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Variational iteration method for solving two-point boundary value problems

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Abstract

Variational iteration method is introduced to solve two-point boundary value problems. Numerical results demonstrate that the method is promising and may overcome the difficulty arising in Adomian decomposition method.

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Keywords: Variational iteration method; Boundary value problems; Singular point; Adomian decomposition method

1. Introduction

The numerical solution of two-point boundary value problems (BVPs) is of great importance due to its wide application in scientific research. Adomian et al. solved a generalization of Airy's equation by decomposition method [3]; Ravi Kanth et al. dealt with singular two-point BVPs by cubic spline [12]; Caglar et al. applied B-spline interpolation to two-point BVPs and compared results with finite difference, finite element and finite volume methods [4].

The variational iteration method, which was proposed originally by He [7] in 1999, has been proved by many authors to be a powerful mathematical tool for various kinds of nonlinear problems. It was successfully applied to delay differential equations [5], to Duffing equation with non-linearity of fifth order and mathematical pendulum [7], to Burger's equation and coupled Burger's equation [1], to generalized KdV and coupled Schrodinger–KdV [2], to construct solitary solution and compaction-like solution [11], and other problems [10,9].

In this paper, we intend to extend the use of variational iteration method to two-point BVPs and compare with other methods. The results demonstrate that the method has many merits and may overcome the difficulty arising in Adomian decomposition method. Furthermore, it is possible to derive the exact solution by using one iteration only.

2. Analysis of the variational iteration method

According to the variational iteration method [7,1], we consider the following differential equation:

$$Lu + Nu = g(x), \tag{1}$$

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where L is a linear operator, N is a non-linear operator, and $g(x)$ is an inhomogeneous term. Then, we can construct a correct functional as follows:

$$u_{n+1}(x)=u_n(x)+\int_0^x \lambda\{Lu_n(\xi)+N\tilde{u}_n(\xi)-g(\xi)\}\,\mathrm{d}\xi, \tag{2}$$

where λ is a general Lagrangian multiplier [6–8], which can be identified optimally via variational theory. The second term on the right is called the correction and $\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta\tilde{u}_n=0$.

3. Examples

Now we apply the variational iteration method to solve some two-point boundary problems.

Example 1. We consider a simple one-dimensional example:

$$\begin{aligned}&u''(x)=2,\\&u(0)=1,\quad u(1)=2.\end{aligned} \tag{3}$$

According to the variational iteration method, we derive a correct functional as follows:

$$u_{n+1}(x)=u_n(x)+\int_0^x \lambda\{u_n''(\xi)-2\}\,\mathrm{d}\xi, \tag{4}$$

where λ is the Lagrange multiplier and can be easily identified as:

$$\lambda=\xi-x. \tag{5}$$

Therefore, we have the following iteration formula:

$$u_{n+1}(x)=u_n(x)+\int_0^x (\xi-x)\{u_n''(\xi)-2\}\,\mathrm{d}\xi. \tag{6}$$

Now, we begin with an arbitrary initial approximation:

$$u_0(x)=A+Bx, \tag{7}$$

where A and B are constants to be determined. By the variational iteration formula (6), we have

$$\begin{aligned}u_1&=u_0+\int_0^x (\xi-x)\{0-2\}\,\mathrm{d}\xi\\&=A+Bx+x^2.\end{aligned} \tag{8}$$

By imposing the boundary conditions at $x=0$ and $x=1$ yields $A=1$ and $B=0$, thus

$$u_1(x)=1+x^2, \tag{9}$$

which is the exact solution. It is important to note that the initial approximation can be freely selected with unknown constants, which can be determined by the boundary conditions.

Example 2. We consider the one-dimensional non-linear example of [3]:

$$\begin{aligned}&u''(x)-40xu(x)=2,\\&u(-1)=u(1)=0.\end{aligned} \tag{10}$$

The large numerical coefficient of xu makes the equation relatively stiff. According to the variational iteration method [7], the non-linear terms have to be considered as restricted variation. Similarly, the multiplier can be easily identified as $\lambda=\xi-x$, leading to the following iteration formula:

$$u_{n+1}(x)=u_n(x)+\int_0^x (\xi-x)\{u_n''(\xi)-40\xi u_n(\xi)-2\}\,\mathrm{d}\xi. \tag{11}$$

Table 1
Solutions by variational iteration method and comparisons

x	u_1	ϕ_6	G–M
−1.0	0.0	0.0	0.0
−0.8	0.3487	0.2538	0.0513
−0.6	0.3920	0.2968	0.0840
−0.4	0.2859	0.1473	0.1151
−0.2	0.1458	−0.0250	0.1377
0.0	0.0471	−0.1188	0.1228
0.2	0.0251	−0.1354	0.0306
0.4	0.0748	0.1141	−0.1418
0.6	0.1508	−0.0839	−0.2942
0.8	0.1677	−0.5241	−0.2544
1.0	0.0	0.0	0.0

We begin with an arbitrary initial approximation:

$$u_0(x) = A + Bx, \tag{12}$$

where A and B are constants to be determined. By the variational iteration formula (11), we have

$$\begin{aligned} u_1 &= u_0 + \int_0^x (\xi - x)\{-40\xi(A + B\xi) - 2\}\,\mathrm{d}\xi \\ &= A + Bx + x^2 + \frac{20}{3}Ax^3 + \frac{10}{3}Bx^4. \end{aligned} \tag{13}$$

By imposing the boundary conditions at $x = -1$ and $x = 1$ yields $A = 9/191$ and $B = -60/191$. Now, we can compare the results obtained by the variational iteration method for $u = u_1$ with those of Adomian's for $u = \phi_6$ and the Grey–Mangeot Sales (G–M) results given in [3]. Table 1 shows the results by different methods.

It is important to note that we derive the solution by only one iteration step and more the steps iterated the more accurate results can be obtained.

Example 3. Now, we deal with the following singular two-point BVP:

$$\begin{aligned} &u''(x) + \frac{1}{x}u'(x) + u(x) - \frac{5}{4} - \frac{x^2}{16} = 0, \\ &u'(0) = 0, \quad u(1) = \tfrac{17}{16}. \end{aligned} \tag{14}$$

In order to determine the Lagrange multiplier in as simple a manner as possible, we treat the singular terms as restricted variations. So, we construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda \left\{ u_n''(\xi) + \frac{1}{\xi}\tilde{u}'(\xi) + \tilde{u}(\xi) - \frac{5}{4} - \frac{\xi^2}{16} \right\} \mathrm{d}\xi. \tag{15}$$

The Lagrange multiplier, therefore, can be readily identified as $\lambda = \xi - x$. As a result, we obtain the following iteration formula:

$$u_{n+1}(x) = u_n(x) + \int_0^x (\xi - x) \left\{ u_n''(\xi) + \frac{1}{\xi}u'(\xi) + u(\xi) - \frac{5}{4} - \frac{\xi^2}{16} \right\} \mathrm{d}\xi. \tag{16}$$

Begin with $u_0(x) = A$, by the above iteration formula we have the following approximate solution:

$$u_1 = A - \frac{4A - 5}{8}x^2 + \frac{1}{192}x^4. \tag{17}$$

Table 2
Solving singular boundary value problem and comparisons

x	Exact	u_1
0.0	1.0000	0.8646
0.1	1.0006	0.8665
0.2	1.0025	0.8723
0.3	1.0056	0.8820
0.4	1.0100	0.8956
0.5	1.0156	0.9131
0.6	1.0225	0.9346
0.7	1.0306	0.9603
0.8	1.0400	0.9901
0.9	1.0506	1.0241
1.0	1.0625	1.0625

By imposing the boundary conditions at $x=1$ yields $A=\frac{83}{96}$. Now, we can compare the results obtained by the variational iteration method for $u=u_1$ with the exact solution $u(x)=1+x^2/16$ in Table 2.

4. Conclusions

In this paper, we have demonstrated the applicability of the variational iteration method for solving two-point BVPs with the help of some concrete examples. The results show that: (1) the initial approximation can be freely selected with unknown constants, which can be determined by the boundary conditions; (2) it is possible to derive the exact solution by using one iteration only and this method is also valid for large coefficient; (3) the application of restricted variations in correction functional makes it much easier to deal with singular problems and this method does not need discretization and interpolation which are required in [12].

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