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Optimal fault-tolerant routings with small routing tables for k -connected graphs

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Abstract

We study the problem of designing fault-tolerant routings with small routing tables for a k -connected network of n processors in the surviving route graph model. The surviving route graph $R(G, \rho)/F$ for a graph G , a routing ρ and a set of faults F is a directed graph consisting of non-faulty nodes of G with a directed edge from a node x to a node y iff there are no faults on the route from x to y . The diameter of the surviving route graph could be one of the fault-tolerance measures for the graph G and the routing ρ and it is denoted by $D(R(G, \rho)/F)$. We want to reduce the total number of routes defined in the routing, and the maximum of the number of routes defined for a node (called route degree) as least as possible. In this paper, we show that we can construct a routing λ for every n -node k -connected graph such that $n \geq 2k^2$, in which the route degree is $O(k\sqrt{n})$, the total number of routes is $O(k^2n)$ and $D(R(G, \lambda)/F) \leq 3$ for any fault set F ($|F| < k$). In particular, in the case that $k = 2$ we can construct a routing λ' for every biconnected graph in which the route degree is $O(\sqrt{n})$, the total number of routes is $O(n)$ and $D(R(G, \lambda')/\{f\}) \leq 3$ for any fault f . We also show that we can construct a routing ρ_1 for every n -node biconnected graph, in which the total number of routes is $O(n)$ and $D(R(G, \rho_1)/\{f\}) \leq 2$ for any fault f , and a routing ρ_2 (using ρ_1) for every n -node biconnected graph, in which the route degree is $O(\sqrt{n})$, the total number of routes is $O(n\sqrt{n})$ and $D(R(G, \rho_2)/\{f\}) \leq 2$ for any fault f .

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Keywords: Fault tolerance; Fixed routing; Routing table; Diameter; Distributed computing

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1. Introduction

Consider a communication network or an undirected graph G in which a limited number of link and/or node faults F might occur. A *routing* ρ for a graph defines at most one path called *route* for each ordered pair of nodes. A routing is said to be *minimal-length* if any route from x to y is assigned to a shortest path from x to y . We assume that it must be chosen without knowing which components might be faulty.

Given a graph G , a routing ρ and a set of faults F , the *surviving route graph* $R(G, \rho)/F$ is defined to be a directed graph consisting of all nonfaulty nodes in G , with a directed edge from a node x to a node y iff the route from x to y is intact. The diameter of the surviving route graph (denoted by $D(R(G, \rho)/F)$) could be one of the fault-tolerance measures for the graph G and the routing ρ [2,4]. In a network with a fixed routing, the time required to send a message along a route is often dominated by the message processing time at the endpoints of the route. Thus, the total message transmission time is proportional to the diameter of the surviving route graph. Also, in some distributed environment the diameter of the surviving route graph affects the number of phases required for each round of certain distributed protocols such as Byzantine agreement protocol [4]. Therefore, we need routings that minimize these diameters. The routing ρ on G is called (d, f) -tolerant if $D(R(G, \rho)/F) \leq d$ for any set F with at most f faults.

When we consider the fault tolerance of ATM and/or optical networks, routings must satisfy several constraints such as the number of routes defined for a node (*route degree of the node*) and the total number of routes defined in the routing [12]. Since the size of the routing table is dominated by the route degree of the node and the edge-load of the routing (the maximum number of routes passing through the edge over all edges) is dependent on the total number of routes, the route degree and the total number of routes should be as least as possible [3]. Moreover, if there is an edge between two nodes for which a route must be defined, the route should be defined as the edge (we call such routings *edge-routings*). Edge routings decrease edge-load of the routing and the size of the routing tables.

Many results have been obtained for the diameter of the surviving route graph [6,10,11,13]. As far as we use minimal-length routings for general graphs, we can not expect good behavior for the diameter of the surviving route graph, say constant diameter [4]. It is also shown that the graph connectivity does not help to reduce the diameter of the surviving route graph if only minimal-length routings are considered [4]. Therefore, we must consider non-minimal-length routings to obtain efficient fault-tolerant ones for k -connected graphs. For n -node k -connected graphs, a $(5, k - 1)$ -tolerant routing and a $(3, k - 1)$ -tolerant routing can be constructed if $n \geq k^2$ and $n \geq 2k^2$, respectively [9]. A $(2, k - 1)$ -tolerant routing can be constructed for every n -node k -connected graph such that $n \geq 7k^3 \lceil \log_2 n \rceil$ [15]. However, in these routings, the route degree of most nodes is $n - 1$ and it is undesirable. Stronger results have been known for n -node biconnected graphs [12]; We can construct a $(2, 1)$ -tolerant routing with $O(n)$ routes for every n -node biconnected graph [12]. It can be shown that the routing is optimal in the sense that not only the diameter of the surviving route graph but also the total number of routes in the routing are the minimum.

In this paper, we show the following results which improve the previous ones with respect to the route degree and the total number of routes.

- (1) For every n -node k -connected graph such that $n \geq 2k^2$, we can construct a $(3, k - 1)$ -tolerant routing λ in which the route degree is $O(k\sqrt{n})$ and the total number of routes is $O(k^2n)$.
- (2) For every n -node biconnected graph, we can construct a $(3, 1)$ -tolerant edge-routing λ' in which the route degree is $O(\sqrt{n})$ and the total number of routes is $O(n)$.
- (3) For every n -node biconnected graph, we can construct a $(2, 1)$ -tolerant routing ρ_1 in which the total number of routes is $O(n)$ and a $(2, 1)$ -tolerant routing ρ_2 in which the route degree is $O(\sqrt{n})$ and the total number of routes is $O(n\sqrt{n})$.

We improve the $(3, k - 1)$ -tolerant routing shown in [9] to the routing λ so that the route degree of λ is reduced to $O(k\sqrt{n})$ with preserving the total number of routes. We also show that the diameter of the surviving route graph for λ is optimal among routings with the route degree $O(k\sqrt{n})$ if $n \geq 2k^2$ and $k = o(n^{1/6})$. In the case that $k = 2$, we obtain a stronger result than the general case.

The routing ρ_1 does not improve the previous result. However, the idea to define ρ_1 is different from the previous ones and it induces the routing ρ_2 that is the first $(2, 1)$ -tolerant routing with route degree $O(\sqrt{n})$ for biconnected graphs. We also show that the total number of routes in the routing ρ_2 is the minimum among $(2, 1)$ -tolerant routings with route degree $O(\sqrt{n})$.

2. Preliminary

In this section, we give definitions and terminology. We refer readers to [8] for basic graph terminology.

Unless otherwise stated, we deal with an undirected graph $G = (V, E)$ that corresponds to a network. For a node v of G , $N_G(v) = \{u \mid (v, u) \in E\}$ and $\deg_G(v) = |N_G(v)|$. $\deg_G(v)$ is called *degree* of v and if G is apparent it is simply denoted by $\deg(v)$. For a node set $U \subseteq V$, the subgraph induced by U is the maximal subgraph of G with the node set U and denoted by $G\langle U \rangle$. A graph G is *k-connected* if there exist k node-disjoint paths between every pair of distinct nodes in G . For a node $v \in V$ and a node set $U \subseteq V - \{v\}$, *v-U fan* is a set of $|U|$ node-disjoint paths from v to all nodes of U . Usually 2-connected graphs are called *biconnected graphs*.

The *distance* between nodes x and y in G is the length of the shortest path between x and y and is denoted by $\text{dis}_G(x, y)$. The *diameter* of G is the maximum of $\text{dis}_G(x, y)$ over all pairs of nodes in G and is denoted by $D(G)$. Let $P(u, v)$ and $P(v, w)$ be a path from u to v and a path from v to w , respectively. In general, even if both $P(u, v)$ and $P(v, w)$ are simple, the concatenation of $P(u, v)$ and $P(v, w)$ is not always simple. Thus we consider two kinds of concatenation: one is a usual concatenation (denoted by $P(u, v) \cdot P(v, w)$) and the other is a special concatenation (denoted by $P(u, v) \odot P(v, w)$), which is defined as the shortest path from u to w in the graph $P(u, v) \cup P(v, w)$ to make the concatenated path simple.

Let $G = (V, E)$ be a graph and let x and y be nodes of G . Define $P_G(x, y)$ to be the set of all simple paths from the node x to the node y in G , and $P(G)$ to be the set of all simple paths in G . A *routing* is a partial function $\rho: V \times V \rightarrow P(G)$ such that

$\rho(x, y) \in P_G(x, y) (x \neq y)$. The path specified to be $\rho(x, y)$ is called the *route from x to y* . For the routes $\rho(x_{i-1}, x_i) (1 \leq i \leq p)$, define $[\rho(x_0, x_1), \rho(x_1, x_2), \dots, \rho(x_{p-1}, x_p)]$ to be $\rho(x_0, x_1) \cdot \rho(x_1, x_2) \cdots \rho(x_{p-1}, x_p) (p \geq 1)$. We call $[\rho(x_0, x_1), \rho(x_1, x_2), \dots, \rho(x_{p-1}, x_p)]$ a *route sequence of length p from x_0 to x_p* .

For a graph $G = (V, E)$, let $F \subseteq V \cup E$ be a set of nodes and edges called a set of *faults*. We call $F \cap V (= F_V)$ and $F \cap E (= F_E)$ the set of *node faults* and the set of *edge faults*, respectively. If an object such as a route or a node set does not contain any element of F , the object is said to be *fault free*.

For a graph $G = (V, E)$, a routing ρ on G and a set of faults $F (= F_V \cup F_E)$, the *surviving route graph*, $R(G, \rho)/F$, is a directed graph with node set $V - F_V$ and edge set $E(G, \rho, F) = \{(x, y) \mid \rho(x, y) \text{ is defined and fault free}\}$. In what follows, unless confusion arises we use notations for directed graphs as the same ones for undirected graphs.

In the surviving route graph $R(G, \rho)/F$, when $F = \emptyset$ the graph is called the route graph. In the route graph, the outdegree of a node v is called *the route degree of a node v* and the maximum of the route degree of all nodes is called *the route degree of the routing ρ* . The number of directed edges in the route graph corresponds to the total number of routes in the routing ρ . If the number of edges in the route graph is m , the routing ρ is called *m -route-routing* or simply *m -routing*.

For a graph $G = (V, E)$ and a routing ρ , if for any edge (x, y) in G such that $\rho(x, y)$ is defined, the route $\rho(x, y)$ is assigned to the edge, ρ is called *edge-routing*.

A routing ρ is *bidirectional* if $\rho(x, y) = \rho(y, x)$ for any node pair (x, y) in the domain of ρ . If a routing is not bidirectional, it is called *unidirectional*. Note that if the routing ρ is bidirectional, the surviving route graph $R(G, \rho)/F$ can be represented as an undirected graph.

Given a graph G and a routing property P , a routing ρ on G is *optimal* with respect to P if $\max_{F \text{ s.t. } |F| \leq k} (D(R(G, \rho)/F))$ is minimum over all routings on G satisfying P . Note that from the definition of the optimality, if $D(R(G, \rho)/F)$ is 2 for any set of faults F such that $|F| \leq k$, the routing is obviously optimal with respect to any property. If the property P is known, we simply call the routing is optimal.

Lemma 2.1 [1]. *Let $G = (V, E)$ be an n -node directed graph. If the maximum outdegree of G is d and the diameter of G is p , then $|E| = \Omega(n^2/d^{p-1})$.*

3. Optimal routing for k -connected graphs

In this section, we show that for n -node k -connected graphs with $n \geq 2k^2$, we can construct a $(3, k - 1)$ -tolerant bidirectional edge-routing λ such that the route degree is $O(k\sqrt{n})$ and the total number of routes in λ is $O(k^2n)$ and we show that the routing λ is optimal with respect to the route degree of $O(k\sqrt{n})$ if $n \geq 2k^2$ and $k = o(n^{1/6})$.

The routings for k -connected graphs are based on the following two properties [9,11,15].

Lemma 3.1 [8]. *Let $G = (V, E)$ be a k -connected graph. Let U be any node set of V such that $|U| = k$ and let v be any node in $V - U$. Then there is a v - U fan.*

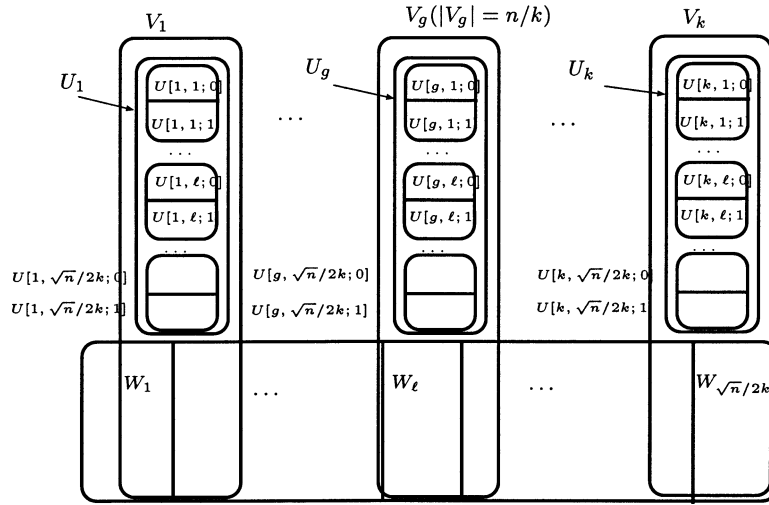


Fig. 1. The partition of nodes in G .

Let $u \in U$. The path from v to u in v - U fan is denoted by $P_{\text{fan}}(v, u; U)$. If there is an edge between v and u in U , the path $P_{\text{fan}}(v, u; U)$ in the v - U fan can be changed to this edge [4].

Lemma 3.2 [7]. *Let $G = (V, E)$ be a k -connected graph. Let v_1, v_2, \dots, v_k be distinct nodes and a_1, a_2, \dots, a_k be positive integers such that $\sum_{i=1}^k a_i = |V|$. Then there is a partition V_1, V_2, \dots, V_k of V such that $v_i \in V_i$, $|V_i| = a_i$ and the induced subgraph $G\langle V_i \rangle$ is connected for $i = 1, 2, \dots, k$.*

First we consider k -connected graphs with at least $4k^2$ nodes, and later we extend the result for k -connected graphs with at least $2k^2$ nodes.

Let $G = (V, E)$ be an n -node k -connected graph such that $n \geq 4k^2$. From Lemma 3.2, there are k disjoint connected graphs G_1, G_2, \dots, G_k which contain disjoint node subsets V_1, V_2, \dots, V_k such that $|V_g| = n/k$ for $g = 1, 2, \dots, k$. Let U_g be a subset of V_g such that $|U_g| = \sqrt{n}$ and let each U_g be partitioned into $\sqrt{n}/2k$ sets with each $2k$ nodes. These sets are denoted by $U[g, \ell]$ ($1 \leq \ell \leq \sqrt{n}/2k$). Furthermore, each $U[g, \ell]$ is partitioned into two sets with cardinalities k . These sets are denoted by $U[g, \ell; 0]$ and $U[g, \ell; 1]$. Let $W = V - \bigcup_{g=1}^k U_g$ and let W be partitioned into $\sqrt{n}/2k$ sets. These sets are denoted by $W_1, W_2, \dots, W_{\sqrt{n}/2k}$. The partition of V is shown in Fig. 1.

A bidirectional routing λ is defined as follows:

- (1) For $x \in W_\ell$ and $y \in U[g, \ell; 0]$ ($1 \leq g \leq k, 1 \leq \ell \leq \sqrt{n}/2k$), $\lambda(x, y) = \lambda(y, x) = P_{\text{fan}}(x, y; U[g, \ell; 0])$.

¹ For simplicity, we assume that $\sqrt{n}/2k$ is an integer.

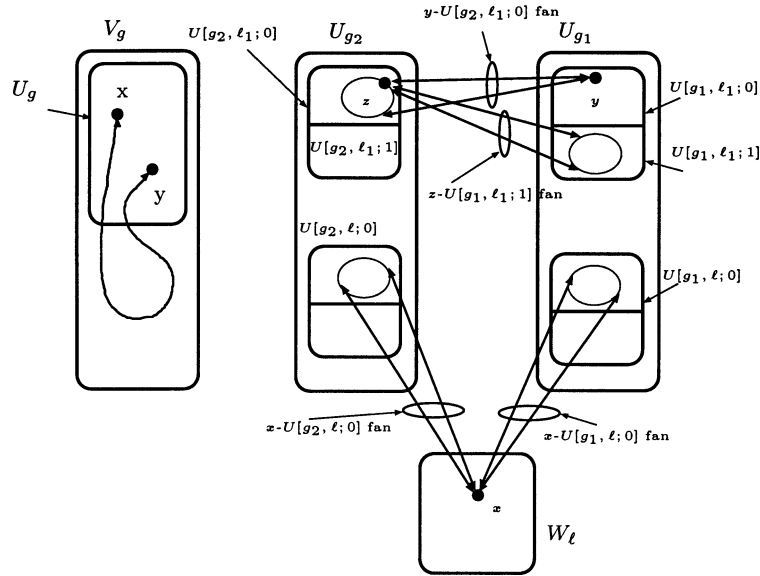


Fig. 2. The routing λ .

No. of def.	Outdegree of node x	No. of routes
1.	k^2 ($x \in W_\ell$) $2k\sqrt{n} - 2k^2$ ($x \in U[g, \ell; 0]$)	$k^2n - k^3\sqrt{n}$ $kn - k^2\sqrt{n}$
2.	$\sqrt{n} - 1$ ($x \in U_g$)	$nk - \sqrt{n}k$
3.	$k^2 - k$ ($x \in U[g, \ell; i]$)	$k^2\sqrt{n} - k\sqrt{n}$
4.	$k^2 - k$ ($x \in U[g, \ell; i]$)	$k^2\sqrt{n} - k\sqrt{n}$

Fig. 3. The outdegree and the number of routes for λ .

- (2) For $x, y \in U_g$ ($1 \leq g \leq k$),
 $\lambda(x, y) = \lambda(y, x) =$ a shortest path between x and y in $G(V_g)$.
- (3) For $x \in U[g_1, \ell; i]$ and $y \in U[g_2, \ell; i]$ ($1 \leq g_2 < g_1 \leq k, 1 \leq \ell \leq \sqrt{n}/2k, i = 0, 1$)
 $\lambda(x, y) = \lambda(y, x) = P_{\text{fan}}(x, y; U[g_2, \ell; i])$.
- (4) For $x \in U[g_1, \ell; i_1]$ and $y \in U[g_2, \ell; i_2]$ ($1 \leq g_2 < g_1 \leq k, 1 \leq \ell \leq \sqrt{n}/2k, i_1 \neq i_2$)
 $\lambda(x, y) = \lambda(y, x) = P_{\text{fan}}(y, x; U[g_1, \ell; i_1])$.

Fig. 2 shows the routes in λ . Fig. 2 does not show $z'-U[g_1, \ell_1; 0]$ fan ($z' \in U[g_2, \ell_1; 1]$) and $y'-U[g_2, \ell_1; 1]$ fan ($y' \in U[g_1, \ell_1; 1]$) for the lack of space. Because of the property of $v-U$ fan, λ is an edge-routing. Fig. 3 shows the outdegrees of λ and the number of routes defined in λ . It can be verified that the node with the maximum degree is in $U[g, \ell; 0]$ and the route degree of λ is $(2k + 1)\sqrt{n} - 2k - 1 = O(k\sqrt{n})$, and the total number of routes in λ is $(k^2 + 2k)n + (-k^3 - k^2 - 2k)\sqrt{n} = O(k^2n)$.

From the definition of λ , the next lemma holds.

Lemma 3.3. *Let $G = (V, E)$ be a k -connected graph on which the routing λ is defined. For any node x and any node set U_g such that $x \notin U_g$, there are k node-disjoint routes from x to k nodes in U_g .*

Theorem 3.4. *Let $G = (V, E)$ be an n -node k -connected graph such that $n \geq 4k^2$. The routing λ on G is $(3, k - 1)$ -tolerant.*

Proof. Let F be a faulty set with $|F| < k$ and let $R = R(G, \lambda)/F$. Since $|F| < k$ and the number of node sets V_g is k , there is a node set V_I such that $G\langle V_I \rangle$ contains no elements of F . Let x and y be arbitrary non-faulty distinct nodes in $V - F$.

Case 1. Suppose that $x, y \in U_I$. Since $U_I \subseteq V_I$ and the route $\lambda(x, y)$ is defined in $G\langle V_I \rangle$, $\lambda(x, y)$ is fault free. Thus, $\text{dis}_R(x, y) = 1$.

Case 2. Suppose that $x \in U_I$ and $y \notin U_I$. From **Lemma 3.3**, there are k node-disjoint routes between y and k nodes in U_I . Since $|F| < k$, there is a fault free route between y and an node, say w , in U_I . Since $\lambda(x, w)$ does not contain any fault in F , $\text{dis}_R(x, y) = 2$.

The case that $x \notin U_I$ and $y \notin U_I$ can be proved similar to case 2 by using **Lemma 3.3** and it holds that $\text{dis}_R(x, y) = 3$. \square

For an n -node k -connected graph, if $2k^2 \leq n \leq 4k^2$, set $|U_g| = 2k$ instead of $|U_g| = \sqrt{n}$ in the definition. In this case the total number of routes in λ is $O(k^4) = O(k^2n)$ and the route degree of λ is $O(k^2) = O(k\sqrt{n})$. Therefore, the following theorem holds.

Theorem 3.5. *Let $G = (V, E)$ be an n -node k -connected graph. If $n \geq 2k^2$, we can construct a $(3, k - 1)$ -tolerant bidirectional edge-routing such that the total number of routes is $O(k^2n)$ and the route degree is $O(k\sqrt{n})$. This routing is optimal with respect to the route degree of $O(k\sqrt{n})$ if $n \geq 2k^2$ and $k = o(n^{1/6})$.*

Proof. The optimality of λ can be shown as follows. If $k = o(n^{1/6})$ the least number of routes in $(2, k - 1)$ -routings with route degree of $O(k\sqrt{n})$ is $\omega(n^{4/3})$ from **Lemma 2.1**. On the other hand, if $k = o(n^{1/6})$, the total number of routes in λ is $o(n^{4/3})$. \square

In the case that $k = 2$, we will show in the next section that for every n -node biconnected graph we can construct a $(3, 1)$ -tolerant bidirectional edge-routing such that the total number of routes is $O(n)$ and the route degree is $O(\sqrt{n})$.

4. Optimal routings for biconnected graphs

4.1. Paths by using s - t numbering

Optimal routings for biconnected graphs are based on s - t numbering [5] which characterize biconnected graphs [12,14].

Given an edge (s, t) of a biconnected graph $G = (V, E)$, a bijective function $g: V \rightarrow \{0, 1, \dots, |V| - 1 = n - 1\}$ is called an s - t numbering if the following conditions are satisfied:

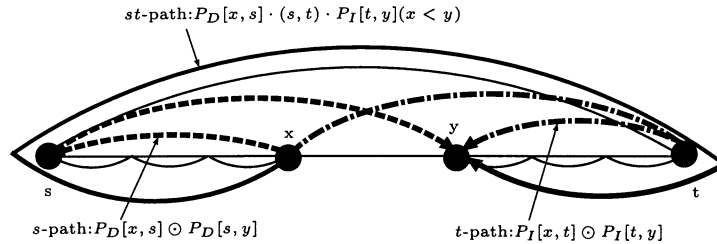


Fig. 4. Three kinds of paths, *s*-path, *t*-path and *st*-path.

- $g(s) = 0, g(t) = n - 1$, and
- every node $v \in V - \{s, t\}$ has two adjacent nodes u and w such that $g(u) < g(v) < g(w)$.

In what follows, we assume that the node set of G is *s*-*t* numbered and it is denoted by $\{0, 1, \dots, n - 1\}$, where $s = 0$ and $t = n - 1$.

For a node v in G , we define two paths $P_I[v, t]$ and $P_D[v, s]$ as follows:

- (1) $P_I[v, t] = (v_0(= v), v_1, \dots, v_p(= t))$, where $v_i = \max\{u \mid u \in N_G(v_{i-1})\}$ ($1 \leq i \leq p$), and
- (2) $P_D[v, s] = (v_0(= v), v_1, \dots, v_q(= s))$, where $v_i = \min\{u \mid u \in N_G(v_{i-1})\}$ ($1 \leq i \leq q$).

Since we treat unidirectional routings, we consider directions for undirected paths. Therefore, for example, $P_I[v, t]$ denotes the path from v to t and the path from t to v of the same one is denoted by $P_I[t, v]$.

Note that if (v, s) and (v, t) are in E , $P_D[v, s] = (v, s)$ and $P_I[v, t] = (v, t)$ from the definition.

From the definition of the *s*-*t* numbering, two paths $P_I[v, t]$ and $P_D[v, s]$ are well defined and $P_I[x, t]$ and $P_D[x, s]$ are node-disjoint for any node $x (\neq s, t)$.

We define the following concatenated paths with P_I s and P_D s. Let x and y be arbitrary distinct nodes. $P_s[x, y]$, $P_t[x, y]$ and $P_{st}[x, y]$ are defined as $P_D[x, s] \odot P_D[s, y]$, $P_I[x, t] \odot P_I[t, y]$ and $P_D[x, s] \cdot (s, t) \cdot P_I[t, y]$ (if $x < y$) and $P_I[x, t] \cdot (t, s) \cdot P_D[s, y]$ (if $x > y$), respectively, and they are called *s*-path, *t*-path and *st*-paths, respectively (Fig. 4).

4.2. Optimal (3, 1)-tolerant routing λ'

In this section, for every biconnected graph we construct an $O(n)$ -routing λ' with route degree $O(\sqrt{n})$ such that the diameter of the surviving route graph is three for any one fault. We also show that the routing is optimal in $O(n)$ -routings with route degree $O(\sqrt{n})$.

Theorem 4.1. *Let a routing σ on G be an $O(n)$ -routing. If the route degree of σ is at most $o(n)$, $D(R((G, \sigma)/\{f\})) \geq 3$ for any fault f .*

Proof. We show that this statement holds even if there is no fault in G . Assume that the diameter of the route graph is at most 2. The least number of edges of a graph of n nodes with degree k and diameter 2 is $\Omega(n^2/k)$ from Lemma 2.1. Since the route degree of σ is $o(n)$, the number of edges in the route graph must be $\omega(n)$. It is a contradiction. \square

We construct an $O(n)$ -routing λ' with route degree $O(\sqrt{n})$ which attains the lower bound in Theorem 4.1. The routing λ' is a hierarchical one based on the optimal routing ρ in [14].

We assume that a biconnected graph $G = (V = \{s = 0, \dots, t = n - 1\}, E)$ has at least 5 nodes. We divide the nodes of G into $\ell = \lfloor n/p \rfloor$ sections of size p each except the last section. Note that the last section contains at most $2p - 1$ nodes. For each section denoted by V_i ($1 \leq i \leq \ell$), the least numbered node and the largest numbered node are denoted by s_i and t_i , respectively. Note that $s = s_1$ and $t = t_\ell$.

routing λ'

- (1) For $x \in V_i$ ($1 \leq i \leq \ell$) such that $s_i < x < t_i$,
 $\lambda'(x, s_i) = P_s[x, s_i], \lambda'(s_i, x) = P_s[s_i, x],$
 $\lambda'(x, t_i) = P_t[x, t_i], \lambda'(t_i, x) = P_t[t_i, x],$ and
 $\lambda'(x, s_{i+1}) = P_s[x, s_{i+1}], \lambda'(s_{i+1}, x) = P_s[s_{i+1}, x]$ (if $i < \ell$) and
 $\lambda'(x, t_{i-1}) = P_t[x, t_{i-1}], \lambda'(t_{i-1}, x) = P_t[t_{i-1}, x]$ (if $1 < i$).
- (2) For i, j ($1 \leq i < j \leq \ell$),
 $\lambda'(s_i, s_j) = P_s[s_i, s_j], \lambda'(s_j, s_i) = P_s[s_j, s_i]$ and
 $\lambda'(t_i, t_j) = P_t[t_i, t_j], \lambda'(t_j, t_i) = P_t[t_j, t_i].$
- (3) For i, j ($1 \leq i \leq j \leq \ell$),

$$\lambda'(s_i, t_j) = \begin{cases} (s_1, t_\ell) & \text{if } i = 1 \text{ and } j = \ell, \\ P_s[s_1, t_j](= P_D[s_1, t_j]) & \text{if } i = 1 \text{ and } j \neq \ell, \\ P_t[s_i, t_\ell](= P_I[s_i, t_\ell]) & \text{if } i \neq 1 \text{ and } j = \ell, \\ P_{st}[s_i, t_j] & \text{otherwise,} \end{cases}$$

and

$$\lambda'(t_j, s_i) = \begin{cases} (t_\ell, s_1) & \text{if } i = 1 \text{ and } j = \ell, \\ P_s[t_j, s_1](= P_D[t_j, s_1]) & \text{if } i = 1 \text{ and } j \neq \ell, \\ P_t[t_\ell, s_i](= P_I[t_\ell, s_i]) & \text{if } i \neq 1 \text{ and } j = \ell, \\ P_{st}[t_j, s_i] & \text{otherwise.} \end{cases}$$

- (4) For x and y such that the routes $\lambda'(x, y) = \lambda'(y, x)$ are defined in (1)–(3), if $(x, y) \in E$ then $\lambda'(x, y) = \lambda'(y, x)$ is changed to (x, y) .

It is easily verified that λ' is a bidirectional edge-routing and an $O(n + \ell^2)$ -routing with route degree $O(p + \ell)$. Thus, if $\ell = \lfloor \sqrt{n} \rfloor$ then λ' is an $O(n)$ -routing with route degree $O(\sqrt{n})$.

Except that $(x, y) \in E$ and $f = (x, y)$, we can assume that f is a node because if f is an edge we can consider that one of the endpoints of f is faulty. We write $f \in [a, b]$ if $f \in V$ and $a \leq f \leq b$.

In the case that $(x, y) \in E$ and $f = (x, y)$: if $\lambda'(x, y)$ is not defined then it is trivial. Otherwise, from the definition of λ' we have two cases, (1) $x \in V_i$ and $y \in \{s_i, t_i, s_{i-1}, t_{i-1}\}$ for some i and (2) $x, y \in \{s_i, s_j, t_i, t_j\}$ for $i \neq j$. We can easily find a route sequence of length 3 that are fault free.

Theorem 4.2. *Let G be a biconnected graph with at least 5 nodes. Then $D(R(G, \lambda')/\{f\}) \leq 3$ for any fault f in G .*

Proof. Let $R = R(G, \lambda')/\{f\}$. Let x and y be any pair of distinct nonfaulty nodes in G .

(1) Suppose that $x = s$ and $y = t$. If f is not the edge (s, t) then $\text{dis}_R(s, t) = 1$. Otherwise, for any $z \in \{s_i, t_i\}$ ($1 < i < \ell$), the route sequence $[\lambda'(s, z), \lambda'(z, t)]$ cannot contain f . Thus, $\text{dis}_R(s, t) \leq 2$.

(2-1) Suppose that $x = s$ and $y \in \{s_j, t_j\}$ ($1 \leq j \leq \ell$ and $y \neq t$) or $x \in \{s_i, t_i\}$ ($1 \leq i \leq \ell$ and $x \neq s$) and $y = t$. For the former case, since there are two node-disjoint route sequences $\lambda'(s, y)$ and $[\lambda'(s, t), \lambda'(t, y)]$, $\text{dis}_R(x, y) \leq 2$. The latter case can be proved similarly.

(2-2) Suppose that $x \in \{s_i, t_i\}$ and $y \in \{s_j, t_j\}$ ($1 \leq i \leq j \leq \ell$, $x \neq s$ and $y \neq t$).

If $f = (s, t)$ or $f \in [s, x]$, $\lambda'(x, t = t_\ell)$ and $\lambda'(y, t = t_\ell)$ are fault free. Thus, $\text{dis}_R(x, y) \leq 2$. Similarly, if $f \in [y, t]$ since $\lambda'(x, s = s_1)$ and $\lambda'(y, s = s_1)$ are fault free $\text{dis}_R(x, y) \leq 2$. Otherwise ($f \in [x, y]$), $\lambda'(x, s_1)$, $\lambda'(s_1, t_\ell)$ and $\lambda'(y, t_\ell)$ are fault free. Thus, $\text{dis}_R(x, y) \leq 3$.

(3) Suppose that $x \in V_i - \{s_i, t_i\}$ and $y \in V_j - \{s_j, t_j\}$ ($1 \leq i \leq j \leq \ell$) such that $g(x) < g(y)$.

If $f = (s, t)$ or $f \in [s, x]$, since $\lambda'(x, t_i)$ and $\lambda'(y, t_j)$ are fault free, $\text{dis}_R(x, t_i) \leq 1$ and $\text{dis}_R(y, t_j) \leq 1$. Also since $\lambda'(t_i, t_j)$ does not contain the fault, $\text{dis}_R(x, y) \leq 3$. Similarly, $\text{dis}_R(x, y) \leq 3$ holds for the case that $f \in [y, t]$. Otherwise ($f \in [x, y]$), $\lambda'(x, s_i)$, $\lambda'(s_i, t_j)$ and $\lambda'(y, t_j)$ are fault free. Thus, $\text{dis}_R(x, y) \leq 3$.

(4) Otherwise, $x \in V_i$ and $y \in V_j$ ($1 \leq i \leq j \leq \ell$) such that either $x \in \{s_i, t_i\}$ or $y \in \{s_j, t_j\}$. The cases that $x = s_i$ and $s_j < y < t_j$ and that $s_i < x < t_i$ and $y = t_j$ can be proved similar to the case (3). For the case that $x = t_j$ and $s_j < y < t_j$, except that $f \in [x, y]$ we can prove similar to the case (3). In the case that $f \in [x, y]$, if $1 < i$ and $i < \ell$ then $\lambda'(x, t_{i-1})$, $\lambda'(t_{i-1}, s_{j+1})$ and $\lambda'(y, s_{j+1})$ are fault free. Thus, $\text{dis}_R(x, y) \leq 3$. If $i = 1$ or $i = \ell$, then $\lambda'(x = s_1, t_\ell)$, $\lambda'(t_\ell, t_j)$ and $\lambda'(y, t_j)$ (if $i = 1$) and $\lambda'(x = s_i, s_1)$, $\lambda'(s_1, t_\ell)$ and $\lambda'(y, t_\ell)$ (if $i = \ell$) are fault free, respectively. Thus, $\text{dis}_R(x, y) \leq 3$.

The case that $s_i < x < t_i$ and $y = t_j$ can be treated symmetrically. \square

For a biconnected graph G with at most 4 nodes, it is easily proved that an edge-routing for G is (3, 1)-tolerant. Therefore, the following theorem holds.

Theorem 4.3. *For every n -node biconnected graph G , we can construct a (3, 1)-tolerant edge-routing on G in which the total number of routes is $O(n)$ and the route degree is $O(\sqrt{n})$. This routing is optimal with respect to $O(n)$ -routings with route degree $O(\sqrt{n})$.*

4.3. Optimal routing ρ

As far as the authors have known, there do not exist $(2, k - 1)$ -tolerant routings for k -connected graphs with their route degree $O(k\sqrt{n})$. Although the case that $k \geq 3$ is still open, we show that we can construct a $(2, 1)$ -tolerant edge-routing for biconnected graphs such that its route degree is $O(\sqrt{n})$ and the total number of routes is $O(n\sqrt{n})$. This routing is optimal because it is $(2, 1)$ -tolerant. From Lemma 2.1, in order to define $(2, 1)$ -tolerant routings with route degree $O(\sqrt{n})$, the total number of routes must be $\Omega(n\sqrt{n})$. Thus, the total number of routes in the routing shown here attains the lower bound.

Let $G = (V, E)$ be a biconnected graph with n nodes. Assume that $n \geq 18$ and V is divided into $\lfloor n/18 \rfloor$ groups of 18 nodes each and the last group made up of the remaining $(n \bmod 18)$ nodes. Each group except the last one is divided into two parts with 9 nodes each.

For a node $v \in V$, let $q = v \text{ div } 18$, $r = v \bmod 18$, $g = r \text{ div } 9$ and $\ell = r \bmod 9$. Each node v is represented as $[q; g, (i, j)]$, where $0 \leq q \leq \lfloor n/18 \rfloor$, $g = 0, 1$ and (i, j) is the ternary representation of ℓ ($0 \leq i, j \leq 2$).

We define the routing ρ for G based on the ternary representation (i, j) of ℓ of each node as follows: Let x and y be represented as $[q_x; g_x, (i_x, j_x)]$ and $[q_y; g_y, (i_y, j_y)]$, respectively. The route $\rho(x, y)$ is defined as shown in Fig. 5. The route from x to y is determined based on j_x and i_y . For example, if $j_x = 0$ and $i_y = 2$ then the t -path from x to y is used to define $\rho(x, y)$ and $j_x = 2$ and $i_y = 0$ then the st -path from x to y is used to define $\rho(x, y)$ and so on. The routing ρ is well-defined. It is a unidirectional and $n(n - 1)$ -routing and its route-degree is $n - 1$.

The intuitive idea of the routing ρ is as follows: Let u and v be arbitrary distinct nodes and we consider the route sequence from u to v . Since in each interval there are two nine-node sets, at least one of them is fault-free and each node in this fault-free nine-node set plays a role of an intermediate node z in the route sequence from u to v according to the location of a fault. Since surviving routes $\rho(u, z)$ and $\rho(z, v)$ are s -path, t -path or st -path, the all combinations of these paths are prepared and the intermediate node z is determined according to u and v . For example, if $j_u = 0$, $i_v = 0$, $\rho(u, z)$ is s -path and $\rho(z, v)$ is t -path, the intermediate node z is chosen such that $i_z = 0$ and $j_z = 1$. Lemma 4.4 will show that the definition of ρ depicted in Fig. 5 is sufficient to prove that ρ is $(2, 1)$ -tolerant.

Let $I[q, g] = \{[q; g, (i, j)] \mid 0 \leq i \leq j \leq 2\}$, where $0 \leq q \leq \lfloor n/18 \rfloor - 1$ and $g = 0, 1$, and let $I[q] = I[q, 0] \cup I[q, 1]$.

	$i_y = 0$	$i_y = 1$	$i_y = 2$
$\rho(x, y) =$	$j_x = 0 \quad P_s[x, y]$	$P_{st}[x, y]$	$P_t[x, y]$
	$j_x = 1 \quad P_t[x, y]$	$P_s[x, y]$	$P_{st}[x, y]$
	$j_x = 2 \quad P_{st}[x, y]$	$P_t[x, y]$	$P_s[x, y]$

where $x = [q_x; g_x, (i_x, j_x)]$ $y = [q_y; g_y, (i_y, j_y)]$.

Fig. 5. The routing ρ .

Lemma 4.4. Let $G = (V, E)$ be a biconnected graph on which the routing ρ is defined. For arbitrary distinct nodes x and y , arbitrary two kinds of paths a -path and b -path ($a, b \in \{s, t, st\}$) and any interval $I[q, g]$, there exists a node $z \in I[q, g]$ such that $\rho(x, z)$ is a -path and $\rho(z, y)$ is b -path.²

Proof. Let $x = [q_x; g_x, (i_x, j_x)]$ and $y = [q_y; g_y, (i_y, j_y)]$. For example, if $j_x = 0, i_y = 1, a = s$ and $b = st$, then z is defined as $[q; g, (0, 0)] \in I[q, g]$. From the definition of ρ , we can see that $\rho(x, z)$ is defined as s -path and $\rho(z, y)$ is defined as st -path. Formally, we define i_z and j_z of $z = [q; g, (i_z, j_z)] \in I[q, g]$ according to a and b as follows:

$$i_z = j_x \text{ (if } a = s\text{)}, j_x + 2 \pmod{3} \text{ (if } a = t\text{)} \text{ and } j_x + 1 \pmod{3} \text{ (if } a = st\text{)},$$

$$j_z = i_y \text{ (if } b = s\text{)}, i_y + 1 \pmod{3} \text{ (if } b = t\text{)} \text{ and } i_y + 2 \pmod{3} \text{ (if } b = st\text{)}.$$

Then it can be verified that the route $\rho(x, z)$ is defined as a -path and $\rho(z, y)$ is defined as b -path for $z \in I[q, g]$ from the definition of ρ . \square

Theorem 4.5. Let G be an n -node biconnected graph such that $n \geq 18$. The routing ρ on G is $(2, 1)$ -tolerant.

Proof. Let f be any fault and let $R = R(G, \rho) \setminus \{f\}$. Let x and y be arbitrary distinct nonfaulty nodes in G .

Since there is one fault in G , either $I[0, 0] = \{[0; 0, (i, j)] \mid 0 \leq i \leq j \leq 2\}$ or $I[0, 1]$ does not contain f . Without loss of generality, we can assume that $I[0, 1]$ is fault free. Note that $I[0, 1] = \{9, 10, \dots, 17\}$. There are 12 cases according to the locations of x, y and $I[0, 1]$ and they can be similarly proved by using Lemma 4.4. We show one case that $x < I[0, 1] < y$ (it means that $x < 9$ and $17 < y$).

Suppose that $x < I[0, 1] < y$. If $f = (x, y) \in E$, then from Lemma 4.4 there is a node $z \in I[0, 1]$ such that $\rho(x, z)$ is an st -path and $\rho(z, y)$ is a t -path and they do not contain f . If $f \in [s = 0, x - 1]$ then from Lemma 4.4 there is a node $z \in I[0, 1]$ such that both $\rho(x, z)$ and $\rho(z, y)$ are t -paths and their routes are fault free. If $f \in [x + 1, 7]$, then from Lemma 4.4 there is a node $z \in I[0, 1]$ such that $\rho(x, z)$ is an s -path and $\rho(z, y)$ is a t -path and they are fault free. The cases that $f \in [18, y - 1]$ and $f \in [y + 1, t = n - 1]$ can be proved similarly. Therefore $\text{dis}_R(x, y) \leq 2$. \square

The routing ρ is not an edge-routing because there is a case that $(x, y) \in E$ and $\rho(x, y)$ is defined as an st -path. However, it can be changed into an edge-routing as follows. Since both an s -path and a t -path from x to y become an edge if $(x, y) \in E$ from the definition, if $\rho(x, y)$ is defined by an st -path and $(x, y) \in E$, then $\rho(x, y)$ is defined as the edge (x, y) . We can show that the modified ρ is $(2, 1)$ -tolerant.

In the proof of Theorem 4.5, we only use the routes between nodes in $I[0]$, from nodes in $I[0]$ to other nodes and from nodes not in $I[0]$ to nodes in $I[0]$. Thus, we can obtain a

² It may be possible that $x = z$ or $y = z$. It can occur that $x, y \in I[q, g]$. In this case an empty path is considered to be $a(b)$ -path, respectively.

(2, 1)-tolerant unidirectional edge-routing ρ_1 in which the total number of routes is $O(n)$ as follows.

routing ρ_1

Let x and y be represented as $[q_x; g_x, (i_x, j_x)]$ and $[q_y; g_y, (i_y, j_y)]$, respectively same as in ρ . $\rho(x, y)$ is defined as shown in Fig. 5 if $(q_x = q_y = 0)$, $(q_x \neq 0$ and $q_y = 0)$ or $(q_x = 0$ and $q_y \neq 0)$

Theorem 4.6. *Let G be an n -node biconnected graph such that $n \geq 18$. The routing $\overline{\rho_1}$ on G is (2, 1)-tolerant edge-routing with $O(n)$ routes.*

4.4. Optimal routing with route degree $o(n)$

We construct a (2, 1)-tolerant routing ρ_2 with route degree $O(\sqrt{n})$ for n -node biconnected graphs by modifying the routing ρ .

Let $G = (V, E)$ be an n -node biconnected graph. We assume that each node x in G is denoted by $[q_x; g_x, (i_x, j_x)]$ same as in ρ . We can assume that there is an integer ℓ such that $(n/18 = \ell^2)$.³ Thus, q_x can be represented by (q_x^L, q_x^R) , where $1 \leq q_x^L, q_x^R \leq \ell = \sqrt{n/18}$.

routing ρ_2

Let x and y be represented as $[q_x = (q_x^L, q_x^R); g_x, (i_x, j_x)]$ and $q_y = [(q_y^L, q_y^R); g_y, (i_y, j_y)]$, respectively. $\rho_2(x, y)$ is defined as shown in Fig. 5 if $q_x = q_y$ or $q_x^R = q_y^L$.

In the routing ρ_2 , the route $\rho_2(x, y)$ is defined if x and y are in the same interval $I[q]$ or the right part q_x^R of q_x and the left part q_y^L of q_y are equal. From the definition of ρ_2 , we can verify that the route degree is $O(\sqrt{n})$. We can show that the routing ρ_2 is (2, 1)-tolerant by using Lemma 4.4 and the following lemma.

Lemma 4.7. *Let $x = [q_x = (q_x^L, q_x^R); g_x, (i_x, j_x)]$ and $y = [q_y = (q_y^L, q_y^R); g_y, (i_y, j_y)]$ be arbitrary distinct nodes of G on which ρ_2 is defined. Then, one of the following conditions holds.*

- (1) $q_x = q_y$, that is, x and y are in the same group $I[q_x = q_y]$.
- (2) $q_x^R = q_y^L$, that is, the routes from x to nodes in $I[q_y]$ are defined.
- (3) There is a group $I[q_z]$ such that $q_x^R = q_z^L$ and $q_z^R = q_y^L$, that is, the routes from x to nodes in $I[q_z]$ and from nodes in $I[q_z]$ to nodes in $I[q_y]$ are defined.

The total number of routes defined in ρ_2 is $O(n\sqrt{n})$, because the route degree of each node is (\sqrt{n}) . From Lemma 2.1 the total number of routes is at least $\Omega(n\sqrt{n})$ to define (2, 1)-tolerant routings with route degree $O(\sqrt{n})$. The routing ρ_2 attains the lower bound of the total number of routes.

Since in the case that $n < 18$ we can construct a (2, 1)-tolerant edge-routing for n -node biconnected graphs [12], the following theorem holds.

³ Otherwise, we choose $\ell^2 \leq n/18 < (\ell + 1)^2$ and consider ℓ^2 groups. Each of the remaining at most $2\ell + 1$ groups is merged into each of ℓ^2 groups, where at most $2\ell + 1$ groups contain 36 nodes instead of 18 nodes.

Theorem 4.8. *Let G be an n -node biconnected graph. We can construct $(2, 1)$ -tolerant edge-routing on G with $O(n\sqrt{n})$ routes and route degree $O(\sqrt{n})$.*

5. Concluding remarks

We have shown three optimal edge-routings with smaller routing tables. It is an interesting open question whether or not there exists an $(2, k - 1)$ -tolerant routing with route degree $O(k\sqrt{n})$ for n -node k -connected graphs ($k \geq 3$). It is also an interesting open question whether or not there exists an $(2, 1)$ -tolerant bidirectional routing with route degree $O(\sqrt{n})$ for n -node biconnected graphs.

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