Appl. Math. Lett. Vol. 6, No. 2, pp. 79–81, 1993
Printed in Great Britain.

0893-9659/93 \$6.00 + 0.00

RANDOM FLUCTUATIONS AT AN EQUILIBRIUM OF A NONLINEAR REACTION-DIFFUSION EQUATION

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(Received and accepted December 1992)

Abstract—The expectation of a dynamical variable satisfying a general nonlinear diffusion equation with small random fluctuations about an equilibrium point is found to order ϵ^2. The dependence of the magnitude and direction of shift of the mean from the equilibrium on the properties of the nonlinear term is established.

1. INTRODUCTION

There has been much interest recently in stochastic effects in nonlinear dynamical systems [1]. Nonlinear partial differential equations with random forcing terms have particularly attracted much attention. Such equations have occurred in various areas of physics and biophysics. One important application in physics has been in the stochastic quantization of field theories [2–4], whereas similar equations, and in particular the Ginzburg-Landau equation, have arisen in stastistical mechanics [5,6]. In biophysics, nonlinear random diffusions arise when considering randomly occurring conductance changes in nerve cell membranes. Deterministically, the voltage changes accompanying conductance changes at the microscopic or macroscopic level can be predicted using the Hodgkin-Huxley equations [7]. A system of equations with similar properties is that of Fitzhugh-Nagumo, whose reduced form coincides with a Ginzburg-Landau equation. The reduced form of that equation with white noise stimulation has been considered previously [8].

In this paper, we will give a result for the general nonlinear diffusion driven by two-parameter white noise:

$$u_t = u_{xx} + g(u) + \epsilon W_{xt}, \tag{1}$$

where $\{W(x,t),\ x \in (-\infty,\infty),\ t \geq 0\}$ is a standard 2-parameter Wiener process, i.e., a Gaussian process with

$$E(W(x,t)) = 0,$$
$$E(W(x,t)W(y,s)) = \min(x,y)\min(s,t),$$

ϵ is a small real constant, and g is a function at least twice differentiable at equilibrium points, u_0, where $g(u_0) = 0$. The domain of $u(x,t)$ is $-\infty \leq a \leq x \leq b \leq \infty$ and $t \leq 0$. The second mixed partial derivative $w = W_{xt}$ is called 2-parameter white noise and we have formally

$$W(x,t) = \int_0^x \int_0^t w(y,s)\,ds\,dy.$$

We will demonstrate that when a dynamical system is randomly perturbed, as in equation (1), about an asymptotically stable point u_0, then the mean value of $u(x,t)$, denoted by $E(u(x,t))$, is shifted above or below the equilibrium point according to whether the second derivative $g''(u_0)$ is positive or negative. This phenomenon occurs even though the additive noise has zero mean. Expressions will be given from which the displacement can be calculated.

2. DERIVATION AND DISCUSSION

Since ϵ is small, we put

$$u(x,t) = u_0 + \sum_{k=1}^{\infty} \epsilon^k u_k(x,t), \tag{2}$$

assuming that initially $u(x,t) = u_0$ for all $x \in [a,b]$, and that suitable boundary conditions are given at $x = a$ and $x = b$. It is further assumed that u_0 is an asymptotically stable solution with $g'(u_0) < 0$. We note that the added noise in (1) has zero mean but that an added constant simply has the effect of changing u_0. To obtain $E(u)$ to order ϵ^2 we need $E(u_1)$ and $E(u_2)$.

Substituting (2) in (1) and equating coefficients of powers of ϵ gives a sequence of stochastic partial differential equations for the u_k, whose first two members are

$$u_{1,t} = u_{1,xx} + g'(u_0)\, u_1 + W_{xt}, \tag{3}$$

$$u_{2,t} = u_{2,xx} + g'(u_0)\, u_2 + \frac{g''(u_0)}{2!} u_1^2. \tag{4}$$

Each equation in the sequence is linear and has the same kernel. Existence and uniqueness of solutions of (1) and a mode of convergence of the perturbative expansion (2) have been established previously [9].

Equation (3), which is linear, has been studied in detail [10]. Its solution, since $u_1(x,0) = 0$, almost surely, is

$$u_1(x,t) = \int_0^t \int_a^b G(x,y;t-s)\, dW(y,s), \tag{5}$$

where the integral is a stochastic integral with respect to a two-parameter Wiener process [11] and G is the Green's function for

$$u_t = u_{xx} + g'(u_0)u.$$

Thus, $G = e^{g'(u_0)t} G_H$ where G_H is the Green's function for the usual heat equation $u_t = u_{xx}$. From the properties of stochastic integrals, it follows that $E(u_1(x,t)) = 0$.

In addition, we have, with probability one, $u_2(x,0) = 0$. Hence, from (4) one gets

$$u_2(x,t) = \frac{g''(u_0)}{2!} \int_0^t \int_a^b G(x,y;t-s)\, u_1^2(y,s)\, dy\, ds.$$

Hence, we have

$$E(u(x,t)) = u_0 + \frac{1}{2}\epsilon^2 g''(u_0) \int_0^t \int_a^b G(x,y;t-s)\, E[u_1^2(y,s)]\, dy\, ds + O(\epsilon^3). \tag{6}$$

Since $E(u_1^2)$ can be found from the integral representation of u_1, an explicit expression for $E(u)$ to order ϵ^2 is readily obtained.

Now $G(x,y;t-s) > 0$ for all x and for all $t > 0$. Furthermore, u_1^2 is nonnegative with probability 1, so the integral in the expression for u_2 is positive and so is its expectation. Thus, $E(u_2)$ has the sign of $g''(u_0)$. This enables one to write

$$E(u) = u_0 + \frac{1}{2}\epsilon^2 \operatorname{sgn}\, (g''(u_0))|F| + O(\epsilon^3),$$

where

$$F = |g''(u_0)| \int_0^t \int_a^b G(x,y;t-s)\, E[u_1^2(y,s)]\, dy\, ds.$$

This shows that the direction of shift of the mean is up from the equilibrium point if g has a positive second derivative at the equilibrium point and down if the second derivative is negative. The magnitude of the shift is given by the above expression.

It is apparent that this result is true if the noise is restricted to any subset A of $[a,b]$. That is, u satifies

$$u_t = u_{xx} + g(u) + I_A(x)\, W_{xt},$$

where $I_A(x)$ is the indicator of A defined by $I_A = 1$, $x \in A$, $I_A = 0$, $x \notin A$. Similar statements apply to any restriction of the time interval of application of the noise to any subset of $[0,\infty)$.

We may also consider the effects of a random impulsive forcing term such as that which arises when the Wiener process in (1) is replaced by the difference of two Poisson processes. Thus, let $\{\Pi^+(x,t),\ x \in (-\infty,\infty),\ t \geq 0\}$ and $\{\Pi^-(x,t),\ x \in (-\infty,\infty),\ t \geq 0\}$ be two independent 2-parameter Poisson processes with the same intensity, λ, such that $\Pi^{\pm}(x,t)$ are Poisson random variables with means λxt. Then a similar result is true for the mean of solutions of

$$u_t = u_{xx} + g(u) + \Pi^+_{xt}(x,t) - \Pi^-_{xt}(x,t),$$

in relation to an equilibrium point. This applies also to any case in which the Poisson processes are restricted to any subset of $[a,b] \times [0,\infty)$. These conclusions follow by replacing dW in expression (5) for $u_1(x,t)$ by $d(\Pi^+ - \Pi^-)$. A further extension is to the case of a combination of independent Gaussian and Poisson white noises.

In the case of (1) with

$$g(u) = u(u-a)(1-u),$$

where $0 < a < 1$, we obtain the Ginzburg-Landau equation of statistical mechanics [5] or the Fitzhugh-Nagumo equation of biophysics [12] with random perturbations. Explicit expressions for the mean to order ϵ^2 have been derived and evaluated on bounded intervals with Neumann conditions at the end points [13].

The displacement of the mean of $u(x,t)$ from u_0, when there is present a small random white noise term, has important ramifications for the measurement of the positions of equilibrium points in physical, chemical, and biological systems. Thus, any estimate of an equilibrium point of a distributed nonlinear system satisfying a diffusion equation, obtained in the presence of background noise which may be approximated by any of those considered here, and made by averaging an experimental recording, will give a value displaced away from the true equilibrium point in a direction which depends on the sign of the second derivative of the nonlinear term.

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