W$. Then by (9), $[W_{\lambda}, Z_{\mu}] \le W$ and $W_{\alpha+1} = W_{\lambda}W$ since $\lambda \in \Lambda$. Hence also $W_{\alpha+1} = W_{\alpha+2}W$, and (8) shows that $[W_{\lambda}, Z_{\mu}] = R \le V_{\alpha+1}$ since $Z_{\mu} \le V_1 Z_{\alpha+4}$ and $[Z_{\alpha+4}, W_{\alpha+2}] = 1$. This gives $[W_{\alpha+1}, Q] \le V_{\alpha+1}$, which contradicts (10) and (11).

Assume finally that $Z_{\alpha'} \not\le W$. Then $(W_{\lambda} \cap W)C$ is normal in $G_{\alpha+1}$ and $W \le V_{\alpha+1}$, which contradicts (12).

10.

In this section we finish the discussion started in Section 9. More precisely, in this section we assume Hypothesis 2, $[Z_{\alpha}, Z_{\alpha'}] = 1$, and $b = 3$.

Recall from (9.3) that $G_{\alpha+2}/Q_{\alpha+2} \cong SL_2(2)$, $|Z_{\alpha+2}| = 4$, and $Z_{\alpha+2} = Z_{\alpha+1} \times Z_{\alpha'} = \Omega_1(Z(Q_{\alpha+2}))$. Since $G_{\alpha+2}$ is 2-transitive on $\Delta(\alpha+2)$ the path $(\alpha+1, \alpha+2, \alpha')$ is a $G_{\alpha+2}$-conjugate of $(\alpha', \alpha+2, \alpha+1)$. Hence, we have symmetry in $\alpha+1$ and α'; i.e., $V_{\alpha+1} \not\le Q_{\alpha'}$ and $V_{\alpha'} \not\le Q_{\alpha+1}$.

Note further that by (7.7)(b), $C_{G_{\alpha'}}(V_{\alpha'}) \le Q_{\alpha'}$. We use the following notation:

$$\overline{V}_{\alpha'} = V_{\alpha'}/Z_{\alpha'} \quad \text{and} \quad \overline{G}_{\alpha'} = G_{\alpha'}/Q_{\alpha'}.$$

(10.1) *Let* $W = \langle (V_{\alpha+1} \cap Q_{\alpha'})^{G_{\alpha+2}} \rangle$ *and* $W_0 = (\bigcap_{\rho \in \Delta(\alpha+2)} Q_{\rho}) \cap W_{\alpha+2}$. *Then*

$$W'_{\alpha+2} = V_{\alpha+1} \cap V_{\alpha'}, \qquad |W_0/W| = 2, \quad \text{and} \quad |W_{\alpha+2}/W| = 2^3,$$

and one of the following holds:

(a) $2^6 \le |S| \le 2^7$ *and* $G_{\alpha+1}/Q_{\alpha+1} \cong G_{\alpha+2}/Q_{\alpha+2} \cong SL_2(2)$, *and*

(a_1) $Z_{\alpha+2} = W$ *and* $O_2(E_{\alpha+2}) \cong C_4 \times C_4$,

(a_2) $|V_{\alpha+1}| = 2^3$ *and* $O_2(E_{\alpha+1})$ *is extra special of order* 2^5, *and*

(a_3) *there exists an involution* $a \in Q_{\alpha+2} \setminus Z_{\alpha+2}$ *such that* $C_G(a)$ *is not solvable*.

(b) $2^{11} \leq |S| \leq 2^{12}$, $G_{\alpha+2}/Q_{\alpha+2} \cong SL_2(2)$, *and* $G_{\alpha+1}/Q_{\alpha+1} \cong$ Fb(20), Fb(20) *being the Frobenius group of order* 20, *and*

(b_1) $|V_{\alpha+1}/Z_{\alpha+1}| = |O_2(E_{\alpha+1})/V_{\alpha+1}| = 2^4$ *and* $O_2(E_{\alpha+1})' = V_{\alpha+1}$, *and*

(b_2) $|Z_{\alpha+2}| = |W/V_{\alpha+1} \cap V_{\alpha'}| = |W_{\alpha+2}/W_0| = |O_2(E_{\alpha+2})W_{\alpha+2}/W_{\alpha+2}| = 4$ *and* $|V_{\alpha+1} \cap V_{\alpha'}/Z_{\alpha+2}| = 2$.

Proof. Since $Q_{\alpha+1}$ is transitive on $\Delta(\alpha+2) \setminus \{\alpha+1\}$ we get that $V_{\alpha+1} \cap Q_{\alpha'} = V_{\alpha+1} \cap Q_\rho$ for $\rho \in \Delta(\alpha+2) \setminus \{\alpha+1\}$. It follows that

(1) $W \leq \bigcap_{\rho \in \Delta(\alpha+2)} Q_\rho$.

Since $[V_{\alpha+1} \cap Q_{\alpha'}, V_{\alpha'} \cap Q_{\alpha+1}] \leq Z_{\alpha+1} \cap Z_{\alpha'} = 1$ we conclude that

(2) $\Phi(W) = 1$.

Assume that $O_2(E_{\alpha+1}) \leq V_{\alpha+1}$. Then by (7.6)(b), $V_{\alpha+1} \not\leq Q_{\alpha+2}$ and $b = 2$, a contradiction. Thus, we have:

(3) There exists a noncentral chief factor of $E_{\alpha+1}$ in $O_2(E_{\alpha+1})/V_{\alpha+1}$.

$V_{\alpha'}$ operates quadratically on $V_{\alpha+1}/Z_{\alpha+1}$. Hence, by (1.2) there exists $t \in V_{\alpha+1} \setminus Q_{\alpha'}$ and $A \leq V_{\alpha'}$ such that $|V_{\alpha'}/A| = 2$ and $[A, t] \leq Z_{\alpha+1}$. We get that

(4) $\|[\bar{V}_{\alpha'}, t]\| = 2$ or $\|[\bar{V}_{\alpha'}, t]\| = 4$ and $Z_{\alpha+1} \leq [V_{\alpha'}, t]$.

Suppose that there exists $a \in V_{\alpha+1} \setminus Q_{\alpha'}$ such that $\|[\bar{V}_{\alpha'}, a]\| = 2$. Since $[V_{\alpha'}, a] \leq V_{\alpha+1} \cap V_{\alpha'}$ we get from (9.5) that either

(5) $G_{\alpha'}/Q_{\alpha'} \cong SL_2(2) \wr C_2$, $|V_{\alpha'}| = 2^5$, and $|V_{\alpha+1} \cap V_{\alpha'}| = 2^3$, or

(6) $G_{\alpha'}/Q_{\alpha'} \cong SL_2(2)$ and $|V_{\alpha'}| = 2^3$.

Assume that (5) holds. Then $\langle \bar{a} \rangle$ is not normal in $\bar{Q}_{\alpha+2}$ and so $|\bar{V}_{\alpha+1}| = 4$ and $V_{\alpha+1} \cap Q_{\alpha'} = V_{\alpha+1} \cap V_{\alpha'}$. It follows that $[Q_{\alpha'} \cap Q_{\alpha+2}, V_{\alpha+1}] \leq V_{\alpha'}$. Since $|Q_{\alpha'}/Q_{\alpha'} \cap Q_{\alpha+2}| = 2$ this contradicts (3) and $|\bar{V}_{\alpha+1}| = 4$.

Assume that (6) holds. We will show (a). Since $|V_{\alpha+1}| = 8$ we get that $Z_{\alpha+2} = W = V_{\alpha+1} \cap V_{\alpha'}$ and $W'_{\alpha+2} = Z_{\alpha+2}$. Moreover, if $|W_{\alpha+2}/Z_{\alpha+2}| = 4$, then $W_{\alpha+2}$ is abelian which contradicts $b = 3$. Thus, we have $|W_{\alpha+2}/Z_{\alpha+2}| = 2^3$.

It follows that $\|[O_2(E_{\alpha'}), W_{\alpha+2}]V_{\alpha'}/V_{\alpha'}]\| = 2$ and thus $|O_2(E_{\alpha'})/V_{\alpha'}| = 4$. Now $O_2(E_{\alpha'})/Z_{\alpha'}$ is abelian and so $[O_2(E_{\alpha'}), \Phi(O_2(E_{\alpha'}))] = 1$. Hence $O_2(E_{\alpha'}) \cong Q_8 \curlyvee Q_8$.

Note that $[Q_{\alpha+2}, O_2(E_{\alpha'})] \leq W_{\alpha+2}$ and thus by (7.6)(b), $O_2(E_{\alpha+2}) = [W_{\alpha+2}, E_{\alpha+2}]$. Now the structure of $O_2(E_{\alpha'})$ gives (a_1). Let $C = C_{Q_{\alpha+2}}(D)$, where $D \in \mathrm{Syl}_3(G_{\alpha+2})$. Note that $Z(G_{\alpha+2}) = 1$ and thus $C_C(O_2(E_{\alpha+2})) = 1$. By (a_1), $[C, O_2(E_{\alpha+2})] = Z_{\alpha+2}$ and $C_{O_2(E_{\alpha+2})}(c) = Z_{\alpha+2}$ for $1 \neq c \in C$. This gives $|C| \leq 4$ and $|S| \leq 2^7$.

It remains to prove (a_3). We will apply (a_1) and (a_2) without reference. Let $T = G_{\alpha+2} \cap G_{\alpha'}$ and $W^* = C_{Q_{\alpha+2}}(W_0)$. Then either $W^* = W_0$ or $|S| = 2^7$ and $|W^*| = 2^4$. Since $[W^*, E_{\alpha+2}] = Z_{\alpha+2}$ and $Z(G_{\alpha+2}) = 1$ we get in both cases that W^* is an elementary abelian normal subgroup of $G_{\alpha+2}$.

Let $N = C_H(Z_{\alpha+2})$ and $N_0 = O_2(N)$. Then $N_H(Z_{\alpha+2}) = G_{\alpha+2}N$ and $Q_{\alpha+2} \in \mathrm{Syl}_2(N)$. Clearly $C_N(N_0/Z_{\alpha+2}) \leq N_0$ since N is of characteristic 2 type. Moreover, $N_0 \leq Q_{\alpha+2}$ and $Q_{\alpha+2}/Z_{\alpha+2}$ is abelian. Hence $N_0 = Q_{\alpha+2}$ and $N \leq N_H(Q_{\alpha+2})$.

Assume that $W_0 = W^*$. Then $|Q_{\alpha+2}/Z_{\alpha+2}| = 2^3$ and $N_H(Q_{\alpha+2})/Q_{\alpha+2}$ is a solvable subgroup of $L_3(2)$ containing Σ_3. This gives $N_H(Q_{\alpha+2}) = G_{\alpha+2}$.

Assume that $W_0 \neq W^*$. Note that $V_{\alpha'}$ is a maximal elementary abelian subgroup of $Q_{\alpha+2}$. Hence, W^* is the only elementary abelian subgroup of order 2^4 in $Q_{\alpha+2}$, and W^* is normal in $N_H(Q_{\alpha+2})$. Let c be an element of odd order in $C_H(Q_{\alpha+2}/W^*)$. Then $Q_{\alpha+2} = W^* C_{Q_{\alpha+2}}(c)$ and $Z_{\alpha+2} = \Phi(C_{Q_{\alpha+2}}(c))$. The 3-subgroup lemma gives $[W^*, \langle c \rangle, C_{Q_{\alpha+2}}(c)] = 1$ and thus $[W^*, \langle c \rangle] \leq Z_{\alpha+2}$. Since c has odd order this yields $[W^*, c] = 1$ and $[Q_{\alpha+2}, c] = 1$. We conclude that $C_H(Q_{\alpha+2}/W^*)$ is a 2-group and $G_{\alpha+2} = N_H(Q_{\alpha+2})$. This implies

(7) $N_H(Z_{\alpha+2}) = G_{\alpha+2}$.

Assume that $W^* \neq W_0$. Suppose that $[W^*, T] \leq Z_{\alpha+2}$. Then $[W^*, O_2(E_{\alpha'})] \leq Z_{\alpha+2} \leq V_{\alpha'}$ and thus $W^* \leq Q_{\alpha'}$. This gives $[W^*, O_2(E_{\alpha'})] \leq Z_{\alpha'}$ and $|W^*/C_{W^*}(O_2(E_{\alpha'}) \cap O_2(E_{\alpha+2}))| \leq 2$. However, the action of D on W^* and $O_2(E_{\alpha+2})$ implies

$$C_{W^*}(O_2(E_{\alpha'}) \cap O_2(E_{\alpha+2})) = C_{W^*}(O_2(E_{\alpha+2})) = Z_{\alpha+2}.$$

This contradicts $|W^*/Z_{\alpha+2}| = 4$. We have shown:

(8) Either $W^* = W_0$ or $T/O_2(E_{\alpha+2}) \cong D_8$.

Let $M = N_H(W^*)$ and set $U = O_{2,2'}(M)T$ and $W_1 = \langle Z_{\alpha'}^U \rangle$. By (1.2),

$$W_1 = \langle C_{W_1}(A) \mid |Q_{\alpha+2}/A| = 2 \rangle$$

since $Q_{\alpha+2}$ is quadratic on W_1. Let A be any subgroup of index 2 in $Q_{\alpha+2}$ and $a \in C_{W_1}(A)$. Then $G_{\alpha+2} = Q_{\alpha+2}C_{G_{\alpha+2}}(a)$ and $\llbracket a, Q_{\alpha+2} \rrbracket \leq 2$. This gives $a \in Z_{\alpha+2}$, a contradiction.

We have shown that $W_1 = Z_{\alpha+2}$, and (7) implies

(9) $N_H(W^*) = G_{\alpha+2}$.

Let $a \in W^* \setminus Z_{\alpha+2}$ and $v \in Z^{\#}_{\alpha+2}$. Note that $C_H(v)$ is of characteristic 2 type. Set $C_a = C_H(a)$, $C_0 = O_{2'}(C_a)$, and $C_1 = O_2(C_a)$. Then

$$C_0 = \langle C_{C_0}(v) \mid v \in Z^{\#}_{\alpha+2} \rangle$$

and the $P \times Q$ lemma applied to $C_{C_0}(v) \times \langle a \rangle$ and $O_2(C_H(v))$ gives $C_0 = 1$.

To prove (a_3) we may assume that $C_{C_\alpha}(C_1) \leq C_1$. According to (9) and the structure of $G_{\alpha+2}$ we have $N_{C_1}(W^*) \leq W^*$ and thus $C_1 = W^*$. This yields, together with (9),

(10) $C_H(a) \leq G_{\alpha+2}$ for every $a \in W^* \setminus Z_{\alpha+2}$.

Next pick $u \in C_a$ such that $\langle u \rangle$ is conjugate to $Z_{\alpha'}$. If $u \in W^*$, then (10) implies that $u \in Z_{\alpha+2}$.

Assume that u is not in W^*. Then $u \notin Q_{\alpha+2}$ since $C_a \cap Q_{\alpha+2} = W^*$, and by (8), $a \in W_0$. We may assume that $u \in G_{\alpha'}$. If $u \notin Q_{\alpha'}$, then $[V_{\alpha'}, u]Z_{\alpha'} = Z_{\alpha+2}$ and $u \in Q_{\alpha+2}$, a contradiction. Thus $u \in Q_{\alpha'}$ and therefore $u \in O_2(E_{\alpha'})$ since $C_{Q_{\alpha'}}(a) \leq O_2(E_{\alpha'})$. Now u is conjugate in $G_{\alpha'}$ to an involution in $aV_{\alpha'}$. On the other hand, every involution in $aV_{\alpha'}$ is in $aZ_{\alpha+2} = aC_{V_{\alpha'}}(a)$. Hence, there exists a conjugate of u in $W^* \setminus Z_{\alpha+2}$, which is impossible as we have seen above. We have shown:

(11) Every conjugate of $Z_{\alpha'}$ in $C_H(a)$ is contained in $Z_{\alpha+2}$.

Let $(\alpha+1, \ldots, \alpha', \ldots, \alpha+9)$ be a path of length 8 and let $u \in Z(\langle a, Z_{\alpha+9} \rangle)^{\#}$. Then $u \in C_a \leq G_{\alpha+2}$. Assume that u is conjugate in H to an element of $Z_{\alpha'}$. Then by (11), $u \in Z_{\alpha+2}$ and thus $\langle u \rangle = Z_\rho$ for some $\rho \in \Delta(\alpha+2)$. On the other hand, $Z_\delta \leq G_{\alpha+5}$ and $Z_\delta \not\leq Q_{\alpha+5}$ for $\delta = \alpha+2, \alpha+9$. This implies that $[Z_\rho, Z_{\alpha+9}] \neq 1$, a contradiction.

Assume that u is not conjugate to an element of $Z_{\alpha'}$. Then, as in step (11), conjugation in $G_{\alpha'}$ shows that u is conjugate to an element of $W^* \setminus Z_{\alpha+2}$. Hence, by (10), $C_H(u) \leq G_\mu$ for some $\mu \in \alpha^G$, but then by (11), $Z_{\alpha+9} \leq Z_\mu$, and $\langle a, Z_{\alpha+9} \rangle$ is abelian since $u \notin Z_\mu$. It follows, again by (11), that $Z_{\alpha+9} \leq Z_{\alpha+2}$. Now $Z_{\alpha+9}$ fixes $\alpha+4$ and $\alpha+6$ and so $Z_{\alpha+9} \leq Q_{\alpha+5}$, a contradiction. This last contradiction shows that (a_3) holds. We may assume now:

(12) No involution in $V_{\alpha+1} \setminus Q_{\alpha'}$ induces a transvection on $\overline{V}_{\alpha'}$.

In particular, by (4) there exists $t \in V_{\alpha+1} \setminus Q_{\alpha'}$ such that $\|[\overline{V}_{\alpha'}, t]\| = 4$ and $Z_{\alpha+1} \le [V_{\alpha'}, t]$. According to (3.6) there exists a subgroup E in $E_{\alpha'}$ such that for $\overline{A}_0 = C_{\overline{V}_{\alpha+1}}(\overline{E})$:

(i) $\overline{E}\langle \bar{t} \rangle$ is dihedral and $E = O^2(E)$,

(ii) $\overline{V}_{\alpha+1} = \langle \bar{t} \rangle \overline{A}_0$, and

(iii) $\langle E, G_{\alpha+2} \cap G_{\alpha'} \rangle = G_{\alpha'}$.

Set $V_1 = [V_{\alpha'}, E]$ and $V_0 = C_{V_{\alpha'}}(E)$. Note that $[V_1, A_0] \le Z_{\alpha'}$ since $V_{\alpha+1}$ is quadratic on $V_{\alpha'}$. Let $y \in Q_{\alpha+2}$ and $t' = t^y$.

Assume that $t' \in A_0$. Then $[V_{\alpha'}, t'] \le [V_0, t']Z_{\alpha'}$. It follows that $Z_{\alpha+1} \le V_0$ and $[Z_{\alpha+2}, E] = 1$. Now (iii) shows that $Z_{\alpha+2}$ is normal in $G_{\alpha'}$, a contradiction.

We have shown that $t' \notin A_0$; i.e., $C_{V_1}(t) = C_{V_1}(t')$ and $[V_1, t]Z_{\alpha'} = [V_1, t']Z_{\alpha'}$. Hence, either $[\overline{V}_1, t] = \overline{Z}_{\alpha+1}$ or $[\overline{V}_{\alpha'}, t] = [\overline{V}_1, t]\overline{Z}_{\alpha+1}$ and $[\overline{V}_{\alpha'}, t] = [\overline{V}_{\alpha'}, t']$. In the first case $|\overline{V}_1| = 4$, and by (1.4), $\overline{E}_{\alpha'}$ is elementary abelian. Since $Z_{\alpha+2} \le V_1$ we conclude that $V_1 = V_{\alpha'}$, a contradiction to (12). Hence, we are in the second case and $\bar{t} = \overline{t'}$. It follows that $\langle \bar{t} \rangle$ is normal in $\overline{Q}_{\alpha+2}$. Together with (3.4) we get

(13) $E_{\alpha'} = [E_{\alpha'}, t]$ and $[V_{\alpha'}, t]Z_{\alpha+2}$ is normal in $G_{\alpha+2}$.

We now apply (1.3). If $\overline{E}_{\alpha'} \cong C_3$, then $|\overline{V}_{\alpha'}| \le 2^3$, which contradicts (12).

Assume that $\overline{E}_{\alpha'}$ is extra special of order 3^3. Let $\langle \bar{e} \rangle = Z(\overline{E}_{\alpha'})$ and $R = [V_{\alpha'}, t]$. Note that $[\bar{e}, \bar{t}] = 1$. Hence by (13), $RZ_{\alpha+2}$ is normal in $\langle G_{\alpha+2}, e \rangle$, and it is easy to see that

$$N_G(RZ_{\alpha+2})/C_G(RZ_{\alpha+2}) \cong L_3(2),$$

which contradicts Hypothesis 2. From (1.3) we conclude:

(14) $\overline{E}_{\alpha'} \cong C_3 \times C_3$ or C_5, $|V_{\alpha'}| = 2^5$, and $|V_{\alpha+1} \cap V_{\alpha'}| = |C_{V_{\alpha+1}}(V_{\alpha'})| = 2^3$.

By (3) there exists a noncentral chief factor of $G_{\alpha'}$ in $O_2(E_{\alpha'})/V_{\alpha'}$, and by (13) and (14), t does not induce a transvection on that chief factor. Hence $[O_2(E_{\alpha'}) \cap Q_{\alpha+2}, V_{\alpha+1}] \not\le V_{\alpha'}$ and thus

(15) $|V_{\alpha+1}/V_{\alpha+1} \cap O_2(E_{\alpha'})| = 2$; in particular, $W \le O_2(E_{\alpha'})$.

Set $Y = C_W(V_{\alpha'})\Omega_1(Z(W_{\alpha+2}))$. By (14), $[C_{Q_{\alpha'}}(V_{\alpha'}), V_{\alpha+1}] \le V_{\alpha'}$ and thus $[Y, E_{\alpha'}] \le V_{\alpha'}$. Let $D \in \mathrm{Syl}_3(E_{\alpha'})$. Then

$$YV_{\alpha'} = Y_0 V_{\alpha'} \qquad \text{for } Y_0 = C_{YV_{\alpha'}}(D).$$

On the other hand, by (7.6)(b), $|O_2(E_{\alpha'})/O_2(E_{\alpha'}) \cap Q_{\alpha+2}| = 2$ and

$$\left[Y, O_2(E_{\alpha'}) \cap Q_{\alpha+2}\right] \le Z(W_{\alpha+2}) \cap V_{\alpha'} \le V_{\alpha+1} \cap V_{\alpha'}.$$

Thus $|[\langle y\rangle, O_2(E_{a'})]Z_{\alpha'}/Z_{\alpha'}| \le 2^3$ for $y \in Y$, and (14) gives $[Y, O_2(E_{\alpha'})] \le Z_{\alpha'}$; in particular, Y_0 is normal in $E_{\alpha'}$. Now (12) shows that $Y_0 \le \Omega_1(Z(W_{\alpha+2})) \le Y$. It follows that

$$Y = Y_0(V_{\alpha+1} \cap V_{\alpha'}) \quad \text{and} \quad Y = \Omega_1(Z(W_{\alpha+2})).$$

Again from $[Y, O_2(E_{\alpha'})] \le Z_{\alpha'}$ and from (7.6)(b) we get that $[Y, E_{\alpha+2}] = Z_{\alpha+2}$. Assume that there exists $y \in Y_0 \setminus Z_{\alpha'}$. Then $C_{G_\delta}(y)$ is transitive on $\Delta(\delta)$ for $\delta = \alpha + 2, \alpha'$, and (7.2) yields a contradiction. We have shown that $Y_0 = Z_{\alpha'}$ and thus

(16) $C_W(V_{\alpha'}) = \Omega_1(Z(W_{\alpha+2})) = V_{\alpha+1} \cap V_{\alpha'}$.

Note that $[W, O_2(E_{\alpha'})] \le C_W(V_{\alpha'})$ since $[V_{\alpha'}, O_2(E_{\alpha'})] \le Z_{\alpha'}$. Hence (16) gives

(17) $|W/V_{\alpha+1} \cap V_{\alpha'}| = 4$ and $[W, G_{\alpha+2} \cap G_{\alpha'}] \le V_{\alpha'}$.

Let $U = \langle W^{E_{\alpha'}}\rangle$. Then $U/V_{\alpha'}$ is elementary abelian. Note that $|[O_2(E_{\alpha'}), t]V_{\alpha'}/V_{\alpha'}| \le 4$. Hence (3) and (14) show that $U = O_2(E_{\alpha'})$ and

(18) $|O_2(E_{\alpha'})/V_{\alpha'}| = 2^4$.

Assume now that $\overline{E}_{\alpha'} \cong C_3 \times C_3$. Let $D = D_1 \times D_2$ [$\in \mathrm{Syl}_3(E_{\alpha'})$], such that $D_i \cong C_3$ and $W_i := C_{V_{\alpha'}}(D_i) \ne Z_{\alpha'}$. Then $V_{\alpha'} = W_1W_2$, $W_1 \cap W_2 = Z_{\alpha'}$, and $[W_i, D_j]Z_{\alpha'} = W_i$.

Suppose that $[O_2(E_{\alpha'}), D_i] = O_2(E_{\alpha'})$. Then the 3-subgroup lemma gives $[O_2(E_{\alpha'}), W_i] = 1$ and thus $[O_2(E_{\alpha'}), V_{\alpha'}] = 1$, which contradicts (16).

Hence $O_2(E_{\alpha'}) = Q_1Q_2$, where $Q_i = C_{O_2(E_{\alpha'})}(D_i)$. Note that $[Q_i, D_j]Z_{\alpha'} = Q_i$. Hence, another application of the 3-subgroup lemma yields $[Q_1, Q_2] \le W_1 \cap W_2 = Z_{\alpha'}$. Since $|Q_i/W_i| = 4$ we conclude that $Q_i' \le Z_{\alpha'}$ and thus $O_2(E_{\alpha'})' \le Z_{\alpha'}$. However, by (7.6)(b), $O_2(E_{\alpha'}) \not\le Q_{\alpha+2}$ and so $[W, O_2(E_{\alpha'})] \not\le Z_{\alpha+2}$. This contradicts (15). Together with (14) we have shown that $\overline{E}_{\alpha'} \cong C_5$.

Note that $|O_2(E_{\alpha'})/O_2(E_{\alpha'}) \cap Q_{\alpha+2}| = 2$ and $[O_2(E_{\alpha'}) \cap Q_{\alpha+2}, V_{\alpha+1}] \le V_{\alpha+1} \cap Q_{\alpha'} \le W$. By (17), $|W/W \cap V_{\alpha'}| = 2$ and so $O_2(E_{\alpha'}) \cap Q_{\alpha+2} \not\le V_{\alpha'}Q_{\alpha+1}$. This implies:

(19) $G_{\alpha+1}/Q_{\alpha+1} \cong \mathrm{Fb}(20)$.

Since $V_{\alpha+1}$ does not induce transvections on $O_2(E_{\alpha'})/V_{\alpha'}$ we also have $|W_{\alpha+2}/W| = 2^3$. Let $W \leq \tilde{W} \leq W_{\alpha+2}$ such that $[\tilde{W}, O^2(E_{\alpha+2})] \leq W$ and $|\tilde{W}/W| = 2$. Then $|[\tilde{W}, O_2(E_{\alpha'})]V_{\alpha'}/V_{\alpha'}| \leq 2$ and thus $\tilde{W} \leq Q_{\alpha'}$. Hence $\tilde{W} = W_0$.

To prove (b) it remains to show $|Q_{\alpha'}/O_2(E_{\alpha'})| \leq 2$ since then $|Q_{\alpha+2}/W_{\alpha+2}| \leq 2^3$ and $|O_2(E_{\alpha+2})W_{\alpha+2}/W_{\alpha+2}| = 4$. We apply ($b_1$) and ($b_2$) without reference.

Let $C = C_{Q_{\alpha'}}(O_2(E_{\alpha'}))$. Note that $C \leq C_{G_{\alpha+2}}(W) \leq Q_{\alpha+2}$. Since $V_{\alpha'} \not\leq C$ we get that $C \cap V_{\alpha'} = Z_{\alpha'}$ and $[C, V_{\alpha+1}] \leq C \cap V_{\alpha+1} = Z_{\alpha'}$. Hence (12) implies $C \leq C_{Q_{\alpha+1}}(V_{\alpha+1})$ and $[C, O_2(E_{\alpha+1})] \leq V_{\alpha+1}$. On the other hand, $[C, O_2(E_{\alpha+1}) \cap Q_{\alpha+2}] \leq C \cap V_{\alpha+1} = Z_{\alpha'}$ and so $|[c, O_2(E_{\alpha+1})]| \leq 4$ for $c \in C$. Thus $[C, O_2(E_{\alpha+1})] \leq Z_{\alpha+1}$ and $[C, O_2(E_{\alpha+1}) \cap Q_{\alpha+2}] = 1$. Now $CZ_{\alpha+2}$ is normal in $G_{\alpha+2}$ and $[CZ_{\alpha+2}, O_2(E_{\alpha+2})] = 1$. From (7.5)(c) we get that

(20) $C = Z_{\alpha'}$.

Let $C_1 = C_{Q_{\alpha'}}(V_{\alpha'})$. Then $Q_{\alpha'} = C_1 O_2(E_{\alpha'})$ and $C_1 \cap O_2(E_{\alpha'}) = V_{\alpha'}$. Hence it suffices to prove that $|C_1/V_{\alpha'}| \leq 2$. Since $[V_{\alpha'} \cap V_{\alpha+1}, C_1] = 1$ it follows that $C_1 \leq V_{\alpha'} Q_{\alpha+1}$. Let $(\alpha - 1, \alpha, \alpha + 1, \ldots, \alpha')$ be a path of length 4 such that

$$Z_{\alpha'} \not\leq Q_{\alpha-1} \quad \text{and} \quad Z_{\alpha-1} \not\leq Q_{\alpha'},$$

and let $C_2 = C_1 \cap Q_{\alpha-1}$.

Note that $[C_2, V_{\alpha-1}] \leq V_{\alpha+1}$ and thus $[E_{\alpha+1}, C_2] \leq V_{\alpha+1}$. It follows that $C_2 \leq C_{Q_{\alpha+1}}(V_{\alpha+1})$ and $[C_2, V_{\alpha+1} \cap Q_{\alpha'}] = 1$. This implies that $[C_2, O_2(E_{\alpha'})] \leq Z_{\alpha'}$.

Assume that $C_2 \not\leq V_{\alpha'}$. Then $C_{C_2 V_{\alpha'}}(O_2(E_{\alpha'})) \not\leq Z_{\alpha'}$ by (ii) since $|C_2 V_{\alpha'}/C_2 V_{\alpha'} \cap Q_{\alpha-1}| = 4$, a contradiction to (12).

Let $C_0 = C_{C_2 V_{\alpha'}}(D)$. Then $[C_0, O_2(E_{\alpha'})] = 1$ and by (20), $C_0 = Z_{\alpha'}$. Hence $C_2 \leq V_{\alpha'}$ and $|C_1/V_{\alpha'}| \leq 2$ since $V_{\alpha'} \cap Q_{\alpha+1} \not\leq Q_\alpha$ and $|C_1 \cap Q_\alpha/(C_1 \cap Q_{\alpha-1})Z_{\alpha'}| = 2$. This proves (b).

DEFINITION. Let H be a finite group which contains two subgroups $G_{\alpha+2}$ and $G_{\alpha+1}$ such that $G_{\alpha+2} \cap G_{\alpha+1} = S \in \mathrm{Syl}_2(H)$ and $O_2(\langle G_{\alpha+2}, G_{\alpha+1}\rangle) = 1$. Then H is of type M_{12} and ${}^2F_4(2)'$, respectively, if (10.1)(a) and (b), respectively, hold for $G_{\alpha+2}$ and $G_{\alpha+1}$.

Note that M_{12} and ${}^2F_4(2)'$ are examples for such groups H.

11.

In this section we prove Theorems 1 and 2. Let H be a finite group. Suppose that

(i) H satisfies the hypothesis of Theorem 1 or

(ii) H satisfies the hypothesis of Theorem 2, but not (d) or (e) of its conclusion.

Then in both cases H satisfies Hypothesis 2 of Section 5. If (i) holds, then (8.2), (8.6), and (9.1), (9.10), and (10.1) prove Theorem 1. Hence, we may assume that (ii) holds.

If H also satisfies the hypothesis of Theorem 1, then (a) of Theorem 2 is a direct consequence of Theorem 1. Thus, we may assume, in addition, that S_0 is contained in a unique maximal 2-local subgroup of H. In particular, the subgroups S, P_1, and P_2 of Hypothesis 2 are as in (5.1)(c). Now (8.2) shows that either

(I) $P_1 \cong P_2 \cong \Sigma_4$ and $S \cong D_8$, or

(II) $P_1 \cong P_2 \cong C_2 \times \Sigma_4$ and $S \cong C_2 \times D_8$.

Let $Q = O_2(P_1)$ and $N = N_H(Q)$. Then (5.1)(c_2) implies that $S \in \mathrm{Syl}_2(N)$, and the solvability of N and $C_N(Q) \leq Q$ gives $N = P_1$.

Assume case (I). Let $x \in Q \setminus Z(S_0)$. Then $Q = C_{S_0}(x)$ and

$$4^{-1}|S_0| = |S_0/C_{S_0}(x)| = \left|\{[s,x] | s \in S_0\}\right|.$$

It follows that $S_0' = \Phi(S_0)$ and $|S_0/\Phi(S_0)| = 4$. Now [4, 5.4.5] shows that (b) of Theorem 2 holds.

In case (II), S contains exactly two elementary abelian subgroups of order 8 and $|N_{S_0}(S)/S| = 2$. On the other hand, $C_H(Z(P_1))$ is of characteristic 2 type, and so $C_H(Z(P_1)) = P_1$. It follows that $J(S) = J(N_{S_0}(S))$ and $S_0 = N_{S_0}(S)$. Now (c) of Theorem 2 holds.

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REFERENCES

1. B. Baumann, Über endliche Gruppen mit einer zu $L_2(2^n)$ isomorphen Faktorgruppe, *Proc. Amer. Math. Soc.* **74** (1979), 215–222.
2. D. Goldschmidt, Automorphisms of trivalent graphs, *Ann. of Math.* **111** (1980), 377–404.

3. K. Gomi, On the 2-local structure of groups of characteristic 2 type, *J. Algebra* **108** (1987), 492–502.
4. D. Gorenstein, ''Finite Groups,'' Harper & Row, New York, 1968.
5. D. Gorenstein and R. Lyons, Nonsolvable finite groups with solvable 2-local subgroups, *J. Algebra* **38** (1976), 453–522.
6. Z. Janko, Nonsolvable finite groups all of whose 2-local subgroups are solvable, I, *J. Algebra* **21** (1972), 458–517.
7. F. Smith, Finite simple groups all of whose 2-local subgroups are solvable, *J. Algebra* **34** (1975), 481–520.
8. B. Stellmacher, Pushing up, *Arch. Math.* **46** (1986), 8–17.
9. J. G. Thompson, Nonsolvable finite groups all of whose local subgroups are solvable, I, *Bull. Amer. Math. Soc.* **74** (1968), 383–437; II, *Pacific J. Math.* **33** (1970), 451–536; III, *Pacific J. Math.* **39** (1971), 483–534; IV, *Pacific J. Math.* **48** (1973), 511–592; V, *Pacific J. Math.* **50** (1974), 215–297; VI, *Pacific J. Math.* **51** (1974), 573–630.