Existence of multiple positive solutions for nonlinear $m$-point boundary value problems

Chuanzhi Bai $^{a,b}$ and Jinxuan Fang $^{a,*}$

$^a$ Department of Mathematics, Nanjing Normal University, Nanjing 210097, People’s Republic of China
$^b$ Department of Mathematics, Huaiyin Normal College, Huaiyin 223001, People’s Republic of China

Received 7 January 2002
Submitted by Z.S. Athanassov

Abstract

In this paper, we afford some sufficient conditions to guarantee the existence of multiple positive solutions for the nonlinear $m$-point boundary value problem for the one-dimensional $p$-Laplacian

\[
\left(\phi_p(u')\right)' + a(t)f(t,u) = 0, \quad t \in (0, 1),
\]

\[
u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i).
\]

© 2003 Elsevier Science (USA). All rights reserved.

Keywords: Multi-point boundary value problem; Positive solution; Fixed point theorem in cones; One-dimensional $p$-Laplacian

1. Introduction

In this paper, we are concerned with the existence of multiple positive solutions to the $m$-point boundary value problem (MBVP) for the one-dimension $p$-Laplacian

\[
\left(\phi_p(u')\right)' + a(t)f(t,u) = 0, \quad t \in (0, 1),
\]

\[
u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i),
\]

* Corresponding author.
E-mail address: jxfang@pine.njnu.edu.cn (J. Fang).

0022-247X/03/$ – see front matter © 2003 Elsevier Science (USA). All rights reserved.
doi:10.1016/S0022-247X(02)00509-7
where \( \phi_p(s) = |s|^{p-2}s, \ p > 1, \ 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, \ a_i \geq 0 \) for \( i = 1, \ldots, m - 3 \) and \( a_{m-2} > 0 \). We also assume the following:

(H1) \( \sum_{i=1}^{m-2} a_i \xi_i < 1. \)

(H2) \( f \in C([0, 1] \times [0, \infty], [0, \infty]). \)

(H3) \( a \in C([0, 1], [0, \infty)) \) and there exists \( x_0 \in (\xi_{m-2}, 1) \) such that \( a(x_0) > 0. \)

The study of multi-point boundary value problems for linear second order ordinary differential equations was initiated by Il’in and Moiseev [5]. Since then, there is much current attention focused on the study of nonlinear multi-point boundary value problems, see [1,2,4,9] to name a few. Equation (1.1) with Dirichlet boundary condition has been studied extensively, see, for example, [6,10]. When \( p = 2 \) and \( f(t,u) \equiv f(u) \), (1.1) reduces to

\[
 u'' + a(t)f(u) = 0, \quad t \in (0, 1).
\] (1.3)

Recently, Ma [8] showed the existence of at least of one positive solution to (1.3), (1.2) under the conditions that \( f \) is either superlinear or sublinear. Inspired and motivated by the recent work in [7,8], our purpose here is to give some existence results for one or multiple positive solutions to MBVP (1.1), (1.2). Our theorems generalizes and extends the main result in [8].

By the positive solution of (1.1), (1.2) we understand a function \( u(t) \) which is positive on \( 0 < t < 1 \) and satisfies (1.1) and (1.2).

In obtaining positive solutions of (1.1), (1.2), the following fixed point theorem in cones will be fundamental.

**Lemma 1.1** [3,7]. Let \( K \) be a cone in a Banach space \( X \). Let \( D \) be an open bounded subset of \( X \) with \( D_K = D \cap K \neq \emptyset \) and \( \overline{D} \notin K \). Assume that \( A : D_K \to K \) is a compact map such that \( x \neq Ax \) for \( x \in \partial D_K \). Then the following results hold:

1. If \( \|Ax\| \leq \|x\|, \ x \in \partial D_K \), then \( i_K(A, D_K) = 1 \).
2. If there exists \( e \in K \setminus \{0\} \) such that \( x \neq Ax + \lambda e \) for all \( x \in \partial D_K \) and all \( \lambda > 0 \), then \( i_K(A, D_K) = 0 \).
3. Let \( U \) be open in \( X \) such that \( \overline{U} \subset D_K \). If \( i_K(A, D_K) = 1 \) and \( i_K(A, U_K) = 0 \), then \( A \) has a fixed point in \( D_K \setminus \overline{U} \). The same results holds if \( i_K(A, D_K) = 0 \) and \( i_K(A, U_K) = 1 \).

2. Main results

In this paper, we always assume that (H1)–(H3) hold. We know that \( \phi_q \) is the inverse function to \( \phi_p \) (\( p > 1 \)), where \( \phi_q(s) = |s|^{q-2}s, \ q = p/(p-1) > 1 \). Similar to the Lemma 1 in [8], it is easy to check that (1.1), (1.2) has a solution \( u = u(t) \) if and only if \( u \) solves the operator equation
\[ u(t) = - \int_0^t \phi_q \left( \int_0^s a(\tau) f(\tau, u(\tau)) d\tau \right) ds \\
- \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q \left( \int_0^{\xi_i} a(\tau) f(\tau, u(\tau)) d\tau \right) ds}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \\
+ t \int_0^1 \phi_q \left( \int_0^t a(\tau) f(\tau, u(\tau)) d\tau \right) ds \\
:= Au(t). \quad (2.1) \]

For convenience, let
\[ \gamma_1 = \min \left\{ a_{m-2}(1 - \xi_m - 2), a_{m-2}\xi_m - 2, \xi_m - 2 \right\} \]
and
\[ \gamma_2 = \min \left\{ \max \left\{ \sum_{i=1}^{m-2} a_i \xi_1, a_{m-2}\xi_m - 2 \right\}, \xi_m - 2 \right\} \frac{\int_0^{1} \phi_q \left( \int_0^{\xi_m - 2} a(\tau) d\tau \right) d\tau}{\int_0^{1} \phi_q \left( \int_0^{\xi_m - 2} a(\tau) d\tau \right) d\tau}. \]

From (H3), there exists \([c, d] \subset (\xi_m - 2, 1)\) such that \(a(t) > 0\) for \(t \in [c, d]\). So,
\[ \int_{\xi_m - 2}^{1} \phi_q \left( \int_0^{\xi_m - 2} a(\tau) d\tau \right) ds \geq \int_{c}^{d} \phi_q \left( \int_0^{a(\tau)} d\tau \right) \geq \int_{c}^{d} \phi_q \left( \int_0^{a(\tau)} d\tau \right) \geq 0. \quad (2.2) \]

Hence, by (H1) and (2.2), we have that \(0 < \gamma_1, \gamma_2 < 1\). Denote
\[ K = \left\{ u \mid u \in C[0, 1], u(t) \geq 0, \min_{\xi_m - 2 \leq t \leq 1} u(t) \geq \gamma \| u \| \right\}, \quad (2.3) \]
where \(\gamma = \gamma_1 \gamma_2\). It is obvious that \(K\) is a cone in \(C[0, 1]\). By (2.1), we have
\[ (Au)'(t) = -\phi_q \left( \int_0^t a(\tau) f(\tau, u(\tau)) d\tau \right) \\
- \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q \left( \int_0^{\xi_i} a(\tau) f(\tau, u(\tau)) d\tau \right) ds}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \\
+ \int_0^1 \phi_q \left( \int_0^t a(\tau) f(\tau, u(\tau)) d\tau \right) ds \\
:= Au(t). \]

Thus, from (H2) and (H3), we have
\[ (Au)'(t_2) \leq (Au)'(t_1), \quad \text{for any } t_1, t_2 \in [0, 1] \text{ with } t_1 \leq t_2. \]
Hence, \((Au)'(t)\) is a decreasing function on [0, 1], that is, the graph of \(Au(t)\) is concave down on (0, 1). Since

\[ Au(0) = 0, \quad Au(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \]

thus, by Lemmas 2 and 4 in [8], we have

\[ Au ≥ 0 \quad \text{and} \quad \inf_{t \in [\xi_{m-2}, 1]} Au(t) ≥ \gamma_1 \|Au\| > \gamma_2 \|Au\|, \quad \text{for} \; u \in K, \]

that is, \(AK \subset K\). It is easy to check that \(A : K \to K\) is completely continuous.

We define \(K_\rho = \{x \in K : \|x\| < \rho\}\). Furthermore, we define a set \(\Omega_\rho\) as

\[ \Omega_\rho = \left\{ x \in K : \min_{\xi_{m-2} \leq t \leq 1} x(t) < \gamma_1 \rho \right\}. \]

Lemma 2.1 [7]. \(\Omega_\rho\) defined above has the following properties:

(a) \(\Omega_\rho\) is open relative to \(K\).
(b) \(K_{\gamma_1 \rho} \subset \Omega_\rho \subset K_\rho\).
(c) \(x \in \partial \Omega_\rho\) if and only if \(\min_{\xi_{m-2} \leq t \leq 1} x(t) = \gamma_1 \rho\).
(d) If \(x \in \partial \Omega_\rho\), then \(\gamma_1 \rho \leq x(t) \leq \rho\) for \(t \in [\xi_{m-2}, 1]\).

Now, for the convenience, we introduce the following notations. Let

\[ f^\rho_{\gamma_1 \rho} = \min \left\{ \min_{t \in [\xi_{m-2}, 1]} \frac{f(t, u)}{\phi_p(\rho)} : u \in [\gamma_1 \rho, \rho] \right\}, \]

\[ f^{\gamma_1 \rho}_0 = \max \left\{ \max_{t \in [0, 1]} \frac{f(t, u)}{\phi_p(\rho)} : u \in [0, \rho] \right\}, \]

\[ f^a = \lim_{u \to a} \sup_{t \in [0, 1]} \frac{f(t, u)}{\phi_p(u)}, \]

\[ f_a = \lim_{u \to a} \inf_{t \in [\xi_{m-2}, 1]} \frac{f(t, u)}{\phi_p(u)} \] \((\alpha := \infty \text{ or } 0^+)\),

\[ m = 1 - \sum_{i=1}^{m-2} a_i \xi_i \int_0^1 \phi_q \left( \int_0^s a(\tau) d\tau \right) ds \]

and

\[ M = \frac{1 - \sum_{i=1}^{m-2} a_i \xi_i}{\min \left[ \max \{ \sum_{i=1}^{m-2} a_i \xi_1, a_{m-2} \xi_{m-2}, \xi_{m-2} \} \right] \int_{\xi_{m-2}}^1 \phi_q \left( \int_{\xi_{m-2}}^s a(\tau) d\tau \right) ds}. \]

Remark 2.1. By (H_1) and (2.2), it is easy to see that \(0 < m, M < \infty\) and \(M \gamma_1 \gamma_2 = M \gamma_1 < m\).

Now, we impose conditions on \(f\) which assure that \(i_K(A, K_\rho) = 1\).
Lemma 2.2. If \( f \) satisfies the following condition
\[
f_0 \rho \leq \phi_p(m) \quad \text{and} \quad u \neq Au \quad \text{for } u \in \partial K_\rho,
\]
then \( i_K(A, K_\rho) = 1 \).

Proof. By (2.1) and condition (2.4) we have for \( y \in \partial K_\rho \)
\[
Ay(t) \leq \frac{t \int_0^1 \phi_q \left( \int_0^x a(\tau) f(\tau, y(\tau)) d\tau \right) ds}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \leq \phi_q \left( \phi_p(\rho) \phi_p(m) \right) \frac{\int_0^1 \phi_q \left( \int_0^x a(\tau) d\tau \right) ds}{1 - \sum_{i=1}^{m-2} a_i \xi_i}
\]
\[
= \rho m \sum_{i=1}^{m-2} a_i \xi_i = \rho = \|y\|.
\]
This implies that \( \|Ay\| \leq \|y\| \) for \( y \in \partial K_\rho \). By Lemma 1.1(1), we have \( i_K(A, K_\rho) = 1 \). \( \square \)

Next, we impose conditions on \( f \) which assure that \( i_K(A, \Omega_\rho) = 0 \).

Lemma 2.3. If \( f \) satisfies the following condition
\[
f_0 ^\rho \rho \geq \phi_p(M M') \quad \text{and} \quad u \neq Au \quad \text{for } u \in \partial \Omega_\rho,
\]
then \( i_K(A, \Omega_\rho) = 0 \).

Proof. Let \( e(t) \equiv 1 \) for \( t \in [0, 1] \); then \( e \in \partial K_1 \). We claim that
\[
y \neq Ay + \lambda e, \quad y \in \partial \Omega_\rho, \lambda > 0.
\]
In fact, if not, there exist \( y_0 \in \partial \Omega_\rho \) and \( \lambda_0 > 0 \) such that \( y_0 = Ay_0 + \lambda_0 e \). From the process of proof of Lemma 4 in [8] and (2.1) we have
\[
\min_{t \in [\xi_{m-2}, 1]} Ay_0(t) = \min \{ Ay_0(1), Ay_0(\xi_{m-2}) \}.
\]
For \( i = 1, \ldots, m - 2 \), we obtain
\[
(1 - \xi_i) \int_0^{\xi_i} \phi_q \left( \int_0^x a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds \leq \xi_i \int_0^{\xi_i} \phi_q \left( \int_0^x a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds.
\]
(2.7)

So, by (2.7)
\[
\xi_i \int_0^{\xi_i} \phi_q \left( \int_0^x a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds - \int_0^{\xi_i} \phi_q \left( \int_0^x a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds
\]
\[
= \xi_i \int_0^{\xi_i} \phi_q \left( \int_0^{\xi_i} a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds
\]

\[
\int_0^{\xi_i} a(\tau) f(\tau, y_0(\tau)) d\tau + \int_0^{\xi_i} a(\tau) f(\tau, y_0(\tau)) d\tau
\]
\[
= \xi_i \int_0^{\xi_i} \phi_q \left( \int_0^{\xi_i} a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds
\]
\[ + \xi_i \int_0^{\xi_i} \phi_q \left( \int_0^s a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds - \int_0^{\xi_i} \phi_q \left( \int_0^s a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds \]

\[ \geq \xi_i \int_0^{\xi_i} \phi_q \left( \int_0^s a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds + \xi_i \int_0^{\xi_i} \phi_q \left( \int_0^s a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds \]

\[ - (1 - \xi_i) \int_0^{\xi_i} \phi_q \left( \int_0^s a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds \]

\[ \geq \xi_i \int_0^{\xi_i} \phi_q \left( \int_0^s a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds, \quad i = 1, \ldots, m - 2. \quad (2.8) \]

By (2.1), Lemma 2.1(d), condition (2.5) and (2.8) we have

\[
A y_0(1) = - \int_0^1 \phi_q \left( \int_0^s a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds
\]

\[
- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q \left( \int_0^s a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds
\]

\[
+ \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q \left( \int_0^s a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds}{1 - \sum_{i=1}^{m-2} a_i \xi_i}
\]

\[
= \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q \left( \int_0^s a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds}{1 - \sum_{i=1}^{m-2} a_i \xi_i}
\]

\[
- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q \left( \int_0^s a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds
\]

\[
= \frac{1}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \sum_{i=1}^{m-2} a_i \left( \xi_i \int_0^{\xi_i} \phi_q \left( \int_0^s a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds \right.
\]

\[
- \int_0^{\xi_i} \phi_q \left( \int_0^s a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds
\]

\[
\geq \frac{1}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \sum_{i=1}^{m-2} a_i \xi_i \int_0^{\xi_i} \phi_q \left( \int_0^s a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds
\]

\[
\geq \frac{1}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \max \left\{ \sum_{i=1}^{m-2} a_i \xi_1, a_{m-2} \xi_{m-2} \right\}
\]
\[
\phi_q\left(\int_{\xi_{m-2}}^{1} \phi_q \left( \int_{\xi_{m-2}}^{s} a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds \right) \\
\geq \frac{\phi_q(\phi_p(M\gamma)\phi_p(\rho))}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \max \left\{ \sum_{i=1}^{m-2} a_i \xi_1, a_m^{-2} \xi_{m-2} \right\} \int_{\xi_{m-2}}^{1} \phi_q \left( \int_{\xi_{m-2}}^{s} a(\tau) d\tau \right) ds \\
= \frac{M\gamma\rho}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \max \left\{ \sum_{i=1}^{m-2} a_i \xi_1, a_m^{-2} \xi_{m-2} \right\} \int_{\xi_{m-2}}^{1} \phi_q \left( \int_{\xi_{m-2}}^{s} a(\tau) d\tau \right) ds.
\]

(2.9)

For \( t > 0 \)
\[
\left( \int_{0}^{t} \phi_q \left( \int_{0}^{s} a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds \right)'
= \frac{t \phi_q \left( \int_{0}^{t} a(\tau) f(\tau, y_0(\tau)) d\tau \right) - \int_{0}^{t} \phi_q \left( \int_{0}^{s} a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds}{t^2} \\
\geq 0,
\]

which implies that
\[
\int_{0}^{\xi_{m-2}} \phi_q \left( \int_{0}^{s} a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds \geq \int_{0}^{\xi_{m-2}} \phi_q \left( \int_{0}^{s} a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds \xi_i,
\]

(2.10)

for \( i = 1, 2, \ldots, m - 2 \). Thus, by Lemma 2.1(d), (2.1), (2.8) and (2.10) we have
\[
A_{y_0}(\xi_{m-2})
= - \int_{0}^{\xi_{m-2}} \phi_q \left( \int_{0}^{s} a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds \\
- \xi_{m-2} \sum_{i=1}^{m-2} a_i \int_{0}^{\xi_{m-2}} \phi_q \left( \int_{0}^{s} a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds \\
+ \xi_{m-2} \int_{0}^{1} \phi_q \left( \int_{0}^{s} a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds \\
= \xi_{m-2} \int_{0}^{1} \phi_q \left( \int_{0}^{s} a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds - \int_{0}^{\xi_{m-2}} \phi_q \left( \int_{0}^{s} a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds \\
+ \frac{1}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \sum_{i=1}^{m-2} a_i \left( \xi_i \int_{0}^{\xi_{m-2}} \phi_q \left( \int_{0}^{s} a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds \\
- \xi_{m-2} \int_{0}^{\xi_{m-2}} \phi_q \left( \int_{0}^{s} a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds \right)
\]
\[
\geq \frac{\xi_{m-2} \int_{0}^{1} \phi_q \left( \int_{0}^{s} a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds - \int_{0}^{\xi_{m-2}} \phi_q \left( \int_{0}^{s} a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \geq \frac{\xi_{m-2} \int_{0}^{1} \phi_q \left( \int_{0}^{s} a(\tau) f(\tau, y_0(\tau)) d\tau \right) ds}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \geq \phi_q \left( \phi_p(M'\gamma) \phi_p(\rho) \right) \frac{\xi_{m-2} \int_{0}^{1} \phi_q \left( \int_{0}^{s} a(\tau) d\tau \right) ds}{1 - \sum_{i=1}^{m-2} a_i \xi_i} = \frac{M'\gamma \rho}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \int_{0}^{1} \phi_q \left( \int_{0}^{s} a(\tau) d\tau \right) ds + \lambda_0.
\]

Hence, from (2.6), (2.9) and (2.11), we have that for \( t \in [\xi_{m-2}, 1] \)
\[
y_0(t) = Ay_0(t) + \lambda_0 e(t) \geq \min_{t \in [\xi_{m-2}, 1]} Ay_0(t) + \lambda_0
\geq \min \{ Ay_0(1), Ay_0(\xi_{m-2}) \} + \lambda_0
\geq \frac{M'\gamma \rho}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \min \left\{ \max \left\{ \sum_{i=1}^{m-2} a_i \xi_1, a_{m-2} \xi_{m-2} \right\}, \xi_{m-2} \right\}
\times \int_{\xi_{m-2}}^{1} \phi_q \left( \int_{\xi_{m-2}}^{s} a(\tau) d\tau \right) ds + \lambda_0
= \gamma \rho + \lambda_0.
\]
This implies that \( \gamma \rho \geq \gamma \rho + \lambda_0 \), a contradiction. Hence, by Lemma 1.1(2), it follows that 
\[i_K(A, \Omega_\rho) = 0.\]

We now give our results on the existence of multiple positive solutions of (1.1), (1.2).

By Lemmas 2.2, 2.3 and 1.1, we have

**Theorem 2.4.** Assume that one of the following conditions holds:

\((H_4)\) There exist \( \rho_1, \rho_2, \rho_3 \in (0, \infty) \) with \( \rho_1 < \gamma \rho_2 \) and \( \rho_2 < \rho_3 \) such that
\[
f_{\rho_1} \leq \phi_p(m), \quad f_{\gamma \rho_2} \geq \phi_p(M'\gamma), \quad u \neq Au \quad \text{for} \quad u \in \partial \Omega_{\rho_2}
\] and
\[
f_{\rho_3} \leq \phi_p(m).
\]

\((H_5)\) There exist \( \rho_1, \rho_2, \rho_3 \in (0, \infty) \) with \( \rho_1 < \rho_2 < \gamma \rho_3 \) such that
\[
f_{\gamma \rho_1} \geq \phi_p(M'\gamma), \quad f_{\rho_2} \leq \phi_p(m), \quad u \neq Au \quad \text{for} \quad u \in \partial K_{\rho_2}
\] and
\[
f_{\rho_3} \geq \phi_p(M'\gamma).
\]
Then (1.1), (1.2) has two positive solutions. Moreover, if in (H4) \( f^{\rho_1}_0 \leq \phi_p(m) \) is replaced by \( f^{\rho_1}_0 < \phi_p(m) \), then (1.1), (1.2) has a third positive solution \( y_3 \in K_{\rho_1} \).

The proof is similar to that given for Theorem 2.10 in [7]; we omit it here.

Theorem 2.4 can be generalized to obtain many positive solutions; we also omit it here.

As a special case of Theorem 2.4 we obtain the following result.

**Corollary 2.5.** If there exists \( \rho > 0 \) such that one of the following conditions holds:

(H6) \( 0 \leq f^0 < \phi_p(m) \), \( f^\rho \gamma \geq \phi_p(M\gamma) \), \( u \neq Au \) for \( u \in \partial \Omega_\rho \) and \( 0 \leq f^\infty < \phi_p(m) \),

(H7) \( \phi_p(M) < f^0 \leq \infty \), \( f^\rho_0 \leq \phi_p(m) \), \( u \neq Au \) for \( u \in \partial K_\rho \) and \( \phi_p(M) < f^\infty \leq \infty \),

then (1.1), (1.2) has two positive solutions.

**Proof.** We show that (H6) implies (H4). It is easy to verify that \( 0 \leq f^0 < \phi_p(m) \) implies that there exists \( \rho_1 \in (0, \gamma \rho) \) such that \( f^{\rho_1}_0 < \phi_p(m) \). Let \( k \in (f^\infty, \phi_p(m)) \). Then there exists \( r > \rho \) such that \( \max_{t \in [0,1]} f(t,u) \leq k \phi_p(u) \) for \( u \in [r, \infty) \) since \( 0 \leq f^\infty < \phi_p(m) \). Let 

\[
\beta = \max \left\{ \max_{t \in [0,1]} f(t,u) : 0 \leq u \leq r \right\}
\]

and 

\[
\rho_3 > \max \left\{ \frac{\beta}{\phi_p(m) - k}, \rho \right\}.
\]

Then we have

\[
\max_{t \in [0,1]} f(t,u) \leq k \phi_p(u) + \beta \leq k \phi_p(\rho_3) + \beta < \phi_p(m) \phi_p(\rho_3) \quad \text{for} \quad u \in [0, \rho_3].
\]

This implies that \( f^{\rho_3}_0 < \phi_p(m) \) and (H4) holds. Similarly, (H7) implies (H5). \( \square \)

By a similar argument to that of Theorem 2.4, we obtain the following results on existence of at least one positive solution of (1.1), (1.2).

**Theorem 2.6.** Assume that one of the following conditions holds:

(H8) There exist \( \rho_1, \rho_2 \in (0, \infty) \) with \( \rho_1 < \gamma \rho_2 \) such that

\[
f^{\rho_1}_0 \leq \phi_p(m) \quad \text{and} \quad f^{\rho_2}_\gamma \geq \phi_p(M\gamma).
\]

(H9) There exist \( \rho_1, \rho_2 \in (0, \infty) \) with \( \rho_1 < \rho_2 \) such that

\[
f^{\rho_1}_\gamma \geq \phi_p(M\gamma) \quad \text{and} \quad f^{\rho_2}_0 \leq \phi_p(m).
\]

Then (1.1), (1.2) has a positive solution.

As a special case of Theorem 2.6, we obtain the following result:
Corollary 2.7. Assume that one of the following conditions holds:

\((H_{10})\) \(0 \leq f^0 < \phi_p(m)\) and \(\phi_p(M) < f_\infty \leq \infty\).

\((H_{11})\) \(0 \leq f_\infty < \phi_p(m)\) and \(\phi_p(M) < f^0 \leq \infty\).

Then \((1.1), (1.2)\) has a positive solution.

Remark 2.2. For \(p = 2\) and \(f(t,u) \equiv f(u)\), Corollary 2.7 generalizes Theorem 1 in [8].

Acknowledgment

The authors express their thanks to Professor Zhivko S. Athanassov for his careful reading and useful suggestions in improving this paper.

References