Max-algebra and pairwise comparison matrices, II

L. Elsner a,*, P. van den Driessche b,1

a Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany
b Department of Mathematics and Statistics, University of Victoria, Victoria, BC, Canada V8W 3R4

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ABSTRACT

This paper is a continuation of our 2004 paper “Max-algebra and pairwise comparison matrices”, in which the max-eigenvector of a symmetrically reciprocal matrix was used to approximate such a matrix by a transitive matrix. This approximation was based on minimizing the maximal relative error. In a later paper by Dahl a different error measure was used and led to a slightly different approximating transitive matrix. Here some geometric properties of this approximation problem are discussed. These lead, among other results, to a new characterization of a max-eigenvector of an irreducible nonnegative matrix. The case of Toeplitz matrices is discussed in detail, and an application to music theory that uses Toeplitz symmetrically reciprocal matrices is given.

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1. Introduction

As introduced in [13], an (entrywise) positive \( n \)-by-\( n \) matrix \( A = (a_{ij}) \) is called a symmetrically reciprocal matrix (SR-matrix) if \( a_{ij}a_{ji} = 1 \) for \( i, j = 1, \ldots, n \). In [10], an SR-matrix \( B = (b_{ij}) \) is called transitive if there is a positive \( n \)-vector \( w = (w_1, \ldots, w_n) \) such that \( b_{ij} = w_i/w_j \) for \( i, j = 1, \ldots, n \). Approximating an SR-matrix \( A \) by a transitive matrix \( B \) is an important step in the analytic hierarchy process (AHP method) for decision making by attaching a ranking to the SR-matrix; see, e.g. [13,14]. A transitive (thus rank one) matrix \( B \) is sought that minimizes some distance to \( A \); for example, in [9] this is taken as the Frobenius norm \( ||A - B||_F \).

For an SR-matrix \( A \) and a positive vector \( w = (w_1, \ldots, w_n) \) define the functionals

\[
E_1(w) = \max_{ij} |a_{ij} - w_i/w_j|/a_{ij}
\]  

(1)
and

\[ E_2(w) = \max_{i,j} a_{ij} w_j / w_i. \]  

(2)

The functional \( E_1 \) is introduced in [7], and minimizes the maximal relative error, while \( E_2 \) is treated in [3], in which the notation \( E_2(w) = \delta(A, B) \) is used where \( B = (w_i / w_j) \) is a transitive matrix. As obviously \( E_1(w) = E_2(w) - 1 \), with these distances the approximation problem mentioned above boils down to the problem of minimizing \( E_2(w) \).

The functional in (2) motivates our study in Section 2 of the set of vectors

\[ C_{A,r} = \{ w > 0, E_2(w) \leq r \} \]  

(3)

and its normalized subset

\[ D_{A,r} = \{ x \in C_{A,r}, x_1 = 1 \}, \]  

(4)

where \( A = (a_{ij}) \) is any irreducible nonnegative matrix with \( E_2(w) \) as defined in (2) and \( r \) is a positive number. The set \( D_{A,r} \) leads to a new characterization of a max-eigenvector (statement 4 of Theorem 1 below). In Section 3 we prove a perturbation result and specialize this to SR-matrices. In Section 4 we specialize to Toeplitz matrices, and in Section 5 we discuss a problem from music theory that uses Toeplitz SR-matrices.

As was outlined in [7], it makes sense to treat this approximation problem in the framework of the max-algebra, and we refer a reader to Section 2 of [7] and references therein. Briefly, the max-algebra consists of the set \( \mathbb{R}_+ \) of nonnegative numbers, with \( a \oplus b = \max\{a, b\} \) and \( a \otimes b = ab \), the usual product, for \( a, b \in \mathbb{R}_+ \). In the notation of [7], \( \mu(A) \) denotes the max-eigenvalue of a nonnegative irreducible matrix \( A \), i.e., \( A \otimes x = \mu(A)x \) for a nonnegative max-eigenvector \( x \). The eigenvalue \( \mu(A) \) is the maximum cycle geometric mean in the weighted directed graph associated with \( A \). If \( A \) is irreducible, then

\[ \mu(A) = \min_{w > 0} E_2(w) \]  

(5)

with \( E_2(w) \) given by (2); see [7,8]. The critical matrix of \( A \), denoted by \( A^C = (a_{ij}^C) \), is formed from the principal submatrix of \( A \) on rows and columns corresponding to vertices on cycles with geometric mean equal to \( \mu(A) \), by setting \( a_{ij}^C = a_{ij} \) if \( i,j \) is on such a cycle, and \( a_{ij}^C = 0 \) otherwise. The vertex set so identified is denoted by \( V^C \). It is well known (see, e.g. [6, Theorem 4.1] that for a matrix \( A \) having \( \mu(A) = 1 \), the set \( V^C(A) \) consists of those \( i \) for which \( (A^+)_{ii} = 1 \). Here we use the notation

\[ A^+ = A \oplus A^2 \oplus \cdots A^n \]  

(6)

and

\[ A^* = I \oplus A \oplus A^2 \oplus \cdots A^{n-1}. \]  

(7)

For \( \mu(A) \leq 1 \), it follows that

\[ I \oplus A \otimes A^* = A^*. \]  

(8)

Thus for any \( i \in V^C(A) \) the \( i \)th column of \( A^* \) is a max-eigenvector of \( A \) [6].

2. Properties of \( C_{A,r} \) and \( D_{A,r} \)

We first collect properties of the set \( C_{A,r} \) defined in the Introduction above, under the assumptions that \( A \) is irreducible and nonnegative. From (5), it follows that

\[ C_{A,r} = \{w > 0, E_2(w) \leq r\} = \{w > 0, A \otimes w \leq rw\}. \]  

(9)

Using the irreducibility assumption, equivalently

\[ C_{A,r} = \{x \geq 0, x \neq 0, A \otimes x \leq rx\}, \]  

(10)

i.e. there is no need to impose the restriction of positive \( x \). Also
Theorem 1. Let $A$ be an $n$-by-$n$ nonnegative irreducible matrix with $C_{A,r}$, $C_A$, $D_{A,r}$, $D_A$ as defined in (3), (4) and (12). Then the following statements hold.

1. $x \in C_{A,r} \iff x^{-1} \in C_{A,\gamma r}$.
2. If $x, y \in C_{A,r}$ then
   
   (a) $\gamma x + (1 - \gamma)y \in C_{A,r}$ for all $\gamma \in [0,1]$.
   (b) $x' \circ y^{1-\gamma} \in C_{A,r}$ for all $\gamma \in [0,1]$.
   (c) $(x^p + y^p)^{1/p} \in C_{A,r}$ for all $p \in R, p \neq 0$.
   (d) $x \oplus y \in C_{A,r}$.
   (e) $z \in C_{A,r}$ where $z = (z_i) = (\min(x_i, y_i))$.
3. $D_{A,r}$ is compact and $D_A$ is a convex polytope.
4. The vector $x = (x_i)$, where $x_i = \min\{y_i, y \in D_A\}$ is in $D_A$. If in addition $1 \in V^C$, then $x$ is a max-eigenvector satisfying $A \otimes x = \mu(A)x$.
5. Define the vector $y = (y_i)$, where $y_i = \max\{z_i, z \in D_A\}$. Then $y^{-1} \in D_{A,r}$. If in addition $1 \in V^C$, then $y$ satisfies $A^T \otimes y^{-1} = \mu(A)y^{-1}$.
6. $D_A$ consists of one vector if and only if $A^C$ is irreducible and $V^C = \{1, \ldots, n\}$.

Statement 4 gives a new characterization for a max-eigenvector (up to scaling); for example, if

$$A = \begin{pmatrix}
1 & 1 \\
1 & 2
\end{pmatrix},$$

then $\mu(A) = 1$ and $(1, 1)^T$ is the max-eigenvector of $A$ since $x_2 = 1 = \min\{y_2, y \in D_A\}$. However, $D_A = \{(1, y_2), 1 \leq y_2 \leq 2\}$ and so contains more than one vector, as required by statement 6 because $V^C = \{1\}$ for $A$ in (13).

We now give another representation of $C_{A,r}$.

Lemma 2. Let $r \geq \mu(A)$. Then $x \in C_{A,r}$ if and only if there is a vector $h \geq 0$ such that $x = (A/r)^* \otimes h$.

Proof. Replacing $A$ by $A/r$, assume that $\mu(A) \leq 1$ and $r = 1$. Thus (8) holds and gives that $A \otimes A^* \otimes h \leq A^* \otimes h$ for $h \geq 0$. Setting $x = A^* \otimes h \geq 0$, this inequality becomes $A \otimes x \leq x$, that is, $x \in C_{A,1}$. Conversely, if $x \in C_{A,1}$, i.e. $A \otimes x \leq x$ with $x \geq 0$, then $x = x \oplus A \otimes x \oplus \cdots \oplus A^{n-1} \otimes x = A^* \otimes x$. Now take $h = x$. □

From Lemma 2 and (7), it follows that step 3 in Algorithm 3.4 of [6] calculates a vector $y \in C_A$. For completeness, we recall here these three steps.

Algorithm. For $A \geq 0$ an $n$-by-$n$ irreducible matrix and $x_0 \geq 0, x_0 \neq 0$, step 3 below produces a vector $y \in C_A$. 

$x \in C_{A,r} \iff \max_{i,k} a_{ik} x_k / x_i \leq r$.
1. Calculate \( x_i = A \otimes x_{i-1} \) for \( i = 1, \ldots, n \).

2. Denoting \( x_i = (x_{1i}, \ldots, x_{ni})^T \), find
   \[
   \mu(A) = \max_{i=1, \ldots, n} \min_{k=0, \ldots, n-1} \left( \frac{x_{ki}}{x_{ik}} \right)^{1/(n-k)}.
   \]

3. Calculate
   \[
   y = x_0 \oplus \frac{1}{\mu(A)} x_1 \oplus \cdots \oplus \frac{1}{\mu(A)^{n-1}} x_{n-1} = (A/\mu(A))^* \otimes x_0.
   \]

Vector \( y \) calculated in step 3 of this algorithm is given by \((A/\mu(A))^* \otimes x_0\) and hence from Lemma 2, \( y \in C_A \) and satisfies \( A \otimes y \leq \mu(A)y \). We note that steps 4–6 of Algorithm 3.4 of [6] generate a max-eigenvector.

3. Perturbation results

Returning to the approximation problem described in the Introduction, we first consider a general perturbation problem (without assuming that \( A \) is an SR-matrix). The following result states that perturbing a nonnegative irreducible matrix \( A \) with a relative error \( \epsilon < 1 \) yields a perturbation of the max-eigenvalue with a relative error bounded also by the same number \( \epsilon \). Here \( |A - B| \) denotes the modulus of the entry-wise difference of matrices \( A \) and \( B \).

**Theorem 3.** Let \( A, B \) be two nonnegative \( n \times n \) matrices and \( A \) be irreducible. Let
   \[
   |A - B| \leq \epsilon A
   \]
   for some \( \epsilon < 1 \). Then \( B \) is also irreducible, and
   \[
   |\mu(A) - \mu(B)| \leq \epsilon \mu(A).
   \]

**Proof.** We follow the pattern of the proof of the analogous result for the Perron root, as given in [4, Theorem 1].

Clearly, \( A \) and \( B \) have the same digraph, so \( B \) is irreducible and hence \( \mu(B) \) is well defined. By (14)
   \[
   -\epsilon A \leq B - A \leq \epsilon A
   \]
and hence
   \[
   (1 - \epsilon)A \leq B \leq (1 + \epsilon)A.
   \]
Now, as \( \mu(A) \) is a monotone function of the entries of \( A \),
   \[
   (1 - \epsilon)\mu(A) \leq \mu(B) \leq (1 + \epsilon)\mu(A),
   \]
from which (15) follows. \( \square \)

**Remark:** In the special case that \( A \) is an SR-matrix and \( B = (w_i/w_j) \) with \( w > 0 \) is a transitive matrix, the assumption (14) means exactly that \( E_1(w) \leq \epsilon \), with \( E_1(w) \) given by (1). In this case from (5)
   \[
   \mu(A) = \min_{w>0} E_2(w) = 1 + \min_{w>0} E_1(w) \leq 1 + \epsilon,
   \]
while (15) gives the weaker bound \( |\mu(A) - 1| \leq \epsilon \mu(A) \) or equivalently \( \mu(A) \leq (1 - \epsilon)^{-1} \).

The above theorem shows that the relative error of the max-eigenvalue is bounded by the relative error of the matrix, but this is not so simple for the max-eigenvector. Under some assumptions, the next result shows that its relative error is bounded by \( 2n - 2 \) times the relative error of the matrix (for small errors, with higher order terms neglected).
Theorem 4. Under the assumptions of Theorem 3, let $V^C(A) \cap V^C(B) \neq \phi$. Then there are max-eigenvectors $x(A), x(B)$ of $A$ and $B$ resp. such that

$$|x(A) - x(B)| \leq \left( \left( \frac{1 + \epsilon}{1 - \epsilon} \right)^n - 1 \right)x(A).$$

(20)

Proof. Introduce the matrices $\bar{A} = A/\mu(A)$ and $\bar{B} = B/\mu(B)$ and let $C = A/\mu(B)$. Then $|\bar{A} - \bar{B}| \leq |\bar{A} - C| + |\bar{B} - C|$ and from (15) the first term is bounded by $\epsilon\mu(A)/\mu(B)\bar{A}$. From (14) it follows that $|\bar{B} - C|$ has the same bound. As (15) also implies that $\mu(A)/\mu(B) \leq (1 - \epsilon)^{-1}$ these bounds give

$$|\bar{A} - \bar{B}| \leq 2\epsilon/(1 - \epsilon).$$

(21)

For $\eta < 1$ and nonnegative numbers $c_i, d_i, i \in I$ satisfying $|c_i - d_i| \leq \eta c_i$, it is easy to see that

$$\max_{i \in I} c_i - \max_{i \in I} d_i \leq \eta \max_{i \in I} c_i.$$  

(22)

If the set of indices $I$ has $m$ elements, then

$$\left| \prod_{i \in I} c_i - \prod_{i \in I} d_i \right| \leq ((1 + \eta)^m - 1) \prod_{i \in I} c_i.$$  

(23)

In the directed graph associated with $\bar{A}$, any path of length $\ell$ connecting vertices $i$ and $j$ and given by the sequence $i = i_1, i_2, \ldots, i_{\ell+1} = j$ leads to the path product $a_{i_1,i_2} \cdots a_{i_\ell,i_{\ell+1}}$. By (21) and (23) this gives

$$|\bar{a}_{i_1,i_2} \cdots \bar{a}_{i_\ell,i_{\ell+1}} - \bar{b}_{i_1,i_2} \cdots \bar{b}_{i_\ell,i_{\ell+1}}| \leq ((1 + \eta)^\ell - 1)\bar{a}_{i_1,i_2} \cdots \bar{a}_{i_\ell,i_{\ell+1}},$$

(24)

in which $\eta = 2\epsilon/(1 - \epsilon)$.

Observe now that for fixed $i,j$ the maximum over all path products of all lengths $\ell \leq n - 1$ connecting $i$ and $j$ is just the $(i,j)$ entry of $\bar{A}^\ast$. Hence by (24)

$$|(\bar{A}^\ast)_{ij} - (\bar{B}^\ast)_{ij}| \leq \max_{\ell \leq n-1} ((1 + \eta)^\ell - 1)(\bar{A}^\ast)_{ij},$$

(25)

which is equivalent to

$$|\bar{A}^\ast - \bar{B}^\ast| \leq \left( \left( \frac{1 + \epsilon}{1 - \epsilon} \right)^n - 1 \right)|\bar{A}^\ast|,$$

(26)

since the right side of (25) is monotone in $\ell$. Now take $x(A) = \bar{A}^\ast e_i, x(B) = \bar{B}^\ast e_i$, where $i \in V^C(A) \cap V^C(B)$ and $e_i$ is the $n$-vector with 1 in entry $i$ and 0 elsewhere. Then $x(A), x(B)$ are max-eigenvectors of $A, B$, resp., and (20) follows. $\square$

For small values of $\epsilon$ by using the binomial theorem and ignoring terms in $\epsilon^2$ and higher powers, the bound in (20) gives

$$|x(A) - x(B)| \leq 2\epsilon(n - 1)x(A).$$

(27)

4. The Toeplitz case

For certain Toeplitz matrices the Perron vector and the max-eigenvector coincide and are both of the special form $w = (1, s, \ldots, s^{n-1})^T$. We now study this case, and then in Section 5 give an application of our results.

Let $n \in N$ and $t_i$ for $i = 1, n, \ldots, n - 1$ be $2n - 1$ real or complex numbers. They define an $n$-by-$n$ Toeplitz matrix

$$A = (a_{ij}) = \text{Toeplitz}(t_1, \ldots, t_{n-1})$$

by setting $a_{ij} = t_{j-i}$ for $i, j = 1, \ldots, n$. 

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Theorem 5. Let $A = \text{Toeplitz}(t_{1-n}, \ldots, t_{n-1})$ and $s$ be a complex number. The following are equivalent:

1. $w = (1, s, \ldots, s^{n-1})^T$ is an eigenvector of $A$.
2. $t_{-i} = s^i t_{n-i}$ for $i = 1, \ldots, n - 1$.

Proof. If $s = 0$, then the equivalence of statements 1 and 2 is obvious, since $A$ is upper triangular. So assume $s \neq 0$. For $i = 1, \ldots, n$, define

$$h_i = \sum_{j=1}^{n} t_{j-i} s^{j-i}.$$  \hfill (29)

Then for $i = 2, \ldots, n$ it follows that $h_i - h_{i-1} = t_{1-i} s^{1-i} - t_{n+1-i} s^{n+1-i}$. Now statement 1 holds if and only if $h_i$ is the associated eigenvalue and hence is equivalent to $h_i = h_{i-1}$ for $i = 2, \ldots, n$, which is equivalent to statement 2. \qed

For $s \neq 0$, $A$ has $n$ different (classical) eigenvectors of the form given in 1 of Theorem 5. This also follows from the fact that $A$ is diagonally similar to a circulant matrix, a fact that we use in the proof of the next result, in which $\rho(A)$ denotes the Perron root of $A$.

Theorem 6. In the situation of Theorem 5 assume that $A$ is nonnegative and irreducible, that $s > 0$ and that 1 or 2 holds. Then $w$, as defined in 1, is a Perron vector of $A$, i.e. $Aw = \rho(A)w$, and also a max-eigenvector of $A$, i.e.

$$A \otimes w = \mu(A)w,$$ \hfill (30)

where

$$\mu(A) = \max\{t_i s^i, 0 \leq i \leq n - 1\}.$$ \hfill (31)

Proof. Defining $D = \text{diag}(1, s, \ldots, s^{n-1})$, it follows that $C = D^{-1}AD = (c_{ij})$ is a nonnegative, irreducible circulant matrix with entries $c_{ij} = t_{j-i} s^{j-i}$. Hence $C$ has $\rho(C) = \sum_{j=1}^{n} t_{j-1} s^{j-1}$ with Perron vector $e$, where $e = (1, \ldots, 1)^T$. Thus $A$ has Perron root $\rho(A) = \rho(C)$ with Perron vector $De = w$. Also in the max algebra $C \otimes e = \mu(C)e$, with $\mu(C) = \mu(A)$ given by (31). Since $w = D \otimes e$ and $A \otimes D \otimes e = \mu(A)D \otimes e$, Eq. (30) holds and the result follows. \qed

In the case described above, two choices for a weight vector $w$, namely the Perron vector and the max-eigenvector, coincide. We remark that Theorem 6 gives only a sufficient condition that $w^T = (1, s, \ldots, s^{n-1})$ is a max-eigenvector of a Toeplitz matrix. The next theorem gives a condition that is necessary and sufficient.

Theorem 7. Let $A = \text{Toeplitz}(t_{1-n}, \ldots, t_{n-1}) \succeq 0$ be irreducible and $s > 0$. The following are equivalent:

1. $w = (1, s, \ldots, s^{n-1})^T$ is a max-eigenvector of $A$.
2. (i) $\max\{t_i s^i, i \leq 0\} = \max\{t_i s^i, i \geq 0\}(= \mu, \text{say})$
   (ii) There exist $i, k$ satisfying $i \leq 0 < k$ and $k - i \leq n$ with $\mu = t_i s^i = t_k s^k$.

In this case $\mu = \mu(A)$.

Proof. Statement (1) is equivalent to

$$\max\{t_{k-i} s^{k-1}, k = 1, \ldots, n\} = \mu(A) s^{i-1}, \quad i = 1, \ldots, n$$ \hfill (32)

and this is the same as

$$\max\{t_{k-i} s^{k-i}, k = 1, \ldots, n\} = \mu(A), \quad i = 1, \ldots, n,$$ \hfill (33)

which is equivalent to the statements 2. \qed
In the situation of the above theorem the max-eigenvector and the number $s$ are determined in the following way. Define $g(s) = \max_{i \neq 0} t_is_i^T$, where $s > 0$. As $A$ is irreducible there is a positive $i$ and a negative $j$ such that $t_i > 0$ and $t_j < 0$. Hence $g(s)$ goes to $\infty$ as $s$ goes to 0 and to $\infty$. So the minimum of $g(s)$ exists, say at $s_1$ with $g(s_1) = \min g(s) = \mu_1$. Note that $g(s)$ minimal is a necessary condition for $2(i)$ in Theorem 7 to hold. Then $\mu(A) = \max(t_0, \mu_1)$. As long as $t_0 \leq \mu_1$, the max-eigenvector is uniquely defined by 1 in Theorem 7 with $s = s_1$. Otherwise for $t_0 > \mu_1$ the max-eigenvector is not unique, since for any $s$ such that $g(s) < t_0$, statement 2(i) of Theorem 7 is satisfied, hence $(1, s, \ldots, s^{n-1})^T$ is a max-eigenvector.

Consider the case of a Toeplitz matrix $A$ that is also SR, i.e. $t_0 = 1$ and $t_it_{-i} = 1$ for $i = 1, \ldots, n - 1$. In this case it is easy to see that $t_0 = 1 \leq \mu_1$, but in general $A$ need not have a max-eigenvector of the form $(1, s, \ldots, s^{n-1})^T$. The following matrix illustrates this.

**Example.** Let $A = \text{Toeplitz}(1/3, 3, 1/2, 1, 2, 1/3, 3)$, which is a 4-by-4 SR-matrix with $\mu_1 = g(1) = 3$, but $A \otimes w \neq 3w$. The two maxima in part 2(i) of Theorem 7 are equal, but as $i = -2, k = 3$, the inequality in part 2(ii) does not hold. In fact $\mu(A) = 2.7108 = 54^{1/4}$ (from the 4-cycle on vertices 1, 4, 2, 3) and the max-eigenvector is $(1, 0.8165, 1.1067, 0.9036)^T$. If $t_0$ in $A$ is changed from 1 to 4, then $t_0 > \mu_1$, and the max-eigenvector is not unique. In this case, $D_A$ consists of more than one vector, because $A^k$ is reducible; see statement 6 of Theorem 1.

We conclude this section by giving another nontrivial class of matrices for which the Perron vector and the max-eigenvector coincide.

**Example.** Let $A = (a_{ij})$ be a 3-by-3 SR-matrix. In [14, p. 313] it is claimed that $A$ has the Perron vector $w = (w_i)$ where $w_i = (a_{i1}a_{i2}a_{i3})^{1/3}$ for $i = 1, 2, 3$. Letting $c = (a_{12}a_{23}a_{31})^{1/3}$ and $W = \text{diag}(w_1, w_2, w_3)$, it follows that
\[
W^{-1}AW = \begin{pmatrix}
1 & c & c^{-1} \\
c^{-1} & 1 & c \\
c & c^{-1} & 1
\end{pmatrix}
\] (34)
is a Toeplitz SR-matrix that is also a circulant. Hence $w = We$ is a Perron vector of $A$ with Perron root $(1 + c + c^{-1})$. In addition $w$ is a max-eigenvector of $A$ with $\mu(A) = \max\{c, c^{-1}\}$. By statement 6 of Theorem 1, $D_A$ consists of one vector, confirming that $w$ is unique (up to scaling).

### 5. An example from music theory

For the basic ideas of music theory used in the following application, we refer a reader to [2,11,12,15]. Consider a pure pitched scale. As an example, take Zarlino’s scale, which can be described in the following way. The ratios of the frequencies of the tone and the frequency of the fundamental tone are given by
\[
z_3 = 9/8, z_5 = 5/4, z_6 = 4/3, z_8 = 3/2, z_{10} = 5/3, z_{12} = 15/8.
\] (35)

We want to attach frequencies $f_i$ to the keys $i: i = \ldots, 1, \ldots, 12, \ldots$ (of a piano, say), thus explaining the numbering of the $z_i$. If the frequencies are scaled such that key 1 has frequency 1, then for $i = 3, 5, 6, 8, 10, 12$, the number $z_i$ as above is the frequency of the key $i$. By requiring the octave condition
\[
f_{i+12} = 2f_i
\] (36)
it suffices to consider only the keys 1, \ldots, 12.

For all scales, starting from any of the above keys, say $i$, we want the ratios of the frequency of the $j$th tone of the scale and the frequency of the fundamental tone $i$ to be the same. In other words it is required that
\[
f_j/f_i = f_{j+r}/f_{i+r}
\] (37)
for all integers $r$. Also it is required that
\[
f_i = z_i, \quad i = 3, 5, 6, 8, 10, 12
\] (38)
as in (35). It is well known that (35), (36), (37) and (38) cannot be satisfied simultaneously, hence we can only require that the \( f_i \) satisfy it in an approximate way. So instead of (37) the ratios \( f_j/f_i \) should be near to numbers \( a_{ij} \) satisfying \( a_{ij} = a_{i+j+i+j} \) for \( i, j = 1, \ldots, 12 \). This means that the matrix \( A = (a_{ij}) \) is a Toeplitz SR-matrix approximating a transitive matrix. Matrix \( A \) can be described by the parameters \( t_i \) for \( i = -11, \ldots, 11 \) where
\[
a_{ij} = t_{i-j}, \quad i, j = 1, \ldots, 12.
\]

The requirements on \( t_i \) are:

1. \( t_0 = 1 \)
2. \( t_i = z_{i+1}, i = 2, 4, 5, 7, 9, 11; \) see (35)
3. \( t_{i+12} = 2t_i \) (octave condition), \( i = -1, \ldots, -11 \)
4. \( t_{-i} = 1/t_i, i = 0, \ldots, 11 \), since \( a_{ij}a_{ij} = 1 \)

From 3 and 4, it follows that \( t_{12-i}t_i = 2t_{-i}t_i = 2 \). From 2 this gives \( t_i \) for \( i = 1, 3, 8, 10 \) and \( t_2^2 = 2 \), hence \( t_6 = \sqrt{2} \). Defining \( d = (d_0, \ldots, d_{11}) \) by
\[
d = (1, 16/15, 9/8, 6/5, 5/4, 4/3, \sqrt{2}/3, 2/5, 5/3, 16/9, 15/8),
\]
then the resulting \( t_i \) are given by \( t_i = d_i \) for \( i = 0, \ldots, 11 \), and \( t_i = 1/d_{-i} \) for \( i = -1, \ldots, -11 \). In this case we have constructed the SR-matrix \( A = \text{Toeplitz}(t_{-11}, \ldots, t_{11}) \). It is easily seen from the octave condition that Theorem 6 statement 2 holds for \( A^T \) with \( s^{12} = 2 \).

Following the approach by Saaty [13] and [7], the Perron vector or the max-eigenvector of \( A^T \) is taken as an approximation to the frequencies. By Theorems 6 and 7 these vectors coincide and are given by \( w \), where
\[
w = (1, s, \ldots, s^{11})^T
\]
with \( s = 2^{1/12} \). This is the case of the equal temperament scale system; see, e.g. [2, p. 16].

We remark that for the matrix \( A \) so constructed \( \mu(A) = (3/5)^{1/4}(6/5)^{3/4} = 1.00907 \ldots \) (from 3 different 4-cycles, one of which is on vertices 1, 4, 7, 10). Hence the maximal relative error that occurs when \( a_{ij} \) is replaced by \( (w_i/w_j) \) is 0.00907. We should however remark that Theorem 7 does not say that the max-eigenvector is unique. The max-eigenvector is unique, but in this case \( A^C \) is reducible with three strongly connected components, thus there are three linearly independent max-eigenvectors; see, e.g. [1, Theorem 5].

From the above proof it follows that the same approach works for any given positive \( z_i \) for \( i = 3, 5, 6, 8, 10, 12 \), as long as \( z_6z_8 = 2 \) holds. So if the scale named after Pythagoras is taken, namely
\[
z_3 = 9/8, z_5 = 81/64, z_6 = 4/3, z_8 = 3/2, z_{10} = 27/16, z_{12} = 243/128,
\]
then the best approximation is again given by the equal temperament system satisfying the requirements 1–4 above. The first row of the matrix \( A \) is now given (instead of (39)) by
\[
d = (1, 256/243, 9/8, 32/27, 81/64, 4/3, \sqrt{2}/3, 2/128, 81/27, 16/9, 243/128),
\]
and the max-eigenvalue \( \mu(A) \) is 1.00566 \ldots = 2^{1/12}243/256 (from the Hamilton cycle on vertices 1, 12, 11, 10, \ldots, 2, 1). This shows that the equal temperament system is nearer to the scale of Pythagoras than to the Zarlino scale.

**Appendix**

**Proof of Theorem 1:** Statement 1 follows immediately from the characterization in Eqs. (10) and (11), as do statements 2(a), (b) and (c) for \( p > 0 \). For \( p < 0 \), statement 2(c) then follows by using 1, and 2(d) and (e) follow from 2(c) by letting \( p \) tend to \( +\infty \) and \( -\infty \), respectively.

For statement 3, observe that, as \( A \) is irreducible, the matrix \( A^+ \) as defined in (6) is strictly positive. Also \( x \in D_{A^+} \) implies that \( A^+ \otimes x \leq \max\{r, \ldots, r^n\} x \). Since \( x \geq 0 \) and using \( x_1 = 1 \) in (11) gives that \( x \) is bounded from above and below. Since \( D_{A^+} \) is closed, it follows that \( D_{A^+_+} \) is compact. Also \( A_+ \) is a convex polytope, as the equation \( A \otimes x \leq \mu(A)x \) is the set of linear inequalities \( a_{ij}x_j \leq \mu(A)x_i \) for \( i, j = 1, \ldots, n \).
For statement 4, we first show that $x \in D_A$. As $D_A$ is compact there exist vectors $z^i \in D_A$ such that $z^i_i = x_i, i = 2, \ldots, n$. Since $x = \min\{z^i, i = 2, \ldots, n\}$, statement 2(e) for $C_A$ gives $x \in D_A$. Let $I = \{i, (A \otimes x)_i = \mu(A)x_i\}$. As $1 \in V^C$, it follows that $1 \in I$. Assume that there is some $k \notin I$, i.e. $\max_l a_{lk}x_l < \mu(A)x_k$. Then this component can be decreased but $x$ remains in $D_A$. This contradicts the definition of $x$, so $x$ is a max-eigenvector. Statement 5 follows from 4 by using 1.

We turn now to the proof of statement 6. By a suitable diagonal similarity scaling and division by $\mu(A)$, it can be assumed that $\mu(A) = 1$ and all entries of $A = (a_{ij})$ satisfy $a_{ij} \leq 1$, i.e. $A \otimes e \leq e$. The existence of such a scaling is implied by (5) in the Introduction. Firstly we show that $A^C$ irreducible and $V^C = \{1, \ldots, n\}$ implies that $D_A$ consists of only one vector, namely $e$. In this case $A^C$ is $n$-by-$n$, all entries of $A^C$ are either 1 or 0 and $A^C \leq A$. Hence $A \otimes e = e$. If $x \in D_A$ then $x_1 = 1$ and $A^C \otimes x \leq A \otimes x \leq x$. Hence if the $(i,j)$ entry of $A^C$ is 1, then $x_j \leq (A \otimes x)_i \leq x_i$. By the irreducibility of $A^C$ it follows that all components of $x$ are equal, thus $x = e$. For the proof of the converse statement, we assume that $D_A$ consists of one vector only and (by the above scaling and division) this is the vector $e$. It follows that $A^C$ is irreducible, since otherwise, e.g. by Theorem 1.1 of [5] there would be at least two max-eigenvectors contained in $D_A$. All entries of $A$ are less than or equal to 1 and hence $A^C \otimes e \leq e$. In fact there is equality here, because otherwise $D_A$ would have more than one vector. Hence each row and each column of $A$ has at least one entry 1. Let $B$ be the matrix formed from $A$ by replacing all entries by 0 that are not equal to 1. Consider the Frobenius normal form of $B$, a block upper triangular with diagonal blocks that are either irreducible or zero 1-by-1 blocks. Now any cycle in $B$ is also a cycle in $A$ and hence as a cycle of 1s it is a critical cycle in $A^C$. Also any critical cycle of $A$ is a cycle in $B$. It follows that $A$ and $B$ have the same critical vertices. Moreover there is at most one irreducible block in $B$, since $A^C$ is irreducible. Also there is no zero 1-by-1 block, as this would imply that $B$ has a zero row or column. So $B$ is irreducible and hence $V^C(A) = V^C(B) = \{1, \ldots, n\}$. □

References