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A note on the Harris–Kesten Theorem

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Abstract

A short proof of the Harris–Kesten result that the critical probability for bond percolation in the planar square lattice is $1/2$ was given in [B. Bollobás, O.M. Riordan, A short proof of the Harris–Kesten Theorem, Bull. London Math. Soc. 38 (2006) 470–484], using a sharp-threshold result of Friedgut and Kalai. Here we point out that a key part of this proof may be replaced by an argument of Russo [L. Russo, An approximate zero–one law, Z. Wahrscheinlichkeitstheor. Verwandte Geb. 61 (1982) 129–139] from 1982, using his approximate zero–one law in place of the Friedgut–Kalai result. Russo's paper gave a new proof of the Harris–Kesten Theorem that seems to have received little attention.

Let $\mathbb{Z}^2$ be the planar square lattice, i.e., the graph with vertex set $\mathbb{Z}^2$ in which each pair of nearest neighbours is joined by an edge. Let $X = E(\mathbb{Z}^2)$ be the edge-set of $\mathbb{Z}^2$, and let $\Omega = \{-1, +1\}^X$. We write $\omega = (\omega_e)_{e\in X}$ for an element of Ω, and say that the edge e is *open* (in the state ω) if $\omega_e = +1$, and *closed* if $\omega_e = -1$. An event $A \subset \Omega$ is *local* if it depends on only finitely many coordinates. As usual, let Σ be the sigma-field generated by local events, and let $\mathbb{P}_p$ be the probability measure on (Ω, Σ) in which each edge is open with probability p, and these events are independent. Let $\theta(p)$ be the $\mathbb{P}_p$-probability that the origin is in an *infinite open cluster*, i.e., an infinite connected subgraph C of $\mathbb{Z}^2$ with every edge of C open. In 1960, Harris [3] proved that $\theta(1/2) = 0$; in 1980, Kesten [5] showed that $\theta(p) > 0$ for $p > 1/2$, establishing that $p_c = 1/2$ is the 'critical probability' for this model. A short proof of these results was given in [1], using a sharp-threshold result of Friedgut and Kalai [2], itself based on a result of Kahn, Kalai and Linial [4].

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In 1982, Russo [6] proved a general sharp-threshold result (weaker than the more recent results described above) and applied it to percolation, to give a new proof of the 'equality of critical probabilities' for site percolation in $\mathbb{Z}^2$. Although Russo does not explicitly say this, his application applies equally well to bond percolation, giving a new proof of the Harris–Kesten Theorem that seems not to be well known. Here we shall present Russo's general sharp-threshold result, and then give a complete version of his application, to bond percolation in $\mathbb{Z}^2$.

Replacing the appropriate section of [1] with this argument gives an even simpler proof of the Harris–Kesten Theorem; we are grateful to Professor Ronald Meester for bringing this to our attention.

An event $A \subset \Omega$ is *increasing* if $\omega \in A$ and $\omega_e \leq \omega'_e$ for every e imply $\omega' \in A$, i.e., if A is preserved when the state of one or more edges is changed from closed to open. An edge e is *pivotal* for an event A if changing the state of e affects whether or not A holds. Let $\delta_e A$ be the event that e is pivotal for A, so $\omega \in \delta_e A$ if and only if exactly one of ω^+, ω^- is in A, where $\omega^\pm$ are the states that agree with ω on all edges other than e, with $\omega_e^+ = 1$ and $\omega_e^- = -1$. In [6], Russo proved the following result about the product measure $\mathbb{P}_p$; in this result the structure of $\mathbb{Z}^2$ is irrelevant, i.e., the ground-set X can be any countable set.

Theorem 1. *For every $\varepsilon > 0$ there is an $\eta > 0$ such that if A is an increasing local event with*

$$\mathbb{P}_p(\delta_e A) < \eta$$

for every $e \in X$ and every $p \in [0, 1]$, then there is a $p_0 \in [0, 1]$ with

$$\mathbb{P}_{p_0-\varepsilon}(A) \leq \varepsilon \quad \textit{and} \quad \mathbb{P}_{p_0+\varepsilon}(A) \geq 1 - \varepsilon.$$

As in [1], by a *k by ℓ rectangle* we mean a rectangle $[a, b] \times [c, d]$ with $a, b, c, d \in \mathbb{Z}$ and $b - a = k$, $d - c = \ell$. We identify a rectangle with the corresponding subgraph of $\mathbb{Z}^2$, including the boundary. A rectangle R has a *horizontal open crossing* if there is a path in R consisting of open edges, joining a vertex on the left-hand side of R to one on the right; we write $H(R)$ for this event. Our starting point will be the following consequence of the Russo–Seymour–Welsh Lemma (see [1] and the references therein): there is a constant $c > 0$ such that

$$\mathbb{P}_{1/2}(H(R)) \geq c, \tag{1}$$

for any $3n$ by n rectangle R. This is essentially the case $\rho = 3$ of Corollary 7 in [1]. (The latter result has an irrelevant restriction to n even; the present statement is immediate from the case $\rho = 4$ of this result.)

Our aim is to deduce Lemma 11 of [1], restated below.

Lemma 2. *Let $p > 1/2$ be fixed. If R_n is a $3n$ by n rectangle, then $\mathbb{P}_p(H(R_n)) \to 1$ as $n \to \infty$.*

It is well known that Lemma 2 implies Kesten's Theorem; see [1]. We shall deduce Lemma 2 from (1) using Theorem 1 and Harris's result, that $\theta(1/2) = 0$. We shall need the concept of the *dual lattice* $(\mathbb{Z}^2)^*$: this is the planar dual of the graph $\mathbb{Z}^2$, having a vertex for each face of $\mathbb{Z}^2$, and an edge e^* for each edge e of $\mathbb{Z}^2$, joining the two vertices corresponding to the faces of $\mathbb{Z}^2$ in whose boundary e lies. We take e^* to be open if and only if e is closed. The following argument is based on that of Russo [6].

Proof of Lemma 2. Let $p_1 > 1/2$ be fixed. Let D be a constant to be chosen below, and let R be a $3n$ by n rectangle with $n \geq 2D + 1$. Suppose that $\omega \in \delta_e H(R)$, and define $\omega^\pm$ as above. Note that e must be an edge of R, as $H(R)$ depends only on such edges. Then, in ω^+ there is an open path in R from the left-hand side to the right using the edge e. Hence, in ω, the endpoints of e are joined by open paths to the left- and right-hand sides of R. One of these paths must have length at least $(3n-1)/2 \geq D$. Thus, for any p,

$$\mathbb{P}_p(\delta_e H(R)) \leq 2\mathbb{P}_p(0 \to D), \tag{2}$$

where $0 \to D$ is the event that there is an open path of length D starting at the origin. Our assumption that e is pivotal also implies that $H(R)$ does not hold in ω^-. It follows (by Lemma 3 of [1]) that in ω^- there is an open path in the dual lattice joining the top of R to the bottom, using the edge e^*. Hence, in the dual lattice, one of the endpoints of e^* is in an open path of length at least D. As edges of the dual lattice are open independently with probability $1-p$, it follows that

$$\mathbb{P}_p(\delta_e H(R)) \leq 2\mathbb{P}_{1-p}(0 \to D). \tag{3}$$

Let $0 < \varepsilon < \min\{(p_1 - 1/2)/2, c\}$ be arbitrary, where $c > 0$ is a constant for which (1) holds. Let $\eta = \eta(\varepsilon)$ be as in Theorem 1. For any p we have $\mathbb{P}_p(0 \to D) \searrow \theta(p)$ as $D \to \infty$. Hence, by Harris's Theorem (Theorem 8 in [1]), $P_{1/2}(0 \to D) \to 0$, so we may choose D such that $\mathbb{P}_{1/2}(0 \to D) \leq \eta/3$. As the event $0 \to D$ is increasing, for $p \leq 1/2$ we have

$$\mathbb{P}_p(0 \to D) \leq \mathbb{P}_{1/2}(0 \to D) \leq \eta/3.$$

Using (2) for $p \leq 1/2$ and (3) for $p \geq 1/2$, it follows that for any $p \in [0, 1]$ and any edge e in R we have

$$\mathbb{P}_p(\delta_e H(R)) \leq 2\eta/3 < \eta.$$

As $H(R)$ is an increasing local event, and $\delta_e H(R)$ is empty for edges outside R, the conditions of Theorem 1 are satisfied. Hence, $\mathbb{P}_p(H(R))$ increases from at most $\varepsilon < c$ to at least $1-\varepsilon$ in some interval of width at most $2\varepsilon < p_1 - 1/2$. As $\mathbb{P}_{1/2}(H(R)) \geq c$ by (1), it follows that $\mathbb{P}_{p_1}(H(R)) \geq 1-\varepsilon$. In other words, we have shown that for $p_1 > 1/2$ and $\varepsilon > 0$ fixed and R_n a $3n$ by n rectangle, we have $\mathbb{P}_{p_1}(H(R_n)) \geq 1-\varepsilon$ if n is large enough. As $\varepsilon > 0$ is arbitrary, this completes the proof. □

In Section 5 of [1], the Friedgut–Kalai sharp-threshold result is used to deduce from (1) a result (Lemma 9 in [1]) that is somewhat stronger than Lemma 2. This stronger form was used in the first proof of Kesten's Theorem given in [1]; however, in [1] two more very simple proofs are given, both of which need only Lemma 2.

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