Computing Modular and Projective Character Degrees of Soluble Groups

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To Tim Wall, my teacher, colleague and longtime friend, on his 65th birthday

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Let $G$ be a finite soluble group and $r$ a rational prime or zero. Let $Z$ be a central cyclic subgroup of $G$; if $r > 0$, then the order of $Z$ is relatively prime to $r$. Let $F$ be an algebraically closed field of characteristic $r$. Let $\lambda$ be a faithful linear character of $Z$ in $F$. Such a $\lambda$ gives rise to a factor system $f$ for $H = G/Z$ and any factor system for $H$ in $F$ so arises. An algorithm for determining the degrees of those irreducible representations of $G$ which restrict to $Z$ to give the scalar representation $\lambda$ is presented. If $Z$ is the trivial subgroup, the algorithm can be used to compute the degrees of all $FG$-irreducibles (together with multiplicities); in particular, the number of conjugacy classes of $G$ and for any prime $r(>0)$, the number of $r$-regular conjugacy classes of $G$ are determined. If $Z$ is nontrivial, the same results are obtained for the twisted group algebra $F\tilde{H}$ with respect to $f$. The starting point is a power-commutator presentation for $G$; only the supposed characteristic $r$ of field $F$ is used and the calculations are performed in sections of $G$. Clifford's theorem is used as the basic reduction tool.

1. Clifford Theory

Let $G$ be a finite soluble group and let $F$ be an algebraically closed field of characteristic $r$; thus $r$ is a prime or zero. In this paper an algorithm for determining the degrees of the irreducible characters of the group algebra $FG$ is presented.

The principal theoretical tool is Clifford's theorem (see section 11A of Curtis & Reiner, 1981). This describes how an irreducible character $\gamma$ of a finite group $G$ behaves upon restriction to a normal subgroup $N$ and how the original character can be described in terms of an irreducible character $\nu$ of $N$.

More precisely, the restriction $\gamma|_N$ of $\gamma$ to $N$ is the sum of conjugates of a single irreducible character $\nu$ of $N$. For $n$ in $N$ and $g$ in $G$, the element $gng^{-1}$ will be written as $gn$. Then the conjugate $\nu \otimes g$ of $\nu$ takes the value $\nu(n)$ on the element $n$ of $N$ ($\nu \otimes g)n = \nu(n) \otimes g$). The stabilizer $S$ of $\nu$ in $G$ is the subgroup of $G$ consisting of those elements $g$ of $G$ such that $\nu \otimes g = \nu$ (i.e. $\nu(n) = \nu(n)$ for all $n$ in $N$). The character $\gamma$ can then be written as an induced character $\sigma^g$ for a certain (irreducible) character $\sigma$ of $S$. Hence, it suffices to describe $\sigma$.

The character $\nu$ of $N$ may be extended to a projective character $\tilde{\nu}$ of $S$; thus, if $N$ and $\tilde{N}$ are the representations corresponding to $\nu$ and $\tilde{\nu}$, then $\tilde{N}(n) = \tilde{N}(n)$ for all $n$ in $N$ and, if $s, s' \in S$, then $\tilde{N}(s) \tilde{N}(s') = f(s, s') \tilde{N}(ss')$, where $f(s, s')$ is a (nonzero) scalar from the coefficient field $F$. The map $f: S \times S \rightarrow F$ is known as a cofactor system on $S$ and may be used to define
a twisted group algebra $F_0 S$ on $S$ with product $s \# s' = f(s, s)ss'$. The associativity of this multiplication comes from the (2-)cocycle condition

$$f(s, s')f(ss', s') = f(s, s's')f(s', s'),$$

which is a consequence of the associativity of matrix multiplication:

$$(\tilde{N}(s)\tilde{N}(s'))\tilde{N}(s') = \tilde{N}(s)\tilde{N}(s')\tilde{N}(s').$$

Given a transversal for $N$ in $S$, $s'$ will denote the representative for the coset $Ns$. If a matrix $\tilde{N}(s')$ is associated with each coset $Ns$, and the matrix $\tilde{N}(ns') = N(n)\tilde{N}(s')$ is assigned for each $n$ in $N$, it then follows that $f(ns, ns') = f(s, s')$ for all $s, s' \in S$ and $n, n' \in N$, i.e. $f$ is essentially a factor system on $\bar{L} = S/N$.

The inverse cofactor system $f^{-1}$ to $f$ is defined by the rule $f^{-1}(s, s') = (f(s, s'))^{-1}.$

The representation $S$ of $S$ corresponding to the character $\sigma$ above corresponds to the Kronecker product representation

$$S = \tilde{N} \otimes L.$$  \hspace{1cm} (1)

Here, $L$ is a projective (irreducible) representation of $\bar{L} = S/N$; its cofactor system is the inverse $f^{-1}$ of $f$. (Thus, $S$ has the trivial cofactor system $f, f^{-1}$ and this is consistent with $S$ being an ordinary (i.e. nonprojective) representation of $S$.) Let $\lambda$ be the character of $\bar{L}$. Let $\gamma$ be the character of $F_{\bar{L}}L$, which corresponds to the representation $L$.

Thus,

$$\text{degree}(\gamma) = (G:S)\text{degree}(\sigma) = (G:S)\text{degree}(\nu)\text{degree}(\lambda).$$

For each conjugacy class representative $\nu$ of the characters of $N$, the degrees (and respective multiplicities) of the possible associated characters $\gamma$ of $G$ are determined by the degrees of the irreducible characters $\lambda$ of $F_{\bar{L}}$. The last transition from $FG$-characters to $F_{\bar{L}}$-characters provides the recursive step in reducing the size of the problem. However, it has seemingly given rise to a more general problem, that is, to find the possible irreducible character degrees for twisted group algebras. How, then, may twist in group algebra multiplication be handled?

Suppose $G$ has a central cyclic subgroup $Z$ and write $G = G/Z$. Let $\zeta$ be a faithful linear $F$-character of $Z$. If the characteristic $r$ of $F$ is nonzero, then $Z$ must be chosen to be an $r'$-group, i.e. the order of $Z$ is coprime to $r$. Then $\zeta$ gives rise to a factor system for $FG$ in $F$. For if a transversal for $Z$ in $G$ is chosen and if $x^s$ denotes the coset representative of $Zx$, then

$$x_1^sx_2^s = x(z_1x_2^s)$$

for a uniquely defined $z$ in $Z$. The formula

$$Zx_1 \neq Zx_2 = \zeta(z)Zx_1x_2$$

defines a twisted group algebra $F_0 \bar{G}$ on $\bar{G}$, where

$$f(Zx_1, Zx_2) = \zeta(z) = z(x_1x_2^s)^{-1}(x_1x_2^s)^{-1}$$

is the corresponding factor system $f$ on $\bar{G}$.

Conversely, given a twisted group algebra $F_0 \bar{G}$, it is possible to construct a finite group $G$ with a central cyclic subgroup $Z$ and with a faithful linear $F$-character $\zeta$, which realises $F_0 \bar{G}$ as in the previous paragraph. To obtain the representation theory of $F_0 \bar{G}$, it is sufficient to restrict attention to those $FG$-representations $G$ which, when restricted to $Z$, give the scalar representation $\zeta$, i.e.

$$G(z) = \zeta(z)I \quad (z \in Z),$$
where $I$ is the unit matrix. This ensures that only $F \tilde{G}$-representations are considered.

So far, Clifford's theory has been used in its usual sense, i.e. that of describing an irreducible representation $\tilde{G}$ of $G$ in terms of an irreducible representation $N$ of a normal subgroup $N$ of $G$. But every irreducible representation $\tilde{N}$ of $N$ arises in this way; for take the composition factors of the induced representation $\tilde{N}^G$, and restrict them back to $N$.

It turns out that when calculating the character degrees of finite soluble groups, all the twisted group algebras associated with sections of $G$ are realisable from larger sections with suitable central cyclic subgroups and so it is not necessary to consider groups other than sections of $G$, nor to calculate cofactor systems at any stage. Also, it is not necessary to find $\tilde{\nu}$.

A subgroup $L$ of $G$ is called a complement for $N$ in $G$ relative to $Z$ if $G = NL$ and $N \cap L = Z$.

The above gives rise to the following algorithm in skeletal form:

Given: A finite soluble group $G$ and a central cyclic subgroup $Z$.

Aim: To find the degrees of the irreducible characters of $G$ which lie over a faithful linear character of $Z$.

If $G$ is abelian then
Return the character degree 1 with multiplicity $|G/Z|$.
Stop.

Else (G nonabelian)

Find a subgroup $N$ of $G$ with $N/Z$ a chief factor.

If $N$ is abelian then

Find the $G$-orbits of extensions of the faithful character of $Z$ to $N$.
For each such $G$-orbit do
Form the $G$-stabilizer $S$ of one member of the orbit.
Set $K$ equal to the kernel of this chosen member.
Apply the algorithm recursively to $S/K$ with central cyclic subgroup $N/K$.
Multiply each character degree by $(G:S)$.
Output the union of all the character degree sets coming from the various orbits.
Stop.

Else (N nonabelian)

Construct a complement $L$ for $N$ in $G$ relative to $Z$ such that $L \geq C_G(N)$.
Apply the algorithm recursively to $L$ with central cyclic subgroup $Z$.
Set $e$ equal to the square-root of the order of $N/Z$.
Multiply each character degree of $L$ obtained by $e$.
Output the set of these modified character degrees.
Stop.

In the alternate cases, the groups $S/K$ and $L$ provide realisations of the twisted group algebras which arise in the Clifford analysis. It will be seen in section 8 that $L$ is unique up to conjugacy in $G$.

A realisation of this algorithm has been implemented in the CAYLEY language (Cannon, 1984). The program is written recursively with input parameters the group $G$, the central cyclic subgroup $Z$ of $G$ and the characteristic $r$ of the field $F$, over which the irreducible character degrees are to be found. The output is the sequence of characters.
degrees (together with their multiplicities) over the field $F$ of characteristic $r$ which, upon restriction to $Z$, give a faithful scalar representation $\zeta$.

As the recursion proceeds, the index $(G:Z)$ decreases and the recursion can be terminated when $G = Z$. Alternatively, it can be terminated when $G$ is abelian; for then each irreducible character of $G$ is 1-dimensional and the number of different irreducibles which extend a given faithful character of $Z$ is $|G/Z|/|G/Z|_r$ when $r > 0.$ (Here, if $h$ is a natural number, then $h_\nu$ denotes the greatest divisor of $h$, which is coprime to $r$.)

Suppose $Z$ is taken to be the trivial subgroup of $G$. If $r$ is 0 or if $r$ does not divide $|G|$, then the nonmodular irreducible character degrees are returned. If $r$ is nonzero and $r$ divides $|G|$, the $r$-modular irreducible character degrees are produced.

If it is desired to handle the projective irreducible representations of a twisted group algebra $F_\ell G$, it is necessary to realise $\tilde{G}$ as $G/Z$ for a suitable central cyclic subgroup $Z$ of $G$. Again, depending upon the value of $r$, either the nonmodular or modular irreducible character degrees of $F_\ell G$ are returned.

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2. Selection of a Suitable Normal Subgroup $N$

Let $G$ be a finite soluble group, let $Z$ be a central cyclic subgroup ($Z$ may be the trivial subgroup) and let $r$ be the characteristic (a prime or zero); if $r$ is nonzero then $Z$ is an $r'$-group. The symbol $\zeta$ will denote a hypothetical faithful linear $F$-character of $Z$.

Since $G$ is soluble, it may be defined using a power-commutator presentation, see Laue, Neubüser & Schoenwaelder (1984). This involves a chain of subgroups of $G$:

$$G = G_0 > G_1 > \ldots > G_n = \langle id \rangle,$$

such that each $G_i$ is normal and of prime index in its predecessor, and a set of generators $g_1, \ldots, g_n$ such that

$$G_{i-1} = \langle g_i, G_i \rangle$$

for each $i$. It will be assumed that $G$ is given by such a pc-presentation. Such a presentation permits the efficient calculation of quotients (difficult in the case of a permutation or matrix representation). Further pc-presentations may be quickly found for any subgroup or quotient group. This is automatically done in the group theory system CAYLEY, see Cannon (1982). All programs discussed here have been implemented in Version 3.6 of the CAYLEY language.

In order to apply Clifford's theorem, a normal subgroup $N$ of $G$ is chosen such that $N$ properly contains $Z$. As $G$ is soluble, it is possible to find an $N$ such that $\tilde{N} = N/Z$ is a minimal elementary abelian normal subgroup of $G/Z = \tilde{G}$. First, a procedure NELSG is used to produce a normal elementary subgroup $N$ of $G$. Then the Cambridge MEAT-AXE (Parker, 1984) is used by a procedure PCMEANS to refine $N$ to a minimal elementary abelian normal subgroup of $\tilde{G}$. The required subgroup $N$ is the preimage of $\tilde{N}$ in $G$. Then $\tilde{N}$ is an elementary abelian $p$-group for some prime $p$.

For each conjugacy class representative $v$ of irreducible characters of $N$, which restrict to the scalar representation $\zeta$ on $Z$, the stabilizer $S$ is calculated. The twisted group algebra $F_{\ell-1} \tilde{L}$ on the quotient group $\tilde{L} = S/N$ must be "realized" on a suitable section of $G$. Here the treatment varies according to the structure of $N$.

If $r$ is nonzero and $\tilde{N}$ is an $r$-group ($p = r$), then the Sylow $r$-subgroup $N$ of $N$ is normal in $G$ and $G$ is replaced by $G/N$; this is discussed in section 3. The case, where $N/Z$ is an elementary $p$-group with $p \neq r$ and $N$ is abelian, is discussed in section 4.
nonabelian $N$ is analysed in section 5. A complement $L$ to $N$ in $G$, relative to $Z$, with $L \geq C_G(N)$ is discussed in section 6 and this subgroup $L$ of $G$, which contains $Z$, turns out to give the necessary realization of the new twisted group algebra in the recursive step. The construction of $L$ for $p$ odd is described in section 7, while the case $p = 2$ is discussed in section 8.

The following notation will be adopted for the remaining sections. Suppose $Z$ is a central cyclic subgroup of a finite group $G$. Let $N$ be a normal subgroup containing $Z$. Suppose that $\overline{N} = N/Z$ is a minimal elementary abelian normal $p$-subgroup of $\overline{G} = G/Z$. Write $P$ and $\overline{Z}$ for the Sylow $p$-subgroups of $N$ and $Z$ respectively. Thus $Z = \overline{Z} \times U$, where $U$ is the product of the Sylow $p'$-subgroups of $Z$. As $\overline{N}$ is a $p$-group, any $p'$-subgroup of $N$ lies in $Z$ and so in $U$. Thus, $N = P \times U$ and so $P$ is normal in $G$ and $U$ is a central cyclic $p'$-subgroup of $G$. The derived subgroup $N'$ of $N$ lies in $P \cap Z = \overline{Z}$. If $x, y \in N$, then $x^p \in Z$ and so $1 = (x^p, y) = (x, y)^p$; thus, $N' \leq Z_p$, where $Z_p$ is the subgroup of order $p$ in $Z$ if $p$ divides the order of $Z$, or trivial otherwise.

The symbol $\zeta$ will denote a hypothetical faithful linear character of $Z$, while $N$ will denote an irreducible representation of $N$ which restricts to $Z$ to give the scalar representation $\zeta I$. Finally, $\nu$ is a character of the representation $N$ of $G$. $N$ can be written as the outer tensor product $N_p \otimes \zeta$, where $N_p$ is the restriction to $P$. The corresponding character formula is $\nu(nu) = \nu(n)\zeta(u)$, for $n \in P$ and $u \in U$. It is necessary to consider all extensions $\nu_p$ of the scalar character $\zeta$ to the normal subgroup $P$ of $G$, i.e. all irreducible representations $N_p$ which are extensions of the scalar representation $\zeta \otimes I$ of $\overline{Z}$.

3. $N/Z$ an $r$-Group

In this case the characteristic $r$ of $F$ is nonzero and the elementary abelian section $N/Z$ is an $r$-group. Then $N$ is abelian and has Sylow $r$-subgroup $rN \approx N/Z$. Then $N$ is normal in $G$ and the ideal of $FG$ generated by the augmentation ideal $I(N)$ of the group algebra $F(N)$ is a nilpotent ideal of $FG$ and so lies in the Jacobson radical $J(FG)$ of $FG$. Hence, as far as irreducible $FG$-modules are concerned, it suffices to consider the quotient group $G/N$. (This is equivalent to taking $\nu$ to be the particular irreducible representation of $N$, which is the product of $\zeta$ with the trivial representation of $r$.) As $Z \cap N$ is trivial, $Z$ is isomorphic to its image $rNZ/rN$ and the hypothetical faithful character $\zeta$ of $Z$ passes to this quotient.

4. $N$ Abelian

Here, $\overline{N} = N/Z$ is an elementary abelian $p$-group with $p$ a prime not equal to the characteristic $r$ of $F$, $N$ is abelian and $\nu$ is an irreducible character (linear representation) of $N$ which extends the character $\zeta$ of $Z$. Let $S$ be the stabilizer of $\nu$ in $G$. Let $K$ be the kernel of $\nu$ so that $K$ is a normal subgroup of $S$ and $N/K$ is cyclic. Furthermore, $K$ avoids $Z$, as the restriction $\zeta$ of $\nu$ to $Z$ is faithful. The characters $\nu$ and $\zeta$ are naturally defined on the images $N/K$ and $ZK/K$ of $N$ and $Z$, respectively, in $S/K$. Again, as $S$ stabilizes $\nu$ and as $\nu$ represents $N/K$ faithfully, so $N/K$ is central and cyclic in $S/K$. As far as those representations of $S$, which restrict to give copies of $\nu$, are concerned, one can pass to the quotient $S/K$.

For some fixed transversal for $N$ in $S$ let $s^t$ be the coset representative for the coset $Ns$. In particular, take $1^t = 1$. Extend $\nu$ to a projective linear representation $\hat{\nu}$ of $S$ by defining $\hat{\nu}(s^t) = 1$ and $\hat{\nu}(ns^t) = \nu(n)$, for $n \in N$. Suppose $s^t_1 s^t_2 = n(s_1 s_2)^t$ with $n \in N$. Then $\hat{\nu}(s_1^t s_2^t) = \nu(n)$.
On the other hand, the factor system $f$ is given by
\[
\tilde{v}(s_1)\tilde{v}(s_2) = f(s_1, s_2)\tilde{v}(s_1^2 s_2),
\]
i.e.
\[
1 = f(s_1, s_2)v(n), \quad \text{or} \quad f^{-1}(s_1, s_2) = v(n).
\]
The twisted group algebra $\mathcal{G} = \mathcal{G}/\mathcal{G}$ on $S = S/N$ has the factor system $f^{-1}$ and so the quotient group $S/K$ gives a "realization" of this twisted group algebra. For $K$ is the kernel of $v$ and $v$ induces a faithful representation of the central cyclic subgroup $ZK/K$, which induces this twist, as in section 1.

In this situation the inductive step of passing from the representation $S$ of $S$ in equation (1) is achieved by taking $L$ to be an ordinary (i.e. nonprojective) irreducible representation of $S/K$, which restricts faithfully to the central cyclic subgroup $N/K$ (hence yielding the hypothetical linear character $v$).

(a) Suppose first that $Z$ is trivial, i.e. $Z = U$ is a $p'$-cyclic group. Then $P$ is elementary abelian. The kernel $K$ of any nontrivial $v$ is a maximal subgroup of $P$ so that it is necessary to consider each maximal subgroup $M$ of $P$. If $P$ is regarded as a vector space over the field $F_p$ of $p$ elements, then $M$ is a maximal subspace. The set of maximal subspaces $M$ of $P$ form the points of the dual vector space $P^*$. Hence all points of $P^*$ must be considered.

However, a $v$ must be chosen from each conjugacy class of characters of $N$ under the action of $G$. In particular, $G$ must permute the maximal subspaces $M$ of $P$ among themselves. The conjugation action of an element $g \in G$ on the elements of $P$ induces an $F_p$-linear mapping, i.e. $(n_1 n_2) = (q_1 q_2)^g$ for $n_1, n_2 \in P$ and where $q_1, q_2$ are taken as integers modulo $p$ lying in $F_p$. So $G$ has a $F_p$-linear representation on the vector space $P$. If $\{p_1, \ldots, p_l\}$ is a basis of $P$ (set of group generators for $P$) and if $p! = \Pi_i p_i^{\alpha_i}$, where the integers $q_{ij}$ (modulo $p$) are taken elements of $F_p$, set $T(g) = (q_{ij})$, an $1 \times l$ matrix with coefficients in $F_p$. Then $T: G \to M(F_p)$, $g \to T(g)$ defines the representation of $G$ on $P$, giving a right $F_pG$-module $P$.

What then is the $G$-action on the dual space $P^*$? By taking transposes a left $F_pG$-module $P^*$, given by $g \to (T(g))^*$, is obtained. In order to retain right modules, it is usual to pass to the contragredient representation $T^*: g \to T^*(g) = (T(g^{-1}))^*$. Then $T^*$ gives a right $F_pG$-module action on $P^*$.

The matrix group $T^*(G)$ is constructed as a subgroup of $Gl_p(P^*)$ (the group of all nonsingular matrices on the vector space $P^*$). The CAYLEY function ORBITS returns a sequence of nonzero vectors in $P^*$, one vector $v$ from each 1-dimensional subspace from each orbit of such subspaces under the action of $T^*(G)$.

The CAYLEY function MATRIX STABILIZER may then be used to find the matrix group stabilizer $S$ of the 1-dimensional subspace $\langle v \rangle$ in $T^*(G)$.

The next step is to find the preimage in $G$ of $S$, by applying $T^*$. The subgroup $S$ is given by a set of matrix generators. Here, the fact that $G$ is given by a pc-presentation must be used. As this pullback operation is performed for each orbit, it is first necessary to calculate $Ker(T^*)$. This is done while calculating the images $T^*(g_i)$ of the generators $g_i$ of $G$ in the following way.

The image of the lowest or last generator of $G$ is found first. If $T^*(g_i)$ is already known, then the image $T^*(g_i)$ is examined to see if it lies in $T^*(G_i)$. If so, then an element of $Ker(T^*)$ is obtained by first replacing $g_i$ by $g_i * g_{i+1}$, so that $T^*(g_i * g_{i+1})$ lies in $T^*(G_{i+1})$, and then modifying to $g_{i+1} * g_{i+2} * g_{i+3}$, to obtain an image in $T^*(G_{i+2})$, etc., so as to finally get an element with image in the trivial subgroup of $T^*(G)$, i.e. an element of $Ker(T^*)$.

Similarly, for each of the matrix generators $S_j$ of $S$ an integer $i$ is found so that $S_j$ lies in
An integer $a$ is found so that $S_j \ast (T^*(g_{i+1}))^a$ lies in $T^*(G_{i+1})$. Then $b$ is found so that $S_j \ast (T^*(g_{i+1}))^a \ast (T^*(g_{i+2}))^b$ lies in $T^*(G_{i+2})$, etc. Then $g_{i+1} \ast g_{i+2} \ast \cdots$ maps under $T^*$ to the group inverse of $S_j$. By performing this for each matrix generator of $S$, a set of elements is constructed which, together with $\text{Ker}(T^*)$, generates the preimage $S_1$ of $S$ under $T^*$.

Next, the subgroup $K$, which is the maximal subgroup of $P$ corresponding to the vector $v$ of $P^*$, is found (as the set of those vectors in $P$ having zero inner product with the vector $v$). The quotient $V = S_1/K$, with quotient map $Q$, is formed. For an element $x \in P \setminus K$ the image $W = Q(x)$ is constructed. If $W$ is not central in $V$, then $V$ is replaced by the centralizer $C_V(W)$ of $W$ in $V$. Consider $Q(U)$, the image of $U$ as lying in $C_V(W)$. Then $X = \langle Q(U), W \rangle$ is cyclic and central in $C_V(W)$. The recursion now proceeds to the group $C_V(W)$, with its central cyclic subgroup $X$.

After processing each nonzero orbit representative $v$ in this way, the trivial character $v$ of $P$ must be considered as a separate case, by passing in the recursive step to the quotient by the normal subgroup $P$ itself, with the central cyclic subgroup being the corresponding image of $U$.

(b) Suppose that $Z$ is nontrivial, but that the exponent of $pZ$ equals that of $P$. Then $P$ may be written as a direct product $P = pZ \times V$, with $V$ of exponent $p$. Write $Z_p$ for the subgroup of $Z$ of order $p$. Then $Z_p \times V$ is $\Omega_1(P)$ (the subgroup of $P$ generated by all elements of order $p$) and so is characteristic in $P$ and normal in $G$.

Again, the contragredient representation of $G$ on the dual $(Z_p \times V)^*$ is formed to obtain orbits and stabilizers of the maximal subgroups of $Z_p \times V$. (These maximals will correspond to the kernels of the various $v$s to be considered.) However, as $Z_p$ is central in $G$, all the maximals in any one such orbit either contain $Z_p$ or none do. Only those maximals which avoid $Z_p$ are considered, as $v$ must remain faithful on $Z_p$. There is no need to choose an $x \in P \setminus K$, as in (a), since $Z_p$ fulfills the role that $W$ played before. Stability in $S_1$ is assured as $Z_p$ is central. The recursive step involves passing directly to $S_1/K$, with the central cyclic subgroup being the image of $Z$.

(c) Finally, if $pZ$ is nontrivial and the exponent of $P$ is greater than that of $pZ$, then $P$ must be cyclic. For if not, $\langle Z, \Omega_1(P) \rangle$ is a proper normal subgroup of $N$ which contains $Z$, contrary to the choice of $N$ being minimal normal in $G$. Thus, $Z$ is of index $p$ in the cyclic group $N$.

In this case the faithful character $\zeta$ of $Z$ must be extended to the character $\zeta'$ of $N$, and the algorithm passes recursively to the centralizer $S$ of $N$ in $G$, which is the stabilizer of $\zeta'$ in $G$. Care must be taken to take one extension $\zeta'$ of $\zeta$ from each orbit of such $\zeta'$s under the action of $G$.

For $|N/Z| = p$ or $|\Omega_1(N)/Z_p| = p$ the situation is simpler and the generation of the contragredient representation may be avoided; these cases are singled out and dealt with directly in separate procedures. This saves time, as the small cases are frequently encountered in the recursive process.

5. The Structure of Nonabelian $N$}

The notation at the end of section 2 is continued with the additional assumption that $N$ is nonabelian. Then $Z(N) = Z$, for clearly $N > Z(N) \geq Z$; if $Z(N) > Z$, then $Z(N)/Z$ would be a nontrivial normal subgroup of $G/Z$ properly contained in $N/Z$, contrary to the minimality of $N$. Also $(N, G)/Z = N$; for otherwise $(N, G) \leq Z$, by the minimality of $N$; then $N$ is central in $G$ and so $N \approx C_p$, making $N$ abelian.
The groups $P$ and $pZ$ are the Sylow $p$-subgroups of $N$ and $Z$ respectively. Write $\tilde{P} = P/pZ$. The commutation operator on $P$ gives rise to a map $f: \tilde{P} \times \tilde{P} \to \mathbb{Z}_p$. As $N$ (or $P$) is not abelian, $\mathbb{Z}_p$ is nontrivial. If the cyclic group $\mathbb{Z}_p$ of order $p$ is represented as the set of integers modulo $p$, and so identified with $\mathbb{F}_p$, and if the elementary abelian group $\tilde{P}$ is interpreted as a vector space over $\mathbb{F}_p$, then $(,)$ gives a bilinear form $f: \tilde{P} \times \tilde{P} \to \mathbb{F}_p$.

Since $pZ$ is the centre of $\tilde{P}$, $f$ must be a nondegenerate form (i.e. $f(xZ, P) = 0$ implies that $xZ = 1$ is the zero element of the vector space $\tilde{P}$). The identity $f(xZ, xZ) = 0$, implies that $f$ is a symplectic form on $\tilde{P}$. Furthermore, as $(x, x') = (x, x')p = (x^p, x^{p'})$, this form respects conjugation action by elements $g$ of $G$. Hence, $G$ acts symplectically on $\tilde{P}$ and there is a group homomorphism $\psi: G \to Sp(\tilde{P})$, where $Sp(\tilde{P})$ is the symplectic group on the vector space $\tilde{P}$ over $\mathbb{F}_p$ with respect to the form $f$. Note that $ker\psi \geq N$, as $\tilde{N}$ is abelian.

In these circumstances $\tilde{P}$ is a direct sum of hyperbolic planes. This means that there is a basis $f_1, \ldots, f_l, g_1, \ldots, g_n$ of $\tilde{P}$ so that $f(f_1, g_1) = \delta_{ij}1_{\mathbb{F}_p}$. Since this decomposition is needed explicitly in the sequel, a construction of such a basis will be presented here, although the construction is well known.

Suppose first that some basis $\tilde{e}_1, \ldots, \tilde{e}_l$ of $\tilde{P}$ is given. Set $f_1 = \tilde{e}_1$. Run through the remaining $\tilde{e}_i$s until an $\tilde{e}_i$ is found such that $f(f_1, \tilde{e}_i) \neq 0_{\mathbb{F}_p}$; such an $\tilde{e}_i$ necessarily exists, for otherwise $f_1$ is orthogonal to all the $\tilde{e}_i$s and so its preimage $f_1$ in $P$ is central, contrary to $f$ being nondegenerate. Take $\tilde{g}_1$ equal to a power of $\tilde{e}_i$ (i.e. an $\mathbb{F}_p$-multiple of $\tilde{e}_i$) so that $f(f_1, \tilde{g}_1) = 1_{\mathbb{F}_p}$. Next, multiply each remaining $\tilde{e}_i$ by a power of $\tilde{g}_1$ so that $f(f_1, \tilde{e}_i) = 0_{\mathbb{F}_p}$ and then the result by a suitable power of $f_1$ so that $f(f_1, \tilde{g}_1) = 0_{\mathbb{F}_p}$. Thus, the remaining $\tilde{e}_i$s span a subspace $\tilde{P}_1$ of $\tilde{P}$ orthogonal to $\langle f_1, \tilde{g}_1 \rangle$. Furthermore, $f$ is nondegenerate on $\tilde{P}_1$, for if some element $\tilde{x}$ of $\tilde{P}_1$ is orthogonal to each element of $\tilde{P}_1$, as it is also orthogonal to $\langle f_1, \tilde{g}_1 \rangle$, it would contradict the nondegenerateness of $f$ on $\tilde{P}$. Hence, the same construction can be applied to $\tilde{P}_1$ to yield

$$\tilde{P} = \langle f_1, \tilde{g}_1 \rangle \oplus \langle f_2, \tilde{g}_2 \rangle \oplus \tilde{P}_2,$$

etc. This process gives $\tilde{P} = \langle f_1, \tilde{g}_1 \rangle \oplus \ldots \oplus \langle f_n, \tilde{g}_n \rangle$, and $l = 2h$; a single dimension left over would be orthogonal to all of $\tilde{P}$, contrary to $f$ being nondegenerate. Each $\langle f_i, \tilde{g}_i \rangle$ is termed a hyperbolic plane. Choose preimages $f_i, g_i$ in $P$ for $f_i, \tilde{g}_i$, for all $i$. The corresponding subgroups $\langle f_i, g_i \rangle$ will now be examined.

Consider the map $\phi: P \to pZ$, $x \mapsto x^p$. Since $P' = pZ$ is central and of order $p$, so $(xy)^p = x^p y^p(x, y)^p$.

For odd $p$, divides $(\frac{p}{2})$ and so $(xy)^p = x^py^p$ or $\phi$ is a group homomorphism. Clearly, $im\phi$ contains $pZ^p$, the subgroup of $pZ$ generated by elements of the form $z^p$ for $z \in pZ$, and $pZ^p$ has index $p$ in $pZ$. But $im\phi$ cannot equal $pZ$; for otherwise the inverse image $\phi^{-1}(pZ^p)$ contains $pZ$ and has index $p$ in $P$; as every subgroup is characteristic in $P$, this gives a normal subgroup of $G$ and so contradicts the minimality of $N$ in $G$. Hence $im\phi = pZ^p$, and the exponent of $P$ equals the order of $pZ$. The set of representatives for the cosets of $pZ$ in $P$ can now be modified so that each representative has order $p$. Then the subgroup $\langle f_i, g_i \rangle$ becomes a nonabelian $p$-group of order $p^3$, exponent $p$ and with centre $pZ$. The subgroup $E$ generated by such subgroups is equal to $\Omega_1(P)$ and is an extraspecial $p$-group of exponent $p$, being the central amalgamated product of the $\langle f_i, g_i \rangle$. Since $E = \Omega_1(P)$ is characteristic in $P$, $E$ is normal in $G$.

When $p = 2$, $(xy)^2 = x^2 y^2(x, y)$. Then $(xy)^4 = x^4 y^4$ and so $\phi^2: x \mapsto x^4$ is also a group homomorphism, $\phi^2: P \to 2Z$. If $2Z$ has order 2, then $P$ is an extraspecial 2-group $E$ and is normal in $G$. If $2Z$ has order greater than 2, then clearly $im\phi^2$ contains $2Z^2$ and lies in $2Z^2$. If $im\phi^2 = 2Z^2$, the inverse image $\phi^{-1}(2Z^2)$ contains $2Z$ and has index 2 in $P$; again, this
inversive image would be characteristic in $P$ and so normal in $G$, giving a contradiction to the choice of $\bar{N}$ in $G$. Thus $im\varphi^2 = \mathbb{Z}^2$. Hence $im\varphi = \mathbb{Z}^2$, and the exponent of $P$ equals the order of $\mathbb{Z}_p$. Any coset representative for $\mathbb{Z}_p$ in $P$ can be adjusted so it has order 2. Assuming that the preimages $f_t, g_t$ are chosen in this manner, then each $\langle f_t, g_t \rangle$ is dihedral of order 8. These last generate an extraspecial subgroup $E$. (However, the subgroup of order 4 in $\mathbb{Z}$ lies in $\Omega_1(P)$ and so $E \neq \Omega_1(P)$. The group $P$ is the central amalgamated product $\mathbb{Z}_p E$ of $\mathbb{Z}_p$ and $E$. The Lemma in section 8 shows that $E$ can be modified so that it becomes a normal extraspecial subgroup $D$ of $G$.

Hence, $N$ is the central amalgamated product $E \times \mathbb{Z}$ of the extraspecial group $E$ with $\mathbb{Z}$. An irreducible character of $E$ which restricts faithfully to its centre $\mathbb{Z}_p$ arises from a faithful representation of $E$ and its isomorphism class is determined by its restriction to its centre $\mathbb{Z}_p$; this is the scalar representation $\zeta_{\mathbb{Z}_p}$. An irreducible representation $\tilde{N}$ of $N$ which restricts to this representation of $E$ is an extension of it by scalar matrices, these last representing the remaining elements of $\mathbb{Z}$. Thus, the isomorphism class of the corresponding character $\psi$ of $N$ is determined by its restriction to the central subgroup $\mathbb{Z}$. In particular, $\psi$ is stable in $G$. When $|N/\mathbb{Z}| = |E|/p = p^2$, the degree of the faithful representation $\tilde{N}$ of $N$ is $p^4$.

6. The Use of a Complement $L$ to $N$ in $G$ Relative to $\mathbb{Z}$

The notation at the end of section 2 will be continued. Here $N$ is assumed nonabelian; thus the notation introduced in section 5 is also relevant.

To perform the recursive step, it is first necessary to find a subgroup $L \leq G$ such that $NL = G$, $N \cap L = \mathbb{Z}$ (i.e. $L$ is the complement of $N$ in $G$ relative to $\mathbb{Z}$) and $L \geq C_\varphi(N)$, where $C_\varphi(N)$ is the centralizer of $N$ in $G$.

The representation $\tilde{N}$ of $N$, with its character $\psi$, is faithful, giving a realization of $N$ in $M_{p^4}(F)$. For each $l \in L$, as $\psi \otimes l = \psi$, there exists a matrix $\tilde{N}(l)$ such that

$$\tilde{N}(l)N(n)(\tilde{N}(l))^{-1} = N(n)$$

for all $n \in N$, and $\tilde{N}(l)$ is unique up to a scalar multiple from the field $F$. As before, a set $T$ of coset representatives of $N$ in $G$ is chosen. Because $L$ is a complement to $N$ in $G$ (modulo $\mathbb{Z}$), these can be chosen in $L$; this makes them unique up to a central multiple from $\mathbb{Z}$. Write $l'$ for the element of $T$ which lies in the coset $Nl$ for $l \in L$. If a particular matrix $\tilde{N}(l')$ is chosen for each coset $Nl$, then this can be extended to the entire coset $Nl$ by defining $\tilde{N}(nl') = \tilde{N}(n)\tilde{N}(l')$ for $n \in N$ and as before this gives a factor system $f$ such that

$$f(nl', n'l') = f(l', l')$$

for all $n, n' \in N$ and $l, l' \in L$.

The map $\theta : L/\mathbb{Z} \to M_{p^4}(F)$, $Zl' \mapsto \tilde{N}(l')$ gives a projective representation of $L/\mathbb{Z}$.

The second ingredient, required to show that the subgroup $L$ (which contains $\mathbb{Z}$) gives a realization of the required twisted group algebra on $S = L/\mathbb{Z}$, is that $\theta$ can be adjusted to an ordinary representation of $L/\mathbb{Z}$, i.e. $\theta(Zl') = \theta(Zl)\theta(Zl')$, without scalar multiples. The adjustment requires the condition $L \geq C_\varphi(N)$.

To see that this does indeed provide the recursive step, suppose that $l[l'] = z(l_1 l_2)$. (It is easy to construct examples where $L$ is not a split extension of $\mathbb{Z}$ and so no choice of coset representatives $l'$ of $Z$ in $L$ can avoid nontrivial central factors $z$.) Then

$$\tilde{N}(l_1 l_2) = N(z)\tilde{N}(l_1 l_2) = \xi(z)\theta(Zl_1 l_2).$$

On the other hand,

$$\tilde{N}(l_1)\tilde{N}(l_2) = f(l_1, l_2)\tilde{N}(l_1, l_2),$$

i.e.

$$\theta(Zl_1)\theta(Zl_2) = f(l_1, l_2)\xi(z)\theta(Zl_1 l_2).$$
and so
\[ 1 = f(l_1, l_2)\zeta(z), \]
or
\[ f^{-1}(l_1, l_2) = \zeta(z). \]

The twisted group algebra \( F \cdot S \) on \( S = G/N \approx L/Z \) has the factor system \( f^{-1} \) and so the subgroup \( L \) gives a "realization" of this twisted group algebra, as \( \zeta \) gives the correct faithful linear representation of the central cyclic subgroup \( Z \), as in section 1. This underlines the fact that the complement \( L \) to \( N \) in \( G \) must contain \( Z \).

The existence and construction of the complement \( L \) of \( N \) in \( G \) relative to \( Z \) satisfying \( L \geq C_N(Z) \) and the existence of the representation \( \theta \) of \( L/Z \) will be discussed in the following two sections, section 7 dealing with the \( p \) odd case and section 8 the \( p = 2 \) case.

Two constructions for \( L \) are given. The first in section 7 is valid only for the \( p \) odd case. The second is valid for all \( p \) but is only used in the \( p = 2 \) case and is discussed in section 8; for this reason it is written down in terms of general \( p \) and not \( p = 2 \); it requires other assumptions on \( G \), but solubility suffices. For \( p \) odd, the first construction requires less group theoretic calculation and so is preferred; it does not assume the solubility of \( G \). For \( p = 2 \), there is no apparent alternative to the second construction for \( L \).

7. Complement \( L \) when \( p \) is Odd

The notation at the end of section 2 is continued, use is made of the structure of the nonabelian normal subgroup \( N \) obtained in section 5, and \( p \) is assumed to be an odd prime. Thus, \( N \) is the amalgamated product \( Z \cdot P \) of \( Z \) and its Sylow \( p \)-subgroup \( P \) of \( N \), while \( P \) is the amalgamated product \( Z \cdot Y \cdot E \) of the Sylow \( p \)-subgroup \( P \) of \( Z \) and \( E \), where \( E \) is a normal extraspecial subgroup of exponent \( p \), order \( p^{2h+1} \) and centre \( Z_p \), the subgroup of \( Z \) of order \( p \). The subgroup \( E \) has a "basis" \( f_1, \ldots, f_h, g_1, \ldots, g_h \) with \( f_1, g_1 = z_1 \), where \( z_1 \) is a generator of \( Z_p \). Write \( \bar{E} = E/Z_p \) and let \( f: \bar{E} \times \bar{E} \rightarrow F_p \) be the symplectic form on \( \bar{E} \); thus \( f(f_1, g_1) = \delta_{ij} f_{ij} \). The subgroup \( N \) has a faithful representation \( N \) (with character \( \nu \)) in \( M_{2h}(F) \), \( F \) being an algebraically closed field of characteristic \( r \neq p \).

The following construction of a complement \( L \) to \( N \) in \( G \) relative to \( Z \) is modelled on Bolt, Room & Wall (1961), Part I. This discussion is divided into four sections.

(i) An analysis of the group \( Zaut \) of those automorphisms of \( E \) which leave its centre pointwise fixed.

(ii) The structure of \( G \), coming from its automorphic action on \( E \).

(iii) The realization of these actions as extensions of the faithful representation \( N \) of \( E \).

(iv) The computational algorithm.

(i) \( Zaut \). \( Zaut \) is the group of those automorphisms \( \alpha \) of \( E \) which leave its centre \( Z_p \) fixed. Thus, for \( x, y \in E \), \( (x, y) = (x, y)^\alpha = (x^\alpha, y^\alpha) \) and so \( \alpha \) induces an automorphism \( \bar{\alpha} \) of \( \bar{E} = E/Z_p \) which leaves the symplectic form \( f \) on \( \bar{E} \) invariant, i.e. \( f(\bar{x}, \bar{y}) = f(\bar{x}^\alpha, \bar{y}^\alpha) \). Define \( f \) on \( E \) by \( f(x, y) = f(\bar{x}, \bar{y}) \). Let \( \text{Symp}(\bar{E}) \) denote the group of symplectic automorphisms of \( \bar{E} \). Clearly, \( \zeta : Zaut \rightarrow \text{Symp}(\bar{E}), \alpha \mapsto \bar{\alpha} \), is a group homomorphism.

Given any symplectic automorphism \( \bar{\alpha} \) of \( \bar{E} \), does this come from an automorphism \( \alpha \) of \( E \) in \( Zaut \)? To define a suitable \( \alpha \) from \( \bar{\alpha} \) it is convenient to choose particular coset representatives \( x^* \) from each coset \( Z_p \cdot x \) in \( E \) as follows:
Set \( f_i^* = f_i \) and \( g_i^* = g_i \) for \( i = 1, \ldots, h \). Choose \( (f_i^* g_i^*)^* = z_i \) \( i = 1 \). Then for \( x, y \in \langle f_i, g_i \rangle \), it is immediately verified that

\[
x^* y^* = z_i^{(x, y)(x, y)^*}.
\]

Finally, suppose \( x = x_1 \cdots x_h \) with \( x_i \in \langle f_i, g_i \rangle \). Then set \( x^* = x_1^* \cdots x_h^* \). If also \( y = y_1 \cdots y_h \) then \( f(x, y) = f(x_1, y_1) + \cdots + f(x_h, y_h) \). As the different components \( x_i \) of \( x \) commute, formula (3) remains true for all \( x, y \in E \), and so it gives an explicit formula for the multiplication in \( E \) in terms of the chosen coset representatives \( X^* \). For \( \bar{x} \in \bar{E} \) define \( (\bar{x})^* = x^* \), where \( x \) is any element in the coset \( \bar{x} \) (i.e. \( \bar{x} = Z_p x \)).

Suppose \( \beta \in \text{Symp}(\overline{E}) \). Define \( \beta^* \) on \( E \) by \( \beta^*: x^* \rightarrow (\bar{x})^* \), \( \beta^* \) acting identically on the centre \( Z_p \). Then \( \beta^* \) respects the multiplication formula (3) and so is an endomorphism of \( E \). As \( \beta \) is nonsingular, \( \beta^* \) maps onto \( E \) modulo \( Z_p \) and so is an epimorphism of \( E \). Hence, \( \beta^* \) is an automorphism of \( E \). The map \( \eta: \text{Symp}(\overline{E}) \rightarrow \text{Zaut}, \beta \mapsto \beta^* \), is a group homomorphism. Also \( \xi(\eta(\beta)) = \xi(\beta^*) = \beta^* = \beta \) and so \( \beta \circ \eta = id \). Thus, \( \eta \) embeds \( \text{Symp}(\overline{E}) \) into \( \text{Zaut} \) and \( \xi \) is onto, i.e. every symplectic automorphism \( \beta \) arises from an automorphism \( \beta^* \) of \( E \) lying in \( \text{Zaut} \).

What is \( \ker(\xi) \)? For \( e \in E \), write \( i_e \) for the inner automorphism of \( E \), i.e. \( x \rightarrow x^e = e^{-1} x e \) for \( x \in E \). These \( i_e \) form the subgroup \( \text{Iaut} \) of inner automorphisms of \( E \). Further, if \( \alpha \in \text{Zaut} \), then \( \alpha^{-1} i_e \alpha = i_{e^\alpha} \) and so \( \text{Iaut} \) is a normal subgroup of \( \text{Zaut} \). As \( E \) is abelian, \( i_e \) is the identity automorphism for each \( e \in E \) and so \( \text{Iaut} \leq \ker(\xi) \). But any automorphism of \( E \), which induces the identity both on the centre \( Z_p \) and on the quotient \( \overline{E} = E/Z_p \), is an inner automorphism of \( E \) (see, for instance, (3C) of Winter, 1972). So \( \ker(\xi) = \text{Iaut} \).

Hence, \( \text{Zaut} \) is the semidirect product of the normal subgroup \( \text{Iaut} \) with a copy of \( \text{Symp}(\overline{E}) \). This choice of complement \( \text{Symp}(\overline{E}) \) depends on the choice of “basis” \( f_{i_1}, \ldots, f_h, g_{i_1}, \ldots, g_h \) of \( E \).

(ii) Action of \( G \) on \( E \). The subgroup \( E \) is normal in \( G \) and the centre \( Z_p \) of \( E \) is central in \( G \). For \( g \in G \), conjugation by \( g \) gives an automorphism \( i_g: x \rightarrow x^g \) of \( G \), whose restriction to \( E \) lies in \( \text{Zaut} \). The map \( \iota: G \rightarrow \text{Zaut}, g \rightarrow i_g \) is a group homomorphism and \( \ker \iota = C_G(E) \), the centralizer of \( E \) in \( G \). (Note \( C_G(E) = C_G(\overline{E}) \).)

Clearly, \( \im \iota \geq \text{Iaut} = \im(i_{i_1}) = \im(i_{i_h}) \), which is normal in \( \text{Zaut} \). So \( \im \iota = \text{Iaut} \). \( L_1 \) for a subgroup \( L_1 \leq \text{Symp}(\overline{E}) \). Set \( L = \iota^{-1}(L_1) \); thus \( L \geq C_G(\overline{N}) \). Then \( Z \leq N \cap L \). But if \( g \in N \cap L \) so \( i_g \in \text{Iaut} \cap L_1 = \langle i_{i_1} \rangle \) and so \( g \) must be central in \( N \) and lie in \( Z \). Thus \( Z = N \cap L \). Finally, \( \im \iota = \text{Iaut} \cdot L_1 = \text{Iaut} \cdot \im(i_{i_1}) \); thus, an element \( g \) of \( G \) differs from an element in \( NL \) by an element which does not move \( E \) (or \( N \)) and so lies in \( \ker \iota = C_G(\overline{E}) \leq L \), i.e. \( g \in NL \) or \( G = NL \). Hence, \( L \) is a complement to \( N \) in \( G \) relative to \( Z \) and \( L \geq C_G(\overline{N}) \), as specified in section 6.

(iii) The existence of an extension of the representation. As set out in section 6, there is a projective extension \( \overline{N} \) of the faithful representation \( N \) of \( N \) to \( L \). What is required is an adjustment of the map \( \theta: L/Z \rightarrow M_{\rho}(F), Z \mapsto \overline{N}(e) \) to give an ordinary representation of \( L/Z \). Now \( \iota: L \rightarrow L_1 \leq \text{Symp}(\overline{E}) \) and so it suffices to find a representation of \( \text{Symp}(\overline{E}) \), which gives the correct action on the elements \( \overline{N}(e) (e \in E) \).

This is exactly the context of Bolt, Room & Wall (1961), except that the complex field \( C \) is used instead of \( F \). However, all constructions and arguments can be done in an algebraic number field (finite dimensional extension of the rationals \( Q \)); moreover, all matrices involved can be realized in the ring of integers \( R \) in this field; thus it is possible to do “decomposition”, i.e. pass modulo a suitable prime ideal of \( R \) onto a field which can be
embedded into the coefficient field $F$ (algebraically closed). Thus, all the argument of their paper can be "decomposed" into the present context.

Bolt et al. call the group of all scalar multiples of those matrices representing symplectic actions on the realization $N(E)$ of $E$ in $M_{p^h}(C)$ the Clifford similarity group $CS(p^h)$. The derived subgroup $CS'$ is also isomorphic to $\text{Symp}(E)$ and this gives the required representation of $\text{Symp}(E)$, called the Weil representation in later literature, for instance, see Howe (1973). (When $p^h = 3$, the derived subgroup is not isomorphic to $\text{Symp}(E)$; but in this case there are three other subgroups of $CS$ isomorphic to $\text{Symp}(E)$.)

A direct construction of the Weil representation involving calculation in the field $F$ is given in Ward (1972).

(iv) The computational algorithm. The calculation of $L$ proceeds as follows. Pass to the quotient $G/N$; this will give a sequence of pc-generators for $G/N$. Each preimage $l \in G$ of such a generator is now modified by multiples of basis elements of $N$ to lie in $L$ as follows. For each $i$, form $f_i = x$. Now form the difference $x_i^* = x^*x^{-1}$. Replace $l$ by $g_l \l$, and so $f_l^H = x^*$. Similarly, if $g_l = y$, form the difference $y_i^* = y(y^*)^{-1}$ and replace $l$ by $f_l^H$ to get $g_l^H = y^*$. (Given $x \in E$, a procedure must be given for the calculation of $x^*$.) This ensures that for any $e \in E$, $(e^*)^H = (e^*)^*$ and so $l_i \in \text{Symp}(E)$ in $Z_{aut}$. Thus, $l_i \in L_1$ and $l \in L$, as $L$ is the preimage of $L_1$ under the map $\iota$. Finally, $L$ is generated in $G$ by all the modified $l_i$ together with $Z$ and the recursive algorithm proceeds to the subgroup $L$ with its central cyclic subgroup $Z$.

Alternatively, the recursive step could proceed by replacing $G$ by the quotient $G_1 = G/E. C_0(E)$, then modifying the preimages in $G$ of a set of pc-generators in $G_1$, as in the last paragraph. These elements, together with $C_0(E)$, generate $L$. This involves less modifications of elements, but it requires the calculation of $C_0(E)$, which involves looking at all generators of $G$ "above" $E$ and so the above treatment was considered more economical overall.

8. The Alternative Construction of the Complement $L$

An alternative construction of the complement $L$ to $N$ in $G$ relative to $Z$ can be based on the proof of Theorem 5.2 in Dade (1976). In particular, this construction is valid when $p = 2$. The notation at the end of section 2 and that of section 5, when $N$ is nonabelian, will be continued. In particular, $N = N/Z$ is minimal normal in $\bar{G} = G/Z$, and $\bar{N}$ is an elementary $p$-group. Also $(N, G)Z = N$. These are the basic requirements needed in the construction. Use is made of the solubility of $G$.

(1) Find a chief section $H/J$ of $G$ above $N$ so that $(N, H)Z = N$ and $(N, J) \leq Z$.

First, take $J = N$ and use NELSG to obtain a normal elementary abelian subgroup $H/J$ in $G/J$. Then $(N, H)$ is normal in $G$ and $\leq N$. By definition of $N$, $(N, H)Z = N$ or $(N, H) \leq Z$. If $(N, H) \leq Z$, take $J = H$ and seek a new $H > J$ with $H/J$ normal elementary abelian in $G/J$. Continue extending $H$ in this manner until $(N, H)Z = N$. The inequality $(N, H) \leq Z$ cannot remain true as $(N, G)Z = N$.

The final section $H/J$ must be an elementary abelian $q$-group for a prime $q \neq p$. For suppose $q = p$. The subgroup $J$ acts trivially on $\bar{N}$ and so $\bar{N}$ is an irreducible $F_p(G/J)$-module. As $H/J$ is a normal $p$-subgroup of $G/J$, the ideal of $F_p(G/J)$ generated by the augmentation ideal $I(H/J)$ of $F_p(H/J)$ lies in the Jacobson radical $J(F_p(G/J))$ and so acts
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trivially on the irreducible module $N$. Thus, $H/J$ acts trivially on $N$, contrary to $(N, H)Z = N$.

In section 5 it was shown that $N$ is the amalgamated central product $Z \times Y E$, where $E$ is an extraspecial $p$-group. When $p$ is odd it was noted that $E = \Omega_1(P)$, where $P$ is the Sylow $p$-group of $N$, and so $E$ is normal in $G$. To justify Dade’s construction it is necessary for $E$ to be normal when $p = 2$, and so an alternative construction of a subgroup $D$ replacing $E$ is given. The proof of this follows closely part of the analysis in section 6 of Dade (1976).

**Lemma.** The group $D = (N, H)$ is a normal extraspecial $p$-subgroup of $G$ and $N = Z \times Y D$, the amalgamated central product.

**Proof.** Let $P$ be the Sylow $p$-subgroup of (nilpotent) subgroup $N$ and let $U$ be the product of the Sylow $p'$-subgroups of $Z$. Then $N = U \times P$ with $P$ and $U$ normal in $G$. Thus, it suffices to show that $D = (N, H) = (P, H)$ is an extraspecial $p$-group and that $P = Z(P) \times Y D$, where $Z(P)$ is the centre of $P$ and is equal to the Sylow $p'$-subgroup $Z$ of $Z$. Then $D$ is normal as both $N$ and $H$ are normal.

First of all, if $g \in G$ acts trivially on $P/Z(P)$ by conjugation, then $g$ acts trivially on $P/P'$. For if not, there is an $x \in P$ such that $x^g = xy$, with $y \in Z(P)$ and $y \notin P' = Z_p$. Thus $y^g \neq 1$, as $Z_p$ is the subgroup of order $p$ in $Z(P)$. On the other hand, $x^g \in Z(P)$, as $P/Z(P)$ is an elementary abelian $p$-group and so $g$ acts trivially on $x^g$, i.e.

$$x^g = (x^g)^g = (xy)^g = x^g y^g,$$

contrary to $y^g \neq 1$.

The quotient $H/J$ acts nontrivially on $P/Z(P)$, as $N = (N, H)Z$ and so $P = (P, H)Z(P)$. Write $J_1 = C_{p'}(P/Z(P))$ (normal in $G$) and so $J_1 \geq J$ and $H/J_1$ is a nontrivial $p'$-group acting faithfully on $P/Z(P)$ of $P/Z(P)$. By the last paragraph, $J_1$ acts trivially on $P/P'$ or $(P, J_1) \leq P'$. Hence $H/J_1$ is a $p'$-group acting faithfully on the abelian $p$-group $P/P'$. As $P = (P, H)Z(P)$, so the part of $P/Z(P)$ centralized by $H/J_1$ is smaller than $P/Z(P)$ and is normal in $G$. By the minimality of $N(\approx P/Z(P))$ in $G$, so $C_{p'}(P/Z(P)) = 1$. Thus, $C_{p'}(H/J_1) \leq Z(P)/P'$. But $Z(P)/P'$ is clearly centralized by $H/J_1$ and so $C_{p'}(H/J_1) = Z(P)/P'$. By Satz III.13.4(b) of Huppert (1967),

$$P/P' = Z(P)/P' \times (P/P', H/J_1).$$

The preimage in $G$ of $(P/P', H/J_1)$ under the homomorphism $G \to G/P'$ is $D = (P, H)$. As $P/Z(P)$ is elementary, so is $P/P'$ (as $(Z(P)/P')/P/Z(P) \approx P/Z(P)$). Also $Z_p = P' = D' = Z(D)$ has order $p$. Thus, $P$ is the amalgamated central product $Z(P) \times Y D$, with $D$ a normal extraspecial $p$-subgroup, as required.

**Corollary.** $C_G(N) = C_G(D)$.

(2) Pass to the quotient $G_0 = G/C_G(N)$.

Write $H_0$ and $N_0$ for the images of $H$ and $N$ in $G_0$, etc. Thus, $N_0 = P_0 = D_0$.

It should be noted that the image $J_0$ of $J$ in $G_0$ is equal to $N_0$. For $(D, J) \leq Z_p = D'$, and so $J$ acts trivially on $D = D/Z_p$, as well as trivially on $Z_p = D'$. As $D$ is extraspecial, by (3C) of Winter (1972), each $j \in J$ acts as an inner automorphism $t_j$ of $D$ for some $j \in D$. Then $j^{-1}t_j \in C_D(D)$ or $j \in D$. $C_D(D) = N_0$. $C_G(N)$, i.e. $J_0 \leq N_0$ and so $J_0 = N_0$, as $J \geq N$.

Thus, $H_0/N_0 = H_0/J_0$, a quotient of $H/J$, and so is an (elementary) $q$-group. Also $N_0$ is a normal elementary abelian $p$-subgroup of $H_0$ and $(N_0, H_0) = N_0$. 

(3) Construct a Sylow $q$-subgroup $K_0$ of $H_0$.

Then $H_0$ will be the semidirect product $N_0K_0$.

An algorithm for constructing the Sylow subgroup is described in Glasby (1988), where the general problem of constructing Hall $\pi$-subgroups, normalizers and conjugating elements sending one Hall $\pi$-subgroup into another is discussed. The situation here is a simple case of this as $H_0$ is a $(p, q)$-group and Sylow's theorem is all that is needed; but a conjugating algorithm is required (see middle paragraph of (4) below) for the inductive construction.

(4) Construct the normalizer $L_0 = N_{G_0}(K_0)$ of $K_0$ in $G_0$.

To do this it should first be noted that $C_{N_0}(K_0) = (1)$; this follows from Satz III.13.4(a) of Huppert (1967), upon noting that $(N_0, H_0) = N_0$. Thus, $N_{H_0}(K_0) = N_{N_0K_0}(K_0) = K_0C_{N_0}(K_0) = K_0$. Pass to the quotient group $G_0/H_0$. Consider each of the preimages $g_0$ in $G_0$ of a set of $p$-generators of $G_0/H_0$. Adjust $g_0$ by an element $n_0$ of $N_0$ to lie in $N_{G_0}(K_0)$. These modified elements $g_0n_0$ of $G_0$, together with $K_0$, generate $L_0$.

To calculate the adjusting element $n_0$, consider the Sylow subgroup $K_0^g$ of $H_0$, then by Sylow's theorem, there exists $h_0 \in H_0$ such that $K_0^{h_0} = K_0$. Write $h_0 = n_0k_0$ and then $K_0 = K_0^{n_0k_0} = K_0^{n_0k_0} = K_0^{n_0k_0}$ and so $g_0n_0 \in N_{G_0}(K_0)$. The conjugating element $h_0$ or $n_0$ is also given explicitly by Glasby's algorithm.

The group $G_0 = N_0L_0$ is a semidirect product. For if $g_0$ is any element of $G_0$, $K_0^g$ is also a Sylow $q$-subgroup of $H_0$, and so there exists an element $n_0k_0$ of $H_0 = N_0L_0$ such that $K_0 = K_0^{n_0k_0} = K_0^{n_0k_0}$. Hence $g_0n_0 \in N_{G_0}(K_0) = L_0$ or $g_0 = l_0n_0^{-1} \in N_0L_0$. Hence $G_0 = N_0L_0$. Take $g_0 \in N_0 \cap L_0$. Then $g_0$ normalizes $K_0$, as $g_0 \in L_0 = N_{G_0}(K_0)$. Also $g_0 \in N_0 \leq H_0$ and so $G_0 \in N_{H_0}(K_0) = K_0$. Thus $g_0 \in N_0 \cap K_0 = (1)$. Thus $N_0 \cap L_0 = (1)$, as required.

(5) Set $L$ equal to the preimage of $L_0$ in $G$.

Then $G$ is the preimage of $G_0$ (= $N_0L_0$) and so equals $NL$. Also, if $x \in N \cap L$, then $x_0 \in N_0 \cap L_0$, which is trivial and so $x \in C_G(N)$, i.e. $x \in C_G(N) \cap N = Z$. Thus $N \cap L = Z$. As $L$ is the preimage of the subgroup $L_0$ of $G_0 = G/C_G(N)$, $L$ certainly contains $C_G(N)$. Thus $L$ is the required complement to $N$ in $G$ relative to $Z$, satisfying $L \geq C_G(N)$.

It should be noted that this complement $L$ is unique up to conjugacy in $G$. For $N = Z \neq D$ and $L \geq C_G(N) \geq Z$ and so $D \cap L = Z(D) = Z_p$ and $G = NL = DL$. Thus $L$ is also a complement to the normal extraspecial subgroup $D$ of $G$ relative to $Z(D)$ and $L \geq C_G(D)$ ( = $C_G(N)$). The uniqueness now follows from Proposition 5.2 of Dade (1976).

This completes the inductive algorithm construction. It remains to verify that the faithful representation $N$ of $N$ can be suitably extended. For complex representations the result follows directly from (5.10) of Dade (1976).

To match the treatment with that of section 6, a representation $\theta$ of $L/Z$ is required which gives the correct action on the faithful representation $N$ of $N$ or its restriction to $D$, this action corresponding to the action of elements of $L$ on $N$ (by conjugation) in $G$. But $C_G(N)(\geq Z)$ also acts trivially on $D$ and so it is sufficient to do this for the image $L_0$ of $L$ in $G_0 = G/C_G(N)$. Thus, it is necessary to consider the abstract semidirect product of $D$ by $L_0$. Theorem 1 of Howlett (1978) can then be applied to give the extension of the representation of $D$ to the semidirect product, with the subgroup $K_0$ providing the (elementary) $p'$-section, required in his theorem. Howlett's paper also deals with complex representations. However, his proof carries over without change to an algebraically closed field $F$ of characteristic $r(\neq p)$. The results, cited in his Lemma 2, remain true for $F$.
9. Trials

Trials have been carried out on many groups, but only a selection are reported here. In each case, whenever there is a nontrivial centre, all possible cyclic subgroups of it are taken by suitable choice of the generator $z_{\text{gen}}$ for $Z$. As the centre in all examples is cyclic, it is sufficient to give the order $z$ of $Z = \langle z_{\text{gen}} \rangle$. Care must be taken to ensure that the characteristic $r$ (either 0 or a prime) of the representations being considered is either 0 or coprime to $z$.

The first examples are groups of upper-triangular matrices. Thus, $\text{Trigp}(d, q)$ stands for the group of all nonsingular, upper-triangular, $d \times d$ matrices with coefficients in the finite field of order $q$. This group has order $(q - 1)^d (d - 1)/2$ and centre of order $(q - 1)$. These groups are constructed by generating the actual matrix group $MG$ by typing in the generating matrices, and using the CAYLEY command $G = \text{PCREPRESENTATION}(MG)$, to obtain the group with a pc-presentation.

Next, the Borel subgroup (upper-triangular matrices) of the symplectic group $\text{Symp}(8, 3)$ of $8 \times 8$ matrices of the field of order 3 is considered. This is denoted by $Bsym(8, 3)$ and has order $2^4 3^8$. Its centre has order 6. It is constructed from the appropriate matrices as in the preceding paragraph.

The wreath products $D_{10} \wr S_4$ and $S_4 \wr S_4$ are constructed using standard CAYLEY functions and then pc-presentations are obtained.

Finally, various quotients of Glasby's group $SG$ of order $2^{11} 3^{13}$ are considered, see Glasby (1989). The chief factors of this group have orders 2, 3, 2, 2, 3, 3, 2, 3, 3. The initial quotients are $C_2, S_3, S_4$ and $\text{GL}(2, 3)$. The results for the whole group $SG$ and the quotient $QSG$ of order $2^{11} 3^{14}$ are given. The latter has centre of order 2 and so a nontrivial $Z$ can be considered. The quotient group $QSGZ = QSG/Z$ is also given.

In each case when the characteristic $r$ is zero and centre is trivial, it is verified that the sum of the squares of the character degrees does equal the group order.

The results are given in the following format: group, the group order ($o$); size of centre ($z$), the characteristic of the field ($r$); running time in seconds ($t$), total number of irreducible characters ($\# ch$) and the character degrees. The character degrees are presented as two sequences, one immediately above the other, the top element being the character degree, with the multiplicity to which this degree occurs immediately below it. The following results were obtained on a SUN 3/260 using CAYLEY V3.6.

$$\text{Trigp}(4, 7), \quad o = 2^4 3^4 7^6; \quad z = 1, r = 0: \quad t = 1091, \quad \# ch = 2856$$

$$\begin{array}{cccccccc}
1 & 6 & 36 & 42 & 216 & 252 & 294 & 1512 & 1764 \\
\end{array}$$

$$\text{Trigp}(4, 7), \quad o = 2^4 3^4 7^6; \quad z = 1, r = 2: \quad t = 1089, \quad \# ch = 312$$

$$\begin{array}{cccccccc}
1 & 6 & 36 & 42 & 216 & 252 & 294 & 1512 & 1764 \\
81 & 81 & 27 & 54 & 3 & 27 & 27 & 3 & 9
\end{array}$$

$$\text{Trigp}(4, 7), \quad o = 2^4 3^4 7^6; \quad z = 1, r = 3: \quad t = 1099, \quad \# ch = 96$$

$$\begin{array}{cccccccc}
1 & 6 & 36 & 42 & 216 & 252 & 294 & 1512 & 1764 \\
16 & 24 & 12 & 16 & 2 & 12 & 8 & 2 & 4
\end{array}$$

$$\text{Trigp}(4, 7), \quad o = 2^4 3^4 7^6; \quad z = 1, r = 7: \quad t = 943, \quad \# ch = 1296$$

$$\begin{array}{cccccccc}
1 & 1296
\end{array}$$
\[ \text{Trigp}(4, 7), o = 2^4 3^4 7^6; \quad z = 2, r = 0: \quad t = 1134, \ # ch = 1428 \]

\[
\begin{array}{cccccccc}
1 & 6 & 36 & 42 & 216 & 252 & 294 & 1512 \\
648 & 324 & 54 & 216 & 3 & 54 & 108 & 3 & 18
\end{array}
\]

\[ \text{Trigp}(4, 7), o = 2^4 3^4 7^6; \quad z = 2, r = 3: \quad t = 1149, \ # ch = 48 \]

\[
\begin{array}{cccccccc}
1 & 6 & 36 & 42 & 216 & 252 & 294 & 1512 \\
8 & 12 & 6 & 8 & 1 & 6 & 4 & 1 & 2
\end{array}
\]

\[ \text{Trigp}(4, 7), o = 2^4 3^4 7^6; \quad z = 2, r = 7: \quad t = 994, \ # ch = 648 \]

\[
\begin{array}{cccccccc}
1 & 6 & 36 & 42 & 216 & 252 & 294 & 1512 \\
648 & 324 & 54 & 216 & 3 & 54 & 108 & 3 & 18
\end{array}
\]

\[ \text{Trigp}(4, 7), o = 2^4 3^4 7^6; \quad z = 3, r = 0: \quad t = 1143, \ # ch = 952 \]

\[
\begin{array}{cccccccc}
1 & 6 & 36 & 42 & 216 & 252 & 294 & 1512 \\
432 & 216 & 36 & 144 & 2 & 36 & 72 & 2 & 12
\end{array}
\]

\[ \text{Trigp}(4, 7), o = 2^4 3^4 7^6; \quad z = 3, r = 2: \quad t = 1156, \ # ch = 104 \]

\[
\begin{array}{cccccccc}
1 & 6 & 36 & 42 & 216 & 252 & 294 & 1512 \\
27 & 27 & 9 & 18 & 1 & 9 & 9 & 1 & 3
\end{array}
\]

\[ \text{Trigp}(4, 7), o = 2^4 3^4 7^6; \quad z = 3, r = 7: \quad t = 992, \ # ch = 432 \]

\[
\begin{array}{cccccccc}
1 & 6 & 36 & 42 & 216 & 252 & 294 & 1512 \\
432 & 216 & 36 & 144 & 2 & 36 & 72 & 2 & 12
\end{array}
\]

\[ \text{Trigp}(4, 7), o = 2^4 3^4 7^6; \quad z = 6, r = 0: \quad t = 1151, \ # ch = 476 \]

\[
\begin{array}{cccccccc}
1 & 6 & 36 & 42 & 216 & 252 & 294 & 1512 \\
216 & 108 & 18 & 72 & 1 & 18 & 36 & 1 & 6
\end{array}
\]

\[ \text{Trigp}(4, 7), o = 2^4 3^4 7^6; \quad z = 6, r = 7: \quad t = 990, \ # ch = 216 \]

\[
\begin{array}{cccccccc}
1 & 6 & 36 & 42 & 216 & 252 & 294 & 1512 \\
216 & 108 & 18 & 72 & 1 & 18 & 36 & 1 & 6
\end{array}
\]

\[ \text{Trigp}(5, 4), o = 2^{10} 3^5; \quad z = 1, r = 0: \quad t = 2295, \ # ch = 1923 \]

\[
\begin{array}{cccccccc}
1 & 3 & 9 & 12 & 27 & 36 & 48 & 81 \\
108 & 144 & 192 & 324 & 432 & 576 & 1296 & 1728 & 2304
\end{array}
\]

\[ \text{Trigp}(5, 4), o = 2^{10} 3^5; \quad z = 1, r = 2: \quad t = 1375, \ # ch = 343 \]

\[
\begin{array}{cccccccc}
1 & 3 & 9 & 12 & 27 & 36 & 48 & 81 \\
108 & 144 & 192 & 324 & 432 & 576 & 1296 & 1728 & 2304
\end{array}
\]

\[ \text{Trigp}(5, 4), o = 2^{10} 3^5; \quad z = 1, r = 3: \quad t = 2418, \ # ch = 61 \]

\[
\begin{array}{cccccccc}
1 & 3 & 9 & 12 & 27 & 36 & 48 & 81 \\
108 & 144 & 192 & 324 & 432 & 576 & 1296 & 1728 & 2304
\end{array}
\]

\[ \text{Trigp}(5, 4), o = 2^{10} 3^5; \quad z = 3, r = 0: \quad t = 2530, \ # ch = 641 \]

\[
\begin{array}{cccccccc}
1 & 3 & 9 & 12 & 27 & 36 & 48 & 81 \\
108 & 144 & 192 & 324 & 432 & 576 & 1296 & 1728 & 2304
\end{array}
\]

\[ \text{Trigp}(5, 4), o = 2^{10} 3^5; \quad z = 3, r = 2: \quad t = 1366, \ # ch = 81 \]

\[
\begin{array}{cccccccc}
1 & 3 & 9 & 12 & 27 & 36 & 48 & 81 \\
108 & 144 & 192 & 324 & 432 & 576 & 1296 & 1728 & 2304
\end{array}
\]

\[ \text{Bsyp}(8, 3), o = 2^4 3^{16}; \quad z = 1, r = 0: \quad t = 3131, \ # ch = 4160 \]

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 6 & 8 & 9 & 12 \\
\end{array}
\]
Computing Character Degrees of Soluble Groups

\( B_{\text{sym}}(8, 3), o = 2^4 3^{16} \); \( z = 1, r = 2 \): \( t = 3109, \ # ch = 625 \)

1 2 3 4 6 8 9 12 18 24 27 36 54 72 81 108 162 216 243 324 486 648 729 972 1458 1944 2916
3 9 6 9 25 3 6 9 13 18 50 62 23 12 9 3109 16 1 1

\( B_{\text{sym}}(8, 3), o = 2^4 3^{16} \); \( z = 1, r = 3 \): \( t = 27, \ # ch = 16 \)

\( B_{\text{sym}}(8, 3), o = 2^4 3^{16} \); \( z = 2, r = 0 \): \( t = 3341, \ # ch = 2080 \)

1 2 3 4 6 8 9 12 18 24 27 36 54 72 81 108 162 216 243 324 486 648 729 972 1458 1944 2916
2 4 6 8 10 12 14 16 18 20 24 36 48 72 108 144 192 288 2916
3 6 12 18 24 36 72 108 144 216 288 432 648 729 972 1458 1944 2916

\( B_{\text{sym}}(8, 3), o = 2^4 3^{16} \); \( z = 2, r = 3 \): \( t = 33, \ # ch = 8 \)

\( B_{\text{sym}}(8, 3), o = 2^4 3^{16} \); \( z = 3, r = 0 \): \( t = 673, \ # ch = 784 \)

27 54 81 108 162 243 324 486 729 972 1458
45 48 96 12 128 96 40 72 192 12 40

\( B_{\text{sym}}(8, 3), o = 2^4 3^{16} \); \( z = 3, r = 2 \): \( t = 655, \ # ch = 79 \)

27 54 81 108 162 243 324 486 729 972 1458
3 6 6 3 16 6 10 9 12 3 5

\( B_{\text{sym}}(8, 3), o = 2^4 3^{16} \); \( z = 6, r = 0 \): \( t = 689, \ # ch = 392 \)

27 54 81 108 162 243 324 486 729 972 1458
24 96 6 6 48 48 20 36 96 6 20

\( D_{10 \ wr S_4}, o = 2^7 3^{16} \); \( z = 1, r = 0 \): \( t = 1181, \ # ch = 105 \)

1 2 3 4 6 8 16 24 32 48 64 96 128
4 2 4 4 4 10 8 24 10 12 8 13 2

\( D_{10 \ wr S_4}, o = 2^7 3^{16} \); \( z = 1, r = 2 \): \( t = 80, \ # ch = 24 \)

1 2 8 16 24 32 48 64 96 128
1 1 2 4 2 4 1 4 3 2

\( D_{10 \ wr S_4}, o = 2^7 3^{16} \); \( z = 1, r = 3 \): \( t = 108, \ # ch = 89 \)

1 3 4 6 8 16 24 32 48 64 96
4 4 4 8 4 4 8 4 24 8 12 4 13

\( D_{10 \ wr S_4}, o = 2^7 3^{16} \); \( z = 1, r = 5 \): \( t = 47, \ # ch = 20 \)

1 2 3 4 6 8
4 2 4 4 2

\( S_4 \ wr S_4, o = 2^1 3^{16} \); \( z = 1, r = 0 \): \( t = 387, \ # ch = 190 \)

1 2 3 4 6 8 12 16 24 32 48 64 72 81 96 108 144 162 192 216 243 324 432 486 648
4 2 4 4 4 8 4 4 8 4 4 16 2 8 4 4 16 10 2 2 25 4 12 6 4 6

\( S_4 \ wr S_4, o = 2^1 3^{16} \); \( z = 1, r = 2 \): \( t = 78, \ # ch = 9 \)

1 2 8 16 24 32 64
1 1 1 2 1 2 1

\( S_4 \ wr S_4, o = 2^1 3^{16} \); \( z = 1, r = 3 \): \( t = 249, \ # ch = 89 \)

1 3 4 6 12 36 54 81 108 216 243 324 486
4 4 4 8 8 16 4 16 1 4 12 4
Note that for characteristic zero or three, the sequence of multiplicities of degrees for $QSGZ$ with $z = 1$ and $QSG$ with $z = 2$ together add up to give the sequence of degrees for the group $QSG$ with $z = 1$. This gives a decomposition of the set of characters of $QSG$ as the union of the set of those which are trivial on the central element of order 2, and the set of those which are faithful on this element.

In the case of the group $SG$ of order $2^{113 \cdot 3} = 3265173504$, the number of conjugacy classes was deduced from the total number (181) of irreducibles. The number of 2-regular
conjugacy classes (40) and the number of 3-regular conjugacy classes (22) were also readily
determined.

When the characteristic $r$ is three, the normal subgroup of $SG$ of order $3^9$ is immediately
factored out, being accounted for in the Jacobson radical of the group algebra, as in
section 3 above. Thus, the character degrees for $SG$ and $QSG$ coincide in this case.

10. Concluding Remarks

1. When $G$ is a $p$-group $(p \neq r)$, then $|N/Z| = p$ and $N$ is abelian. In the recursive step the
stabilizer $S$ is the centralizer $C_G(N)$ of $N$ in $G$, which is either (a) $G$ itself or (b) is maximal in
$G$. If $N$ is cyclic, then $Z$ is enlarged to $N$, and (a) all or (b) one of the (hypothetical) $p$
extensions of $\zeta$ to $N$ need be considered. If $N$ is noncyclic, then there are $p$ subgroups of
order $p$ in $N$ which avoid $Z$; (a) all or (b) one of these must be taken to be the kernel $Ker$
and $\zeta$ (hypothetically) induces a faithful representation of $NKer/Ker$. This is the algorithm
used by Slattery (1986) to construct the character degrees of a $p$-group.

2. Again, in the $p$-group case, if the algorithm is continued until the final section $G^*$ has
its central cyclic subgroup $Z^*$ equal to $G^*$, then the faithful irreducible character $\xi^*$ of $Z^*$
can be induced directly up to the global group $G$ to obtain the corresponding irreducible
character $\gamma$ of $G$. This utilizes the fact that every irreducible character of a $p$-group is
monomial (Theorem 11.3 of Curtis & Reiner, 1981). It has also been exploited by the
author (Conlon, 1990) to obtain the character table of $G$.

3. Whenever $G/Z$ is a $p$-group $(p \neq r)$, then $G$ is the direct product of the $p'$-subgroup $p'Z$
of $Z$ and the Sylow $p'$-subgroup $p'G$ of $G$, i.e. $G = p'Z \times p'G$. In this case the program may
take advantage of Slattery’s procedure RELDEGS (Slattery, 1986) on $p'G$ relative to the
central subgroup $p'Z$. This avoids the use of NELSG and PCMEANS as described in
section 2, as the $p$-step generating series $(p\text{s}g)$ for $p'G$ immediately provides a suitable
subgroup $N$, by remark 1 above. Furthermore, the $p$-group routines in CAYLEY, based on
a $psg$-series, are much faster than the pc-presentation routines, e.g. for the calculation of
centralizers (stabilizers $S$).

4. When $r$ divides $|G|$ the Fong–Swan–Rukolaine theorem (22.1 in Curtis & Reiner,
1981) applies and each modular irreducible arises from some complex character by going
modulo $r$ in the decomposition map. In the algorithm presented here the modular
characters are obtained by homing in on or confining attention to these particular
characters. As in section 3 above, when $r$ divides $|N/Z|$, then $N$ is abelian, as $Z$ is an
$r'$-prime group, and $N = Z \times N$, where $N$ is the Sylow $r$-subgroup of $N$. The general
(1-dimensional) character $\psi$ of $N$, which extends the character $\zeta$ of $Z$, has the form
$\lambda = \zeta \otimes \psi$, where $\psi$ is a character of $N$. When $r = 0$ or $r$ does not divide $|N/Z|$, all possible
characters $\psi$ need to be considered; when $r$ divides $|N/Z|$ only the trivial character $\psi$ of $N$
is considered.

This amounts to giving a proof of this theorem. Subsequently, an alternative version of
the program was developed, exploiting this approach. It calculates the character degrees
for all characteristics (or for any given set of characteristics) simultaneously. The times
required for such calculations are increased by about 5% above the straightforward
characteristic zero cases, reflecting the overhead in keeping track of the viability of the
respective nonzero characteristics at any point on the general (characteristic zero) tree of
the characters.

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References