On the computation of the Ratliff–Rush closure

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Abstract

Let $R$ be a Cohen–Macaulay local ring with maximal ideal $m$. In this paper we present a procedure for computing the Ratliff–Rush closure of an $m$-primary ideal $I \subset R$.

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0. Introduction

Let $R$ be a Cohen–Macaulay Noetherian local ring of dimension $d \geq 1$ with maximal ideal $m$ and residue field $k$ that we may assume infinite. Given an $m$-primary ideal $I \subset R$ in Ratliff and Rush (1978) the Ratliff–Rush closure of $I$ is defined by $	ilde{I} = \bigcup_{k \geq 1} (I^{k+1} : I^k)$, and it holds that

$$
\tilde{I} = \bigcup_{k \geq 1} (I^{k+1} : (x_1^k, \ldots, x_d^k)),
$$

where $x_1, \ldots, x_d$ is a minimal reduction of $I$.

Although the Ratliff–Rush behaves badly under most of the basic operations of commutative algebra it is a basic tool in the study of the Hilbert functions of primary ideals; see for example Rossi and Swanson (2002) and its reference list.

Shah defined in Shah (1991) a finite chain of ideals between $I$ and its integral closure $\bar{I}$:

$$
I \subset I_{[d]} \subset \cdots \subset I_{(1)} \subset \bar{I}
$$

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where \( I_i \) is the \( i \)-coefficient ideal of \( I \), and \( I_{[d]} = \bar{I} \) is the Ratliff–Rush closure of \( I \). The computation of the integral closure of \( I \) can be performed under several hypotheses on \( R \) and \( I \); see Bruns and Kock (2001), Corso et al. (1998) and Delfino et al. (2002). On the other hand, few results are known about the explicit computation of coefficient ideals; see Heinzer and Lantz (1997). Ciupercă in Ciupercă (2001) computed the first coefficient ideal of an ideal \( I \) in an \( S_2 \) ring \( R \), by considering the \( S_2 \)-ification of the extended Rees algebra of \( I \).

The aim of this paper is to present a procedure for the computation of Ratliff–Rush closure. In the Section 1 we prove some results on superficial sequences that enable us to describe, in Section 2, a procedure for computing the Ratliff–Rush closure. We end the paper with some explicit computations of the Ratliff–Rush closure of ideals using the procedure of this paper.

We will use freely Bruns and Herzog (1993) as a general reference for the algebraic concepts appearing in this paper.

1. On superficial sequences

Let \( I \) be an \( m \)-primary ideal of \( R \). We denote by \( \text{gr}_I(R) = \bigoplus_{k \geq 0} I^k/I^{k+1} \) the associated graded ring of \( I \).

Let \( h_I(n) = \text{length}_R(R/I^{n+1}) \) be the Hilbert–Samuel function of \( I \), \( n \in \mathbb{N} \). Hence there exist integers \( e_j(I) \in \mathbb{Z} \) such that

\[
p_I(X) = \sum_{j=0}^{d} (-1)^j e_j(I) \binom{X+d-j}{d-j}
\]

is the Hilbert–Samuel polynomial of \( I \), i.e. \( h_I(n) = p_I(n) \) for \( n \gg 0 \). The integer \( e_j(I) \) is the \( j \)th Hilbert coefficient of \( I \), \( j = 0, \ldots, d \). Shah proved that coefficient ideals are the largest ideals \( I_{[d]} \) containing \( I \) and such that

(i) \( e_i(I) = e_i(I_{[t]}) \), for \( i = 0, \ldots, t \),

(ii) \( I \subset I_{[d]} \subset \cdots \subset I_{[t]} \subset \overline{T} \),

where \( \overline{T} \) is the integral closure of \( I \), (Shah, 1991). Notice that \( \bar{I} \) is the largest ideal containing \( I \) and such that \( e_i(I) = e_i(\bar{I}) \) for \( i = 0, \ldots, d \).

We say that \( x \in I \) is a superficial element of \( I \) if there exists an integer \( k_0 \) such that \( I^{k+1} : x = I^k \) for \( k \geq k_0 \). A set of elements \( x_1, \ldots, x_s \in I \) is a superficial sequence of \( I \) if \( x_j \) is a superficial element of \( I/(x_1, \ldots, x_{j-1}) \) for \( i = 1, \ldots, s \). A superficial sequence \( x_1, \ldots, x_s \) of \( I \) is called tame if \( x_j \) is a superficial element of \( I \), for all \( i = 1, \ldots, s \).

If \( p \in \mathbb{Q}[X] \) is a polynomial then we denote the first formal derivative of \( P \) by \( \Delta p = p(X) - p(X-1) \in \mathbb{Q}[X] \). It is well known that an element \( x \in I \) is a superficial element of \( I \) if and only if \( \frac{\Delta p}{p_T} = \Delta p_I \), where \( T = I/(x) \). If this is the case then \( \Delta p_I = \sum_{j=0}^{d-1} (-1)^j e_j(I) \binom{X+d-j-1}{d-j} \) and \( e_i(\overline{T}) = e_i(I) \) for \( i = 0, \ldots, d - 1 \).

It is well known that there exist Cohen–Macaulay local rings with finite residue field for which there are no superficial elements. For instance, the maximal ideal of
$R = (\mathbb{Z}/(2)[X, Y])/(XY(X + Y))$ has no superficial elements. We will show that if the residue field is infinite then there exist tame superficial sequences.

Let $S = \oplus_{n \geq 0} S_n$ be a Noetherian standard $S_0$-algebra; i.e. $S$ is generated by the degree one piece of $S$, where $S_0$ is an Artinian local ring with residue field $k$. We denote by $S_+ = \oplus_{n > 0} S_n$ the irrelevant ideal of $S$. Let $p_1, \ldots, p_t$ be the associated prime ideals of $S$ such that $\text{rad}(p_i) \neq S_+$. Notice that $S_1/mS_1$ is a finite dimensional $k$-vector space, so we can consider the Zariski open subset of $S$:

$$W(S) = \frac{S_1}{mS_1} \bigg/ \bigcup_{i=1, \ldots, t} \frac{[p_i]_1 + mS_1}{mS_1},$$

where $[p_i]_1$ denotes the homogeneous degree one piece of $p_i$, $i = 1, \ldots, t$.

The following result is well known—we include it here for the reader’s convenience; see Zariski and Samuel (1975) and Sally (1978).

**Proposition 1.1.** Let $S = \oplus_{n \geq 0} S_n$ be a Noetherian standard $S_0$-algebra, where $S_0$ is an Artinian local ring with residue field $k$.

(i) Let $z$ be an element of $S_1$ such that its coset belongs to $W(S)$. Then for $n \gg 0$,

$$[(0 : z)]_n = 0.$$

(ii) If the residue field is infinite, then $W(S) \neq \emptyset$.

From this result it is easy to prove:

**Corollary 1.2.** Let $R$ be a Cohen–Macaulay Noetherian local ring of dimension $d \geq 1$ with maximal ideal $m$ and residue field $k$ that we assume infinite. A set elements $x_1, \ldots, x_s \in I$, $1 \leq s \leq d$, such that their cosets $\overline{x}_1, \ldots, \overline{x}_s \in I/mI$ are generic, form a tame superficial sequence of $I$.

**Proof.** We will prove the claim by induction on $s$. Since $R$ is Cohen–Macaulay, an element $x \in I \setminus I^2$ is superficial if its initial form $x^* = x + I^2/I^2 \in \text{gr}_I(R)$ obeys $[(0 : x^*)]_n = 0$ for $n \gg 0$. Hence the $s = 1$ case follows from Proposition 1.1 (ii) with $S = \text{gr}_I(R)$.

Let us assume $s \geq 2$ and that $x_1, \ldots, x_{s-1}$ is a tame superficial sequence. We denote by $\overline{T} = \overline{I} + (x_1, \ldots, x_{s-1})/(x_1, \ldots, x_{s-1})$ the ideal generated by $I$ in $\overline{R} = R/(x_1, \ldots, x_{s-1})$. Then there exists a $k$-vector space epimorphism

$$\pi: \frac{I}{mI} \xrightarrow{\overline{T}} \frac{I + (x_1, \ldots, x_{s-1})}{mI + (x_1, \ldots, x_{s-1})}.$$ 

If we pick an element $x_s \in I$ such that its coset $\overline{x}_s \in I/mI$ belongs to $W(\text{gr}_I(R)) \cap \pi^{-1}(W(\text{gr}_{\overline{R}}(\overline{T})))$, then $x_s$ is a superficial element of $I$ and $x_1, \ldots, x_s$ is a superficial sequence of $I$. \qed

Since the residue field is assumed to be infinite, the following hold:

(1) a set of elements $x_1, \ldots, x_d \in I$, such that their cosets $\overline{x}_1, \ldots, \overline{x}_d \in I/mI$ are generic, form a tame superficial sequence of $I$. **Corollary 1.2**,
(2) if \( x_1, \ldots, x_d \in I \) is a set of elements such that \( p_{I/(x_1, \ldots, x_i)} = \Delta^i p_I, \) and \( p_{I/(x_i)} = \Delta p_I, \) for all \( i = 1, \ldots, d, \) then \( x_1, \ldots, x_d \) is a tame superficial sequence of the ideal \( I. \)

(3) if \( x_1, \ldots, x_d \) is a superficial sequence of \( I, \) then \( J = (x_1, \ldots, x_d) \) is a minimal reduction of \( I \) (Swanson, 1994).

We define the postulation number \( p_n(I) \) of \( I \) as the smallest integer \( n \) such that 
\[
\nu_I(1) = p_n(I) \quad \text{for all } t \geq n.
\]
Given a superficial sequence \( x_1, \ldots, x_s \) of \( I \) we denote by 
\[
p_n(I; x_1, \ldots, x_s) = \max \{ \nu_I(1), \ldots, \nu_I(s) \},
\]
the maximum among \( p_n(I) \) and \( p_n(I/(x_i)), i = 1, \ldots, j. \)

**Proposition 1.3.** Let \( I \) be an \( \mathfrak{m} \)-primary ideal of \( R \) and \( x \) a superficial element of \( I. \) For all \( k \geq p_n(I; x) + 1, \)
\[
(I^{k+1} : x) = I^k.
\]

**Proof.** We denote by \( \overline{I} = I/(x) \) the ideal of \( \overline{R} = R/(x). \) Let us consider the exact sequence
\[
0 \rightarrow \frac{(I^{k+1} : x)}{I^k} \rightarrow R/I^k \rightarrow R/I^{k+1} \rightarrow \overline{R}/I^{k+1} \rightarrow 0,
\]
so
\[
\text{length}_R \left( \frac{(I^{k+1} : x)}{I^k} \right) = h_I(k-1) - h_I(k) + h_{\overline{I}}(k).
\]
If \( k \geq p_n(I; x) + 1 \) then we have that \( h_I(k) = p_I(k), h_I(k-1) = p_I(k-1) \) and \( h_{\overline{I}}(k) = p_{\overline{I}}(k), \) so
\[
\text{length}_R \left( \frac{(I^{k+1} : x)}{I^k} \right) = p_I(k-1) - p_I(k) + p_{\overline{I}}(k).
\]
On the other hand, since \( x \) is a superficial element of \( I \) we have that \( p_{\overline{I}}(X) = p_I(X) - p_I(X-1); \) then \( (I^{k+1} : x) = I^k \) for all \( k \geq p_n(I; x) + 1. \) \( \square \)

We will show that for the explicit computations of coefficient ideals it is enough to consider the number \( p_n(I; x_1, \ldots, x_d). \) **Theorem 2.1(i),** but if we look for an explicit formula for the Ratliff–Rush closure avoiding the computation of superficial sequences we have to consider the Castelnuovo–Mumford regularity, **Theorem 2.1(ii).**

Given a standard \( A_0 \)-algebra \( A = A_0 \oplus A_1 \oplus \cdots \) with \( A_0 \) an Artin ring, we denote by \( \text{reg}(A) \) the Castelnuovo–Mumford regularity of \( A, \) i.e. the smallest integer \( m \) such that \( H_{A_m}(A)_n = 0 \) for all \( i = 0, \ldots, d \) and \( n \geq m - i + 1, \) where \( A_+ = A_1 \oplus \cdots \) is the irrelevant ideal of \( A. \)

We denote by \( f: \mathbb{N}^2 \rightarrow \mathbb{N} \) the numerical function defined by
\[
f(e, d) = \begin{cases} 
1 & \text{if } d = 1 \\
2^{e(d-1)!-1}(e-1)^{(d-1)!} & \text{if } d \geq 2.
\end{cases}
\]
Rossi, Trung and Valla prove that \( f(e, d) \) is an upper bound of the Castelnuovo–Mumford regularity of the associated graded ring of \( I. \)
Given a minimal reduction $J$ of $I$ we denote by $r_J(I)$ the reduction number of $I$ with respect to $J$, i.e. the smallest integer $r$ such that $I^{r+1} = JI'$.

In the next result we relate some of the numerical characters that we already defined in this paper.

**Proposition 1.4.** Let $R$ be a Cohen–Macaulay local ring of dimension $d \geq 1$. Let $I$ be an $m$-primary ideal of $R$ and $J$ a minimal reduction of $I$ generated by a tame superficial sequence $x_1, \ldots, x_d$. Then

(i) $r_J(I) \leq \text{reg}(\text{gr}_I(R)) \leq f(e_0(I), d) + 1$.

(ii) $p_n(I; x_1, \ldots, x_d) \leq f(e_0(I), d) + 1$.

**Proof.** (i) The first inequality comes from Trung (1987, Proposition 3.2); see also Brodman and Sharp (1998, Theorem 18.3.12). The second inequality is due to Rossi, Trung and Valla (Rossi et al., 2002, Corollary 3.4).

(ii) From Serre’s formula (Bruns and Herzog, 1993, Theorem 4.4.3), and the right hand side inequality in (i) we have that $p_n(I) \leq f(e_0(I), d) + 1$ and $p_n(I/(x_i), d - 1) + 1, i = 1, \ldots, d$. Since $e_0(I) = e_0(I/(x_i))$ and $f(e, d - 1) \leq f(e, d), we get the claim. □

In Rossi et al. (2002, Corollary 3.4) the right hand side inequality in (i) of the above result is proved for the maximal ideal $I = m$, but the proof holds also for general $m$-primary ideals.

**Corollary 1.5.** Let $x$ be a superficial element of $I$. For all $k \geq f(e_0(I), d) + 2$,

$(I^{k+1} : x) = I^k$.

**Proof.** This is a consequence of Proposition 1.4(ii) and Proposition 1.3. □


In this section we compute explicitly Ratliff–Rush closure by using Proposition 1.3 and Corollary 1.5. We consider the increasing ideal chain

$\mathcal{L}_1 \subset \mathcal{L}_2 \subset \cdots \subset \mathcal{L}_k \subset \cdots$

where

$\mathcal{L}_k = (I^{k+1} : (x^k_1, \ldots, x^k_d))$.

Notice that $\bar{I} = \bigcup_{k \geq 1} \mathcal{L}_k$ is the Ratlif–Rush closure of $I$.

**Theorem 2.1.** Let $R$ be a Cohen–Macaulay local ring of dimension $d \geq 1$. Let $I$ be an $m$-primary ideal of $R$ and let $x_1, \ldots, x_d$ be a tame superficial sequence of $I$.

(i) For all $k \geq p_n(I; x_1, \ldots, x_d) + 1$,

$\bar{I} = (I^{k+1} : (x^k_1, \ldots, x^k_d))$.

(ii) For all $k \geq (d + 1)(f(e_0(I)) + 2)$,

$\bar{I} = (I^{k+1} : I^k)$.
Proof. (i) We have to prove that for all \( k \geq \text{pn}(I; x_1, \ldots, x_d) + 1, \mathcal{L}_k = \mathcal{L}_{k+1}. \) Notice that for all \( n \geq 1 \) we have \( \mathcal{L}_n \subset \mathcal{L}_{n+1} \) so we only need to prove \( \mathcal{L}_{k+1} \subset \mathcal{L}_k. \) Given \( a \in \mathcal{L}_{k+1} \) we have \( ax_i^{k+1} = x_i(ax_i^k) \in I^{k+2}, \) for all \( i = 1, \ldots, d. \) Since \( k \geq \text{pn}(I; x_1, \ldots, x_d) + 1, \) from Proposition 1.3 we get \( ax_i^k \in I^{k+1}, \) for all \( i = 1, \ldots, d, \) so \( a \in \mathcal{L}_k. \)

(ii) Notice that \( J = (x_1, \ldots, x_d) \) is a minimal reduction of \( I, \) so for all \( k \geq r_J(I) \)

\[
I^{(d+1)k} = I^{dk}(x_1^k, \ldots, x_d^k).
\]

From Proposition 1.4 we have that \( r_J(I) \leq \text{reg}(\text{gr}_I(R)) \leq f(e_0(I), d). \) Let \( n \geq f(e_0(I), d) + 2 \) be an integer and let \( a \in \overline{I} \) be an element of the Ratliff–Rush closure of \( I. \) Hence from (i) we have \( a(x_1^k, \ldots, x_d^k) \subset I^{k+1} \) and since \( I^{(d+1)k} = I^{dk}(x_1^k, \ldots, x_d^k) \) we get

\[
aI^{(d+1)k} \subset aI^{dk}(x_1^k, \ldots, x_d^k) \subset I^{(d+1)k+1}.
\]

In particular we have \( a \in (I^{(d+1)k+1} : I^{(d+1)k}); \) since by definition \( (I^{(d+1)k+1} : I^{(d+1)k}) \subset \overline{I}, \) we get the claim. \( \square \)

From the last result we deduce that the problem of computing the Ratliff–Rush closure can be reduced to the computation of the postulation number of \( I \) and its quotients \( I/(x_i), i = 1, \ldots, d. \) Next we recall how to compute these numbers.

We denote by \( PS_I(X) \in \mathbb{Z}\llbracket X \rrbracket \) the Poincaré series of \( I, \) namely

\[
PS_I(X) = \sum_{i \geq 0} \text{length}_R \left( \frac{I^i}{I^{i+1}} \right) X^i.
\]

It is known that there exists a polynomial \( f(X) = \sum_{i=0}^s a_iX^i \in \mathbb{Z}[X] \) such that

\[
PS_I(X) = \frac{f(X)}{(1-X)^d}.
\]

It is easy to prove that \( e_j(I) = \sum_{i=j}^s \binom{i}{j}a_i, \) for \( j = 0, \ldots, d, \) and that \( \text{pn}(I) = \deg(f) - d. \)

Remark 2.2. It is well known that the computation of the Poincaré series of \( I \) and its quotients \( I/(x_i) \) can be reduced to an elimination of variables process; see for example the library primary.lib of CoCoA (Capani et al., 2003).

A procedure for computing the Ratliff–Rush closure

We assume that the residue field \( \mathbf{k} \) is infinite.

Step 1. Compute the Poincaré series of \( I. \) From this we get the Hilbert coefficients \( e_i(I), i = 1, \ldots, d, \) and the postulation number \( \text{pn}(I) \) of \( I. \)

Step 2. Pick \( d \) random elements \( x_1, \ldots, x_d \) of the \( \mathbf{k} \)-vector space \( I/\mathfrak{m}I \) such that \( p_i/(x_1, \ldots, x_i) = \Delta^i p_i \) and \( p_i/(x_i) = \Delta p_i, \) for \( i = 1, \ldots, d. \) We check these conditions by computing the corresponding Poincaré series. Recall that \( x_1, \ldots, x_d \) is a tame superficial sequence of \( I \) and generates a minimal reduction of \( I, \) Corollary 1.2.
Step 3. From the computation of the Poincaré series \( PS_I(x_i) = f_i(X) / (1 - X)^{d-1} \), for \( i = 1, \ldots, d \), and the fact that \( \text{pn}(I / x_i) = \deg(f_i) - (d - 1) \), we get the number \( \text{pn}(I : x_1, \ldots, x_d) \).

Step 4. For \( k \geq \text{pn}(I : x_1, \ldots, x_d) + 1 \) we get

\[
\tilde{I} = (I^{k+1} : (x_1^k, \ldots, x_d^k)).
\]

**Remark 2.3.** Notice that if \( I \) is a monomial ideal, then Step 4 can be performed without Gröbner basis computation.

We will show how to compute the Ratliff–Rush closure in some explicit examples of Ciupercă (2001), Heinzer and Lantz (1997) and Rossi and Swanson (2002).

The computations are performed by using CoCoA and the ground field is assumed to be of characteristic zero with arbitrary large precision (Capani et al., 2003). In the following examples tame superficial sequences are obtained by taking random elements.

**Example 2.4.** Example 1.10 of Rossi and Swanson (2002). Let \( I = (x^{10}, x^8 y, xy^4, y^5) \) be an ideal of \( R = \mathbb{Q}[x, y]_{(x, y)} \). The Poincaré series of \( I \) is

\[
PS_I(X) = \frac{35 + 4X + 4X^2 + 4X^3 - 2X^4}{(1 - X)^2},
\]

so \( e_0(I) = 45, e_1(I) = 16, e_2(I) = 4 \) and \( \text{pn}(I) = 2 \). A CoCoA computation shows that

\[
PS_{I/(xy^4)}(X) = \frac{35 + 6X + 2X^2 + 2X^3}{1 - X},
\]

and

\[
PS_{I/(y^5+x^8 y+xy^4+x^{10})}(X) = \frac{35 + 5X + 4X^2 + X^3}{1 - X}.
\]

Computing the corresponding Hilbert coefficients we deduce that \( xy^4 \) and \( y^5 + x^8 y + xy^4 + x^{10} \) are superficial elements of \( I \) and \( \text{pn}(I; y^5 + x^8 y + xy^4 + x^{10}, xy^4) = 2 \). Since the length of \( R/(y^5 + x^8 y + xy^4 + x^{10}, xy^4) \) is \( 45 = e_0(I) \) we deduce that \( y^5 + x^8 y + xy^4 + x^{10}, xy^4 \) is a tame superficial sequence of \( I \). Then by Theorem 2.1(i) we get

\[
I \subseteq \tilde{I} = (I^4 : ((y^5 + x^8 y + xy^4 + x^{10})^3, (xy^4)^3))
= (x^{10}, y^5, xy^4, x^7 y^2, x^6 y^3, x^5 y^4, x^8 y).
\]

**Remark 2.5.** Let \( x_1 = y^5 + x^8 y + xy^4 + x^{10}, x_2 = xy^4 \) be the minimal reduction of the ideal \( I \) of the last example. Since \( \text{pn}(I : x_1, x_2) = 2 \) we have that \( \tilde{I} = (I^4 : (x_1^3, x_2^2)) \). On the other hand, Theorem 2.1(ii) gives that \( \tilde{I} = (I^{k+1} : I^k) \) for all \( k \geq 540 \), which is a hard computation.

**Example 2.6.** Example 1.4 of Rossi and Swanson (2002). Let us consider the ideal

\[
I = (y^{22}, x^4 y^{18}, x^7 y^{15}, x^8 y^{14}, x^{11} y^{11}, x^{14} y^8, x^{15} y^7, x^{18} y^4, x^{22})
\]

of the local ring \( R = \mathbb{Q}[x, y]_{(x, y)} \). A similar computation to that we did in the previous
example, with \( x_1 = x^{22} + y^{22} \) and \( x_2 = y^{22} + x^4 y^{18} + x^7 y^{15} + x^8 y^{14} + x^{11} y^{11} + x^{14} y^8 + x^{15} y^7 + x^{18} y^6 + x^{22} \), shows that \( \tilde{I} = (I^4 : (x_1^2, x_2^2)) = I \) and \( I^2 \subseteq \tilde{I}^2 = (I^8 : (x_1^6, x_2^6)) = I^2 + (x^{24} y^{20}, x^{20} y^{24}). \)

**Example 2.7.** Example 3.3 of Ciupercă (2001). Let us consider the ideal \( I = (x^8, x^3 y^2, x^2 y^4, y^8) \) of the local ring \( R = \mathbb{Q}[x, y]_{(x,y)} \). A similar computation to that carried out before shows that \( I = \tilde{I} \); here \( x_1 = x^8 + x^3 y^2 + y^8 \) and \( x_2 = x^8 + y^8 \). Ciupercă in Ciupercă (2001) computed the first coefficient ideal of \( I \):

\[ I = \tilde{I} \subseteq I_{[1]} = (x^8, x^3 y^2, x^2 y^4, xy^6, y^8). \]

**Example 2.8.** Example 3.1 of Heinzer and Lantz (1997). Let us consider the ideal \( I = (x^6, x^2 y^4, y^6, x^2 z^2, z^6, x^4 y^2, x^4 z^2, x^2 y^2 z^2) \) of the local ring \( R = \mathbb{Q}[x, y, z]_{(x,y,z)} \). A similar computation to that carried out before shows that \( I = \tilde{I} = (I^2, (x_1, x_2, x_3)) \), where \( x_1 = y^6, x_2 = z^6, \) and \( x_3 = y^6 + x^6 + x^2 y^2 z^2 \).

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