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Advances in Mathematics 225 (2010) 1893–1913

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An "almost" full embedding of the category of graphs into the category of groups

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Received 23 July 2009; accepted 15 April 2010

Communicated by Ross Street

Abstract

We construct a functor $F : \mathcal{G}raphs \to \mathcal{G}roups$ which is faithful and "almost" full, in the sense that every nontrivial group homomorphism $FX \to FY$ is a composition of an inner automorphism of FY and a homomorphism of the form Ff, for a unique map of graphs $f : X \to Y$. When F is composed with the Eilenberg–Mac Lane space construction $K(FX, 1)$ we obtain an embedding of the category of graphs into the unpointed homotopy category which is full up to null-homotopic maps.

We provide several applications of this construction to localizations (i.e. idempotent functors); we show that the questions:

(1) Is every orthogonality class reflective?
(2) Is every orthogonality class a small-orthogonality class?

have the same answers in the category of groups as in the category of graphs. In other words they depend on set theory: (1) is equivalent to weak Vopěnka's principle and (2) to Vopěnka's principle. Additionally, the second question, considered in the homotopy category, is also equivalent to Vopěnka's principle.

MSC: 18A40; 20J15; 55P60; 18A22; 03E55

Keywords: Category of groups; Localization; Large cardinals

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[1] The author was partially supported by grant N N201 387034 of the Polish Ministry of Science and Higher Education.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2010.04.015

1. Introduction

Matumoto [17] proved that for any graph Γ there exists a group G whose outer automorphism group is isomorphic to the group of automorphisms of Γ. His result received a considerable attention since every group can be realized as the group of automorphisms of some graph.

The main result of this article may be viewed as a functorial version of the above. We construct a functor F from the category of graphs to the category of groups which is faithful and "almost" full, in the sense that the maps

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{G}raphs}(X, Y) \to \mathrm{Hom}_{\mathcal{G}roups}(FX, FY)$$

induce bijections

$$\overline{F}_{X,Y} : \mathrm{Hom}_{\mathcal{G}raphs}(X, Y) \cup \{*\} \to \mathrm{Rep}(FX, FY).$$

Here $\mathrm{Rep}(A, B) = \mathrm{Hom}_{\mathcal{G}roups}(A, B)/B$ where B acts on $\mathrm{Hom}_{\mathcal{G}roups}(A, B)$ by conjugation and $*$ is an additional point which we send to the trivial element of Rep. A *graph* is a set with a binary relation.

Full and faithful functors are convenient tools that allow one to transfer constructions and properties between categories. The category of graphs is very comprehensive and well researched. Adámek and Rosický proved in [1, Theorem 2.65] that every accessible category has a full embedding into the category of graphs. Instead of quoting the complete definition of accessible categories let us mention that these contain, as full subcategories, "most" of the "non-homotopy" categories: the categories of groups, fields, R-modules, Hilbert spaces, posets (i.e. partially ordered sets), simplicial sets, metrizable spaces or CW-spaces and continuous maps, the category of models of some first-order theory, and many more. In fact, under a large cardinal hypothesis that the measurable cardinals are bounded above, any concretizable category fully embeds into the category of graphs [19, Chapter III, Corollary 4.5].

In this article we describe several applications of the functor F, constructed in Section 4; the choice of the applications is strongly affected by the interests of the author.

A *localization* may be defined as a functor from a category $\mathcal{C}$ to itself that is a left adjoint to inclusion of a subcategory $\mathcal{D} \subseteq \mathcal{C}$; it is an idempotent functor which may be viewed as a projection of $\mathcal{C}$ onto the subcategory $\mathcal{D}$. A more common definition of localization can be found in Section 8. Libman [16] inspired a question of whether the values of localization functors at finite groups can have arbitrarily large cardinalities. For all finite simple groups such localizations were constructed by Göbel, Rodríguez and Shelah in [11], [10], and for some such groups by the author in [18]. In Section 10 we see that the functor F immediately produces yet another such construction.

This article was motivated by another application. Adámek and Rosický proved in [1, Chapter 6] that large cardinal axioms called Vopěnka's principle and weak Vopěnka's principle (both formulated in the category of graphs) have many implications related to localizations and the structure of accessible categories. These axioms are believed to be consistent with the standard set theory ZFC while their negations are known to be consistent with ZFC. Casacuberta, Scevenels and Smith [5] extended some of these implications to the homotopy category. In Section 9 we see that a functor which sends a graph Γ to the Eilenberg–Mac Lane space $K(F\Gamma, 1)$ is, up to null-homotopic maps, a full embedding of the category of graphs into the (unpointed) homotopy category. We strengthen the results of [5] by showing that Vopěnka's principle is actually equivalent to its formulation in the homotopy category: every orthogonality class in the homotopy category is a small-orthogonality class in the homotopy category (i.e. it is associated with an f-localization of Bousfield and Dror Farjoun [9]) if and only if this is the case in the category of graphs.

On the other hand, it was hoped that some consequences of Vopěnka's principles in the category of groups might be provable in ZFC. Casacuberta and Scevenels [3] hint that this might be the case for a "long standing open question in categorical group theory" that asks if every orthogonality class $\mathcal{D}$, in the category of groups, is reflective – that is, if the inclusion functor $\mathcal{D} \to \mathcal{G}roups$ has a left adjoint. In Section 8 we find that this question is actually equivalent to weak Vopěnka's principle.

The work presented in this paper has begun during the author's visit to Centre de Recerca Mathemàtica, Bellaterra, at the inspiration of Carles Casacuberta.

2. Definitions

A *graph* Γ is a set of *vertices*, $\mathsf{vert}\,\Gamma$, together with a set of *edges*, which is a binary relation $\mathsf{edge}\,\Gamma \subseteq \mathsf{vert}\,\Gamma \times \mathsf{vert}\,\Gamma$. A morphism $\Gamma \to \Delta$ between graphs is an edge preserving function $\mathsf{vert}\,\Gamma \to \mathsf{vert}\,\Delta$. The category of graphs is denoted $\mathcal{G}raphs$.

An *m-graph* (m for multi-edge) is a category Γ whose objects form a disjoint union of a set of vertices, $\mathsf{vert}\,\Gamma$, and a set of edges, $\mathsf{edge}\,\Gamma$. Each nonidentity morphism of an m-graph Γ has its source in $\mathsf{edge}\,\Gamma$ and its target in $\mathsf{vert}\,\Gamma$. Each edge $e \in \mathsf{edge}\,\Gamma$ is a source of two nonidentity morphisms: one labeled ι_e whose target is the *initial vertex* of e, and the other labeled τ_e whose target is the *terminal vertex* of e. Morphisms between m-graphs are functors that preserve the edges, the vertices and the labeling: $f(\iota_e) = \iota_{f(e)}$ and $f(\tau_e) = \tau_{f(e)}$. The category of m-graphs is denoted $m\text{-}\mathcal{G}raphs$.

A *u-graph* (u for undirected-edge) is an m-graph without the labeling of morphisms. The category of u-graphs is denoted $u\text{-}\mathcal{G}raphs$.

A u-graph is usually visualized as in (4.1) where the nonidentity morphisms are represented by incidence between edges (intervals) and vertices (small circles). A graph or an m-graph is similarly visualized, with arrows on its edges.

We have an obvious full and faithful inclusion functor $I : \mathcal{G}raphs \to m\text{-}\mathcal{G}raphs$ which has a left adjoint (the edge collapsing functor $J : m\text{-}\mathcal{G}raphs \to \mathcal{G}raphs$), that is,

$$\mathrm{Hom}_{\mathcal{G}raphs}(J\Gamma, \Delta) \cong \mathrm{Hom}_{m\text{-}\mathcal{G}raphs}(\Gamma, I\Delta)$$

where Γ is in $m\text{-}\mathcal{G}raphs$ and Δ is in $\mathcal{G}raphs$.

A *graph of groups* is a functor $G : \Gamma \to \mathcal{G}roups$ where Γ is a u-graph and for each morphism i in Γ, $G(i)$ is a monomorphism. Γ is called the underlying u-graph of G.

Convention. If $G : \Gamma \to \mathcal{G}roups$ is a graph of groups and a, b are objects in Γ, we consider the values of G on a and b, that is, G_a and G_b, to be different whenever a and b are different, and G takes morphisms to inclusions. In short, we treat G as the image of an inclusion of Γ into $\mathcal{G}roups$ all of whose morphisms are inclusions. The objects of G are called the *edge* and the *vertex groups*.

A *tree* (a *tree of groups*) is a connected u-graph (graph of groups) without circuits, that is, closed paths without backtracking.

If G is a group, $g \in G$ and $A \subseteq G$ then ${}^{g}A$ denotes gAg^{-1}.

3. Bass–Serre theory

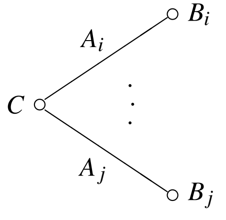
In this section we collect facts concerning groups acting on trees, which will be used later. The key reference is [20]. The symbol $\ast_A G_i$ denotes the *amalgam* of groups G_i along the common subgroup A, and $\operatorname{colim} G$ denotes the *colimit* of a graph of groups G.

Lemma 3.1. *Let $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ and A be a common subgroup of G_1 and G_2. If $H_1 \cap A = B = H_2 \cap A$ then the homomorphism $h : H_1 *_B H_2 \to G_1 *_A G_2$ induced by the inclusions is injective.*

Proof. See [20, §1.3, Proposition 3]. □

As a consequence we obtain

Lemma 3.2. *Let G be a graph of groups consisting of one central vertex group C and vertex groups B_i, $i \in I$, attached to C along edge groups A_i, $i \in I$:*



If $H_i \subseteq B_i$ are subgroups such that $H_i \cap A_i$ is trivial for $i \in I$ then the homomorphism $h : \ast_{i \in I} H_i \to \operatorname{colim} G$ induced by the inclusions is injective and its image trivially intersects C.

Proof. We identify I with an ordinal and proceed by induction. The case when I is a singleton is obvious, as is the case when I is a limit ordinal and the result is established for all $I_0 < I$. Suppose that $I = I_0 \cup \{i_0\}$ and the result is established for I_0. Let G_0 be the graph of groups obtained from G by deleting B_{i_0} and A_{i_0}. We have

$$\operatorname{colim} G = B_{i_0} *_{A_{i_0}} \operatorname{colim} G_0.$$

By the inductive assumption, h is injective on $\ast_{i \in I_0} H_i$ and $h(\ast_{i \in I_0} H_i) \cap C$ is trivial, and therefore Lemma 3.1 implies the result for I. □

The most powerful element of the Bass–Serre theory is the following.

Theorem 3.3. *(See [20, §4.5, Theorem 9].) Let G be a tree of groups and T the underlying u-graph. There exists a u-graph X containing T and an action of $G_T = \operatorname{colim} G$ on X which is characterized (up to isomorphism) by the following properties:*

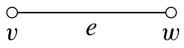
(a) *T is the fundamental domain for X mod G_T and*
(b) *for any v in $\mathsf{vert}\, T$ (resp. e in $\mathsf{edge}\, T$) the stabilizer of v (resp. e) in G_T is G_v (resp. G_e).*

Moreover, X is a tree.

As a corollary of Theorem 3.3 we immediately obtain:

Remark 3.4. Let X and G be as above.

(a) Each vertex group of G is a subgroup of $\operatorname{colim} G$.
(b) The stabilizers of the vertices and edges of X are respectively the $\operatorname{colim} G$ conjugates of the vertex and edge groups of G.
(c) If a subgroup H of $\operatorname{colim} G$ stabilizes two vertices v and w in X then it stabilizes the shortest path from v to w and therefore H is contained in all the vertex and edge stabilizers of this path.
(d) For any edge

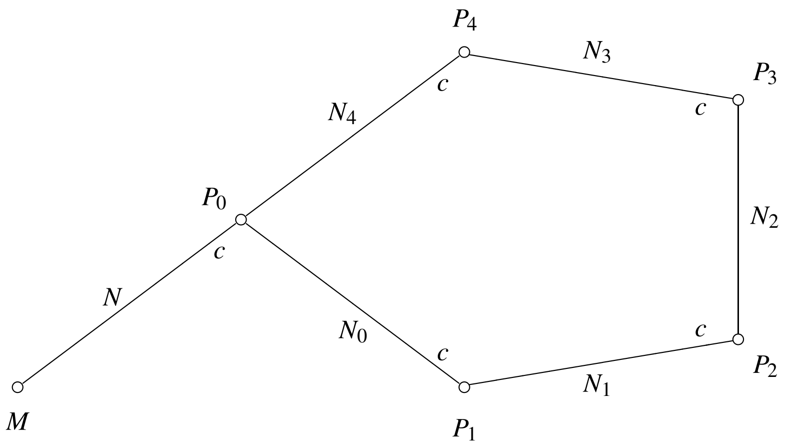


in G we have $G_v \cap G_w = G_e$ in $\operatorname{colim} G$.

Lemma 3.5. *If G is a tree of groups and $H \subseteq \operatorname{colim} G$ is a finite subgroup then H is conjugate in $\operatorname{colim} G$ to a subgroup of some vertex group G_v.*

4. Construction of the functor *F*

We start with the following graph of groups, where some edge to vertex incidences are labeled with c:



(4.1)

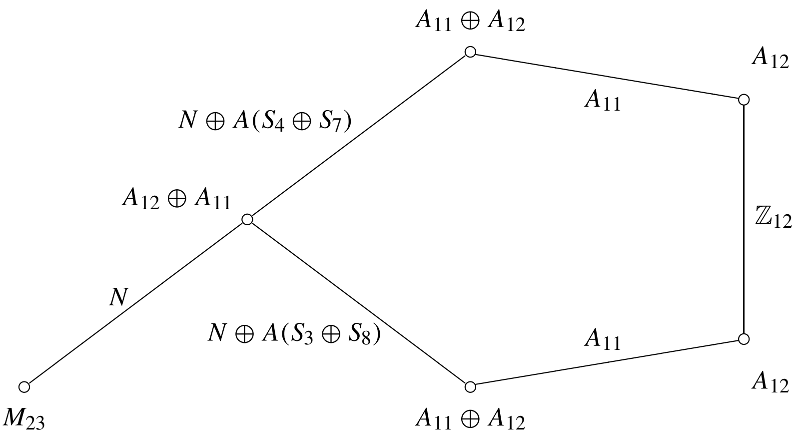
We assume the following conditions:

C1 M is finite, centerless and any homomorphism $f : M \to M$ is either trivial or an inner automorphism.
C2 M admits no nontrivial homomorphisms to P_i for $i = 0, 1, \dots, 4$.

C3 If an inclusion $A \subseteq B$ in (4.1) is labeled c and $f : B \to B$ is a homomorphism which is the identity on A then f is the identity.
C4 If A_1 and A_2 are edge groups ($A_1 \neq N_2$) adjacent to the common vertex group B then A_1 is not conjugate in B to a subgroup of A_2. If $A_1 = A_2$ we require that $N_B(A_1) = A_1$.
C5 $N_1 \cap N_2$ and $N_2 \cap N_3$ are trivial.
C6 $N_1 \cap N_0$ and $N_3 \cap N_4$ are trivial.
C7 If $A \supseteq C \subseteq B$ is an edge in (4.1) and $C \subseteq B$ is labeled c then no homomorphism $f : B \to A$ is the identity on C.
C8 If an inclusion $A \subseteq B$ in (4.1) is labeled c and $K \subseteq B$ is a normal subgroup which contains A then $K = B$.

Lemma 4.2. *There exists a graph of groups* (4.1) *satisfying conditions* C1–C8.

Proof. We have:



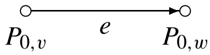
Here M_{23} is the Mathieu simple group, $N \cong \mathbb{Z}_{11} \rtimes \mathbb{Z}_5$ is the normalizer of the Sylow 11-subgroup in M_{23} [12, page 265], S_n and A_n denote the n-th symmetric and the n-th alternating groups. $A(S_p \oplus S_q)$ is the intersection of $S_p \oplus S_q$ and A_{12} in S_{12}. The inclusions are as follows:

(1) $N \subseteq A_{12} \oplus A_{11}$ is determined by any inclusions $N \subseteq A_{12}$ and $N \subseteq A_{11}$.
(2) $N \oplus A(S_p \oplus S_q) \subseteq A_{12} \oplus A_{11}$ equals $(N \subseteq A_{12}) \oplus$ (natural inclusion $A(S_p \oplus S_q) \subseteq A_{11}$).
(3) $N \oplus A(S_p \oplus S_q) \subseteq A_{11} \oplus A_{12}$ equals $(N \subseteq A_{11}) \oplus (A(S_p \oplus S_q) \subseteq A_{12})$.
(4) $A_{11} \subseteq A_{12}$ is the inclusion of a maximal subgroup.
(5) $A_{11} \subseteq A_{11} \oplus A_{12}$ is determined by $\mathrm{id}_{A_{11}}$ and $A_{11} \subseteq A_{12}$.
(6) $\mathbb{Z}_{12} \subseteq A_{12}$ is the inclusion of a transitive subgroup.

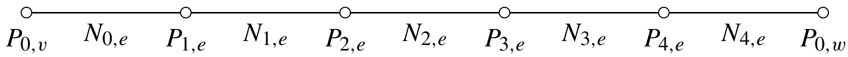
We know [12, page 265] that M_{23} has no outer automorphisms and has an element of order 23. The order of M_{23} is not divisible by 25. Also all the automorphisms of A_{11} and A_{12} come from S_{11} and S_{12}. This and well known properties of symmetric groups make it straightforward to verify that all the conditions C1–C8 are satisfied. □

The construction of $G\Gamma$ and $F\Gamma$

Let Γ be an m-graph. We construct a u-graph $A\Gamma$ as follows. Replace each vertex v in Γ with a vertex $P_{0,v}$, add a new vertex M, connect M to every $P_{0,v}$ with an edge N_v, and finally replace every subgraph



where $e \in \Gamma$ with a subgraph



(4.3)

We say that M, N, N_i, P_i for $i = 0, 1, \ldots, 4$ are *types* of objects M, N_a, $N_{i,a}$, $P_{i,a}$ for $i = 0, 1, \ldots, 4$ and a in $\mathsf{vert}\,\Gamma$ or $\mathsf{edge}\,\Gamma$, respectively. We see that the resulting functor A preserves colimits of connected diagrams.

We construct a graph of groups $G\Gamma$ by taking $A\Gamma$ as the underlying u-graph and sending each object P of $A\Gamma$ to a group isomorphic to the group in (4.1) labeled with the type of P. We send morphisms in $A\Gamma$ to the corresponding inclusions in (4.1). We label c those inclusions in $G\Gamma$ which correspond to similarly labeled inclusions in (4.1). The isomorphisms between the groups in $G\Gamma$ and the groups in (4.1), their inverses and compositions are referred to as *standard isomorphisms*. If $f : \Gamma \to \Gamma'$ is a morphism of m-graphs then we define $Gf : G\Gamma \to G\Gamma'$ in the obvious way using standard isomorphisms. We see that the resulting functor G, from m-graphs to graphs of groups, preserves colimits of connected diagrams.

We define

$$F\Gamma = \operatorname{colim} G\Gamma,$$

in particular $F\emptyset = M$. We obtain $Ff : F\Gamma \to F\Gamma'$ as the colimit homomorphism.

Remark 4.4. Since colimits commute we see that F also preserves colimits of connected diagrams.

5. Properties of the functor *F*

In order to apply Bass–Serre theory we need to construct $F\Gamma$ using colimits of trees of groups rather than colimits of general graphs of groups. Let $G_1\Gamma$ be the subgraph of groups of $G\Gamma$ consisting of the vertices of types M, P_0, P_1, P_4 and the edges of types N, N_0, N_4. Let $G_2\Gamma$ be the subgraph of $G\Gamma$ consisting of the vertices of types P_2, P_3 and the edges of type N_2. Without changing the colimit, we can make $G_2\Gamma$ a tree of groups by adding a trivial vertex group and connecting it to every vertex group of type P_2 with a trivial edge group. Let $G_0\Gamma$ be the subdiagram of $G\Gamma$ consisting of the edges of type N_1 and N_3. Then $G\Gamma$ is the colimit, in the category of diagrams, of the following:

$$G_1\Gamma \leftarrow G_0\Gamma \rightarrow G_2\Gamma.$$

Let $F_i\Gamma = \operatorname{colim} G_i\Gamma$ for $i = 1, 2, 3$. Since colimits commute, we see that $F\Gamma$ is the colimit of

$$F_1\Gamma \leftarrow F_0\Gamma \rightarrow F_2\Gamma.$$

It is clear that

$$F_0\Gamma = \mathop{\ast}_{e \in \mathsf{edge}\,\Gamma} (N_{1,e} * N_{3,e})$$

and

$$F_2\Gamma = \mathop{\ast}_{e \in \mathsf{edge}\,\Gamma} (P_{2,e} *_{N_{2,e}} P_{3,e}).$$

Lemma 5.1. *The homomorphisms $F_0\Gamma \to F_i\Gamma$ for $i = 1, 2$ are injective.*

Proof. This is a consequence of conditions C6 and C5 and Lemma 3.2. □

Lemma 5.2. *The vertex groups of $G\Gamma$ map injectively into $F\Gamma$.*

Proof. This follows from Remark 3.4(a) and the construction of $F\Gamma$ by means of colimits of trees, including Lemma 5.1. □

We need an analogue of Theorem 3.3:

Lemma 5.3. *Let Γ be an m-graph and $A\Gamma$ be the underlying u-graph of $G\Gamma$. There exists a u-graph X and an action of $F\Gamma$ on X which is characterized (up to isomorphism) by the following properties:*

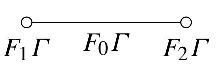
(a) *$A\Gamma$ is the fundamental domain for X mod $F\Gamma$ and*
(b) *for any v in $\mathsf{vert}\,A\Gamma$ (resp. e in $\mathsf{edge}\,A\Gamma$) the stabilizer of v (resp. e) in $F\Gamma$ is $G\Gamma_v$ (resp. $G\Gamma_e$).*

Proof. The proof is similar to the proof of [20, §4.5, Theorem 9]: Since we know from Lemma 5.2 that the vertex groups $G\Gamma_v$ embed into the colimit group $F\Gamma$, it is clear that $\mathsf{vert}\,X$ (resp. $\mathsf{edge}\,X$) is the disjoint union of the $F\Gamma \cdot v \cong F\Gamma/G\Gamma_v$ for $v \in \mathsf{vert}\,A\Gamma$ (resp. the $F\Gamma \cdot e \cong F\Gamma/G\Gamma_e$ for $e \in \mathsf{edge}\,A\Gamma$). The nonidentity morphisms are defined by means of the inclusions $G\Gamma_e \subseteq G\Gamma_{\text{target of } \iota_e}$ and $G\Gamma_e \subseteq G\Gamma_{\text{target of } \tau_e}$. This defines a graph on which the group $F\Gamma$ acts (on the left) in the obvious way, and all the assertions of the lemma are immediate. □

Remark 5.4. A subgroup of $F\Gamma$ stabilizes a vertex or an edge of X if and only if it is conjugate in $F\Gamma$ to a subgroup of a vertex group or an edge group of $G\Gamma$.

Lemma 5.5. *If $H \subseteq F\Gamma$ is a finite subgroup then it stabilizes a vertex of X.*

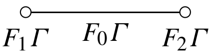
Proof. At the beginning of this section we have presented $F\Gamma$ as the colimit of the following tree of groups:



Lemma 3.5 implies that H is conjugate in $F\Gamma$ to a subgroup of F_1 or F_2, which again are colimits of trees of groups. Remark 5.4 completes the proof. □

Lemma 5.6. *Let X be the u-graph as in Lemma* 5.3. *If N is a subgroup of $F\Gamma$ which stabilizes two vertices P and Q in X then N stabilizes some path connecting these vertices.*

Proof. Let $\tilde{X}$ be the tree as in Theorem 3.3 for the graph of groups G below:



Then (cf. proof of Lemma 5.3) $\mathsf{vert}\,\tilde{X}$ is the disjoint union of the $F\Gamma \cdot v \cong F\Gamma/F_i\Gamma$ for $i = 1, 2$, and $\mathsf{edge}\,\tilde{X} = F\Gamma \cdot e \cong F\Gamma/F_0\Gamma$. We have an $F\Gamma$-equivariant "map" of u-graphs $f : X \to \tilde{X}$ induced by the inclusions $G\Gamma_v \subseteq F_1\Gamma$ or $G\Gamma_v \subseteq F_2\Gamma$ for $v \in \mathsf{vert}\,X$ and $G\Gamma_e \subseteq F_0\Gamma$ for e in $\mathsf{edge}\,X$ and of type N_1 or N_3. We write "map" in quotation marks since it takes edges of type other than N_1 or N_3 to vertices – it is a map of diagrams but not of u-graphs.

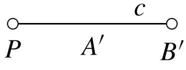
If $e \in \mathsf{edge}\,\tilde{X}$ then $f^{-1}(e)$ is a set of disjoint edges in X. If $v \in \mathsf{vert}\,\tilde{X}$ then $f^{-1}(v)$ is a tree isomorphic to the underlying tree of either $G_1\Gamma$ or $G_2\Gamma$.

Now N stabilizes $f(P)$ and $f(Q)$, and since $\tilde{X}$ is a tree, it stabilizes the shortest path L in $\tilde{X}$, connecting $f(P)$ to $f(Q)$.

If $e \in \mathsf{edge}\,L$ then the stabilizer of e is ${}^gF_0\Gamma$ for some $g \in F\Gamma$, hence $N \subseteq {}^gF_0\Gamma = \mathop{\ast}_{a\in\mathsf{edge}\,\Gamma}({}^gN_{1,a} * {}^gN_{3,a})$. Since the vertex groups of $G\Gamma$ are finite, Remark 5.4 implies that N is finite, hence $N \subseteq {}^gN_{i,a}$ for $i = 1$ or $i = 3$ and some $a \in \mathsf{edge}\,\Gamma$. This means that N stabilizes some edge in $f^{-1}(e) \subseteq X$.

If $v \in \mathsf{vert}\,L$ then the stabilizer of v is ${}^gF_1\Gamma$ or ${}^gF_2\Gamma$ for some $g \in F\Gamma$, hence $N \subseteq {}^gF_i\Gamma$ for $i = 1$ or $i = 2$ and N stabilizes the tree $f^{-1}(v) \subseteq X$. We know that N stabilizes two vertices in $f^{-1}(v)$: if v is an inner vertex of L these are ends of the edges in X, mapped by f to the edges adjacent to v in L, and stabilized by N as seen above; if $v = f(P)$ or $v = f(Q)$ is an end of L then one or both of these two vertices is P or Q respectively. Since $f^{-1}(v)$ is a tree we see that N stabilizes the shortest path connecting these two vertices. By concatenating the paths and edges described above, we obtain the required path that connects P and Q, and is stabilized by N. □

Lemma 5.7. *Let $A \subseteq B$ be an edge-to-vertex inclusion labeled c in* (4.1). *Let X be the u-graph as in Lemma* 5.3 *and*

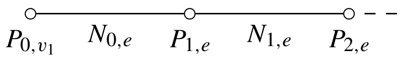


be an edge in $G\Gamma \subseteq X$ where A' and B' are of type A and B respectively. The standard isomorphism $f : A \to A'$ extends uniquely to $\overline{f} : B \to F\Gamma$, and this extension is the standard isomorphism onto B'.

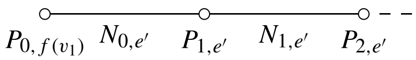
Proof. Only the uniqueness needs to be proved. Lemma 5.5 implies that $f(B)$ stabilizes a vertex V of X. Condition C7 excludes the case $V = P$. Lemma 5.6 implies that A' stabilizes some path connecting V to P. If $V \neq B'$ then A' stabilizes two different edges adjacent to P or to B'. This is excluded by condition C4 as the stabilizers of edges in X adjacent to a vertex W in $G\Gamma$ are the W-conjugates of edges in $G\Gamma$ adjacent to W. We are left with $V = B'$, that is, $f(B) \subseteq B'$, and condition C3 completes the proof. □

Lemma 5.8. *Let Γ and Δ be m-graphs. If $h : F\Gamma \to F\Delta$ is a homomorphism which restricts to the identity on $M = F\emptyset$ then there exists a unique $f : \Gamma \to \Delta$ such that $h = Ff$.*

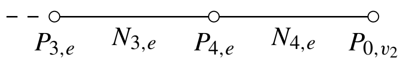
Proof. Lemma 5.7, applied to $N \subseteq P_0$ in (4.1), implies that for any vertex v in Γ there exists a vertex w in Δ such that h takes $P_{0,v}$ in $G\Gamma$ to $P_{0,w}$ in $G\Delta$ via a standard isomorphism. This allows us to define $f(v) = w$. Lemma 5.7, applied to the remaining inclusions, labeled c in (4.1), implies that for any edge $e = (v_1, v_2)$ in Γ there exist edges $e' = (f(v_1), w_2)$ and $e'' = (w_1, f(v_2))$ in Δ such that h takes, via standard isomorphisms, the "half edge subgraphs" of $G\Gamma$ to the "half edge subgraphs" of $G\Delta$ as indicated below:



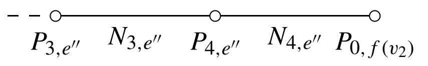
to



and



to



If $e' \neq e''$ then $P_{2,e} \cap P_{3,e} = N_{2,e}$ in $G\Gamma$ goes to $P_{2,e'} \cap P_{3,e''}$ which is trivial, and we have a contradiction. Thus $e' = e''$ and f preserves the edges. □

Lemma 5.9. *If Γ_0 is a sub-m-graph of Γ then $F\Gamma_0$ is a subgroup of $F\Gamma$.*

Proof. It is clear that $F_i\Gamma_0$ is a free factor of $F_i\Gamma$ for $i = 0$ and $i = 2$. It is also clear that $G_1\Gamma_0$ is a subtree of groups of $G_1\Gamma$; hence, inductively applying Lemma 3.1 we see that $F_1\Gamma_0$ is a subgroup of $F_1\Gamma$. We complete the proof by applying Lemma 3.1 to the inclusions $F_i\Gamma_0 \subseteq F_i\Gamma$ for $i = 1, 2$. □

Lemma 5.10. *Let Γ be an m-graph. For any $g \in F\Gamma$ there exists a finite subgraph $\Gamma_0 \subseteq \Gamma$ such that $g \in F\Gamma_0$.*

Proof. This is clear since $F\Gamma$ is generated by the vertex groups of $G\Gamma$ and each of those comes from a single vertex or edge in Γ. □

Lemma 5.11. *Let Γ be an m-graph. For any nontrivial homomorphism $f : M \to F\Gamma$ there exists an inner automorphism c_g of $F\Gamma$ such that the composition $c_g f$ is the identity on M.*

Proof. Lemma 5.5 and Remark 5.4 imply that $f(M)$ is conjugate in $F\Gamma$ to a subgroup of a vertex group V in $G\Gamma$. Condition C2 and the construction of $G\Gamma$ imply that $V = M$, thus $c_g f(M) \subseteq M$ for some g in $F\Gamma$. Condition C1 completes the proof. □

Lemma 5.12. *If Γ is an m-graph, A is a group and $f : F\Gamma \to A$ is a homomorphism which is trivial on M then f is trivial.*

Proof. The result follows from condition C8 since $F\Gamma$ is generated by the vertex groups connected to M by paths whose edges are labeled c as in (4.1). □

If A and B are groups then we define $\mathrm{Rep}(A, B) = \mathrm{Hom}(A, B)/B$, that is, we identify two homomorphisms $f, h : A \to B$ if there exists an inner automorphism c_g of B such that $f = c_g h$. The set $\mathrm{Rep}(A, B)$ contains a *trivial* element corresponding to the trivial homomorphism.

Theorem 5.13. *For all m-graphs Γ, Δ the composition*

$$\mathrm{Hom}_{m\text{-}\mathcal{G}raphs}(\Gamma, \Delta) \cup \{*\} \to \mathrm{Hom}_{\mathcal{G}roups}(F\Gamma, F\Delta) \to \mathrm{Rep}(F\Gamma, F\Delta),$$

where $*$ is sent to the trivial homomorphism, is bijective. The isomorphism is functorial in Γ and Δ.

Proof. This is immediate from Lemmas 5.12, 5.11 and 5.8. □

Let $\overline{\mathrm{Hom}}(A, B)$ denote the set of nontrivial homomorphisms from A to B.

Remark 5.14. $\overline{\mathrm{Hom}}(F\Gamma, F\Delta)$ is functorial in Γ and Δ since $\mathrm{Hom}(F\Gamma, F\Delta)$ is and Lemmas 5.11 and 5.12 imply that if $f : F\Gamma \to F\Delta$ and $h : F\Delta \to F\Phi$ are nontrivial homomorphisms then hf is also nontrivial.

Remark 5.15. Note that $\mathrm{Hom}(\emptyset, \Delta) = \mathrm{Hom}_{\mathcal{G}raphs}(\emptyset, \Delta)$ is a point. Lemmas 5.11 and 5.8 imply that for every $f : \mathrm{Hom}(\emptyset, \Delta) \to \overline{\mathrm{Hom}}(F\emptyset, F\Delta)$ we have a pullback diagram:

$$\begin{array}{ccc} \mathrm{Hom}(\Gamma, \Delta) & \longrightarrow & \overline{\mathrm{Hom}}(F\Gamma, F\Delta) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\emptyset, \Delta) & \xrightarrow{f} & \overline{\mathrm{Hom}}(F\emptyset, F\Delta) \end{array}$$

That is,

$$\overline{\mathrm{Hom}}(F\Gamma, F\Delta) \cong \overline{\mathrm{Hom}}(F\emptyset, F\Delta) \times \mathrm{Hom}(\Gamma, \Delta).$$

The following theorem puts together Remarks 5.14 and 5.15.

Theorem 5.16. *For m-graphs Γ and Δ we have a bijection*

$$\mathrm{Hom}(F\Gamma, F\Delta) \cong \overline{\mathrm{Hom}}(F\emptyset, F\Delta) \times \mathrm{Hom}(\Gamma, \Delta) \cup \{*\},$$

which is functorial in Γ and Δ. The $*$ corresponds to the trivial homomorphism. A nontrivial homomorphism $h : F\Gamma \to F\Delta$ corresponds to a pair $h|_{F\emptyset}$ and $f : \Gamma \to \Delta$ such that $Ff = h$.

6. Colimits and limits

In this section we prove that the functor F preserves directed colimits and countably codirected limits.

We say that a poset X is *directed (resp. countably directed)* if any finite subset (resp. any countable subset) of X has an upper bound in X. A poset is viewed as a category where $a \leqslant b$ corresponds to a morphism $a \to b$. A diagram (i.e. functor) $\Gamma : X \to \mathcal{C}$ and its colimit $\operatorname{colim} \Gamma$ are called *directed* if X is directed. A diagram Γ and its limit $\lim \Gamma$ are called *countably codirected* if the opposite category X^{op} is countably directed.

The results of this section are stated and proved for (countably) directed diagrams, but [1, Theorem 1.5] and [1, Remark 1.21] yield immediate generalizations to the (countably) filtered case.

In this article we use Remark 6.1 only; the remainder of this section is provided for the sake of completeness.

Colimits

We have noticed in Remark 4.4 that $F : m\text{-}\mathcal{G}raphs \to \mathcal{G}roups$ preserves colimits of connected diagrams. Since the inclusion functor $I : \mathcal{G}raphs \to m\text{-}\mathcal{G}raphs$ preserves directed colimits we obtain

Remark 6.1. The composition $FI : \mathcal{G}raphs \to \mathcal{G}roups$ preserves directed colimits.

Limits

The inclusion functor I preserves all limits. We investigate preservation of limits by F.

Lemma 6.2. *If Γ_1 and Γ_2 are subgraphs of an m-graph Γ then $F(\Gamma_1 \cap \Gamma_2) = F\Gamma_1 \cap F\Gamma_2$.*

Proof. Lemma 5.9 implies that the statement of the lemma makes sense. Since $\Gamma_1 \cup \Gamma_2 = \operatorname{colim}(\Gamma_1 \supseteq \Gamma_1 \cap \Gamma_2 \subseteq \Gamma_2)$ Remark 4.4 implies that $F(\Gamma_1 \cup \Gamma_2) = F\Gamma_1 *_{F(\Gamma_1 \cap \Gamma_2)} F\Gamma_2$ hence the result follows from Remark 3.4(d). □

Lemma 6.3. *If $\{\Gamma_\alpha\}_{\alpha \in A}$ is a countably codirected diagram of finite m-graphs then there exist α_0 and β in A such that*

(a) *the projection $p_0 : \lim \Gamma_\alpha \to \Gamma_{\alpha_0}$ is injective,*
(b) *the images of p_0 and $p^\beta_{\alpha_0} : \Gamma_\beta \to \Gamma_{\alpha_0}$ coincide.*

Proof. If S is a set of objects in $\Gamma = \lim \Gamma_\alpha$ then for any pair $s \neq t$ in S there exists $\alpha_{s,t}$ in A such that the projection $p_{s,t} : \Gamma \to \Gamma_{\alpha_{s,t}}$ is injective on $\{s, t\}$. If S is at most countable then there exists α_0 such that each $p_{s,t}$ factors through $p_0 : \Gamma \to \Gamma_{\alpha_0}$, hence p_0 is injective on S. But Γ_{α_0} is finite, hence Γ is finite, and by taking S to be the set of objects of Γ we complete the proof of (a).

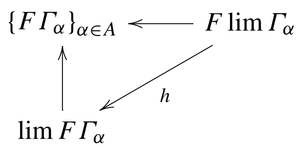
If $B = \{\beta \in A \mid \beta \to \alpha_0\}$ then $\lim_{\alpha \in A} \Gamma_\alpha \to \lim_{\beta \in B} \Gamma_\beta$ is an isomorphism. Clearly $\operatorname{im} p_0 \subseteq \operatorname{im} p^\beta_{\alpha_0}$ for $\beta \in B$. Let $K_\beta = (p^\beta_{\alpha_0})^{-1}(\operatorname{im} p^\beta_{\alpha_0} \setminus \operatorname{im} p_0)$ be viewed as a set of objects. If each K_β is nonempty then, as a codirected limit of finite sets, $\lim K_\beta$ is nonempty, which is a contradiction since $\lim K_\beta \subseteq \lim \Gamma_\beta$ and $p_0(\lim K_\beta) \cap p_0(\lim \Gamma_\beta) = \emptyset$. □

Lemma 6.4. *If $\{\Gamma_\alpha\}_{\alpha\in A}$ is a countably codirected diagram of m-graphs and $\Delta_\alpha \subseteq \Gamma_\alpha$ are finite subgraphs such that for all structure maps $p_\alpha^\beta : \Gamma_\beta \to \Gamma_\alpha$ we have $\Delta_\alpha \subseteq p_\alpha^\beta(\Delta_\beta)$ then there exist finite subgraphs $\overline{\Delta}_\alpha \subseteq \Gamma_\alpha$ such that $\Delta_\alpha \subseteq \overline{\Delta}_\alpha$ for all α and $\{\overline{\Delta}_\alpha\}_{\alpha\in A}$ is a diagram, that is, $p_\alpha^\beta(\overline{\Delta}_\beta) \subseteq \overline{\Delta}_\alpha$.*

Proof. Define $\overline{\Delta}_\alpha$ as the union of $p_\alpha^\beta(\Delta_\beta)$ over all structure maps p_α^β whose target is Γ_α. Only the finiteness of $\overline{\Delta}_\alpha$ needs proof. Suppose that $S = \{s_0, s_1, \ldots\}$ is an infinite subset of objects in $\overline{\Delta}_\alpha$. Then there exist $\alpha_0, \alpha_1, \ldots$ such that $s_i \in p_\alpha^{\alpha_i}(\Delta_{\alpha_i})$ for $i \in \mathbb{N}$. Since $\{\Gamma_\alpha\}_{\alpha\in A}$ is countably codirected there exists α_* in A such that Γ_{α_*} maps to every Γ_{α_i} for $i \in \mathbb{N}$, hence $\Delta_{\alpha_i} \subseteq p_{\alpha_i}^{\alpha_*}(\Delta_{\alpha_*})$ implies $p_\alpha^{\alpha_i}(\Delta_{\alpha_i}) \subseteq p_\alpha^{\alpha_*}(\Delta_{\alpha_*})$ for $i \in \mathbb{N}$, which is a contradiction since Δ_{α_*} is finite. □

Proposition 6.5. *The functor F constructed in Section* 4 *preserves countably codirected limits.*

Proof. Let $\{\Gamma_\alpha\}_{\alpha\in A}$ be a countably codirected diagram of m-graphs. We obtain an extended diagram



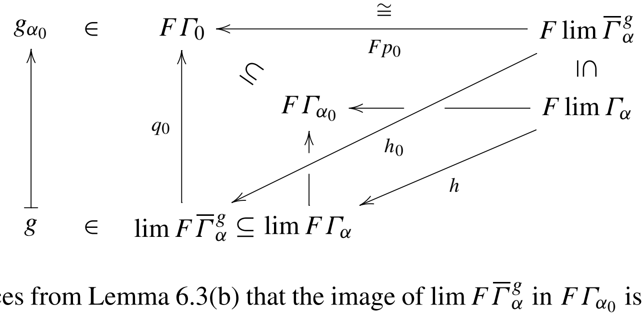
(6.6)

where h comes from the universal property of the limit. We need to prove that h is a bijection.

Injectivity of h. Let g be a nonidentity element of $F \lim \Gamma_\alpha$. Lemma 5.10 implies the existence of a finite subgraph $\Gamma_0 \subseteq \lim \Gamma_\alpha$ such that $g \in F\Gamma_0$. We look at the diagram formed by the images of Γ_0 in Γ_α for $\alpha \in A$, and by Lemma 6.3(a) we obtain α_0 such that Γ_0 maps injectively to Γ_{α_0}; hence Lemma 5.9 implies that $F\Gamma_0 \to F\Gamma_{\alpha_0}$ is one-to-one and therefore $h(g)$ is nontrivial, which proves the injectivity of h.

Surjectivity of h. Let $g \in \lim F\Gamma_\alpha$ and let g_α be the image of g in $F\Gamma_\alpha$. Let $\Gamma_\alpha^g \subseteq \Gamma_\alpha$ be a finite subgraph such that $g_\alpha \in F\Gamma_\alpha^g$ for $\alpha \in A$. Lemma 6.2 implies that we may require Γ_α^g to be the smallest subgraph with $g_\alpha \in F\Gamma_\alpha^g$. The minimality implies that $\Gamma_\alpha^g \subseteq p_\alpha^\beta(\Gamma_\beta^g)$ for all structure maps p_α^β, hence by Lemma 6.4 we obtain a diagram $\{\overline{\Gamma}_\alpha^g\}_{\alpha\in A}$ of finite subgraphs such that $\Gamma_\alpha^g \subseteq \overline{\Gamma}_\alpha^g \subseteq \Gamma_\alpha$.

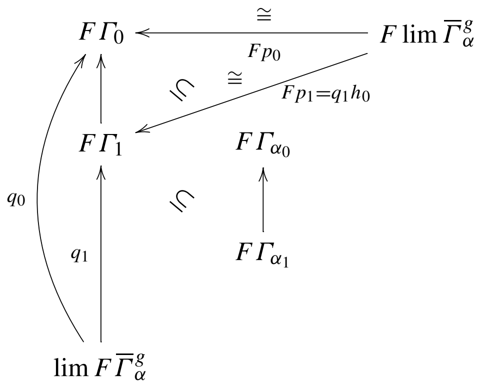
Lemma 6.3(a) gives us α_0 such that $p_0 : \lim \overline{\Gamma}_\alpha^g \to \overline{\Gamma}_{\alpha_0}^g \subseteq \Gamma_{\alpha_0}$ is injective. Let Γ_0 be the image of p_0. We put the above into the following diagram, which is a modification of (6.6).



(6.7)

One easily deduces from Lemma 6.3(b) that the image of $\lim F\overline{\Gamma}_\alpha^g$ in $F\Gamma_{\alpha_0}$ is contained in $F\Gamma_0$, hence q_0 is well defined. Fp_0 is an isomorphism since p_0 is an isomorphism, and therefore q_0 is onto.

To complete the proof it is enough to show that q_0 is one-to-one. Suppose that $\ker q_0$ contains a nonidentity element k. Then we have a structure map $\Gamma_{\alpha_1} \to \Gamma_{\alpha_0}$ such that k is not in the kernel of $\lim F\overline{\Gamma}^g_\alpha \to F\Gamma_{\alpha_1}$. As above, $p_1 : \lim \overline{\Gamma}^g_\alpha \to \overline{\Gamma}^g_{\alpha_1}$ is injective and if $\Gamma_1 = \operatorname{im} p_1$ then the image of $\lim F\overline{\Gamma}^g_\alpha$ in $F\Gamma_{\alpha_1}$ is contained in $F\Gamma_1$. We obtain a modification of (6.7):



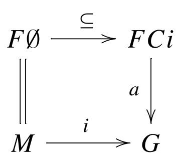
(6.8)

and $k \in \ker q_0 \setminus \ker q_1$, which is a contradiction, since $p_1 : \lim \overline{\Gamma}^g_\alpha \to \Gamma_1$ is an isomorphism. □

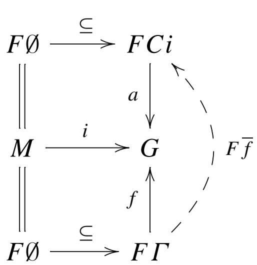
Remark 6.9. The functor F does not preserve codirected limits: Let $\Gamma_n = \mathbb{N}$ for positive integers n. For $n < m$ define $p^m_n : \Gamma_m \to \Gamma_n$ as $p^m_n(k) = \max\{0, k - (m - n)\}$. Then it is easy to see that $\lim \Gamma_n$ is countable while $\lim F\Gamma_n$ is uncountable.

7. Approximations of groups by graphs

Proposition 7.1. *Let G be a group and $M = F\emptyset$ be as in Section* 4. *For every inclusion $i : M \to G$ there exists an m-graph Ci and a diagram*



such that for every m-graph Γ and f as below



there exists a unique $\overline{f} : \Gamma \to Ci$ for which the diagram above commutes.

Proof. The construction of Ci is tautological: Let $N \subseteq P_0$ be the inclusion as in (4.1). The vertices of Ci are homomorphisms $v : P_0 \to G$ such that $v|_N = i|_N$. The edges $v \to w$ of Ci are those maps, of the graph of groups pictured in (4.3) to G, whose restrictions to $P_{0,v}$ and to $P_{0,w}$ are v and w respectively. The existence and uniqueness of $\overline{f}$ is immediate. □

8. Orthogonal subcategory problem in the category of groups

In this section we apply Theorem 5.16 to prove (Proposition 8.7) that if there exists an orthogonal pair in the category of graphs which is not associated with a localization then there exists an orthogonal pair in the category of groups which is not associated with a localization. The premise of the implication above is consistent with the standard set theory ZFC, in fact it is equivalent to the negation of weak Vopěnka's principle. We conclude this section with Proposition 8.8. The converses of Propositions 8.7 and 8.8 follow from [1, Theorem 6.22] and [1, Corollary 6.24(iii)].

In order to make the paper self-contained we begin with a collection of definitions and preliminary facts, most of them extracted from [3].

Orthogonal pairs

Let $\mathcal{C}$ be a category (here $\mathcal{G}roups$ or $\mathcal{G}raphs$). A morphism $f : A \to B$ is *orthogonal* to an object C (we write $f \perp C$) if f induces a bijection

$$\mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C). \tag{8.1}$$

If $\mathcal{M}$ is a class of morphisms and $\mathcal{O}$ is a class of objects in $\mathcal{C}$ then $\mathcal{M}^{\perp} = \{C \in \mathcal{C} \mid f \perp C \text{ for every } f \in \mathcal{M}\}$ and $\mathcal{O}^{\perp} = \{f : A \to B \mid f \perp C \text{ for every } C \in \mathcal{O}\}$. An *orthogonal pair* $(\mathcal{S}, \mathcal{D})$ consists of a class $\mathcal{S}$ of morphisms and a class $\mathcal{D}$ of objects such that $\mathcal{S}^{\perp} = \mathcal{D}$ and $\mathcal{D}^{\perp} = \mathcal{S}$. If $(\mathcal{S}, \mathcal{D})$ is an orthogonal pair then $\mathcal{D}$ is called an *orthogonality class*, $\mathcal{D}$ is closed under limits and $\mathcal{S}$ is closed under colimits. If $\mathcal{M}$ is a class of morphisms and $\mathcal{O}$ is a class of objects then $(\mathcal{M}^{\perp\perp}, \mathcal{M}^{\perp})$ and $(\mathcal{O}^{\perp}, \mathcal{O}^{\perp\perp})$ are orthogonal pairs.

Localizations

A *localization* is a functor $L : \mathcal{C} \to \mathcal{C}$ together with a natural transformation $\eta : Id \to L$ such that $\eta_{LX} : LX \to LLX$ is an isomorphism for every X and $\eta_{LX} = L\eta_X$ for all X.

Every localization functor L gives rise to an orthogonal pair $(\mathcal{S}, \mathcal{D})$ where $\mathcal{S}$ is the class of morphisms f such that Lf is an isomorphism and $\mathcal{D}$ is the class of objects isomorphic to LX for some X. A class $\mathcal{D}$ is called *reflective* if it is part of an orthogonal pair $(\mathcal{S}, \mathcal{D})$ which is associated with a localization.

Remark 8.2. Let $\mathcal{C}$ be a category and $(\mathcal{S}, \mathcal{D})$ an orthogonal pair in $\mathcal{C}$. If for each object X in $\mathcal{C}$ there exists a morphism $\eta_X : X \to LX$ in $\mathcal{S}$ with LX in $\mathcal{D}$ then the assignment $X \mapsto LX$ defines a localization functor associated with $(\mathcal{S}, \mathcal{D})$; this was observed in [4, 1.2].

Weak Vopěnka's principle

Weak Vopěnka's principle is a large cardinal axiom equivalent to the following statements:

(WV1) Every orthogonal pair in $\mathcal{G}raphs$ is associated with a localization.
(WV2) Every orthogonal pair in a locally presentable category ($\mathcal{G}roups$ is such a category) is associated with a localization.

The equivalence to (WV1) is proved in [1, Theorem 6.22] and [1, Example 6.23]. The equivalence to (WV2) is proved in [1, Example 6.25] and stated in remark that precedes it. Weak Vopěnka's principle is believed to be consistent with the standard set theory (ZFC), but it is not provable in ZFC: the negation of weak Vopěnka's principle is consistent with ZFC. Proposition 8.7 and (WV2) imply a new equivalent formulation of weak Vopěnka's principle:

(WV3) Every orthogonal pair in $\mathcal{G}roups$ is associated with a localization.

More details and an interesting historical essay on Vopěnka's principle and its weak version can be found in [1].

Orthogonal subcategory problem in the category of groups

Lemma 8.3. *Let $f : \Gamma \to \Phi$ be a morphism and Δ be an object in m-$\mathcal{G}raphs$. Then $f \perp \Delta$ if and only if $Ff \perp F\Delta$.*

Proof. Theorem 5.16 yields

$$\begin{array}{ccc} \mathrm{Hom}(F\Phi, F\Delta) & \cong & \overline{\mathrm{Hom}}(F\emptyset, F\Delta) \times \mathrm{Hom}(\Phi, \Delta) \cup \{*\} \\ \downarrow & & \downarrow \\ \mathrm{Hom}(F\Gamma, F\Delta) & \cong & \overline{\mathrm{Hom}}(F\emptyset, F\Delta) \times \mathrm{Hom}(\Gamma, \Delta) \cup \{*\} \end{array}$$

which implies the claim (see (8.1) for definition of orthogonality). □

Remark 8.4. Throughout the remainder of this section, for a given orthogonal pair $(\mathcal{S}, \mathcal{D})$ in m-$\mathcal{G}raphs$ we fix an orthogonal pair $(\overline{\mathcal{S}}, \overline{\mathcal{D}})$ in $\mathcal{G}roups$ such that $F\mathcal{S} \subseteq \overline{\mathcal{S}}$ and $F\mathcal{D} \subseteq \overline{\mathcal{D}}$. Such a pair $(\overline{\mathcal{S}}, \overline{\mathcal{D}})$ exists since by Lemma 8.3 we may take $\overline{\mathcal{S}} = F\mathcal{D}^{\perp}$ and $\overline{\mathcal{D}} = \overline{\mathcal{S}}^{\perp}$.

Lemma 8.5. *Let G be a group in $\overline{\mathcal{D}}$ which admits an embedding $i : F\emptyset \to G$. If Ci is the m-graph described in Proposition* 7.1 *then Ci is in $\mathcal{D}$.*

Proof. Let $f : \Gamma \to \Phi$ be in $\mathcal{S}$ and $h : \Gamma \to Ci$ be any map in m-$\mathcal{G}raphs$. Then the composition $F\emptyset \subseteq F\Gamma \to FCi \xrightarrow{a} G$ equals i, and so we obtain

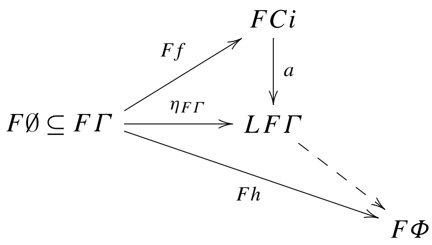
$$\begin{array}{ccc} F\Gamma & \xrightarrow{Fh} & FCi \\ {\scriptstyle Ff}\downarrow & \nearrow{\scriptstyle Fs} & \downarrow{\scriptstyle a} \\ F\Phi & \xrightarrow[t]{} & G \end{array}$$

The unique homomorphism t exists since $Ff \perp G$. The lift Fs exists by Proposition 7.1. Then $aFsFf = tFf = aFh$ and the uniqueness in Proposition 7.1 implies $FsFf = Fh$, hence by

Theorem 5.16 we have $sf = h$. If $s, s' : \Phi \to Ci$ are two maps such that $sf = h = s'f$ then $aFsFf = aFs'Ff$; hence, as $Ff \perp G$, we have $aFs = aFs'$. Uniqueness in Proposition 7.1 yields $Fs = Fs'$, and hence by Theorem 5.16 we obtain $s = s'$. Thus $f \perp Ci$ for any f in $\mathcal{S}$ and therefore Ci is in $\mathcal{D}$. □

Lemma 8.6. *If the orthogonal pair $(\overline{\mathcal{S}}, \overline{\mathcal{D}})$ is associated with a localization L then the pair $(\mathcal{S}, \mathcal{D})$ is also associated with a localization.*

Proof. Remark 8.2 implies that it is enough to find for every m-graph Γ a map $\eta_\Gamma : \Gamma \to \Delta$ in $\mathcal{S}$ such that Δ is in $\mathcal{D}$. We look at the diagram



For every map $h : \Gamma \to \Phi$ with Φ in $\mathcal{D}$ the group $F\Phi$ is in $\overline{\mathcal{D}}$, hence we have a factorization of Fh through $\eta_{F\Gamma}$ and therefore a factorization of h through $f : \Gamma \to Ci$. However, the uniqueness of the map $Ci \to \Phi$ under Γ is problematic. We remedy this through an inductive construction. Let $\Delta_0 = Ci$. If we can choose Φ in $\mathcal{D}$ and two different maps $g_1, g_2 : \Delta_0 \to \Phi$ such that $g_1 f = g_2 f$ then we define Δ_1 to be the limit of the diagram

$$\Delta_0 \underset{g_2}{\overset{g_1}{\rightrightarrows}} \Phi$$

We view Δ_1 as a subgraph of Δ_0, and correspondingly we obtain $f_1 : \Gamma \to \Delta_1$. We repeat this construction along some ordinal λ whose cofinality exceeds the cardinality of Δ_0; for limit ordinals $\gamma < \lambda$ we define Δ_γ to be the limit, that is, the intersection, of $\{\Delta_\alpha\}_{\alpha<\gamma}$. Since $\{\Delta_\alpha\}$ is a strictly decreasing sequence of subgraphs of Δ_0 it has to stabilize at some Δ_β, which implies that every map $\Gamma \to \Phi$ with Φ in $\mathcal{D}$ factors uniquely through $f_\beta : \Gamma \to \Delta_\beta$, hence f_β is in $\mathcal{S}$. Also Δ_β is in $\mathcal{D}$ since Ci is in $\mathcal{D}$ (by Lemma 8.5) and $\mathcal{D}$ is closed under limits. Therefore $\eta_\Gamma = f_\beta$ is the map we were looking for. □

Proposition 8.7. *Assuming the negation of weak Vopěnka's principle, there exists an orthogonal pair in the category of groups which is not associated with any localization.*

Proof. The negation of (WV1) implies the existence of an orthogonal pair $(\mathcal{S}_0, \mathcal{D})$ in $\mathcal{G}raphs$ which is not associated with any localization. We view $\mathcal{S}_0$ and $\mathcal{D}$ as classes of morphisms and objects in m-$\mathcal{G}raphs$. Let $\mathcal{S} = \mathcal{D}^\perp$; since $\mathcal{S}_0 \subseteq \mathcal{S}$ and $\mathcal{D} = \mathcal{S}^\perp$ we see that the orthogonal pair $(\mathcal{S}, \mathcal{D})$ is not associated with any localization in m-$\mathcal{G}raphs$. Lemma 8.6 implies that no pair $(\overline{\mathcal{S}}, \overline{\mathcal{D}})$ as described in Remark 8.4 is associated with a localization in $\mathcal{G}roups$. □

Vopěnka's principle and the existence of generators

We say that an orthogonal pair $(\mathcal{S}, \mathcal{D})$ is *generated* by a set of morphisms $\mathcal{S}_0$ if $\mathcal{D} = \mathcal{S}_0^{\perp}$. If such a set $\mathcal{S}_0$ exists then we say that $\mathcal{D}$ is a *small-orthogonality class*. A class of graphs is *rigid* if it admits no morphisms except the identity morphisms (i.e. the corresponding full subcategory is discrete). A class is *large* if it has no cardinality (i.e. it is bigger than any cardinal number).

Vopěnka's principle is another large cardinal axiom which influences the theory of localizations. Among many equivalent formulations of this principle we have the following ones:

(V1) There exists no large rigid class of graphs.
(V2) Every orthogonality class of graphs is a small-orthogonality class.
(V3) Every orthogonality class of objects in any locally presentable category (among those is $\mathcal{G}$*roups*) is a small-orthogonality class.

Equivalence between these statements follows from [1, Corollary 6.24] and [1, Example 6.12].

The next proposition is a nonconstructive but stronger, in terms of the large cardinal hierarchy [14, page 472], version of [5, Theorem 6.3]. Together with (V3) it yields another characterization of Vopěnka's principle:

(V4) Every orthogonality class of groups is a small-orthogonality class.

Proposition 8.8. *Assuming the negation of Vopěnka's principle there exists an orthogonal pair* $(\overline{\mathcal{S}}, \overline{\mathcal{D}})$ *in the category of groups such that* $\overline{\mathcal{D}}$ *is not a small-orthogonality class.*

Proof. Negation of (V2) implies the existence of an orthogonal pair $(\mathcal{S}, \mathcal{D})$ in $\mathcal{G}$*raphs* such that $\mathcal{D}$ is not a small-orthogonality class. As in Remark 8.4, we have an orthogonal pair $(\overline{\mathcal{S}}, \overline{\mathcal{D}})$ in $\mathcal{G}$*roups* such that $F\mathcal{S} \subseteq \overline{\mathcal{S}}$ and $F\mathcal{D} \subseteq \overline{\mathcal{D}}$. Suppose that $\overline{\mathcal{D}}$ is a small-orthogonality class, that is, there exists a set $\mathcal{S}_0 \subseteq \overline{\mathcal{S}}$ such that $\overline{\mathcal{D}} = \mathcal{S}_0^{\perp}$. Then there exists an uncountable cardinal λ such that $\overline{\mathcal{D}}$ is closed under λ-directed colimits; it is enough that the cofinality of λ is greater than all the cardinalities of domains and targets of maps in $\mathcal{S}_0$. Since $\mathcal{D} = F^{-1}(\overline{\mathcal{D}})$ Remark 6.1 implies that $\mathcal{D}$ is closed under λ-directed colimits. As the orthogonality class $\mathcal{D}$ is closed under arbitrary limits, by [13, Corollary] it is a λ-orthogonality class and thus a small-orthogonality class [1, 1.35 and the following]; this contradiction completes the proof. □

9. Homotopy category

We translate the results of the preceding section to the homotopy category *Ho* and to the pointed homotopy category Ho_*. In this section we obtain an orthogonality preserving embedding of $\mathcal{G}$*raphs* into *Ho* and a characterization of Vopěnka's principle in terms of the homotopy theory. Results of [5] were close to such a characterization. In this section *space* means simplicial set; whenever a space X is a right argument of a Hom or of a mapping space functor we assume that X is fibrant.

The functor $B : \mathcal{G}roups \to Ho_*$ which sends a group G to the Eilenberg–Mac Lane space $K(G, 1)$ is full and faithful. Since $\mathrm{Hom}_{Ho}(X, Y) = \mathrm{Hom}_{Ho_*}(X, Y)/\pi_1(Y)$ Theorem 5.13 implies that the composition BF followed by the forgetful functor $Ho_* \to Ho$ induces the bijections

$$BF_{X,Y} : \mathrm{Hom}_{m\text{-}\mathcal{G}raphs}(X, Y) \cup \{*\} \to \mathrm{Hom}_{Ho}(BFX, BFY) \tag{9.1}$$

where $*$ is sent to the constant map.

We say that a morphism $f : A \to B$ is *orthogonal* to an object X in Ho if it induces an equivalence of the mapping spaces

$$\mathrm{map}(B, X) \to \mathrm{map}(A, X).$$

This notion of orthogonality is used, as in Section 8, to define orthogonal pairs $(\mathcal{S}, \mathcal{D})$ whose right members $\mathcal{D}$ are called orthogonality classes. Analogously we define orthogonality in Ho_* by means of the pointed mapping spaces $\mathrm{map}_*(C, X)$. The fibration $\mathrm{map}_*(C, X) \to \mathrm{map}(C, X) \to X$ for any C shows that for X connected we have $f \perp X$ in Ho if and only if $f \perp X$ in Ho_* for any choice of base points [9, Chapter 1, A.1].

If X is an Eilenberg–Mac Lane space then $\mathrm{map}(A, X)$ is homotopy equivalent to a discrete space whose underlying set is $\mathrm{Hom}_{Ho}(A, X)$. Thus (9.1) yields the following.

Lemma 9.2. *Let $f : \Gamma \to \Phi$ be a morphism and Δ be an object in m-$\mathcal{G}$raphs. Then $f \perp \Delta$ if and only if $BFf \perp BF\Delta$.*

The following strengthens the result of [5].

Theorem 9.3. *The following conditions are equivalent:*

(V2) *Every orthogonality class of graphs is a small-orthogonality class.*
(*ho*V) *Every orthogonality class in the homotopy category is a small-orthogonality class.*

Proof. The implication (V2) $\Longrightarrow$ (*ho*V) is [5, Theorem 5.3].

Assuming the negation of (V2), Proposition 8.8 yields an orthogonal pair $(\mathcal{S}, \mathcal{D})$ in the category of groups such that $\mathcal{D}$ is not of the form $\mathcal{S}_0^{\perp}$ for any set of morphisms $\mathcal{S}_0$. Let $f : S^2 \to *$ be a map from a 2-sphere to a point. It is clear that a space X is orthogonal to f if and only if all the connected components of X are Eilenberg–Mac Lane spaces. Thus $f \in B\mathcal{D}^{\perp}$ and $B\mathcal{D}^{\perp\perp}$ is the class consisting of those spaces all of whose connected components are homotopy equivalent to a member of $B\mathcal{D}$.

The remainder of the proof is similar to the proof of Proposition 8.8. If $B\mathcal{D}^{\perp\perp}$ is a small orthogonality class then it is closed under λ-directed homotopy colimits, for some ordinal λ of sufficiently large cofinality. But then $B\mathcal{D}$ is closed under λ-directed homotopy colimits, hence $\mathcal{D}$ is closed under λ-directed colimits, hence $\mathcal{D}$ is a small orthogonality class, which is a contradiction. □

10. Large localizations of finite groups

In this section we obtain a third construction of a class of localizations which send a finite simple group to groups of arbitrarily large cardinalities. Previous examples of such localizations are described in [11], [10] and [18].

Let M be a group that is part of a graph of groups satisfying conditions C1–C8 stated before Lemma 4.2; we may take $M = M_{23}$, the Mathieu group.

Theorem 10.1. *For any infinite cardinal κ there exists a localization L in the category of groups such that LM has cardinality κ.*

Proof. Let F be the functor constructed in Section 4. We have $M = F\emptyset$. We know [22] that for every infinite cardinal κ there exists a graph Γ of cardinality κ such that the identity is the unique morphism $\Gamma \to \Gamma$. Let $i : \emptyset \to \Gamma$ be the inclusion of the empty set. Clearly i is orthogonal to Γ. Let $\eta = Fi : F\emptyset \to F\Gamma$. Lemma 8.3 implies that $\eta \perp F\Gamma$. By [2, Lemma 2.1] there exists a localization L in the category of groups such that $LF\emptyset = F\Gamma$, which completes the proof. □

11. Closing remarks

It is intriguing to ask the following.

Question: Does there exist a faithful functor F from the category of graphs to the category of abelian groups such that $f \perp \Gamma$ in the category of graphs if and only if $Ff \perp F\Gamma$ in the category of abelian groups?

Some results suggest that the category of abelian groups might be sufficiently comprehensive to allow such a functor: there exists a considerable literature on abelian groups with prescribed endomorphism rings (see for example [15, Chapter V], [8, Chapter XIV], [6]). In fact the example of an orthogonality class of groups that is not a small-orthogonality class, constructed in [5, Theorem 6.3] under the assumption of nonexistence of measurable cardinals, consists of abelian groups. Also there exist arbitrarily large sets $\{A_i\}_{i\in I}$ of abelian groups such that $\mathrm{Hom}(A_i, A_i) = \mathbb{Z}$ and $\mathrm{Hom}(A_i, A_j) = 0$ for $i \neq j$ in I [21] and such that $\mathrm{Hom}(A_i, A_i) = A_i$ and $\mathrm{Hom}(A_i, A_j) = 0$ for $i \neq j$ in I [7].

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