Primary and Secondary Steady Flows of the Taylor Problem

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INTRODUCTION

In this paper we study steady state primary and secondary flows of the Taylor problem. Instead of varying only a "load" parameter, e.g., the Reynolds number, as is often done in a study of the Taylor problem, one may vary also a certain "structure" parameter of the problem so as to obtain more detailed results. The use of such a structure parameter leads e.g., to more complete bifurcation diagrams for the primary and secondary flows than those obtained previously by means of catastrophe theory, amplitude equations, or singularity theory.

Much of the motivation here for seeking complete bifurcation diagrams for steady flows of the Taylor problem is provided by the experimental results of Benjamin [1] and Benjamin and Mullin [2] on "short" fluid-filled cylinders with annular cross section. The experimental results of [1, 2] are summarized in Fig. 1 in the case where the primary and secondary steady flows have two or four cells (see Fig. 5 of [1]) and in Fig. 3 in the case where such flows have one or two cells (see Fig. 9 of [2]); the experimental results are described in terms of a Reynolds number, \( R \), and the variable length, \( l \), of the two cylinders. A primary flow here is one that develops under gradual changes of \( R \) from \( R = 0 \), and a secondary flow is one that cannot be obtained by slowly increasing \( R \) from \( R = 0 \).

The experimental results summarized in Fig. 1 may be described as follows (see [1, pp. 35, 41]). (1) If \( l \) is above the level of the point \( B \) in Fig. 1, the primary flow has four cells and develops continuously with increasing \( R \) whereas the two-cell flow exists only as a secondary mode for \( R \) sufficiently large. (2) If \( l \) is below the level of the point \( B \) but above the level of the cusp-point \( C \), the primary flow shows traits of both a two-cell

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and a four-cell structure and develops continuously with increasing $R$ up to the line $BC$ at which an abrupt transition takes place and a definite two-cell structure evolves. Moreover, for this range of $l$ the pure four-cell structure exists only as a secondary mode for $R$ sufficiently large. (3) If $l$ is below the level of the point $C$, the primary flow has two cells and develops continuously with increasing $R$. The reader is referred to Benjamin [1] and to Benjamin and Mullin [2] for an analysis of these experimental results based upon catastrophe theory and degree theory, to Schaeffer [17] for an analysis based upon singularity theory, and to Hall [8] for an analysis based upon the theory of amplitude equations. The reader is referred also to Cliffe [4] for a closely related numerical investigation of one-cell and two-cell Taylor flows and to Cliffe and Mullin [5] for a detailed experimental and numerical investigation of anomalous flows in the Taylor problem. The above papers consider, in particular, the following fundamental question (see [1, p. 151 and [17, p. 307]): how does the number of cells in the primary flow, an integer, depend upon the length $l$ of the cylinder? Various approaches using the indicated methods are presented in [1, 2, 4, 5, 8, 17] and a number of striking results are derived that partially answer this and other related questions.

One of the main differences between the analytical approach presented here and, e.g., those in [1, 2, 17] is related to the following observation in [1, p. 35]: the "transcritical bifurcation point $B$," at which the two-cell and four-cell flows connect and the mutation of the primary flow occurs, should be included in the results described in (2) above. That is, for fixed $l$ corresponding to the point $B$ of Fig. 1, the primary flow starts out as a flow having traits of both a two-cell and a four-cell structure and evolves with increasing $R$ into a flow having a two-cell structure whereas the pure four-

![Fig. 1. Qualitative sketch of the experimentally determined bifurcation set for two-cell and four-cell steady flows (see [1, Fig. 5]).](image-url)
cell flow exists only as a secondary mode for $R$ sufficiently large. This observation provides much of the motivation for the approach of the present paper in that it suggests that a bifurcation diagram of the type shown in Fig. 2 may hold for fixed $I$ corresponding to the point $B$ of Fig. 1; the ordinate in Fig. 2 is some "norm" of the flow, the solid lines represent stable flows, the dashed lines represent unstable flows, and the labels 2, 4, or $2+4$ indicate whether the flow has essentially a two-cell structure, a four-cell structure, or a combination of two-cell and four-cell structures. Since such a diagram is in very close agreement with the experimental results for the point $B$ in Fig. 1, one would expect that Fig. 2 describes one of the basic bifurcation diagrams at $B$ and that certain other effects in the Taylor problem for $I$ below the level of $B$, such as the "continuous evolution" and the "folding" of the primary flow reported in [1, p. 35], correspond to perturbations that split the basic bifurcation diagram in Fig. 2.

In the case of the experimental results summarized in Fig. 3, one has the following description of primary and secondary flows (see [2, pp. 244-45]).

1. If $I$ is above the level of the point $Q$ in Fig. 3, the primary flow has two cells and develops continuously with increasing $R$ while the single-cell flow exists only as a secondary flow for $R$ sufficiently large.

2. If $I$ is below but near the level of the point $Q$, (a) the primary flow has two cells until $R$...
reaches the line $PQ$ at which a single-cell flow forms abruptly, and (b) at a considerably higher value of $R$ to the right of the line $SQ$, the two-cell flow appears to be reinstated as a stable secondary flow. This last observation in 2(b) is the one of greatest importance for the approach presented here because it suggests that the two-cell flow undergoes at least two secondary bifurcations. In spite of the wealth of information contained in [1, 2, 8, 17], the possibility of multiple secondary bifurcations of steady flows is not treated adequately in these papers, in part, because of the use of oversimplified finite-dimensional models.

The above experimental results for one- and two-cell flows suggest that one of the basic bifurcation diagrams for the "transcritical bifurcation point $Q$" in Fig. 3 is given by Fig. 4, and that some of the other effects described in [2] again correspond to perturbations that split the basic bifurcation diagram in Fig. 4. Figure 4 has been drawn with a supercritical bifurcation at $\lambda = \lambda_2$ but analogous results hold for a subcritical bifurcation at $\lambda = \lambda_2$ (see Remark 3.2).

Motivated by the above experimental results and observations, it is natural to seek the basic bifurcation diagrams for the Taylor problem when $l$ is fixed at a value supposedly representing the value of $l$ at either the transcritical bifurcation point $B$ in Fig. 1 or $Q$ in Fig. 3. In this paper, for such fixed $l$, we succeed in establishing a bifurcation diagram of the type shown in Fig. 2 for steady flows with two and four cells, and one of the type shown in Fig. 4 for steady flows with one and two cells. Moreover, both diagrams are established in the same mathematical setting, a setting involving steady state solutions of a small-gap approximation to the Navier-Stokes equations. Thus, the bifurcation diagrams in Figs. 2 and 4 and their perturbations not only seem to represent a valid theoretical
explanation of the experimental results in [1, 2] but they also arise in the
analysis of an infinite-dimensional model for the Taylor problem that is
based upon a widely accepted type of approximation. In addition, the
qualitative analytical results described in Figs. 2 and 4 seem to complement
the striking numerical results in [4, 5]. For example, what we have labeled
a stable 1-cell flow in Fig. 4 actually has a “weak” 2-cell component as well
and this is in close agreement with the stable single-cell flow shown in [4,
Fig. 4c] except that the streamline patterns are, of course, determined in
detail in [4]. On the other hand, some of the qualitative results described
in Fig. 4 such as the role of the stable (1 + 2)-cell flow and the
reinstatement of the 2-cell flow as a stable secondary flow do not seem to
be obtained directly in the numerical investigation in [4].

The approach used in the present paper may be described as follows.
One assumes that the fluid fills the space between two concentric cylinders
with radii \( R_1 \) and \( R_2 \), \( R_2 > R_1 \), both of length \( l \), and that the inner and
outer cylinders are rotated at constant angular velocities \( \Omega_1 \) and \( \Omega_2 \),
respectively. If we set

\[
\mu = \frac{\Omega_2}{\Omega_1}, \quad \eta = \frac{R_1}{R_2}, \quad d = R_2 - R_1, \quad A = \frac{\Omega_1(\mu - \eta^2)}{1 - \eta^2},
\]
then the Taylor number, $T$, and the "structure" parameter, $\gamma$, used throughout the paper are defined by the formulas

$$T = -\frac{1}{2} (1 + \mu) \left[ \frac{4A\Omega_c d^3}{v^2} \right],$$  \hspace{1cm} (1.1)$$

$$\gamma = \frac{2(\eta^2 - \mu)}{\eta(1 + \eta)(1 + \mu)}. \hspace{1cm} (1.2)$$

The Taylor number $T$ is a typical "load" parameter for the Taylor problem, sometimes called the "better" Taylor number (e.g., see [6, p. 97]; however, the use of the structure parameter $\gamma$ appears to be new. Since the Taylor number is nonnegative, the parameter $\gamma$ is also nonnegative and is a measurement of how far away certain physical parameters are from the Rayleigh line (see, e.g., [10, p. 138] for a discussion of the role of the Rayleigh line in the Taylor problem). We shall see that even in the linear problem the use of the parameter $\gamma$ provides an extension of some well-known results for the critical Taylor number because $\gamma$ is a "richer" parameter than the parameter $\mu$ used in a more standard approach (e.g., compare the formula for the critical Taylor number given in [6, (17.50), p. 98] with that given in Remarks 2.1 and 2.2 in Section 2).

Splitting methods from bifurcation theory can now be used to reduce the Taylor problem to a finite-dimensional problem. Such a reduction is carried out using $\gamma$ as an "amplitude" parameter and leads to a system of bifurcation equations of the type given in (2.1) of section 2. The desired bifurcation diagrams in Fig. 2 and Fig. 4 for fixed $\gamma$ sufficiently small are then established by solving Eqs. (2.1) for $\gamma$ near $\gamma = 0$. Complete bifurcation diagrams are obtained here, in part, because the bifurcation equations (2.1) contain linear as well as both quadratic and cubic terms. This is in contrast to the approach in [17] using singularity theory in which only linear and cubic terms appear in the unfolded equations, or the approach in [8] using amplitude equations in which only linear and quadratic terms appear in the time-independent equations.

The outline of the paper is as follows. In Section 2 we formulate the basic bifurcation equations (2.1) studied in the paper and describe the hypotheses under which they are to be solved. To justify the setting of Section 2, bifurcation equations of the form (2.1) are derived for the Taylor problem in Appendix B in the setting of steady state solutions of a small-gap approximation to the Navier–Stokes equations; it is expected that the use of a small-gap approximation here is a technical rather than a crucial restriction in the analysis. The small-gap approximation is introduced in Appendix A along with the basic assumptions (see (A.27) ff.) on the length $l$ of the cylinders under which we obtain primary and secondary steady
flows having one, two, or four cells. In Section 3 we establish bifurcation diagrams of the type in Figs. 2 and 4 for the "reduced" bifurcation equations obtained by formally setting \( \gamma = 0 \) in (2.1), and in Section 4 we establish the desired diagrams for sufficiently small, positive, fixed \( \gamma \) by means of the implicit function theorem. To show here that the bifurcation diagrams and stability properties are the same for \( \gamma = 0 \) and \( \gamma \neq 0 \), we make use of special symmetry and invariance arguments near the points of secondary bifurcation. Such arguments employ some of the basic ideas of bifurcation and symmetry breaking described in Chow et al. [3], and Sattinger [15, 16] as well as some of the ideas developed in [13] to treat multiple points of secondary bifurcation in Bénard-type convection problems.

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2. THE BIFURCATION EQUATIONS

In this section we formulate a two-dimensional bifurcation problem whose solution leads to the bifurcation diagrams in Figs. 2 and 4. Bifurcation equations of the form studied here are derived for the Taylor problem in Appendix B and various quantities are introduced in Remarks 2.1 and 2.2 so that the results for Eqs. (2.1) may be interpreted in terms of results for the Taylor problem.

The bifurcation equations to be studied are

\[
0 = G_1(\beta_1, \beta_2, \tau, \gamma) = -\tau \beta_1 - b_1 \beta_1 \beta_2 + c_1 \beta_1 \beta_2^3 + d_1 \beta_1^3 + r_1(\beta_1, \beta_2, \tau, \gamma), \tag{2.1a}
\]

\[
0 = G_2(\beta_1, \beta_2, \tau, \gamma) = - (\tau + a) \beta_2 - b_2 E \beta_1^2 + c_2 \beta_1^2 \beta_2 + d_2 \beta_2^3 + r_2(\beta_1, \beta_2, \tau, \gamma) \tag{2.1b}
\]

where the coefficients \( a, b_i, c, \) and \( d_i \) are real \((i = 1, 2)\); for the sake of convenience we write the coefficient of \( \beta_1^2 \) in (2.1b) as \( b_2 E \) with \( E = c_2 - d_1 \).

We assume in (2.1) that (1) \((\beta, \tau, \gamma) \in \mathcal{B} \times (0, \gamma_0)\), where \( \gamma_0 > 0 \) and, for a given \( \rho > 0 \), \( \mathcal{B} \subset \mathbb{R}^3 \) is defined as

\[
\mathcal{B} = \{ (\beta_1, \beta_2, \tau) : |\beta_1| < \rho, |\beta_2| < \rho, |\tau| < \rho \}, \tag{2.2}
\]

(2) for \((\beta, \tau, \gamma) \in \mathcal{B} \times (0, \gamma_0)\) the remainder terms \( r_1 \) and \( r_2 \) satisfy

\[
r_1(\beta_1, \beta_2, \tau, \gamma) = \beta_1 \rho(\beta_1^2, \beta_2, \tau, \gamma), \tag{2.3a}
\]
where $p, q,$ and $s$ are analytic; moreover, $p, q,$ and $s$ and their partial derivatives are uniformly of order $O(\gamma)$ as $\gamma \to 0^+$ in the sense that, e.g.,

$$|p(\beta_1, \beta_2, \tau, \gamma)| \leq c\gamma, \quad 0 < \gamma < \gamma_0. \quad (2.4)$$

where the constant $c$ depends only on $\gamma_0$ and $\rho$.

Given $\rho > 0$ in (1) and (2), $\gamma_0$ is always assumed to be sufficiently small so that various estimates used in the paper hold uniformly in $\mathcal{B}$. For purposes of the analysis, it is important here that $\rho$ can be arbitrarily large provided that $\gamma_0$ is chosen sufficiently small.

Note that, if $b_1 = b_2 = a = \gamma = 0$ in (2.1), the resultant equations are similar to the unperturbed cubic model in [17, p. 319] whereas, if $c_1 = c_2 = d_1 = d_2 = 0$ in (2.1), the resultant equations are similar to the time-independent amplitude equations in [8, p. 582].

We shall essentially regard the coefficients $a, b_i, c_i,$ and $d_i$ ($i = 1, 2$) in (2.1) as real parameters satisfying certain "neccessary" conditions for secondary bifurcation given in hypotheses (H1)–(H3). There are two reasons for such an approach. First, the analysis in Appendix B shows that the bifurcation equations for the Taylor problem have precisely the form given in (2.1), however, the actual coefficients determined in Appendix B are extremely complicated so that comparisons of the relative sizes of the various coefficients are difficult. Second, although a small-gap approximation to the Navier–Stokes equations is used in Appendix B, it is possible even in the general case that the relevant bifurcation equations for the Taylor problem are again of the form given in (2.1); if so, then the analysis of Sections 3 and 4 and the resultant bifurcation diagrams in Figs. 2 and 4 would apply also to the Taylor problem in the more general context of steady state solutions of the full Navier–Stokes equations.

The bifurcation equations (2.1) are solved under the following hypotheses:

(H1) $c_1 > d_2 > 0$ and $c_2 > d_1 > 0,$

(H2) $b_2 > 0$ and $b_1 b_2 > E^{-1} |a| d_1,$ if $a < 0$, $b_2 > 0$ and $b_1 b_2 > 2a,$ if $a > 0,$

(H3) $F b_2^2 - b_1 b_2 + a = 0,$

where $E$ and $F$ are given by

$$E = c_2 - d_1 > 0 \quad \text{and} \quad F = c_1 - d_2 > 0. \quad (2.5)$$
Hypothesis (H1) is essentially a normalization condition in that it ensures the existence of stable supercritical Taylor cells in the special case \( a = b_1 = b_2 = 0 \); a hypothesis of the type (H1) is usually assumed in such problems (e.g., see [17, p. 315]). In the case of the explicit bifurcation equations for the Taylor problem derived in Appendix B, the inequalities \( d_i > 0 \) in (H1) are an immediate consequence of the definition of the \( d_i \) in (B.38), however, the additional inequalities \( E > 0 \) and \( F > 0 \) in (H1) are not so apparent.

We shall see that hypothesis (H2) reflects some of the basic asymmetries in the Taylor problem, e.g., steady flows with two cells and four cells do not occur in a symmetric manner in the experiments in [1] whereas they do occur in essentially a symmetric manner in a pure cubic model with \( b_1 = b_2 = 0 \) (e.g., see [2; 17]). The explicit assumptions on \( b_1 \) and \( b_2 \) in (H2) are chosen for convenience to limit the number of possible bifurcation diagrams (see also Remarks 3.1 and 3.2). The crucial assumption in (H2) and throughout this paper is \( b_2 \neq 0 \).

Hypothesis (H3) is actually a "necessary" condition for secondary bifurcation of certain solutions of (2.1). It is the analog of the "necessary" condition derived in [17, (2.7), p. 327] as part of the universal unfolding of the simpler cubic model obtained from (2.1) by setting \( a = b_1 = b_2 = 0 \). Since many of the experimental results in [1, 2] can be explained in terms of secondary bifurcations, hypothesis (H3) is also in a sense a natural hypothesis for an analysis of the Taylor problem. In the actual physical problem, however, a hypothesis such as (H3) most likely holds only as some sort of "lowest order" approximation that serves to delineate various ranges of the coefficients leading to qualitatively distinct perturbed bifurcation diagrams (see, e.g., the detailed discussions of qualitatively distinct bifurcation diagrams in [7, 17]).

Equations (2.1) may, of course, model a variety of nonlinear problems, however, hypotheses (H1)-(H3) and the conditions on the remainder terms in (2.3) and (2.4) have been formulated specifically with the Taylor problem in mind. To clarify this relationship, the following two remarks summarize a number of properties of the linear problem from Appendix A and Appendix B and are used throughout the paper to interpret results obtained for Eqs. (2.1) in terms of results for the Taylor problem.

**Remark 2.1.** In the Taylor problem the relationship between the variable \( \tau \) in (2.1) and the Taylor number \( T \) in (1.1) is as follows. The smallest characteristic value, \( \mu_0 \), of the linear problem at \( \gamma = 0 \) (see (A.30)) splits for \( \gamma > 0 \) into two eigenvalues of the operator \( L - \gamma M \) (see (B.43) and (B.44)). For \( \gamma \) sufficiently small, these two smallest eigenvalues of \( L - \gamma M \) are of the form
\[ \lambda_k(\gamma) = \mu_0 - \mu_0^3 a_k \gamma^2 + A_k(\gamma), \]
\[ \lambda_{2k}(\gamma) = \mu_0 - \mu_0^3 a_{2k} \gamma^2 + A_{2k}(\gamma), \]

where \(a_k\) and \(a_{2k}\) are defined as in (B.5) and (B.17), and \(A_j\) is real and analytic and of order \(O(\gamma^2)\) as \(\gamma \to 0^+\) \((j = k, 2k)\). The variable \(\tau\) is now defined by (see also (B.46))

\[ \lambda = \mu_0 - \mu_0 \gamma^2 (\mu_0^3 a_k - \tau) \]

so that, up to order \(\gamma^3\), \(\tau\) is a measurement of the distance from \(\lambda\) to \(\lambda_k\). Thus, if \((\beta, \tau)\) is a solution of (2.1) for fixed \(\gamma\) sufficiently small, then (2.8) determines \(\lambda\) and, hence, the Taylor number \(T = \lambda^2\).

**Remark 2.2.** In the Taylor problem the role of the parameter \(a\) in (2.1) may be described as follows. In the context of Remark 2.1, one sets

\[ a = (a_{2k} - a_k) \mu_0^3. \]

If \(a > 0\), then, for \(\gamma\) sufficiently small, \(\lambda_k > \lambda_{2k}\) and \(\lambda_{2k}\) is the critical eigenvalue of the linearized problem for \(\gamma > 0\). Similarly, if \(a < 0\), then, for \(\gamma\) sufficiently small, \(\lambda_k < \lambda_{2k}\) and \(\lambda_k\) is the critical eigenvalue for \(\gamma > 0\). Note that if \(a > 0\) and if \(k = 1\), then \(\tau\) is a measurement of the distance from \(\lambda\) to \(\lambda_1\) in (2.6) but \(\lambda_2\) in (2.7) determines the critical Taylor number, \(T_*(\gamma)\), for steady flows having one cell or two cells. On the other hand, if \(a < 0\) and if \(k = 2\), then \(\tau\) is a measurement of the distance from \(\lambda\) to \(\lambda_2\) except that \(\lambda_2\) in (2.6) now determines the critical Taylor number for steady flows having two cells or four cells. For each fixed \(\gamma\) satisfying \(0 \leq \gamma < \gamma_0\), \(\gamma_0 > 0\) sufficiently small, we shall see that the case \(a < 0\) leads to the bifurcation diagram in Fig. 2 whereas the case \(a > 0\) leads to Fig. 4. Thus, using the above definitions of \(\lambda_2\) and setting \(\lambda_2^2 \equiv T_*(\gamma)\), one sees that the bifurcation diagrams in Fig. 2 for \(a < 0\) and \(k = 2\) and Fig. 4 for \(a > 0\) and \(k = 1\) are in almost total agreement at the transcritical bifurcation points with the experimental results in [1] for steady flows with two cells and four cells and in [2] for steady flows with one cell and two cells.

We proceed in the following sections to solve the bifurcation equations (2.1) and to determine the stability of the resultant solutions. By stability here we mean "linearized" stability. Using the results in Appendix B and the approach in [12, 14], one can show that, if a steady flow of the Taylor problem is generated by a solution \((\beta^*, \tau^*, \gamma^*)\) of the bifurcation equations (2.1), then the stability properties of both the solution \((\beta^*, \tau^*, \gamma^*)\) and the flow are determined by the eigenvalues of the Jacobian matrix, \(J \equiv \delta(G_1, G_2)/\delta(\beta_1, \beta_2)\), of the system (2.1) at \((\beta^*, \tau^*, \gamma^*)\): a solution \((\beta^*, \tau^*, \gamma^*)\) of (2.1) and the flow it generates are both stable if the eigen-
values of $J$ at $(\beta^*, \tau^*, \gamma^*)$ have positive real parts, unstable if at least one of the eigenvalues of $J$ has negative real part, and of indeterminate stability otherwise. Thus, we have the following criteria for stability in terms of the determinant and trace of $J$: (1) if, at a solution $(\beta^*, \tau^*, \gamma^*)$ of (2.1), $\det J > 0$ and $\text{tr} J > 0$, then the solution is stable whereas, if $\det J > 0$ and $\text{tr} J < 0$, then the solution is unstable, (2) if $\det J < 0$, then the solution is always unstable.

3. The Bifurcation Diagrams for $\gamma = 0$

In this section we establish the bifurcation diagrams in Figs. 2 and 4 for the “reduced” bifurcation equations obtained by formally setting $\gamma = 0$ in (2.1). The bifurcation diagrams for $\gamma \neq 0$ are established in Section 4 by methods using the implicit function theorem and some special symmetry arguments.

We consider the reduced bifurcation equations given by

$$0 = -\tau \beta_1 - b_1 \beta_1 \beta_2 + c_1 \beta_1 \beta_2^2 + d_1 \beta_1^3,$$  \hspace{1cm} (3.1a)

$$0 = - (\tau + a) \beta_2 - b_2 E \beta_1^2 + c_2 \beta_1^2 \beta_2 + d_2 \beta_2^3, \quad (\beta_1, \tau) \in \mathbb{R}^2 \times \mathbb{R}^1. \hspace{1cm} (3.1b)$$

Equations (3.1) are not “typical” reduced bifurcation equations for the Taylor problem (e.g., see [10, 16]). The difference arises here because of the scalings used in Appendix B to derive the bifurcation equations in (2.1). Because of the form of Eq. (3.1), it is natural to consider separately solutions with $\beta_1 = 0$ and solutions with $\beta_1 \neq 0$.

3A. Solutions of (3.1) with $\beta_1 = 0$

If $\beta_1 = 0$, the system (3.2) has the “trivial” solution $(\beta_1, \beta_2, \tau) = (0, 0, \tau)$, $\tau \in \mathbb{R}^1$, and nontrivial solution branches determined by the parabola

$$\mathcal{P}_1: \quad \tau = - a + d_2 \beta_2^2, \quad \beta_1 = 0, \beta_2 \in \mathbb{R}^1. \hspace{1cm} (3.2)$$

Note that $\mathcal{P}_1$ contains a “positive” nontrivial solution branch, say $\mathcal{P}_1^+$, where $\beta_2 > 0$ and $\tau > - a$, and a “negative” branch, say $\mathcal{P}_1^-$, where $\beta_2 < 0$ and $\tau > - a$. Note also that $\mathcal{P}_1^+$ and $\mathcal{P}_1^-$ bifurcate form the trivial solution at $(\beta_1, \beta_2, \tau) = (0, 0, - a)$. The Jacobian, $J$, of (3.1) with respect to $\beta_1$ and $\beta_2$, along $\mathcal{P}_1^+$, has the eigenvalues $\nu = 2d_1 \beta_2^2 > 0$ and

$$\nu = F \beta_2^2 - b_1 \beta_2 + a. \hspace{1cm} (3.3)$$

If $a > 0$ and if $\beta_2$ is sufficiently small, then $\nu > 0$ in (3.3) and both branches $\mathcal{P}_1^\pm$ are stable. If $a < 0$, both branches $\mathcal{P}_1^\pm$ are unstable for $\beta_2$ sufficiently small. We determine next the stability of $\mathcal{P}_1^\pm$ for $\beta_2$ not necessarily small.
Solving (3.3) for \( \beta_2 = \beta^\pm_2 \) when \( v = 0 \) and \( a < 0 \), one sees from (H3) that \( \beta^+_2 = b_2 > 0 \) and \( \beta^-_2 = a/Fb_2 < 0 \). Thus, if \( a < 0 \), there is one positive and one negative value of \( \beta_2 \) at which \( v = 0 \) in (3.3). Since \( v \) changes sign at both \( \beta^+_2 \), it follows that \( \mathcal{P}^+_1 \) gains stability at \( \beta^+_2 = b_2 \) and \( \mathcal{P}^-_1 \) gains stability at \( \beta^-_2 = a/Fb_2 \), i.e., \( \mathcal{P}^+_1 \) gains stability at \( (0, b_2, \tau^+ \) and \( \mathcal{P}^-_1 \) gains stability at \( (0, a/Fb_2, \tau^-) \), where \( \tau^+ = -a + d_2 b_2^2 \) and \( \tau^- = -a + d_2(a/Fb_2)^2 \).

If \( a > 0 \), by completing the square in (H3) and using (H2), one sees that

\[
4aF < b^2_1. \tag{3.4}
\]

Thus, there are exactly two positive values of \( \beta_2 \), namely \( \beta^+_2 = b_2 \) and \( \beta^-_1 = a/Fb_2 \), at which \( v = 0 \) in (3.3); note also that (H2) and (H3) imply

\[
a - Fb^2_2 < 0 \tag{3.5}
\]

so that \( 0 < a/Fb_2 < b_2 \). It follows in the case \( a > 0 \) that the branch \( \mathcal{P}^+_1 \) loses stability at \( \beta^+_2 = a/Fb_2 \) but regains stability at \( \beta^-_2 = b_2 \), whereas \( \mathcal{P}^-_1 \) is always stable. This completes the stability analysis of the branches \( \mathcal{P}^\pm_1 \) lying in the plane \( \beta_1 = 0 \).

3B. Solutions of (3.1) with \( \beta_1 \neq 0 \).

(1) We begin by showing that the branch \( \mathcal{P}^+_1 \) determined by the parabola \( \mathcal{P}_1 \) in (3.2) undergoes secondary bifurcation at \( \beta_2 = b_2 \) regardless of whether \( a > 0 \) or \( a < 0 \).

If \( \beta_1 \neq 0 \), then (3.1a) is equivalent to

\[
0 = -\tau - b_1 \beta_2 + c_1 \beta_2^2 + d_1 \beta_1^2, \tag{3.6}
\]

which can be rewritten as

\[
0 = -(\tau + a) + d_1 \beta_1^2 + d_2 \beta_2^2 + (a - b_1 \beta_2 + Fb_2^2). \tag{3.7}
\]

Since (3.1b) can be rewritten as

\[
0 = \beta_2 [-(-\tau + a) + d_1 \beta_1^2 + d_2 \beta_2^2] + E(\beta_2 - b_2) \beta_1^2, \tag{3.8}
\]

it follows that the parabola

\[
\mathcal{P}_2: \quad \tau = -a + d_1 \beta_1^2 + d_2 \beta_2^2, \quad \beta_2 = b_2, \beta_1 \in \mathbb{R}^1, \tag{3.9}
\]

determines nontrivial solution branches of (3.1) for \( \beta_1 \neq 0 \) if and only if \( \beta_2 = b_2 \) satisfies (3.3) with \( v = 0 \). Thus for either \( a > 0 \) or \( a < 0 \), the parabola \( \mathcal{P}_2 \) lying in the plane \( \beta_2 = b_2 \) branches from \( \mathcal{P}^+_1 \) at the point at which \( v = 0 \) in (3.3). We shall not distinguish the branches of \( \mathcal{P}_2 \) for \( \beta_1 < 0 \) and \( \beta_1 > 0 \) since these branches and their subsequent secondary bifurcations are sym-
metric with respect to the plane $\beta_1 = 0$. Note that there are no analogous parabolas of secondary bifurcation lying in the plane $\beta_2 = a/Fb_2$ in either the case $a > 0$ or the case $a < 0$.

We now determine the stability properties of the branches determined by $P_2$. The determinant of the Jacobian matrix $J$ and the trace of $J$, along $P_2$, satisfy
\begin{align}
\det J &= 2d_1 \beta_2^2 [a + E\beta_1^2 - Fb_2^2], \\
\text{tr} J &= (c_2 + d_1) \beta_1^2 + 2d_2 b_2^2 > 0.
\end{align}
Thus, if $a < 0$, $\det J < 0$ for $\beta_1$ near $\beta_1 = 0$ so that $P_2$ bifurcates from $\mathcal{P}^+_1$ at $\beta_2 = b_2$ but both branches of $P_2$ are unstable; the branches of $\mathcal{P}_2$, parameterized by $\beta_1$, remain unstable until
\begin{equation}
\beta_1^2 = E^{-1}(Fb_2^2 - a)
\end{equation}
at which they both gain stability. If $a > 0$, then (3.5) implies that $\det J < 0$ for $\beta_1$ near $\beta_1 = 0$. Thus, both branches of $P_2$ again start out unstable and gain stability when $\beta_1^2$ is given by (3.12). Note that (3.5) implies that there are always real solutions of (3.12) in the case $a > 0$; this is, of course, obvious in the case $a < 0$.

(2) Next we seek solutions of (3.1) with $\beta_1 \neq 0$ that branch from points not on $\mathcal{P}^+_1$. Since $\beta_1$ appears in (3.1) and (3.6) only as $\beta_1^2$, we can eliminate $\beta_1$ and obtain a single equation for $\beta_2$ and $\tau$, namely,
\begin{align}
0 &= E(\beta_2 - b_2) \tau - A\beta_2 - C\beta_2^2 - D\beta_2^3 \\
&= (\beta_2 - b_2) [E\tau - \beta_2(D\beta_2 - Ab_2^{-1})],
\end{align}
where $E$ is defined as in (2.5). Here we have used also that
\begin{align}
A &= ad_1 + Eb_1 b_2, \\
C &= -b_1 c_2 - Eb_2 c_1, \\
D &= c_1 c_2 - d_1 d_2,
\end{align}
and that $D = -b_2^{-2}(b_2 C + A)$ is equivalent to (H3). Thus, the parabola
\begin{align}
\mathcal{P}_3: \quad \tau &= E^{-1}\beta_2(D\beta_2 - Ab_2^{-1}), \quad \beta_1^2 = E^{-1}\beta_2(F\beta_2 - ab_2^{-1}), \quad \beta_2 \in \mathbb{R}^1,
\end{align}
determines nontrivial solutions of (3.1), provided that $\beta_1$ is real; to obtain the equation for $\beta_1^2$, one uses (3.6) and the given equation for $\tau$.

Remark 3.1. For the sake of convenience we have assumed in (H2)
that, e.g., if \( a < 0 \), then \( b_1 b_2 > E^{-1}|a| d_1 \). Under this assumption, the auxiliary parameter \( A \) defined in (3.14) is positive. If \( A < 0 \) and \( a < 0 \), it is still possible to carry out a complete bifurcation and stability analysis of (2.1) along the lines given here, however, in this case there are additional bifurcation diagrams to consider.

In the following paragraphs we discuss the existence and stability of various branches of solutions of (3.1) generated by \( \mathcal{P}_3 \) in (3.17) for the cases \( a < 0 \) and \( a > 0 \). Using (3.13) and (3.17), one finds as a preliminary result that, along \( \mathcal{P}_3 \),

\[
\begin{align*}
\det J &= 2\beta_2^2 [A - Ah_2^{-1}\beta_2 - 2(A + C\beta_2 + D\beta_2^2)] \\
&= -4D\beta_2(\beta_2 - b_2)[\beta_2 - A/(2Db_2)] \\
&\quad \text{(3.18)}
\end{align*}
\]

and

\[
\begin{align*}
\text{tr} J &= 2d_1\beta_1^2 + Fb_2\beta_2 + 2d_2\beta_2^2 - a. \\
&\quad \text{(3.19)}
\end{align*}
\]

To obtain (3.19) we have also made use of (H3) to eliminate \( b_1 \).

If \( a < 0 \), then \( \beta_1^2 \) in (3.17) is positive for \( \beta_2 > 0 \) so that the part of \( \mathcal{P}_3 \) with \( \beta_2 > 0 \) determines a nontrivial branch of solutions of (3.1), say \( \mathcal{P}_3^+ \), i.e.,

\[
\mathcal{P}_3^+: \quad \tau = E^{-1}\beta_2(D\beta_2 - Ab_2^{-1}), \quad \beta_1^2 = E^{-1}\beta_2(F\beta_2 - ab_2^{-1}), \quad \beta_2 > 0.
\]

(3.20)

On the other hand, the part of \( \mathcal{P}_3 \) with \( \beta_2 < 0 \) does not determine a branch of real solutions of (3.1) until \( \beta_2 \) reaches the value \( a/Fb_2 \) at which the parabolas \( \mathcal{P}_1 \) and \( \mathcal{P}_3 \) intersect (in the plane \( \beta_1 = 0 \)); to compare values of \( \tau \) on \( \mathcal{P}_1 \) and \( \mathcal{P}_3 \) at \( \beta_2 = a/Fb_2 \), it is convenient to use (H3) and rewrite \( A \) in (3.14) as

\[
A = ac_2 + EFb_2^2.
\]

(3.21)

Thus, if \( \mathcal{P}_3^- \) is defined as

\[
\mathcal{P}_3^-: \quad \tau = E^{-1}\beta_2(D\beta_2 - Ab_2^{-1}), \quad \beta_1^2 = E^{-1}\beta_2(F\beta_2 - ab_2^{-1}), \quad \beta_2 < a/Fb_2,
\]

(3.22)

it follows that \( \mathcal{P}_3^- \) bifurcates from \( \mathcal{P}_1^- \) when \( \beta_2 = a/Fb_2 \). As in the case of \( \mathcal{P}_2 \) we shall not distinguish the branches of \( \mathcal{P}_3^+ \) for \( \beta_1 < 0 \) and \( \beta_1 > 0 \) since such branches are symmetric with respect to the plane \( \beta_1 = 0 \).

Note that \( A > 0 \) implies that \( \tau < 0 \) on \( \mathcal{P}_1^+ \) for \( (\beta_1, \beta_2) \) near \((0, 0)\) and that \( \mathcal{P}_2^+ \) has negative minima when \( \beta_2 = A/(2Db_2) \), \( \tau = -A^2/(4DEb_2^2) \), and \( \beta_1 \) is given by (3.17). Moreover, making use of (3.21), one sees that

\[
Db_2^2 - A = b_2^2(d_1 F + d_2 E) - ac_2 > 0
\]

(3.23)
so that

\[
\frac{A}{Db_2} < b_2. \tag{3.24}
\]

Thus, it follows from (3.18) that, along \( \mathcal{P}_3^+ \), \( \det J \) is positive for \( A/(2Db_2) < b_2 < b_2 \), and negative for either \( b_2 > b_2 \) or \( 0 < b_2 < A/(2Db_2) \). Since, in addition, (3.19) implies that \( \text{tr} \ J > 0 \) along \( \mathcal{P}_3^+ \), one sees that \( \mathcal{P}_3^+ \) is stable for \( A/(2Db_2) < b_2 < b_2 \) and unstable for either \( b_2 > b_2 \) or \( 0 < b_2 < A/(2Db_2) \). In fact, using (3.12) and (3.17) one sees that \( \mathcal{P}_3^+ \) loses stability at the points in the plane \( \beta_2 = b_2 \), at which the branches of \( \mathcal{P}_2 \) gain stability; the value of \( \tau \) at such intersections is \( E^{-1}(Db_2^2 - A) \) which, by (3.23), is positive. Finally, along \( \mathcal{P}_3^- \), \( \det J < 0 \) for \( b_2 < a/Fb_2 \) so that \( \mathcal{P}_3^- \) is unstable.

If \( a > 0 \), one sees that \( \mathcal{P}_3 \) again determines two nontrivial branches of real solutions of (3.1), say

\[
\mathcal{P}_3^-: \quad \tau = E^{-1}\beta_2(D\beta_2 - Ab_2^{-1}), \quad \beta_1^+ = E^{-1}\beta_2(F\beta_2 - ab_2^{-1}), \quad \beta_2 < 0, \tag{3.25}
\]

\[
\mathcal{P}_3^+: \quad \tau = E^{-1}\beta_2(D\beta_2 - Ab_2^{-1}), \quad \beta_1^- = E^{-1}\beta_2(F\beta_2 - ab_2^{-1}), \quad \beta_2 > a/Fb_2. \tag{3.26}
\]

It is important here that (3.5) holds so that \( \mathcal{P}_3 \) determines the branch \( \mathcal{P}_3^+ \) for \( b_2 < b_2 \). In fact, making use of (3.21) to compare the \( \tau \) values for \( \beta_2 = a/Fb_2 \), one finds that \( \mathcal{P}_3^+ \) bifurcates from \( \mathcal{P}_3^+ \) at the point at which \( \mathcal{P}_3^+ \) loses stability. Moreover, since \( \det J \) and \( \text{tr} \ J \) are still given by (3.18) and (3.19), along \( \mathcal{P}_3^+ \), \( \text{tr} \ J = 2d_1\beta_1^2 + 2d_2\beta_2^2 > 0 \) and \( \det J \) has the same sign as in the case \( a < 0 \). Thus, if \( (a/Fb_2) > A/(2Db_2) \), then \( \mathcal{P}_3^+ \) is stable for \( (a/Fb_2) < b_2 < b_2 \) and unstable for \( b_2 > b_2 \), whereas, if \( (a/Fb_2) < A/(2Db_2) \), then \( \mathcal{P}_3^+ \) is stable for \( A/(2Db_2) < b_2 < b_2 \) and unstable for \( (a/Fb_2) < b_2 < A/(2Db_2) \) and \( b_2 > b_2 \). As in the case \( a < 0 \), it is easy to see that \( \mathcal{P}_3^+ \) loses stability at the points in the plane \( \beta_2 = b_2 \) at which \( \mathcal{P}_2 \) gains stability; however, the value of \( \tau \) at such intersections is not necessarily positive. Since, along \( \mathcal{P}_3^- \), \( \det J < 0 \), one sees that \( \mathcal{P}_3^- \) is always unstable. This completes the bifurcation and stability analysis for the reduced equations in (3.1).

**Remark 3.2.** Combining the above results, we obtain a bifurcation diagram for \( a < 0 \) that is similar to Fig. 2 and one for \( a > 0 \) that is similar to Fig. 4. Figures 2 and 4 can be thought of as the projection of the bifurcation diagrams of this section onto the plane \( \beta_1 = 0 \) so that the ordinate in both these figures is \( \beta_2 \) and the abscissa is \( \tau \). We have drawn Fig. 4 under the assumption \( A/(2Db_2) < a/Fb_2 \) so that \( \mathcal{P}_3^+ \) bifurcates supercritically from \( \mathcal{P}_1^+ \) at \( \beta_2 = a/Fb_2 \); if \( A/(2Db_2) > a/Fb_2 \), then \( \mathcal{P}_3^+ \) bifurcates subcritically.
from $\mathcal{P}_+^*$ at $\beta_2 = a/Fb_2$ and is unstable up to $\beta_2 = A/(2Db_2)$ at which it gains stability.

It is possible that generic perturbations of the bifurcation diagrams for $\gamma = 0$ could be determined by using some of the ideas in [17, Sect. 3] with (H3) as a "necessary" condition for secondary bifurcation. On the other hand, since we require for each fixed $\gamma$ sufficiently small that the perturbed diagrams should fit together near all three points of secondary bifurcation, it seems more natural here to proceed as in the next section using some special symmetry and invariance properties of the Taylor problem.

4. THE BIFURCATION DIAGRAMS FOR $\gamma \neq 0$

The solutions obtained for $\gamma = 0$ in Section 3 vary smoothly for $\gamma \neq 0$ except possibly at the points at which the solutions undergo secondary bifurcation and the ordinary implicit function theorem does not directly apply. In this section we show, however, that even in a neighborhood of the points of secondary bifurcation the bifurcation diagrams for $\gamma = 0$ and $0 < \gamma < \gamma_0$ are qualitatively the same provided that $\gamma_0$ is sufficiently small. We shall first obtain perturbations $\mathcal{P}_k(\gamma)$ for $\gamma \neq 0$ of various parts of the parabolas $\mathcal{P}_k (k = 1, 2, 3)$ determined in Section 3 and then piece together the various perturbations while carrying out a complete stability analysis. It is implicitly assumed throughout this section that $\rho$ in (2.2) is chosen sufficiently large so that all points of secondary bifurcation on the parabolas $\mathcal{P}_k (k = 1, 2, 3)$ are included in the analysis. The section concludes with a summary of the main results for steady flows of the Taylor problem.

We shall see that

(i) the unique perturbation for $\gamma \neq 0$ of the parabola $\mathcal{P}_1$ in (3.2) is essentially determined by solutions of the single equation

$$0 = -\tau - a + d_2 \beta_2^2 + s(\beta_2^2, \tau, \gamma) \equiv H_0(\beta_2^2, \tau, \gamma), \quad (4.1)$$

(ii) that of $\mathcal{P}_2$ in (3.9) by solutions of the system

$$0 = -\tau - b_1 \beta_2 + c_1 \beta_2^2 + d_1 \beta_1^2 + p(\beta_1^2, \beta_2, \tau, \gamma) \equiv H_1(\beta_1^2, \beta_2, \tau, \gamma), \quad (4.2a)$$
$$0 = G_2(\beta_1, \beta_2, \tau, \gamma), \text{ and} \quad (4.2b)$$

(iii) that of $\mathcal{P}_3$ in (3.17) by solutions of the system

$$0 = H_1(\beta_1^2, \beta_2, \tau, \gamma), \quad (4.3a)$$
Here \( p, q, \) and \( s \) are defined as in (2.3a–2.3c) of Section 2. Note that (4.1) follows directly from (2.1) and (2.3a–2.3c), by setting \( \beta_1 = 0 \) and dividing by \( \beta_2 \), and (4.2) also follows directly from (2.1) by dividing (2.1a) by \( \beta_1 \). To obtain (4.3b), one multiplies (4.2a) by \( b_2 \), subtracts the resultant equation from (2.1b), and makes use of hypothesis (H3).

(i) The perturbation of \( \mathcal{P}_1 \). Because of the results in Section 3, it is again convenient to consider separately solutions with \( \beta_1 = 0 \) and \( \beta_1 \neq 0 \). The desired perturbation \( \mathcal{P}_1(\gamma) \) is a solution of (2.1) with \( \beta_1 = 0 \) and is obtained by solving the single equation \( H_0 = 0 \) in (4.1).

**Theorem 4.1.** Given \( \rho > 0 \) there exists \( \gamma_0 > 0 \) such that, for \( 0 < \gamma < \gamma_0 \), equation (4.1) has a solution

\[
\tau = -a + \tau_0(\gamma) + \beta_2^2(d_2 + \tau_1(\beta_2, \gamma))
\]

that is bounded, analytic, and unique in

\[
\mathcal{B}_1 = \{ (\beta_2, \tau, \gamma) : |\tau + a - d_2\beta_2^2| < K_1 \gamma, |\beta_2| < \rho, 0 < \gamma < \gamma_0 \},
\]

where the constant \( K_1 \) depends only on \( \rho \), and \( \tau_0, \tau_1 \) are \( O(\gamma) \) as \( \gamma \to 0^+ \) uniformly for \( |\beta_2| < \rho \).

**Proof.** Let \( \beta_2^* \) satisfying \( |\beta_2^*| < \rho \) be given and set \( \tau^* = -a + d_2(\beta_2^*)^2 \). It follows from (2.4) that \( (\beta_2^*, \tau^*, 0) \) is a solution of (4.1) at which \( \partial H_0/\partial \tau = -1 \). Hence, by the implicit function theorem, Eq. (4.1) has a solution \( \tau = \tau(\beta_2, \gamma) \) that is bounded, analytic and unique in a neighborhood of \( (\beta_2, \tau, \gamma) = (\beta_2^*, \tau^*, 0) \) with \( \tau(\beta_2^*, 0) = \tau^* \). Since, for sufficiently small \( \gamma_0 \),

\[
|\tau + a - d_2\beta_2^2| = |s(\beta_2^2, \tau, \gamma)| \leq s_0 \gamma,
\]

a finite number of such neighborhoods cover \( \mathcal{B}_1 \). Using the fact that \( s \) has the form

\[
s(\beta_2^2, \tau, \gamma) = f(\tau, \gamma) + \beta_2^2 g(\beta_2, \tau, \gamma),
\]

where \( f, g \) are bounded, analytic and \( O(\gamma) \) in \( \mathcal{B}_1 \), it follows that \( \tau \) has the form given in (4.4).

**Remark 4.1.** If we now define \( \mathcal{P}_1^+(\gamma) \) (resp. \( \mathcal{P}_1^-(\gamma) \)) as the curve \( (\beta_1, \beta_2, \tau) \) with \( \beta_1 = 0 \) and \( \tau \) given by (4.4) with \( 0 < \beta_2 < \rho \) (resp.
For $0 < \gamma < \gamma_0$, $\mathcal{P}_1^{\pm} (\gamma)$ are the desired perturbations of the parabolas $\mathcal{P}_1^\pm$ defined by (3.2). Note that the form of $\tau$ in (4.4) implies that, for each $\gamma < \gamma_0$, the point $(0, 0, -a + \tau_0(\gamma))$ lying on the $\tau$-axis is the unique "turning point" of the curve $\mathcal{P}_1(\gamma) = \mathcal{P}_1^+ (\gamma) \cup \mathcal{P}_1^- (\gamma)$ in the plane $\beta_1 = 0$, i.e., the point at which $\partial \tau / \partial \beta_1 = 0$. Thus, one sees, in particular, that the turning point $(0, 0, -a)$ of $\mathcal{P}_1$ in (3.2) does not perturb off the $\tau$ axis for $0 < \gamma < \gamma_0$. This is in agreement with Figs. 2 and 4, where $\lambda_n(\gamma)$ corresponds to $\tau = -a + \tau_0(\gamma)$ (see also Remark 2.1).

Clearly, for $\gamma_0$ sufficiently small, $\mathcal{P}_1^{\pm} (\gamma)$ inherits the stability properties of $\mathcal{P}_1^{\pm}$ in Section 3A, except possibly near the points, if any, of secondary bifurcation on $\mathcal{P}_1^\pm (\gamma)$. To determine and treat the points of secondary bifurcation on $\mathcal{P}_1^{\pm} (\gamma)$, one needs a more refined analysis based upon equations (4.2) (see Theorem 4.2 and Remark 4.2).

(ii) Perturbations of $\mathcal{P}_2$ and $\mathcal{P}_3$ near $\mathcal{P}_1(\gamma)$. Setting $\beta_1 = \gamma = 0$ in (4.2) and eliminating $\tau$, one obtains (3.3) with $v = 0$. Thus, the points of secondary bifurcation on $\mathcal{P}_1^{\pm}$ in Section 3A are nontrivial solutions of (4.2) when $\beta_1 = \gamma = 0$. At such points the determinant of the Jacobian matrix $J_2 = \partial (H_1, G_2) / \partial (\beta_2, \tau)$ is given by

$$\det J_2 = a - F\beta_2^2. \quad (4.8)$$

Thus, making use of the results in Section 3A, if $a < 0$, then $\det J_2 < 0$ at the points corresponding to $\beta_2 = a/Fb_2$ and $\beta_2 = b_2$ whereas, if $a > 0$, then $\det J_2 > 0$ at $\beta_2 = a/Fb_2$ and $\det J_2 < 0$ at $\beta_2 = b_2$. If one now considers $\beta_1^2$ and $\gamma$ as parameters in (4.2), an application of the implicit function theorem yields the following results.

**Theorem 4.2.** Given $\rho > 0$ there exists $\gamma_0 > 0$ and $b_0 > 0$ such that, for $0 < \gamma < \gamma_0$ and $|\beta_1| < b_0$, all of the following hold:

(a) If $a < 0$, then (4.2) has analytic solutions $(\beta_2, \tau) = (b^\pm(\beta_1^2, \gamma), \tau^\pm(\beta_1^2, \gamma))$ such that $(b^\pm, \tau^\pm)$ satisfy $(b^\pm(0, 0), \tau^\pm(0, 0)) = (\beta_1^\pm, -a + d_2(\beta_1^2)^2)$ and are unique near these points.

(b) If $a > 0$, then (4.2) has analytic solutions $(\beta_2, \tau)$ given by $(b'(\beta_1^2, \gamma), \tau'(\beta_1^2, \gamma))$ and $(b''(\beta_1^2, \gamma), \tau''(\beta_1^2, \gamma))$ such that $(b', \tau')$ and $(b'', \tau'')$ satisfy

(i) $(b'(0, 0), \tau'(0, 0)) = (\beta_2, -a + d_2(\beta_2)^2),$

(ii) $(b''(0, 0), \tau''(0, 0)) = (\beta_2^*, -a + d_2(\beta_2^*)^2),$

and are unique near these points. Moreover, for each $\gamma$ satisfying $0 < \gamma < \gamma_0$, if $a < 0$, then the points $(b^\pm(0, \gamma), \tau^\pm(0, \gamma))$ lie on the respective curves $\mathcal{P}_1^{\pm} (\gamma)$ defined in Remark 4.1 and are points of secondary bifurcation for
\( \mathcal{P}_1^+(\gamma) \) whereas, if \( a > 0 \), then \((b'(0, \gamma), \tau'(0, \gamma))\) and \((b''(0, \gamma), \tau''(0, \gamma))\) both lie on \( \mathcal{P}_1^+(\gamma) \) and are points of secondary bifurcation for \( \mathcal{P}_1^+(\gamma) \).

The fact that the given points all lie on \( \mathcal{P}_1(\gamma) \) when \( \beta_1 = 0 \) follows because (4.2b) with \( \beta_1 = 0 \) and \( \beta_2 \neq 0 \) is equivalent to (4.1) which by Theorem 4.1 has unique solutions near the points of secondary bifurcation on \( \mathcal{P}_1 \).

**Remark 4.2.** For each \( \gamma \) satisfying \( 0 < \gamma < \gamma_0 \), we consider the curves defined by

\[ \mathcal{P}_2(\gamma): (\beta_1, \beta_2, \tau) = (\beta_1, b^+ (\beta_1^2, \gamma), \tau^+ (\beta_1^2, \gamma)), \quad |\beta_1| < b_0, \text{ if } a < 0, \quad (4.9) \]

\[ \mathcal{P}_2(\gamma): (\beta_1, \beta_2, \tau) = (\beta_1, b''(\beta_1^2, \gamma), \tau''(\beta_1^2, \gamma)), \quad |\beta_1| < b_0, \text{ if } a > 0. \quad (4.10) \]

Clearly, \( \mathcal{P}_2(\gamma) \) in (4.9) or (4.10) is the unique analytic perturbation near \( \mathcal{P}_1(\gamma) \) of the parabola \( \mathcal{P}_2 \) in (3.9). In a similar manner, for each \( \gamma \) satisfying \( 0 < \gamma < \gamma_0 \), the curves

\[ \mathcal{P}_3^-(\gamma): (\beta_1, \beta_2, \tau) = (\beta_1, b^- (\beta_1^2, \gamma), \tau^- (\beta_1^2, \gamma)), \quad |\beta_1| < b_0, \text{ if } a < 0, \quad (4.11) \]

\[ \mathcal{P}_3^+(\gamma): (\beta_1, \beta_2, \tau) = (\beta_1, b'(\beta_1^2, \gamma), \tau'(\beta_1^2, \gamma)), \quad |\beta_1| < b_0, \text{ if } a > 0, \quad (4.12) \]

are the unique analytic perturbations near \( \mathcal{P}_1(\gamma) \) of the parabola \( \mathcal{P}_3^- \) in (3.22) and \( \mathcal{P}_3^+ \) in (3.26). All of the curves in (4.9)--(4.12) can now be continued locally in \( \beta_1 \) as solutions of (2.1) up until the Jacobian matrix of (2.1), i.e., \( J = \partial \mathcal{G}_1, \mathcal{G}_2) / \partial (\beta_1, \beta_2) \), has a zero eigenvalue.

To determine the stability properties of the perturbed branches in Remark 4.2, we note that \( J \) has only one zero eigenvalue at the points of secondary bifurcation on \( \mathcal{P}_1(\gamma) \), namely \( \nu(\gamma) = H_1(0, \beta_2, \tau, \gamma) \). Thus, one easily sees that the stability properties of the perturbed branches \( \mathcal{P}_4(\gamma) \) are the same as those listed in Section 3 for the parabolas \( \mathcal{P}_i \); e.g., if \( a > 0 \), then \( \mathcal{P}_3^+(\gamma) \) loses stability at \((0, b'(0, \gamma), \tau'(0, \gamma))\) and regains stability at \((0, b''(0, \gamma), \tau''(0, \gamma))\) while \( \mathcal{P}_2(\gamma) \) bifurcates from \( \mathcal{P}_1^+(\gamma) \) at \((0, b''(0, \gamma), \tau''(0, \gamma))\) and is unstable for \( \beta_1 \) sufficiently small.

(iii) **Perturbations of \( \mathcal{P}_3 \).** To determine the continuation for \( \beta_1 \) not necessarily small of \( \mathcal{P}_3^+ \) in (3.20) and \( \mathcal{P}_3^- \) in (3.22), if \( a < 0 \), and of \( \mathcal{P}_3^- \) in (3.25) and \( \mathcal{P}_3^+ \) in (3.26), if \( a > 0 \), we make use of the system (4.3). Because of the form of the equations in (4.3) it is sufficient here to consider \( \beta_1 > 0 \). Substituting \( \beta_1^2 \) from (4.3a) into (4.3b), we obtain

\[ \tau = E^{-1} \beta_2 (D\beta_2 - Ab_2^{-1}) + p - E^{-1} d_1(q - \beta_2 p)/(\beta_2 - b_2). \quad (4.13) \]
One uses here also that

\[ b_1 c_2 - d_1 b_2 F - b_2^{-1}[b_1 b_2 E + d_1 a] = Ab_2^{-1}, \]

which follows from (H3). On the other hand, eliminating \( \tau \) from (4.3), we obtain

\[
\beta_1^2 = E^{-1}\beta_2(F\beta_2 - ab_2^{-1}) - E^{-1}(q - \beta_2 p)/(\beta_2 - b_2). \tag{4.14}
\]

Thus, since \( p \) and \( q \) are of order \( O(\gamma) \) as \( \gamma \to 0^+ \), one sees from (4.13) and (4.14) that solutions of (4.3) provide the desired continuation for \( \gamma \neq 0 \) of all the branches \( \mathcal{P}_3^\pm \) determined by (3.17). In particular, one sees from (4.13) and (4.14) that points \( (\beta_1, \beta_2, \tau) \) on \( \mathcal{P}_3^\pm \) with \( \beta_2 \neq b_2 \) are solutions of (4.3) when \( \gamma = 0 \). Moreover, setting \( \xi = \beta_1^2 \), one finds that at such points the determinant of the Jacobian matrix \( \mathcal{J}_3 = \frac{\partial(H_1, H_2)}{\partial(\xi, \tau)} \) satisfies \( \det \mathcal{J}_3 = E \neq 0 \). Thus, for either \( a < 0 \) or \( a > 0 \), an application of the implicit function theorem as in Theorem 4.1 for compact subsets of \( \mathcal{P}_3^\pm \cap \mathcal{B} \) in which \( \beta_2 \neq b_2 \) yields the existence of solutions \( \beta_1^2(\beta_2, \gamma) \) and \( \tau = \tau(\beta_2, \gamma) \) of (4.3). These solutions, in turn, determine the desired perturbations \( \mathcal{P}_3^\pm(\gamma) \) of \( \mathcal{P}_3^\pm \), \( 0 < \gamma < \gamma_0 \), provided that \( \gamma_0 \) is sufficiently small. In addition, making use of (4.14) and comparing (4.2b) and (4.3b) as \( (\beta_1, \beta_2) \to (0, a/Fb_2) \), one sees that such perturbations agree with those in (4.11) and (4.12) obtained from Theorem 4.2. Thus, for \( 0 < \gamma < \gamma_0 \), we have the existence of the desired perturbations \( \mathcal{P}_3^\pm(\gamma) \) of \( \mathcal{P}_3^\pm \), except possibly as \( \beta_2 \to b_2 \). One can show, however, by a limiting argument for fixed \( \gamma \), that the two segments of \( \mathcal{P}_3^\pm(\gamma) \) for \( \beta_2 > b_2 \) and \( \beta_2 < b_2 \) can be pieced together in a continuous manner. In particular, solving (4.14) for \( (q - \beta_2 p) \), one finds that \( (q - \beta_2 p) \) necessarily vanishes at the point on the extension of \( \mathcal{P}_3^\pm(\gamma) \) corresponding to \( \beta_2 = b_2 \), i.e., the point \( (\bar{\beta}_1, b_2, \bar{\tau}) \) with \( \bar{\tau} \) and \( \bar{\beta}_1 \) determined implicitly by (4.13) and

\[
\bar{\beta}_1^2 = E^{-1}[Fb_2^2 - a - (q_2 - b_2 \bar{\beta}_2) + \bar{p}]. \tag{4.15}
\]

Here and in what follows subscripts on \( p \) or \( q \) denote partial derivatives with respect to \( \beta_i \) and, e.g., \( \bar{p} \) denotes \( p(\beta_1^2, \beta_2, \tau, \gamma) \) evaluated at \( (\beta_1, b_2, \bar{\tau}, \gamma) \).

To piece together the various perturbations determined above in parts (i)–(iii) and to carry out a stability analysis for fixed \( \gamma \), \( 0 < \gamma < \gamma_0 \), we require the determinant of the Jacobian matrix \( J = \frac{\partial(G_1, G_2)}{\partial(\beta_1, \beta_2)} \) evaluated along \( \mathcal{P}_2(\gamma) \) and \( \mathcal{P}_3(\gamma) \). It is sufficient here to consider only the curves \( \mathcal{P}_2(\gamma) \) and \( \mathcal{P}_3^\pm(\gamma) \); the analysis for \( \mathcal{P}_3^\pm(\gamma) \) is similar and shows that \( \mathcal{P}_3^-(\gamma) \) is unstable for either \( a < 0 \) or \( a > 0 \). Subtracting (4.15) from (4.14), we obtain as a preliminary result the identity
\[ q - \beta_2 p - (\beta_2 - b_2)[q_2 - b_2 \bar{p}_2 - \bar{p}] \]
\[ = (\beta_2 - b_2)^2 [F(\beta_2 + b_2) - ab_2 - 1] - E(\beta_2 - b_2)(\beta_2^2 - \beta_2^2), \quad (4.16) \]
whenever \((\beta_1, \beta_2, \tau) \in \mathcal{P}_3^+(\gamma)\); this implies, in particular, that \(q_1 - b_2 \bar{p}_1 = 0\) in the limit as \((\beta_1, \beta_2, \tau) \to (\beta_1, b_2, \bar{\tau})\) along \(\mathcal{P}_3^+(\gamma)\).

We first evaluate \(\det J\) along \(\mathcal{P}_3^+(\gamma)\). By a direct calculation using (2.1) and (4.3), one obtains
\[
\det J = \beta_1^2 [J_{11} J_{22} - J_{12} J_{21}], \quad (4.17)
\]
where \(\Delta = a - b_1 \beta_2 + F\beta_2^2 = (\beta_2 - b_2)(F\beta_2 - ab_2 - 1)\) and
\[
J_{11} = 2d_1 + \beta_1^{-1} p_1, \quad (4.18a)
\]
\[
J_{21} = b_2 (2d_1 + \beta_1^{-1} p_1) + 2c_2 (\beta_2 - b_2) + \beta_1^{-1} (q_1 - b_2 p_1) = b_2 J_{11} + 2c_2 (\beta_2 - b_2) + O(\gamma (\beta_2 - b_2)), \quad (4.18b)
\]
\[
J_{12} = \beta_2^{-1} \Delta + \beta_2^{-1} [(c_1 + d_2) \beta_2^2 - a + \beta_2 p_2], \quad (4.18c)
\]
\[
J_{22} = -a + b_2 F\beta_2 + 2d_2 \beta_2^2 + q_2 - p - (q - \beta_2 p)/(\beta_2 - b_2) = \beta_2 J_{12} - F\beta_2 (\beta_2 - b_2) - \Delta + q_2 - \beta_2 p_2 - q_2 + b_2 \bar{p}_2 + \bar{p} - p = \beta_2 J_{12} - 2F(\beta_2 - b_2)[\beta_2 - a(2Fb_2 - 1)] + O(\gamma (\beta_2 - b_2)), \quad (4.18d)
\]
Here the order estimates hold uniformly in a neighborhood of \(\mathcal{P}_3^+(\gamma) \cap \mathcal{B}, 0 < \gamma < \gamma_0\), the identity (4.16) has been used to estimate \((q_1 - b_2 p_1)\), and the terms \(\beta_1^{-1} p_1\) and \(\beta_1^{-1} q_1\) are well defined as \(\beta_1 \to 0\) because of the evenness of \(p\) and \(q\) in \(\beta_1\). Thus, making use of (H3), one obtains
\[
\det J = \beta_1^2 [(\beta_2 - b_2) [ -4D(\beta_2 - A(2Db_2 - 1)) + O(\gamma)]]. \quad (4.19)
\]

A comparison with (3.18) now shows for each \(\gamma, 0 < \gamma < \gamma_0\), that \(\mathcal{P}_3^+(\gamma)\) inherits the stability properties of \(\mathcal{P}_3^+\); e.g., if \(a > 0\), \(\mathcal{P}_3^+ (\gamma)\) in (4.12) bifurcates from \(\mathcal{P}_3^+(\gamma)\) at \((\beta_1, \beta_2, \gamma) = (0, b'(0, \gamma), \tau'(0, \gamma))\), and eventually gains stability but then loses it at the point \((\beta_1, b_2, \bar{\tau})\) at which \(\det J\) in (4.19) vanishes.

In a similar way, along \(\mathcal{P}_2(\gamma)\), one obtains
\[
\det J = \beta_1^2 [(\beta_2^2 - \beta_2^2) [2d_1 E + O(\gamma)]]. \quad (4.20)
\]

In fact, by calculating \(J\) along \(\mathcal{P}_2(\gamma)\) one can show directly that \(\det J\) vanishes for \(\beta_1 \neq 0\) only at the point \((\beta_1, b_2, \bar{\tau})\) at which \(\mathcal{P}_2(\gamma)\) and \(\mathcal{P}_3^+(\gamma)\) intersect; in addition, multiplying (4.2a) by \(\beta_2\) and subtracting the resultant equation from (4.2b) again leads to (4.14) and, hence, to (4.16). In this way one can show that the point \((\beta_1, b_2, \bar{\tau})\) can be defined implicitly also by
means of $P_2(y)$ rather than $P_2^+(y)$ as in the above analysis. Comparing (4.20) and (4.14) with (3.10) and (3.12), one sees for each $y$, $0 < y < y_0$, that again $P_2(y)$ inherits the stability properties of $P_2$; e.g., if either $a < 0$ or $a > 0$, $P_2(y)$ bifurcates from $P_2^+(y)$ and is unstable but gains stability at the only point along $P_2(y)$ at which $\det J$ in (4.20) vanishes, i.e., the point $(\beta_1, b_2, \bar{t})$ at which $P_2(y)$ and $P_2^+(y)$ intersect. This completes the analysis and shows, in particular, that the bifurcation diagrams in Figs. 2 and 4 are also correct for each fixed $y \neq 0$ provided that $y$ is sufficiently small.

There are, of course, other cases that may be studied here, namely $k = 1$ and $a < 0$ (i.e., $\lambda_c = \lambda_1(y)$) or $k = 2$ and $a > 0$ (i.e., $\lambda_c = \lambda_4(y)$). Since we do not require any results in these other cases to establish the bifurcation diagrams in Figs. 2 and 4, such cases are not considered here.

For the convenience of the reader we summarize in the next two remarks some of the main results for steady flows of the Taylor problem.

**Remark 4.3.** (a) If $k = 1$ and $a > 0$ in (2.9), then $\lambda_c = \lambda_2(y)$ in (2.7) gives the critical Taylor number. In this case two 2-cell flows (i.e., $P_2^+(y)$) bifurcate supercritically from $\lambda_c$ and are stable. One of the 2-cell flows (i.e., $P_2^-(y)$) always remains stable. The other 2-cell flow loses stability at a point of secondary bifurcation at which a stable $(1 + 2)$-cell flow forms (i.e., $P_{2+}(y)$) but it regains stability at a point at which an unstable 1-cell flow (i.e., $P_2(y)$) bifurcates. The $(1 + 2)$-cell flow, in turn, loses stability at a point at which the 1-cell flow gains stability. (What we have called a 1-cell flow here has a 2-cell component as well, however, for small $y$ the 2-cell component is "nearly constant." Note that this is in close agreement with the single-cell flow shown in [4, Fig. 4c].)

(b) If $k = 2$ and $a < 0$ in (2.9), then $\lambda_c = \lambda_2(y)$ in (2.6) gives the critical Taylor number. In this case two 4-cell flows (i.e., $P_2^+(y)$) bifurcate supercritically from $\lambda_4(y)$ in (2.7), $\lambda_c < \lambda_4(y)$, and are unstable. One of the 4-cell flows (i.e., $P_2^+(y)$) gains stability at a point at which an unstable 2-cell flow (i.e., $P_2(y)$) bifurcates whereas the other 4-cell flow gains stability at a point at which an unstable $(2 + 4)$-cell flow (i.e., $P_2^-(y)$) bifurcates. In addition, there is a $(2 + 4)$-cell flow (i.e., $P_2^+(y)$) bifurcating from $\lambda_c = \lambda_2(y)$ which is stable or unstable depending on whether the bifurcation is supercritical or subcritical. In any case, however, the $(2 + 4)$-cell flow eventually gains stability but then loses it at a point at which the 2-cell flow gains stability. (The 2-cell flow here has a 4-cell component as well but one that for small $y$ is "nearly constant.""

**Remark 4.4.** All of the steady flows for the Taylor problem in Remark 4.3 are of the form (B.60) in Appendix B with the corresponding Taylor number determined by (B.61) and Remark 2.1. For example, if $k = 2$ and $a < 0$, then, for fixed $y$ satisfying $0 < y < y_0$, the unstable part of the
2-cell flow is given by (B.60) and (4.9) for \(0 \leq \beta_1^2 \leq \beta_1^2\), i.e., in (B.60) one sets

\[
\beta_1^* = \beta_1, \quad \beta_2^* = \beta^+(\beta_1, \gamma) \quad \text{and} \quad \tau^* = \tau^+(\beta_1, \gamma), \quad 0 \leq \beta_1^2 \leq \beta_2^2,
\]

(4.21)

where \(\beta^+\) and \(\tau^+\) (and their continuations) are defined as in (4.9); the stable part of the 2-cell flow here is determined in a similar way by (B.60) and (4.21) for \(\beta_1^2 \leq \beta_2^2 \leq \rho^2\). (One uses also here that \(\det J \neq 0\) in (4.20) for \(\beta_1^2 \neq 0\), \(\beta_1^2\), so that \(\mathcal{D}(\gamma)\) in (4.9) can be uniquely continued in \(\beta_1\) for \(0 \leq \beta_1^2 \leq \rho^2\).) Analogous representations for the other steady flows in Remark 4.3 are given by (B.60) in terms of the appropriate parameter along \(\mathcal{D}_k(\gamma)\), namely \(\beta_1^2\) or \(\beta_2^2\), \(0 \leq \beta_i^2 \leq \rho^2\) (\(i = 1, 2\)).

**APPENDIX A**

In this Appendix we introduce a small-gap approximation to the Navier–Stokes equations, derive in (A.14) the Boussinesq-type system of partial differential equations used throughout Appendix A and Appendix B, determine an appropriate Hilbert space setting for studying periodic solutions of the problem, and formulate the basic assumptions on the length, \(l\), of the cylinders under which we obtain primary and secondary flows having one, two, or four cells.

In seeking periodic solutions of the Taylor problem one assumes that the fluid fills the space between two infinite concentric cylinders with radii \(R_1\) and \(R_2\), \(R_1 < R_2\). The Navier–Stokes equations in cylindrical coordinates with the fluid velocity vector, \(\vec{u} = (\vec{u}_1, \vec{u}_2, \vec{u}_3)\), and pressure, \(\vec{p}\), depending only on \(\tilde{r}\) and \(\tilde{z}\), are given by (e.g., see [6, p. 89])

\[
\begin{align*}
(\vec{u} \cdot \nabla) \vec{u}_1 - \frac{\vec{u}_2}{\tilde{r}} &= -\frac{\partial}{\partial \tilde{r}} \left(\rho \frac{\vec{u}_1}{\tilde{r}}\right) + v \left(\nabla \vec{u}_1 - \frac{\vec{u}_1}{\tilde{r}^2}\right), \quad (A.1a) \\
(\vec{u} \cdot \nabla) \vec{u}_2 + \frac{1}{\tilde{r}} \frac{\vec{u}_1 \vec{u}_2}{\tilde{r}} &= v \left(\nabla \vec{u}_2 - \frac{\vec{u}_2}{\tilde{r}^2}\right), \quad (A.1b) \\
(\vec{u} \cdot \nabla) \vec{u}_3 &= -\frac{\partial}{\partial \tilde{z}} \left(\rho \frac{\vec{u}_3}{\tilde{r}}\right) + v \Delta \vec{u}_3, \quad (A.1c) \\
\nabla \cdot \vec{u} &= 0, \quad (\vec{u} \cdot \nabla) \left(\tilde{r} \vec{u}_1\right) + \frac{\partial \vec{u}_3}{\partial \tilde{z}} = 0, \quad R_1 < \tilde{r} < R_2, 0 \leq \theta \leq 2\pi, -\infty < \tilde{z} < \infty. \quad (A.1d)
\end{align*}
\]
Here \( v \) is the kinematic viscosity, \( \rho \) is the density of the fluid, \( \mathbf{V} = (\partial/\partial \bar{r}, 0, \partial/\partial \bar{z}) \), \( \bar{u}_i = \bar{u}_i(\bar{r}, \bar{z}) \), \( \bar{p} = \bar{p}(\bar{r}, \bar{z}) \) and

\[
\mathbf{u} \cdot \mathbf{V} = \bar{u}_1 \frac{\partial}{\partial \bar{r}} + \bar{u}_3 \frac{\partial}{\partial \bar{z}},
\]

\[
A v = \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left( \bar{r} \frac{\partial v}{\partial \bar{r}} \right) + \frac{\partial^2 v}{\partial \bar{z}^2}, \quad v = v(\bar{r}, \bar{z}).
\]

The basic Couette flow is given by (e.g., see [6, p. 90] and [10, p. 130]) \( \bar{u}_1 = \bar{u}_3 = 0 \) and

\[
\bar{u}_2 = \bar{V}(\bar{r}) = A \bar{r} + \frac{B}{\bar{r}}, \quad \bar{p} = \bar{P}(\bar{r}) = \rho \int \frac{\bar{V}^2}{\bar{r}} \, d\bar{r},
\]

where

\[
A = \frac{\Omega_1 (\mu - \eta^2)}{1 - \eta^2}, \quad B = \frac{\Omega_1 (1 - \mu) R_1^3}{1 - \eta^2}.
\]

Here

\[
\mu = \frac{\Omega_2}{\Omega_1}, \quad \eta = \frac{R_1}{R_2},
\]

and \( \Omega_1, \Omega_2 \) are the angular velocities of the inner and outer cylinders, respectively. It is assumed throughout the paper that \( \mu > -1 \).

One introduces first the dimensionless variables

\[
\bar{r} = dr, \quad \bar{z} = dz, \quad R_1 = dr_1, \quad R_2 = dr_2 = \delta R_1 r_2, \\
\bar{u}_1 = R_1 \Omega_1 u_1', \quad \bar{u}_2 = R_1 \Omega_1 u_2 + \bar{V}(\bar{r}), \quad \bar{u}_3 = R_1 \Omega_1 u_3',
\]

\[
\bar{V}(\bar{r}) = R_1 \Omega_1 V(r), \quad \bar{p} = (\nu R_1 \Omega_1 \rho/d) p' + \bar{P}(\bar{r}),
\]

where

\[
d = R_2 - R_1 \quad \text{and} \quad \delta = \frac{d}{R_1} = \frac{1}{r_1}.
\]

In the small-gap limit one assumes that \( \delta \to 0 \). Setting \( r = r_1 + x + \frac{1}{2}, \quad -\frac{1}{2} \leq x \leq \frac{1}{2} \), and expanding \( \bar{V}(\bar{r})/\bar{r} \) about \( \bar{r} = R_1 \), one obtains the standard result (e.g., see [6, p. 94])

\[
V = \frac{1}{2} (1 + \mu) \left[ 1 - 2 \left( \frac{1 - \mu}{1 + \mu} \right) x + O(\delta) \right].
\]
The form of $V$ in (A.10) and Eqs. (A.1) suggest a second change of variables
\begin{align}
  u_1' &= (1 + \mu)(\gamma \delta)^{1/2} T^{-1/2} u_1, & u_2' &= (1 + \mu) \gamma T^{-1/2} u_2, \\
  u_3' &= (1 + \mu)(\gamma \delta)^{1/2} T^{-1/2} u_3, & p' &= (1 + \mu)(\gamma \delta)^{1/2} T^{-1/2} p,
\end{align}
where $T$ is the "Taylor number"
\begin{equation}
  T = -\frac{1}{2}(1 + \mu) v^{-2} [4 A \Omega_1 d^4],
\end{equation}
and $\gamma$ is the "structure parameter" given by
\begin{equation}
  \gamma = \frac{2(\eta^2 - \mu)}{\eta(1 + \eta)(1 + \mu)}.
\end{equation}
Note that $T \geq 0$ requires that $A$ in (A.5) satisfies $A \leq 0$ so that $\gamma \geq 0$ when $-1 < \mu \leq \eta^2$. Making use of the changes of variables in (A.8) and (A.11), replacing $\partial/\partial r$ by $\partial/\partial x$ and $(1 - \mu)/(1 + \mu)$ by $\gamma + O(\delta)$, and letting $\delta \to 0$ with $\gamma$ and $T$ kept fixed, one obtains a set of small-gap equations for the Taylor problem, namely
\begin{align}
  -\Delta u - \lambda f(u) + \nabla p &= - (u \cdot \nabla) u + g(u) & \text{in } & \Omega_\infty, \\
  \nabla \cdot u &= 0 & \text{in } & \Omega_\infty, \\
  u &= 0 & \text{at } & x = \pm \frac{1}{2}.
\end{align}
Here $\Omega_\infty = \{(x, z): -\frac{1}{2} < x < \frac{1}{2} \text{ and } -\infty < z < \infty\}$, $v = v(x, z)$, $\nabla = (\partial/\partial x, 0, \partial/\partial z)$, $\lambda = T^{1/2}$, $\Delta w = (\partial^2 w/\partial x^2) + (\partial^2 w/\partial z^2)$, and
\begin{align}
  f(u) &= ((1 - 2\gamma x) u_2, u_1, 0), \\
  g(u) &= (\gamma u_2^2, 0, 0).
\end{align}
We shall see that the advantage of using the small-gap equations in (A.14) is that the role of the parameter $\gamma$ in these equations is analogous to that of a structure parameter in temperature-dependent convection problems so that a complete analysis can be carried out for small $\gamma$. In fact, the Boussinesq-type equations in (A.14) are analogous to those for the generalized Bénard problem studied in [12, 13] with $u_3$ and $\theta$ in [12, 13] replaced by $u_1$ and $u_2$, respectively. In carrying out the analysis below for the Taylor problem we shall make repeated use of this analogy along with the methods introduced in [12, 13].

We next introduce an appropriate Hilbert space setting in which to seek
solutions of (A.14) that are periodic in $z$. Given a positive number $\alpha_k$ (to be specified below in (A.27)), we set

$$\Omega = \left\{ \xi = (x, z) : -\frac{1}{2} < x < \frac{1}{2} \text{ and } 0 < z < \frac{2\pi}{\alpha_k} \right\}. \quad (A.16)$$

The (complex) Hilbert space, $\mathcal{H}$, used throughout the paper is defined as the closure of the set $\{ v = (v_1, v_2, v_3) : v \text{ is smooth, periodic in } z \text{ with period } 2\pi/\alpha_k, \text{ and vanishing in a neighborhood of } |x| = \frac{1}{2} \text{ with } \nabla \cdot v = 0 \}$ in the norm, $\| \|$, associated with the inner product

$$(v, w) = \int_{\Omega} \sum_{j=1}^{3} \nabla v_j \cdot \nabla \bar{w}_j. \quad (A.17)$$

Here and in the sequel a bar over a quantity denotes complex conjugation and, whenever possible, the vector notation $v$ is suppressed when dealing with elements of $\mathcal{H}$.

To formulate the problem as an operator equation in $\mathcal{H}$, we take the scalar product of (A.14a) with $\tilde{w} \in \mathcal{H}$, use (A.14b), (A.14c) and integration by parts to obtain

$$(u, w) - \lambda (L, u, w) = (Q_\gamma(u), w). \quad (A.18)$$

Here, for each $\gamma \in \mathbb{R}^1$, the linear operator $L_\gamma : \mathcal{H} \to \mathcal{H}$ and the quadratic operator $Q_\gamma : \mathcal{H} \to \mathcal{H}$ are given by

$$L_\gamma = L - \gamma M, \quad (A.19a)$$

$$Q_\gamma = F + \gamma G, \quad F(u) = \Phi(u, u), \quad G(u) = \Gamma(u, u), \quad u \in \mathcal{H}, \quad (A.19b)$$

where the operators $L : \mathcal{H} \to \mathcal{H}$ and $M: \mathcal{H} \to \mathcal{H}$ are defined (weakly) by

$$(Lv, w) = \int_{\Omega} (v_2 \bar{w}_1 + v_1 \bar{w}_2), \quad (A.20)$$

$$(Mv, w) = 2 \int_{\Omega} xv_2 w_1, \quad v, w \in \mathcal{H}, \quad (A.21)$$

and the bilinear operators $\Phi : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ and $\Gamma : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ are defined (weakly) by

$$(\Phi(u, v), w) = -\int \left[ (u \cdot \nabla) v \right] \cdot \bar{w} = -\int \left( u_1 \frac{\partial v}{\partial x} + u_3 \frac{\partial v}{\partial z} \right) \cdot \bar{w}, \quad (A.22)$$

$$(\Gamma(u, v), w) = \int u_2 v_2 \bar{w}_1, \quad u, v, w \in \mathcal{H}. \quad (A.23)$$
Since \( w \) in (A.18) is an arbitrary element of \( \mathcal{H} \), one obtains the operator equation
\[
    u - \lambda L_y u = Q_y(u), \quad u \in \mathcal{H}, \; \lambda \in \mathbb{R}, \; y \in \mathbb{R}.
\]  

(\*)

Standard regularity methods (e.g., see [11]) can now be used to show that the problems of finding solutions of (A.14) and of (\*) are equivalent.

We will need the following facts about the linear problem at \( y = 0 \) associated with (\*), namely
\[
    u - \mu L_y u = 0, \quad u \in \mathcal{H}, \; \mu \in \mathbb{R}.
\]  

(A.24)

The linear problem (A.24) is equivalent to the classical problem, for smooth \( u \) and \( p \) periodic with period \( 2\pi/\alpha_k \) in \( z \), obtained by setting \( y = 0 \) and omitting the non-linear terms in (A.14). The solutions of (A.24) are determined by (see [12, (2.11)])
\[
    u_j = e^{i\kappa \cdot k_j \phi_j(x)}, \quad k = (0, k_2), \quad j = 1, 2, 3, \quad (A.25a)
\]

\[
    p = e^{i\kappa \cdot k \sigma^2 D^2 \phi'_1}, \quad \sigma = |k| = |k_2|, \quad (A.25b)
\]

\[
    \phi_3 = i\sigma^{-2} k_2 \phi'_1, \quad (A.25c)
\]

where \( D^2 = (d^2/dx^2) - \sigma^2 \), a prime denotes \( d/dx \), and \( \phi_1 \) and \( \phi_2 \) satisfy
\[
    D^4 \phi_1 - \mu \sigma^2 \phi_2 = 0, \quad (A.26a)
\]

\[
    D^2 \phi_2 + \mu \phi_1 = 0, \quad (A.26b)
\]

\[
    \phi_1 = \phi'_1 = \phi_2 = 0 \quad \text{at} \quad x = \pm \frac{1}{2}. \quad (A.26c)
\]

One can show for \( \sigma > 0 \) (e.g., see [9]) that the eigenvalue problem (A.26) has a countable number of positive, simple eigenvalues, \( 0 < \mu_1(\sigma) < \mu_2(\sigma) < \cdots \), depending continuously on \( \sigma \). Moreover, \( \mu_1(\sigma) \to \infty \) as either \( \sigma \to 0^+ \) or \( \sigma \to \infty \).

We now proceed as in [17, p. 310 ff.] and [2, p. 229 ff.] to choose exceptional values of the wave number \( \alpha_k \), depending upon the length \( l \) of the cylinder, such that the linearized problem (A.24) has a smallest characteristic value corresponding to either one and two cells or two and four cells. That is, for a given fixed length of the cylinder \( l \) we set
\[
    \alpha_k = \frac{k\pi}{l} \quad (k = 1 \text{ or } 2) \quad \text{and} \quad \sigma_p = p\alpha_k \quad (p = 1, 2, \ldots). \quad (A.27)
\]

and make the following assumption (for \( \sigma > 0 \), \( \mu_1(\sigma) \) denotes the smallest, positive eigenvalue of (A.26))
BASIC ASSUMPTION

Assume that $l$ is such that $\mu_1(\sigma_1) = \mu_1(\sigma_2)$ and that $\mu_1(\sigma) > \mu_1(\sigma_1)$ when $\sigma \neq \sigma_1, \sigma_2$.

We now use $a_k$ in (A.27) to define the basic Hilbert space $\mathcal{H}$. Note that since $k$ is the number of Taylor cells (i.e., half-periods) fitted into a cylinder of length $l$, the smallest eigenvalue $\mu_1(\sigma_1) = \mu_1(\sigma_2)$ of the linear problem (A.24) corresponds to both one-cell and two-cell flows when $k = 1$ and to both two-cell and four-cell flows when $k = 2$. Such choices of $k$, and, hence, of $a_k$ are motivated by the existence of the "transcriterial bifurcation points" $B$ and $Q$ in Fig. 1 (i.e., $k = 2$) and Fig. 3 (i.e., $k = 1$) (see also the discussion in [2, 17]).

We may now determine a complete solution of the linear problem (A.24). To minimize the necessary calculations, whenever possible, we will use the notation and results from [12]. Since $e^{k \cdot \xi}$ in (A.25) must have period $2\pi/a_k$ in $z$, it follows that the only wave numbers, $\sigma$, correspond to eigenfunctions having the required period in $z$ are those for which $\sigma^2 = n^2a_k^2$ for some integer $n$. Thus, the $\sigma_p$ ($p = 1, 2, \ldots$) defined in (A.27) are the only admissible wave numbers once $a_k$ has been determined. For each $p$ ($p = 1, 2, \ldots$), the eigenvalue problem (A.26) has an infinite sequence of real, nontrivial solutions

$$(\mu, \phi_1, \phi_2) = (\mu_{pq}, \phi_{1pq}, \phi_{2pq}), \quad p = 1, 2, \ldots; q = \pm 1, \pm 2, \ldots \quad (A.28)$$

Since $(-\mu, \phi_1, -\phi_2)$ is a solution of (A.26) whenever $(\mu, \phi_1, \phi_2)$ is a solution of (A.26), we may order the indices so that

$$\phi_1^{(-q)} = \phi_{1pq}, \quad \phi_2^{(-q)} = -\phi_{2pq}, \quad \mu_p^{(-q)} = -\mu_{pq}, \quad \text{and} \quad 0 < \mu_{p1} < \mu_{p2} < \ldots \quad (A.29)$$

Using this notation, we see from the Basic Assumption above that

$$\mu_0 = \min_p \mu_{p1} = \mu_{11} = \mu_{21} \quad (A.30)$$

and that $\mu_{pq} > \mu_0$ if $(p, q) \neq (1, 1)$ or $(2, 1)$.

The above discussion of the underlying problem (A.26) shows that the full eigenvalue problem (A.24) in $\mathcal{H}$ has the solutions

$$\lambda = \mu_{pq} \quad \text{and} \quad u = \psi_{pq}(\xi) = e^{k_{pq} \cdot \xi} \phi_{pq}(x), \quad p = 1, 2, \ldots; \quad q = \pm 1, \pm 2, \ldots; \quad j = \pm 1, \quad (A.31)$$
where
\[ k_{pj} = (0, n_{pj} \alpha_k), \quad (A.32) \]
\[ \phi^{pq}(x) = \left( \phi_1^{pq}(x), \phi_2^{pq}(x), \frac{i \alpha_k n_{pq}}{\sigma_p^2} \frac{d}{dx} \phi_1^{pq}(x) \right). \quad (A.33) \]

Note that \( \phi^{pq} \) depends upon \( j \) only in the third component. We may assume in (A.32) that \( n_{p1} = p \) when \( n_{p(-1)} = -p \) so that
\[ k_{p(-j)} = -k_{pj} \quad \text{and} \quad \psi^{pq(-j)} = \psi^{pq(j)}. \quad (A.34) \]

It follows as in [12, Appendix] that the eigenfunctions \( \{ \psi^{pq} \} \) may be assumed orthonormal in \( \mathcal{H} \), after rescaling by constants depending upon \( p \) and \( q \) but not \( j \), i.e.,
\[ (\psi^{pq}, \psi^{rm}) = \delta_{pq} \delta_{qr} \delta_{jr}, \quad (A.35) \]

where \( \delta_{jk} \) is the usual Kronecker delta symbol.

The following lemma summarizes some of the basic facts of this Appendix for the linearized problem (A.24). The compactness properties are essentially known (e.g., see [11]) while the characterization (A.36) follows easily from (A.21).

**Lemma A.1.** (i) The linear operator \( L: \mathcal{H} \to \mathcal{H} \) is self-adjoint and compact and its characteristic values and eigenfunctions are given by (A.31). The eigenfunctions \( \{ \psi^{pq} \} \) satisfy (A.34) and (A.35), and are complete in \( \mathcal{H} \).

(ii) The linear operator \( M: \mathcal{H} \to \mathcal{H} \) is compact and its adjoint, \( M^* \), is characterized by
\[ (M^*v, w) = 2 \int_{\Omega} xv_1 \bar{w}_2, \quad v, w \in \mathcal{H}. \quad (A.36) \]

**Appendix B**

In this Appendix we show how the problem of finding nontrivial solutions of Eq. (*) in \( \mathcal{H} \) can be reduced to a finite-dimensional problem. In doing so, we will derive bifurcation equations of the type formulated in (2.1) and establish, in particular, the various properties of the remainder terms listed in (2.3). The reduction is carried out by means of splitting methods using the structure parameter \( \gamma \) as an “amplitude” parameter.

Recall from the discussion in Appendix A that the eigenvalue \( \mu_0 = \mu_{11} = \mu_{21} \) of (A.26) defined in (A.30) is also a characteristic value of \( L \).
of multiplicity $N = 4$. The associated null space, $\mathcal{M}$, of $I - \mu_0 L$ is spanned by

$$\psi^{pqj} \equiv \psi^{pqj}, \quad p = 1, 2, \quad j = \pm 1,$$

(B.1)

and the orthogonal complement, $\mathcal{M}^\perp$, of $\mathcal{M}$ in $\mathcal{H}$ is spanned by $\{\psi^{pqj} : (p, q) \neq (1, 1) \text{ or } (2, 1)\}$.

It will be convenient to represent an element $v$ of $\mathcal{H}$ by its Fourier series, namely

$$v = \sum \beta_{pqj} \psi^{pqj}, \quad \beta_{pqj} \in \mathbb{C},$$

(B.2)

where the sum is extended over the set of integer triples $(p, q, j)$ with $1 \leq p < \infty$, $1 \leq |q| < \infty$, and $j = \pm 1$. Note that if $v = \psi \in \mathcal{M}$, then (B.2) reduces to

$$\psi = \sum_{j=1}^{2} \beta_{pj} \psi^{pj}.$$  

(B.3)

The following lemma summarizes the main calculations of this Appendix; it is the analog of the main lemma in [12, Lemma 3.11]. Here, and in the remainder of the paper, the operator $P: \mathcal{H} \to \mathcal{M}^\perp$ denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}^\perp$, the operator $K: \mathcal{M} \to \mathcal{M}^\perp$ denotes the inverse of the restriction of $I - \mu_0 L$ to $\mathcal{M}^\perp$, $\delta(\xi)$ is zero whenever the (scalar or vector) parameter $\xi$ is not zero whereas $\delta(0) = 1$.

**LEMMA B.1.**  
(i) If $v \in \mathcal{H}$, then $Lv$, $Mv$, and $KPv$ can be obtained from (B.2) by formal calculation, e.g.,

$$Lv = \sum \beta_{pqj} \mu_{pq}^{-1} \psi^{pqj}, \quad KPv = \sum_{(p, q, j) \neq (1, 1, j) \text{ or } (2, 1, j)} \beta_{pqj} \left( \frac{\mu_{pq}}{\mu_{pq} - \mu_0} \right) \psi^{pqj},$$

(B.4)

where $\sum_0$ denotes summation over the same set of integer triples as $\sum$, except that $(p, q, j) \neq (1, 1, j)$ or $(2, 1, j)$. In particular, $K$ is bounded, self-adjoint and positive on $\mathcal{M}^\perp$.

(ii) $M: \mathcal{M} \to \mathcal{M}^\perp$, i.e., for $\psi \in \mathcal{M}$, $(M\psi, \psi^{pqj}) = 0$, $p = 1, 2$, $n = \pm 1$.

(iii) For $u, v, w \in \mathcal{H}$, $(\Phi(u, v), \tilde{v}) = -(\Phi(u, w), \tilde{v})$ and $\Gamma(u, v) = \Gamma(v, u)$.

(iv) $F: \mathcal{M} \to \mathcal{M}^\perp$ and, in addition, $(\Phi(\psi^{ij}, \psi^{lm}), \tilde{\psi}^{mn}) = 0$ for $i, l, p = 1, 2$, $|j| = |m| = |n| = 1$.

(v) If $\psi$ is of the form (B.3), then there are real constants $a_k, a_{2k}, b_p^i$ depending on $p$ but not on $n$ such that

$$(MKM\psi, \tilde{\psi}^{pn}) = a_{pk} b_{p(-n)}, \quad p = 1, 2, \quad |n| = 1,$$

(B.5)
STEADY FLOWS OF THE TAYLOR PROBLEM

\( (M K F(\psi), \vec{\psi}_n) = b_1^1 \beta_{1n} \beta_{2(-n)}, \quad |n| = 1, \) \hspace{1cm} (B.6a)

\( (M K F(\psi), \vec{\psi}^{2n}) = b_2^1 \beta_{1n}^{2(-n)}, \quad |n| = 1, \) \hspace{1cm} (B.6b)

\( (\Phi(\psi, KM\psi), \vec{\psi}_n) = b_1^2 \beta_{1n} \beta_{2(-n)}, \quad |n| = 1, \) \hspace{1cm} (B.7a)

\( (\Phi(\psi, KM\psi), \vec{\psi}^{2n}) = b_2^2 \beta_{1n}^{2(-n)}, \quad |n| = 1, \) \hspace{1cm} (B.7b)

\( (\Phi(KM\psi, \psi), \vec{\psi}_n) = b_1^1 \beta_{1n} \beta_{2(-n)}, \quad |n| = 1, \) \hspace{1cm} (B.8a)

\( (\Phi(KM\psi, \psi), \vec{\psi}^{2n}) = b_2^1 \beta_{1n}^{2(-n)}, \quad |n| = 1, \) \hspace{1cm} (B.8b)

\( (G(\psi, \vec{\psi}_n) = b_1^1 \beta_{1n} \beta_{2(-n)}, \quad |n| = 1, \) \hspace{1cm} (B.9a)

\( (G(\psi, \vec{\psi}^{2n}) = b_2^1 \beta_{1n}^{2(-n)}, \quad |n| = 1. \) \hspace{1cm} (B.9b)

(vi) If \( \psi \) is of the form (B.3), then there are constants \( a_{ip}^m \) and \( b_{ip}^m \) such that

\[ (\Phi(\psi, KF(\psi)), \vec{\psi}^m) = \sum_{|l| = 1}^2 \left[ a_{ip}^m (1 - \delta_{ip} \delta_{jn}) + b_{ip}^m \right] \beta_{ij} \beta_{i(-j)} \beta_{p(-n)}, \quad p = 1, 2, \quad |n| = 1. \] \hspace{1cm} (B.10)

The constants \( a_{ip}^m \) and \( b_{ip}^m \) depend upon \( j \) and \( n \) only through \( \sigma_q = |k_{ij} + k_{pn}| \), are nonnegative when \( p = i \), and satisfy \( a_{ip}^{(i-n)} = a_{ip}^{(-i)n} \) and \( b_{ip}^{(i-n)} = b_{ip}^{(-i)n} \).

(vii) If \( \psi \) is the form (B.3), then there are constants \( c_{ip}^m \) such that

\[ (\Phi(KF(\psi), \psi), \vec{\psi}^m) = \sum_{|l| = 1}^2 c_{ip}^m \beta_{ij} \beta_{pn}. \] \hspace{1cm} (B.11)

The constants \( c_{ip}^m \) depend upon \( j \) and \( n \) only through \( \sigma_q = |k_{ij} + k_{pn}| \) and satisfy \( c_{ip}^{(i-n)} = c_{ip}^{(-i)n} \).

Proof: Items (i) and (iii) are proved in the same way as the corresponding items in [12, Lemma 3.1], and (ii) and (iv) follow from the evenness of \( \phi_{p1}^{\delta} \) and \( \varphi_{1}^{\delta} \) in \( x \) \((p = 1, 2)\).

To prove (B.5), we first calculate

\[ (M \psi_{ij}, \psi_{pq} \phi_{pq}) = \delta_{ip} \delta_{jn} A_{pq}, \] \hspace{1cm} (B.12)

where

\[ A_{pq} = \frac{4\pi}{\alpha_k} \int_{-\frac{1}{2}}^{\frac{1}{2}} x \phi_{pq} \phi_{pq} \phi_{pq} dx. \] \hspace{1cm} (B.13)
The $A_{pq}$ are real and $A_{i(\pm 1)} = 0$ because of the evenness of $\phi_1^{p\dagger}$ and $\phi_2^{p\dagger}$. Thus

$$M\psi^{ij} = \sum_{|q| \geq 2} A_{iq} \psi^{iqj}$$  \hspace{1cm} (B.14)

and, by a similar set of calculations,

$$M^* \bar{\psi}^{pn} = \sum_{|q| \geq 2} A^*_{pq} \bar{\psi}^{pqn},$$  \hspace{1cm} (B.15)

where

$$A^*_{pq} = \frac{4\pi}{\alpha_k} \int_{-1/2}^{1/2} x \phi_2^q \phi_1^{p\dagger} \, dx.$$  \hspace{1cm} (B.16)

Making use of (B.14) and (B.15) together with (A.35) and the definition of $K$, one obtains, for $\psi$ as in (B.3),

$$(MKM\psi, \bar{\psi}^{pn}) = (KM\psi, K^* \bar{\psi}^{pn})$$

$$= \left( \sum_{|q| \geq 2} A^*_{pq} \bar{A}_{pq} \right) \beta_{p(-n)} = a_{pk} \beta_{p(-n)}, \quad p = 1, 2, \quad |n| = 1,$$  \hspace{1cm} (B.17)

where

$$\tilde{A}_{pq} = A_{pq} \left( \frac{\mu_{pq}}{\mu_{pq} - \mu_0} \right), \quad |q| \geq 2.$$  \hspace{1cm} (B.18)

To prove (B.6), we calculate

$$(\Phi(\psi^{ij}, \psi^{lm}), \bar{\psi}^{pqn}) = - \int (\psi^{ij} \cdot \nabla) \psi^{lm} \cdot \bar{\psi}^{pqn}$$

$$= \delta(k_{ij} + k_{lm} + k_{pn}) I_1(i, j, l, m, p, q, n),$$  \hspace{1cm} (B.19)

where

$$I_1 = \frac{2\pi}{\alpha_k} \int_{-1/2}^{1/2} \left[ \frac{\alpha_k^2 n_{ij} n_{lm} \phi_1^{i\dagger}}{\sigma_l^2} \frac{d\phi_1^{i\dagger}}{dx} \left( \phi_1^{i\dagger} \phi_1^{pq} + \phi_2^{i\dagger} \phi_2^{pq} - \frac{\alpha_k^2 n_{pn} n_{lm}}{\sigma_p^2 \sigma_l^2} \frac{d\phi_1^{i\dagger}}{dx} \frac{d\phi_1^{pq}}{dx} \right) \right. $$

$$\left. - \phi_1^{i\dagger} \left( \frac{d\phi_1^{i\dagger}}{dx} \phi_1^{pq} + \frac{d\phi_1^{i\dagger}}{dx} \phi_2^{pq} - \frac{\alpha_k^2 n_{pn} n_{lm}}{\sigma_p^2 \sigma_l^2} \frac{d\phi_1^{i\dagger}}{dx} \frac{d\phi_1^{pq}}{dx} \right) \right] \, dx.$$  \hspace{1cm} (B.20)

We require $I_1$ only when $k_{ij} + k_{lm} + k_{pn} = 0$. If $\sigma_p = |k_{pn}|$, then

$$\sigma_p^2 = |k_{ij} + k_{lm}|^2 = (i^2 + 2n_{ij} n_{lm} + l^2) \alpha_k^2$$  \hspace{1cm} (B.21)
so that
\[ \alpha_k^2 n_n n_{lm} = \frac{1}{2}(\sigma^2_{\rho} - \sigma^2_{\rho} - \sigma^2_{\rho}). \]  
(B.22)

Moreover, making use of (B.22), one obtains
\[ \alpha_k^2 n_{pn} n_{lm} = \frac{1}{2}(\sigma^2_{\rho} - \sigma^2_{\rho} - \sigma^2_{\rho}). \]  
(B.23)

Thus, the quantities on the left hand side in (B.22) and (B.23) are independent of \( j, m, \) and \( n \) except that \( \sigma_{\rho} = |k_{ij} + k_{lm}|. \) Subbing (B.22) and (B.23) into \( I_1, \) one sees that
\[ (\Phi(\psi^u, \psi^m), \overline{\psi}^{pqn}) = \delta(k_{ij} + k_{lm} + k_{pn}) I_2(i, l, p, q), \]  
(B.24)

where
\[ I_2(i, l, p, q) = I_1(i, j, l, m, p, q, n) \]  
(B.25)

is real and depends upon \( j, m, \) and \( n \) only through \( \sigma_{\rho} = |k_{ij} + k_{lm}|. \) Let \( A^*_{pq} \) be defined as in (B.16) and set
\[ d_{pq} = A^*_{pq} \left( \frac{\mu_{pq}}{\mu_{pq} - \mu_0} \right), \quad |q| \geq 2. \]  
(B.26)

Then (B.15) and (B.24) imply
\[ (MKF(\psi), \overline{\psi}^{nn}) = \sum_{i,l=1 \atop i\neq l}^{2} \beta_{ij} \beta_{lm} (\Phi(\psi^u, \psi^m), K M^* \overline{\psi}^{pqn}) \]  
(B.27)

\[ = \sum_{i,l=1 \atop i\neq l}^{2} e^u_{pq} \beta_{ij} \beta_{lm} \delta(k_{ij} + k_{lm} + k_{pn}), \]

where
\[ e^u_{pq} = \sum_{|q| \geq 2} d_{pq} I_2(i, l, p, q). \]  
(B.28)

Since the vectors \( k_{ij}, k_{lm} \) have either length \( \sigma_1 \) or \( \sigma_2 = 2\sigma_1, \) it follows that, if \( p = 1, \) then \( \{i = 1, j = n; l = 2, m = -n\} \) or \( \{i = 2, j = -n; l = 1, m = n\} \) whereas, if \( p = 2, \) then \( i = l = 1 \) and \( j = m = -n. \) Thus, if \( p = 1, \) then
\[ (MKF(\psi), \overline{\psi}^{1n}) = e^u_{11} \beta_{1n} \beta_{2(-n)} + e^u_{12} \beta_{2(-n)} \beta_{1n} \equiv b_{1n} \beta_{2(-n)} \]  
(B.29)

whereas, if \( p = 2, \) then
\[ (MKF(\psi), \overline{\psi}^{2n}) = e^u_{21} \beta_{1(-n)} \beta_{2n} \equiv b_{2n} \beta_{1(-n)}. \]  
(B.30)
This establishes (B.6a) and (B.6b). Similar calculations using (B.22) and (B.23), together with (B.14) rather than (B.15), establish (B.7) and (B.8). Since (B.9) follows directly from the definition of $I$ in (A.23), this establishes part (v) of the lemma.

To prove part (vi), we use part (iii) and the bilinearity of $\Phi$ to obtain

$$
(\Phi(\psi, KF(\psi)), \overline{\psi}^m) = - (\Phi(\psi, \psi^m), KF(\psi))
$$

$$
= - \sum_{\substack{i, j, r = 1 \atop |i| = |m| = |r| = 1}}^2 \beta_{ij} \beta_{im} \beta_{rs} (\Phi(\psi^i, \psi^m), K\Phi(\psi^l, \psi^r)).
$$

(B.31)

The inner product in the right hand side of (B.31) can be calculated by Parseval's equation as follows. One sees from (B.24) that

$$
\Phi(\psi^m, \psi^r) = \sum (\Phi(\psi^m, \psi^r), \overline{\psi}^{pqr}) \overline{\psi}^{pqr}
$$

$$
= \sum \delta(\mathbf{k}^m + \mathbf{k}^r - \mathbf{k}^q) I_2(l, r, q, k) \overline{\psi}^{pqr}.
$$

(B.32)

Thus, writing $\psi^{pqr} = \overline{\psi}^{pqr}(-i)$ and making use of (B.24) once again, one obtains

$$
(\Phi(\psi^i, \psi^m), K\Phi(\psi^l, \psi^r))
$$

$$
= \sum I_2(l, r, q, k) I_2(i, p, q, k) \delta(\mathbf{k}^i + \mathbf{k}^p - \mathbf{k}^q) \delta(\mathbf{k}^m + \mathbf{k}^r + \mathbf{k}^q),
$$

(B.33)

where $I_2 = \mu_{pq}^{-1} I_2$ and $\sigma_q = |\mathbf{k}^i + \mathbf{k}^p| = |\mathbf{k}^m + \mathbf{k}^r|$. The conditions

$$
k^i + k^p - k^q = 0 \quad \text{and} \quad k^m + k^r + k^q = 0
$$

(B.34)

imply that either $k^i = -k^m$ and $k^p = -k^r$ or $k^i = -k^r$ and $k^p = -k^m$. It follows that

$$
\delta(\mathbf{k}^i + \mathbf{k}^p - \mathbf{k}^q) \delta(\mathbf{k}^m + \mathbf{k}^r + \mathbf{k}^q)
$$

$$
= \delta(i - l) \delta(p - r) \delta(j + m) \delta(n + s) + \delta(i - r) \delta(p - l) \delta(j + s) \delta(n + m)
$$

$$
- \delta(j + m) \delta(j + s) \delta(j - n) \delta(i - l) \delta(i - r) \delta(i - p).
$$

(B.35)

Combining (B.31), (B.33), and (B.35), we obtain, for $p = 1, 2$ and $|n| = 1$,

$$
(\Phi(\psi, KF(\psi)), \overline{\psi}^m) = - \sum_{\substack{i, j = 1 \atop |i| = 1}}^2 \left[ a_l^m (1 - \delta(i_p \delta(j_m)) + b_l^m \right] \beta_{i} \beta_{j} \beta_{p(-l)} \beta_{p(-m)}
$$

(B.36)
where the constants $a_{ip}^m$ and $b_{ip}^m$ are given by

$$a_{ip}^m = \sum_{|k| \geq 2} \mu_{qk}(\mu_{qk} - \mu_0)^{-1} [I_2(i, p, q, k)]^2,$$  \hspace{1cm} (B.37)

$$b_{ip}^m = \sum_{|k| \geq 2} \mu_{qk}(\mu_{qk} - \mu_0)^{-1} I_2(p, i, q, k) I_2(i, p, q, k).$$  \hspace{1cm} (B.38)

The constants in (B.37) and (B.38) depend on $j$ and $n$ only through $q$, i.e., only through $\sigma_q = |k_{ij} + k_{pm}|$, and are nonnegative when $p = i$. Moreover, since $|k_{ij} + k_{pm}| = |k_{ij} + k_{p(-n)}|$, it follows that $a_{ip}^{(-n)} = a_{ip}^{(-i)n}$ and $b_{ip}^{(-n)} = b_{ip}^{(-i)n}$. This establishes (B.10).

Finally, to prove (B.11), one uses (B.24) and

$$F(\psi) = \sum \beta_{rs} \beta_{im} \Phi(\psi^r, \psi^m)$$  \hspace{1cm} (B.39)

and proceeds as in the proof of (B.10). The condition $p \neq i$ may be assumed in (B.11) because if $\sigma_q = |k_{ij} + k_{pm}| \neq 0$ and if $p = i$, then $j = n$ also so that, by part (iii) above, the coefficients with $p = i$ are necessarily zero, i.e.,

$$(\Phi(\psi^{qm}, \psi^q), \tilde{\psi}^{pn}) = (\Phi(\psi^{qm}, \psi^q), \tilde{\psi}^q) = 0.$$  \hspace{1cm} (B.40)

Note that one cannot assume that $p \neq i$ for the coefficients $a_{ip}^m$ and $b_{ip}^m$ in part (vi) because in that case (B.40) is replaced by $(\Phi(\psi^q, \psi^{qm}), \psi^{qm})$ which is not necessarily zero even when $\psi^q = \psi^{pn}$. This completes the proof of the lemma.

The following use of group representations is convenient in carrying out the reduction to a finite-dimensional problem (see also [12, 13, 15]). Let $\mathcal{G}$ be the group of translations of $\mathbb{R}^2$ that keep $x$ fixed. If $a = (0, a) \in \mathcal{G}$, then a unitary representation $a \rightarrow T_a$ of $\mathcal{G}$ onto $\mathcal{H}$ is defined by

$$(T_a v)(x, z) = v(x, z - a), \hspace{1cm} v \in \mathcal{H}. \hspace{1cm} (B.41)$$

One can show as in [12, Lemma 4.1] that the operators in Eq. (*) commute with $T_a$, $a \in \mathcal{G}$, and, hence, that Eq. (*) is invariant under the representation $a \rightarrow T_a$.

It is useful, in particular, to consider the action of $T_a$ on the eigenfunctions $\psi^{pq}$ in (A.31), namely

$$(T_a \psi^{pq})(x, z) = e^{i k_{pq} \cdot (x, z - a)} \psi^{pq}(x).$$  \hspace{1cm} (B.42)

Thus, since $k_{pq} = (0, \pm p \alpha_k)$, if $a_z = (0, \pi/\alpha_k)$, then $T_{a_z} \psi^{pq} = T_{a_z} \psi^{pq}$ equals $\psi^{pq}$ whenever $p$ is even and $-\psi^{pq}$ whenever $p$ is odd. If one now defines $\mathcal{H}_\pm = \{ v \in \mathcal{H} : T_{a_1} v = \pm v \}$, then $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Moreover, restricting
$I - \mu_0 L$ to either $H^+_\lambda$ or $H^-_\lambda$, one sees that the null space $H^+_\lambda = H \cap H^+_\lambda$ of $I - \mu_0 L$ on $H^+_\lambda$ is spanned by $\psi^{2l}$, $|l| = 1$, and the null space $H^-_\lambda = H \cap H^-_\lambda$ of $I - \mu_0 L$ on $H^-_\lambda$ is spanned by $\psi^{1l}$, $|l| = 1$. By making use of $H^+_\lambda$ or $H^-_\lambda$ and proceeding as in [12, Lemma 3.2], one can now show for positive $\gamma$ sufficiently small that the characteristic value $\mu_0$ of $L$ in (A.30) splits into two eigenvalues, $\lambda_k(\gamma)$ and $\lambda_{2k}(\gamma)$, of the operator $L - \gamma M$ on $H$, namely

$$\lambda_k(\gamma) = \mu_{11} - \mu_{11}^3 a_k \gamma^2 + \lambda_k(\gamma) = \mu_{0} - \mu_{0}^3 a_k \gamma^2 + \lambda_k(\gamma), \quad (B.43)$$

$$\lambda_{2k}(\gamma) = \mu_{21} - \mu_{21}^3 a_{2k} \gamma^2 + \lambda_{2k}(\gamma) = \mu_{0} - \mu_{0}^3 a_{2k} \gamma^2 + \lambda_{2k}(\gamma), \quad (B.44)$$

where $a_{pk}$ ($p = 1, 2$) is defined as in (B.5) and $A_j$ is real and analytic and of order $O(\gamma^2)$ as $\gamma \to 0^+$ ($j = k, 2k$). Thus, if we set

$$a = (a_{2k} - a_k) \mu_0^3, \quad k = 1 \text{ or } 2, \quad (B.45)$$

then, for positive $\gamma$ sufficiently small, the sign of $a$ determines which of the two eigenvalues $\lambda_k(\gamma)$, $\lambda_{2k}(\gamma)$ is the critical eigenvalue of $L - \gamma M$ (see Remark 2.2). Note that even in the case where $\mu_{11} \neq \mu_{21}$ the formula in (B.43) or (B.44) provides an extension of a well-known formula for the critical Taylor number (e.g., see [6, p. 98]) because the structure parameter $\gamma$ used in the above is a "richer" parameter than the parameter $\mu$ used in [6].

We may now proceed as in [12, Sect. 3] to reduce the problem to a finite-dimensional problem and to derive bifurcation equations of the type in (2.1). For small $\gamma > 0$, we seek a solution $(v, \lambda)$ of Eq. (*) of the form

$$v = \gamma(\psi + \gamma \Psi), \quad \lambda = \mu_0 - \mu_0 \gamma^2 (\mu_0^3 a_k - \tau). \quad (B.46)$$

Here $\psi \in M$, $\Psi \in M^\perp$, and $\tau \in \mathbb{R}^1$ are to be determined, and $a_k$ is defined as in (B.5) and (B.17) with $p = 1$. As observed in Remark 2.1, $\tau$ is a measurement of the distance from $\lambda$ to $\lambda_k$ in (B.43).

Substituting (B.46) into Eq. (*), using the projection $P$ onto $M^\perp$ and $S = I - P$ onto $M$, and making use of (ii) and (iv) in Lemma B.1, one obtains equations on $M^\perp$ and $M$. Since $K = [(I - \mu_0 L)|_{M^\perp}]^{-1}$ is bounded on $M^\perp$, given $\rho_0 > 0$ there exists $\gamma_0 > 0$ such that if $(\psi, \tau) \in M \times \mathbb{R}^1$ with $|\tau| + \|\psi\| < \rho_0$, then the equation on $M^\perp$ can be solved for $\Psi = \Psi(\psi, \tau, \gamma)$ provided that $0 \leq \gamma < \gamma_0$ (see [12, (3.16) ff.]). In fact, $\Psi$ is analytic and of the form

$$\Psi = -\mu_0 KM \psi + K F(\psi) + \gamma \Psi_1, \quad (B.47)$$

where $\Psi_1 = \Psi_1(\psi, \tau, \gamma) \in M^\perp$ is bounded with the bound depending only on $\rho_0$. Substituting $\Psi$ in (B.47) into the indicated equation on $M$, and tak-
ing the inner product with $\tilde{\psi}^{pn}$, one obtains, for $\psi$ of the form in (B.3), the set of four (complex) equations ($p = 1, 2$ and $|n| = 1$),

$$0 = -\tau \beta_{p(-n)} + \mu_0^2 a_k \beta_{p(-n)} - \mu_0^2 (MKM \psi, \tilde{\psi}^{pn}) + \mu_0 (MKF(\psi), \tilde{\psi}^{pn})$$

$$+ \mu_0 (\Phi(\psi, KM \psi), \tilde{\psi}^{pn}) + \mu_0 (\Phi(KM \psi, \psi), \tilde{\psi}^{pn})$$

$$- (G(\psi), \tilde{\psi}^{pn}) - (\Phi(\psi, KF(\psi)), \tilde{\psi}^{pn}) - (\Phi(KF(\psi), \psi), \tilde{\psi}^{pn}) + r_{pn}. \quad (B.48)$$

Here, for $|\tau| + \|\psi\| < \rho_0$ and $0 < \gamma < \gamma_0$, the remainder terms

$$r_{pn}(\beta, \tau, \gamma) = (R(\psi, \tau, \gamma), \tilde{\psi}^{pn}) \quad (B.49)$$

are analytic functions of $(\beta, \tau, \gamma)$ and, for some $r_0$ depending only on $\rho_0$, satisfy

$$|r_{pn}(\beta, \tau, \gamma)| \leq \gamma r_0. \quad (B.50)$$

(For formulas for the various quantities are given in [12, (3.17) ff.] from which the formulas for $R$ and the $r_{pn}$ easily follow.) Equations (B.48) are the full set of four (complex) bifurcation equations for $(\beta, \tau, \gamma, z) \in \mathbb{C}^4 \times \mathbb{R}^4 \times \mathbb{R}^4$.

As pointed out in [12, Sect. 3] (see also [15, 16]), instead of solving the full set of bifurcation equations in (B.48) by direct methods, it is more natural to seek solutions satisfying certain symmetry conditions, i.e., to seek solutions belonging to certain subspaces of $\mathcal{H}$. Let $\pi$ denote the transformation taking $(x, z)$ into $(x, -z)$. Let $T_\pi : \mathcal{H} \to \mathcal{H}$ be given by (see also [12, (4.3)])

$$(T_\pi v)(x, z) = (v_1, v_2, -v_3)(x, -z), \quad v \in \mathcal{H}, \quad (B.51)$$

and define

$$\mathcal{S}_\pi = \{ v \in \mathcal{H} : T_\pi v = v \}.$$

Since $T_\pi$ commutes with the operators in $(\ast)$ (see [12, Sect. 4]), Eq. $(\ast)$ may be studied on $\mathcal{S}_\pi$ by merely restricting the operators in $(\ast)$ to $\mathcal{S}_\pi$. Moreover, since

$$T_\pi \psi^{pqij} = \tilde{\psi}^{pqij}, \quad (B.52)$$

an element $v \in \mathcal{H}$ of the form (B.2) belongs to $\mathcal{S}_\pi$ if and only if $\beta_{pqij} = \beta_{pqij}(-j)$. Thus, if $\psi \in \mathcal{M}$ is of the form (B.3), then $\psi \in \mathcal{M}_\pi \equiv \mathcal{M} \cap \mathcal{S}_\pi$ if and only if $\beta_{ij} = \beta_{ij}(-j)$. It follows that $\dim \mathcal{M}_\pi = 2$ and that if $\psi \in \mathcal{M}_\pi$, then $\psi$ is of the form

$$\psi = \beta_{11}(\psi^{11} + \tilde{\psi}^{11}) + \beta_{21}(\psi^{21} + \tilde{\psi}^{21}). \quad (B.53)$$
While $\mathcal{H}$ is a complex Hilbert space we are interested only in real solutions of (*). For a setting like that of $\mathcal{H}_x$, i.e., considering only $\psi \in \mathcal{M}_x$ of the form (B.53), it can be shown as in [12, Sect. 41 (see also [15]) that the system of four (complex) bifurcation equations in (B.48) may be replaced by an equivalent system of two (real) equations in the two (real) variables

$$\beta_1 = \beta_{11} = \beta_{1(-1)} \quad \text{and} \quad \beta_2 = \beta_{21} = \beta_{2(-1)}, \quad (\beta_1, \beta_2) \in \mathbb{R}^2, \quad (B.54)$$

the real variable $\tau$, and the real variable $\gamma$. In fact, setting $p = 1, 2$ and $n = 1$ in (B.48), and making use of (B.54) and (B.5)–(B.11) in Lemma B.1, one sees that the system of bifurcation equations for the small-gap approximation to Taylor problem is of the form

$$0 = -\tau \beta_1 + \beta_1 \beta_2 + \bar{c}_{1} \beta_1 \beta_2^2 + \bar{d}_1 \beta_1^2 + r_1(\beta, \tau, \gamma), \quad (B.55a)$$

$$0 = - (\tau + a) \beta_2 + \bar{c}_2 \beta_2^2 + \bar{d}_2 \beta_2^3 + r_2(\beta, \tau, \gamma), \quad (B.55b)$$

where $(\beta, \tau, \gamma) \in \mathcal{B}_0 \times (0, \gamma_0) \equiv \{(\beta, \tau, \gamma) : |\beta| + |\tau| < \rho_0, 0 < \gamma < \gamma_0\}$. Here, the remainder terms are given by $r_1 \equiv r_{11}$ and $r_2 \equiv r_{21}$ with $r_{pn}$ defined as in (B.49) for $\psi \in \mathcal{M}_x$, and the coefficients are given by

$$\bar{c}_p = \mu_0(b_1^1 + b_2^2) - b_p^4 \quad \text{and} \quad \bar{d}_p = \beta_{11}^p \quad (p = 1, 2),$$

$$\bar{c}_1 = a_{21}^{11} + a_{21}^{11} + b_{21}^{11} + b_{21}^{11} - c_{21}^{11} - c_{11}^{11},$$

$$\bar{c}_2 = a_{12}^{11} + a_{12}^{11} + b_{12}^{11} + b_{12}^{11} - c_{12}^{11} - c_{12}^{11}.$$}

To derive (B.55), we have used also that $r_{pn} = r_{p(-n)}$ for $\psi \in \mathcal{M}_x$, the proof of which is similar to that of Lemma 4.3 in [12]. Clearly, the system of bifurcation equations in (B.55) is of the same type as that formulated in (2.1), except possibly for the properties of the remainder terms required in (2.3).

To establish the properties of the remainder terms listed in (2.3), we again make use of the representation $a \rightarrow T_a$ defined in (B.41). Proceeding as in [13, Sect. 41, one can show for the operator $R$ defined implicitly by (B.49) that $T_a$ commutes also with $R$ in the sense that

$$T_a R(\psi, \tau, \gamma) = R(T_a \psi, \tau, \gamma), \quad \psi \in \mathcal{M}. \quad (B.56)$$

Making use of (B.42) ff., one sees for $a_i = (0, \pi/\sigma_k)$ that $T_1 \psi^{11} = - \psi^{11}$ and $T_1 \psi^{21} = \psi^{21}$, so that, for $\psi$ of the form in (B.53) and (B.54),

$$T_1 \psi = - \beta_1 (\psi^{11} + \bar{\psi}^{11}) + \beta_2 (\psi^{21} + \bar{\psi}^{21}). \quad (B.57)$$

Thus, the action of $T_1$ on such $\psi$ corresponds to the operation
\((\beta_1, \beta_2) \to (-\beta_1, \beta_2)\). Since \((T_i^{-1})^* = T_1\), it follows from (B.49) and (B.56) that (suppressing the \(\tau\) and \(\gamma\) dependence of the \(r_i\))

\[
r_1(\beta_1, \beta_2) = (R(\psi), \psi^{11}) = (T_1 R(\psi), T_1 \psi^{11})
= - (R(T_1 \psi), \psi^{11}) = - r_1(-\beta_1, \beta_2),
\]

\[
r_2(\beta_1, \beta_2) = (R(\psi), \psi^{21}) = (R(T_1 \psi), \psi^{21}) = r_2(-\beta_1, \beta_2).
\]

Thus, one sees that \(r_1\) is odd in \(\beta_1\) and \(r_2\) is even in \(\beta_1\) from which the desired forms of \(p\) and \(q\) in (2.3a), (2.3b) follow. Finally, to show that \(r_2\) satisfies (2.3c) when \(\beta_1 = 0\), one uses the translation \(a_2 = (0, \pi/2\alpha_k)\). Making use of (B.42) with \(a_2 \to T_2\), one sees that \(T_2 \psi^{21} = - \psi^{21}\) so that, if \(\psi\) is of the form (B.53) and (B.54) with \(\beta_1 = 0\), then \(T_2 \psi = - \beta_2 \psi^{21}\). Proceeding as before with \(T_1\) replaced by \(T_2\), one finds that \(r_2(0, \beta_2) = - r_2(0, -\beta_2)\) which establishes property (2.3c).

Equations of the form (B.55), namely Eqs. (2.1), are solved in Section 2 under the hypotheses given there. Thus, under suitable hypotheses, we may assume that the equations (B.55) have (real) solutions \((\beta, \tau, \gamma) \in \mathcal{G}_0 \times (0, \gamma_0)\) for \(\gamma_0\) sufficiently small. The above construction then leads to real \(\psi \in \mathcal{M}_\gamma\) of the form (B.53) and, hence, real solutions \((\nu, \lambda)\) of Eq. (*) in \(\mathcal{S}_\gamma\): If, for \(\gamma_0 = \gamma_0(\rho_0)\) sufficiently small, \((\beta_0, \beta_2, \tau_0, \gamma)\) is a solution of (B.55) in \(\mathcal{G}_0 \times (0, \gamma_0)\), then

\[
\nu^* = \gamma [\beta_0^*(\psi^{11} + \psi^{11}) + \beta_2^*(\psi^{21} + \psi^{21})] + \gamma^2 \Psi^*,
\]

\[
\lambda^* = \mu_0 - \mu_0 \gamma^2 (\mu_0^2 \alpha_k - \tau^*),
\]

is a solution of Eq. (*) with \(\Psi^* = \Psi^*(\beta^*, \tau^*, \gamma)\) uniformly bounded as \(\gamma \to 0^+\).

REFERENCES


