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CLASSIFYING FINITE GROUP ACTIONS ON SURFACES OF LOW GENUS

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The problem of classifying all finite group actions, up to topological equivalence, on a surface of low genus is considered. Several new examples of construction and classification of actions are given. A program for enumerating all finite group actions on a surface of low genus is outlined and the complete classification is worked out for genus 2 and 3. As a by-product all conjugacy classes of finite subgroups of mapping class groups for genus 2 and 3 are determined.

1. Introduction

In this paper we work out the complete classification of orientation-preserving actions of finite groups, up to topological equivalence, on surfaces of genus 2 and 3. The study of finite groups of automorphisms of surfaces has a long history, starting late in the last century, and many results are known. However, the detailed enumeration of all groups of automorphisms of surfaces of low genus has not been pursued until recently (cf. Kuribayashi et al. [37–41]) because the complexity of the group-theoretic problems rapidly increases with increasing genus. The viewpoint of our study will be to 'detopologize' the problem by first transforming it into an equivalent problem about the enumeration of all finite groups which admit certain presentations. Then, we apply, in a systematic way, the machinery of modern group theory, some of which was not used or not available to previous authors. Our main result, Theorem 4.1, gives the complete classification of finite group actions on surfaces of genus 2 and 3. Explicit lists of actions are given in Table 4 for genus 2 and Table 5 for genus 3.

Before giving an outline of our attack on the problem let us briefly describe two important topological motivations for studying the problem, in particular, why we classify actions rather than just the groups. The first motivation is to better understand the *mapping class group*, i.e., the group of homotopy classes of homeomorphisms of a surface. It turns out that the equivalence classes of group *actions* are in 1-1 correspondence to conjugacy classes of finite subgroups of the mapping class group. Thus, our classification here gives some information on the structure of these

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groups. The second motivation arises from the analysis of the singularities of the moduli space of conformal equivalence classes of Riemann surfaces of a given genus (cf. [54]). This singular algebraic variety may be decomposed into a finite disjoint union of smooth subvarieties each of which corresponds to a unique equivalence class of actions of some finite group. Thus, the enumeration of group actions is a necessary component of the analysis of the singularities of these spaces or, indeed, any classification problem of structures on Riemann surfaces that ultimately depends on the conformal structure of the surfaces. There is a strong connection between these moduli spaces and mapping class groups since the moduli space is the quotient of Teichmüller space (contractible complex manifold) by a natural action of the mapping class group. This stratification and its connection to the mapping class group and actions of finite groups on a surface is the subject of the paper [3].

Now we outline a program for the classification of finite group actions on a surface of given genus. We defer to Section 2 all definitions, details and background results used in this outline. Let S be a surface of genus σ and G a finite group acting orientably and effectively on S.

(1.1) The quotient space S/G is a smooth surface and the quotient projection $S \to S/G$ is a branched covering. This covering may be partially characterized by a vector of numbers $(\varrho : m_1, \dots, m_r)$ where $\varrho \leq \sigma$ is the genus of S/G, $r \leq 2\sigma + 2$ is the number of branch points of the covering and the m_i are the orders of certain elements of G which fix points on S. We call $(\varrho : m_1, \dots, m_r)$ the *branching data* of G on S. The Riemann–Hurwitz equation

$$(2\sigma - 2)/|G| = (2\varrho - 2) + \sum_{j=1}^{r} \left(1 - \frac{1}{m_j}\right)$$

must be satisfied, this imposes restrictions on $|G|$ and the branching data that can occur. To find all finite group actions on surfaces of genus σ we first need a list of all possible branching data. The m_j need only be chosen from a small list of possible orders of automorphisms of a surface of genus σ. These orders may be determined from the results of Harvey [26] and Wiman [67] (see formulae (4.7.i)–(4.7.iii)). The list of possible branching data may then be obtained by running through the list of all possible $(r+1)$-tuples $(\varrho : m_1, \dots, m_r)$ where the m_j are in the given finite list of possible orders and selecting those vectors such that the number $(2\sigma - 2)/(2\varrho - 2 + \sum_{j=1}^{r} (1 - 1/m_j))$ is an integer divisible by all the m_j's.

(1.2) For each $(r+1)$-tuple $(\varrho : m_1, \dots, m_r)$ it turns out that the actions of finite groups G on S with the given branching data can be determined by finding all groups whose order is given by the Riemann–Hurwitz equation and such that there are $a_1, \dots, a_\varrho, b_1, \dots, b_\varrho, c_1, \dots, c_r \in G$ which generate G, satisfy $m_j = \mathrm{o}(c_j)$ ($\mathrm{o}(c_j) =$ order of c_j) and

$$\prod_{i=1}^{\varrho} [a_i, b_i] \prod_{j=1}^{r} c_j = 1.$$

(1.3) For each generating set found in 1.2 an equivalence class of G-actions is determined, and each equivalence class of G-actions determines a class of generating sets. The distinct equivalence classes of G-actions can be found by algebraic manipulation of the generating sets as described in Section 2.

(1.4) If G acts on S and $H \subset G$, then the relative projection $S/H \to S/G$ is a branched covering and has certain numerical invariants associated to it. These invariants can be calculated from the branching data of G and H. The numerical invariants impose restrictions on how H lies in G and hence gives us information about the structure of G. To aid in the classification of groups G acting on S with $|G| = n$ and branching data $(\varrho : m_1, \ldots, m_r)$, we may choose subgroups $H \subseteq G$ (e.g., Sylow subgroups, cyclic subgroups $\mathbb{Z}_{m_j}$ or normal subgroups if G is solvable) and use classification of actions of groups of order $< n$, and the numerical invariants of the map $S/H \to S/G$ to force certain structural properties upon G. Thus, we shall classify actions inductively, i.e., with $|G|$ increasing.

Remark. The class of groups that act on surfaces of genus $\leq \sigma$ is a naturally occurring set of groups that may be interesting to solvable group theorists. This set is closed under the operations of taking subgroups (restrict actions) and quotients (for $N \lhd G$, G/N acts on S/N, a surface of genus $\leq \sigma$). The vast majority of these groups are solvable and their maximum possible order grows linearly with σ (at most $84(\sigma - 1)$ by Hurwitz' Theorem). Moreover, the inductive method of constructing groups suggested in 1.4 would undoubtedly pose a number of interesting problems about of one solvable group lying in another.

The remainder of this paper is organized as follows. In Section 2 we give definitions, introduce notation and gather enough results together to establish the transition from the topological problem to the group-theoretic problem. These background results also establish the method of classification of actions by algebraic manipulation of the generating sets described in 1.3 and the inductive approach of 1.4. In Section 3 we describe some new examples and general techniques of constructing and classifying group actions. Example 3.1 shows how character theory may be used in powerful ways to enumerate group actions of groups whose character tables and subgroup lattices are known (see also [55]). These techniques can especially be applied to simple groups using recently derived information on character tables and subgroup structure. Example 3.4 gives new results on the construction and classification of certain actions of split metacyclic groups. Examples 3.5 and 3.6 demonstrate the use of relative projections and inductive procedure referred to in 1.4. The results in Example 3.6 are new. In Examples 3.2 and 3.3 we recall from the literature classifications of actions of prime-order, cyclic and abelian groups, since these results are used extensively in Section 4. In Section 4 we work out the classification of group actions on surfaces of genus 2 and 3, following the outline

1.1–1.4 and relying heavily on the results and examples of Sections 2 and 3. Our notation for groups is given in statements (4.1)–(4.3). Parts of this classification have been obtained or used by other authors [4, 38–41, 43, 67], though their results only make reference to the groups and not their actions. The classification given here is more complete than has been obtained previously, particularly that for genus 3. For the sake of completeness we have given the full details.

We finish this section by listing a categorized selection of background and related articles about group actions on surfaces.

Foundational material. The early investigators used the combinatorial methods of branched coverings for the constructions of actions. In the important paper [29], Hurwitz proved that $84(\sigma-1)$ is the upper bound for the order of the group of automorphisms of S_σ. Later, Wiman [67] found the upper bound $4\sigma+2$ for the order of an automorphism of S_σ. Modern treatments of groups actions employ these combinatorial and topological methods (cf. [65,70]) as well as the equivalent methods of Fuchsian groups and Teichmüller spaces (cf. [26,27,47,50,55,58,59,68–70]).

Hurwitz groups. A Hurwitz group is a group acting on a surface for which the Hurwitz upper bound of $84(\sigma-1)$ on group orders is realized. This topic has held great fascination ever since Hurwitz proved his theorem. Specific Hurwitz groups, including $\mathrm{PSL}_2(7)$, $\mathrm{PSL}_2(8)$ and Janko's first group, have been exhibited in the articles [34, 49, 55, 62]. General classes of Hurwitz groups such as $\mathrm{PSL}_2(q)$ and alternating groups are described in [5, 6, 10, 42, 48, 51, 55, 63].

Specific types of groups. Besides Hurwitz groups, there are several specific classes of groups that have been studied extensively: cyclic and abelian groups, $\mathrm{PSL}_2(q)$ groups and alternating and symmetric groups. The actions of cyclic and abelian groups were constructed by group-theoretic means in the papers [26] and [44], respectively. In addition, there are many results on single automorphisms such as order bounds [67], and topological equivalence [53, 68]. The projective linear groups, $\mathrm{PSL}_2(q)$, are the most studied class of non-abelian groups, cf. [22, 23, 51, 55]. In a series of papers [8–10], Conder obtained many results on actions of alternating and symmetric groups, in particular, which alternating groups are Hurwitz groups.

Minimal genus actions. A recent topic has been the determination of the minimal genus of a surface on which a given group will act. This is considered generally in [65], the projective linear groups have been done in [22, 23] and the alternating groups in [8–10], as mentioned above.

Subgroups of Fuchsian groups. As explained in Section 2 every group action on a surface arises by selecting a normal torsion-free subgroup of a Fuchsian group. Results on the existence, structure, index and maximality of such groups appear in the papers [7, 19, 35, 36, 45, 48]. Related results on non-normal subgroups and permutation groups are given in [58].

Other classification schemes. Other methods of classification of groups actions generally make use of the G-action on the homology or cohomology of the surface, as in the papers [13–15, 64]. Another classification scheme, from the analytic side, has been recently considered by A. Kuribayashi, I. Kuribayashi and H. Kimura [37–41]. Any subgroup of the automorphism group of a Riemann surface of genus σ acts on the σ-dimensional space of holomorphic differentials on the surface. These authors have worked out the classification of all subgroups of $GL(\sigma, \mathbb{C})$ that occur in this way for $2 \leq \sigma \leq 5$. Results have been announced for all these genera, complete results have been published for $\sigma = 2, 5$ [39, 40].

Maps on surfaces, tesselations and symmetric surfaces. The automorphism group of a map on a surface is a finite group of homeomorphisms of that surface, thus the work on maps on surfaces (also tesselations) is relevant to groups actions. Besides the monographs [11] and [46] there are the papers [16, 17, 32, 56, 61]. A symmetry of a surface is an anti-conformal involution of the surface, a symmetric surface is a surface with at least one symmetry. The symmetries of surfaces are important in the study of maps since they occur as reflections in the edges of polygons of maps. Among others, Singerman has contributed to this theory, notably in [4] and [60].

Automorphism groups of surfaces of low genus. Wiman discovered all groups occurring as automorphism groups of surfaces of genus 2. These results were used and extended by Singerman and Bujalance in [4] in which they classified systems of symmetries (as above) that could occur on a surface of genus 2. The groups that occur in genus 3 were classified by Maclachlan in his thesis, though this work on classification has not been published. In [56], Scherk considered the problem of classifying the regular maps on a surface of genus 3, he needed to take automorphism groups into account. In [37], A. Kuribahashi started a long series of papers, many of which appear in Bull. Fac. Sci. Engrg. Chuo Univ., on the equations of curves of genus 3. Some of the later papers were co-authored with K. Komiya or I. Kuribayashi. In these papers the authors sought to classify the curves of genus three by relating the characteristics of their Weierstrass points, their equations and their automorphism groups. Some summary results are given in [38] and [41]. In addition, there are the results of A. Kuribayashi, I. Kuribayashi and H. Kimura referred to above. None of these works consider the problem of topologically inequivalent actions of groups. In fact, the classification of actions is strictly finer than the classification scheme considered by A. Kuribayashi, I. Kuribayashi and H. Kimura. According to Sah (see Example 3.1) there are 7 inequivalent actions of $J(1)$, Janko's first group, on a surface of genus 2091. From the results of [2] none of these actions can be distinguished by the action of $J(1)$ on the space of holomorphic differentials.

Mapping class groups. The Nielsen realization problem, started in [53] and completely solved in [33], states that a finite subgroup of the mapping class group may be realized as a finite group of homeomorphisms of a surface. Thus, mapping class group problems may sometimes be rephrased in terms of finite group actions on surfaces, for example in [3, 21]. Birman's monograph [1] serves as a good background on the mapping class group.

2. Transition to a group-theoretic problem and résumé of results on group actions

Definition 2.1. Let S be an orientable surface of genus σ, $\mathrm{Hom}^+(S)$ its group of orientation-preserving homeomorphisms and G a finite group. We say that G *acts (effectively and orientably) on* S if there is a monomorphism $\varepsilon : G \to \mathrm{Hom}^+(S)$. If $\varepsilon' : G \to \mathrm{Hom}^+(S)$ is another action, then we say that $\varepsilon, \varepsilon'$ are (*topologically*) *equivalent* if there is an $\omega \in \mathrm{Aut}(G)$ and an $h \in \mathrm{Hom}^+(S)$ such that

$$\varepsilon'(g) = h\varepsilon(\omega(g))h^{-1}, \quad g \in G. \tag{2.1}$$

For $\sigma \geq 2$ the theory of Fuchsian groups gives us an effective vehicle to discuss finite group actions on surfaces. For background on Fuchsian groups one may consult, among others, [26, 27, 47, 50, 69, 70], we shall follow [26, 27] fairly closely. Every action of G on S may be constructed by means of a pair of Fuchsian groups $K \lhd G^* \subset \mathrm{PSL}_2(\mathbb{R})$ acting discontinuously on the upper half complex plane $\mathbb{H}$ (the universal cover of S) and an epimorphism $\eta : G^* \to G$ with kernel K. The group K is torsion-free and isomorphic to $\pi_1(S)$. The map η is constructed from ε and a homeomorphism $\mathbb{H}/K \sim S$. It is well known that G^* has the presentation

$$G^* \simeq \left\langle \alpha_i, \beta_i, \gamma_j : 1 \leq i \leq \varrho, 1 \leq j \leq r,\ \prod_{i=1}^{\varrho} [\alpha_i, \beta_i] \prod_{j=1}^{r} \gamma_j = \gamma_1^{m_1} = \cdots = \gamma_r^{m_r} = 1 \right\rangle. \tag{2.2}$$

Identify the $\alpha_i, \beta_i, \gamma_i$ with their images in G^*, it is also well known that

$$\mathrm{o}(\gamma_j) = m_j. \tag{2.3}$$

As in Section 1 we call the $(r+1)$-tuple $(\varrho : m_1, \ldots, m_r)$ the *branching data* for the action of G on S. The branching data vector is unique if the m_j are listed in non-decreasing order. Though the G^* constructed is not unique, a single G^* can be chosen to serve for all finite group actions with the branching data $(\varrho : m_1, \ldots, m_r)$, in particular, for equivalent actions of G. The term branching data is used since ϱ is the genus of the surface S/G and the m_j are the ramification numbers of the branched covering $S \to S/G$. Since $\varrho = 0$ occurs so often we omit ϱ in this case for notational convenience. Also for convenience we make abbreviations such as $(2^3, 3^2)$ for $(2, 2, 2, 3, 3)$, etc. The possible orders of groups and their branching data are limited by the well-known Riemann–Hurwitz equation:

$$(2\sigma - 2)/|G| = (2\varrho - 2) + \sum_{j=1}^{r} \left(1 - \frac{1}{m_j}\right). \tag{2.4}$$

Define the elements a_i, b_i, c_j of G by

$$\begin{aligned} a_i &= \eta(\alpha_i), \quad 1 \leq i \leq \varrho, \\ b_i &= \eta(\beta_i), \quad 1 \leq i \leq \varrho, \\ c_j &= \eta(\gamma_j), \quad 1 \leq j \leq r. \end{aligned} \tag{2.5}$$

These elements generate G,

$$\prod_{i=1}^{\varrho} [a_i, b_i] \prod_{j=1}^{r} c_j = 1, \tag{2.6}$$

and

$$o(c_j) = m_j, \tag{2.7}$$

because of (2.2), (2.3) and since $\ker(\eta)$ is torsion free.

Definition 2.2. A $(2\varrho + r)$-tuple $(a_1, \dots, a_\varrho, b_1, \dots, b_\varrho, c_1, \dots, c_r)$ of elements of G is called a *$(\varrho : m_1, \dots, m_r)$-vector* if (2.6) and (2.7) are satisfied. The vector is called a *generating $(\varrho : m_1, \dots, m_r)$-vector* if G is generated by the a_i, b_i, c_j.

The formula (2.5) establishes a 1–1 correspondence between the set of generating $(\varrho : m_1, \dots, m_r)$-vectors of G and the set $\{\eta: G^* \to G,\ \eta \text{ surjective, } \ker(\eta) \text{ torsion free}\}$ (cf. [26, Theorem 3]). The above description gives a precise modern treatment of Riemann's Existence Theorem which is the basic theorem translating the topological problem of constructing group actions to a problem in finite group theory. We paraphrase this theorem into our terminology.

Proposition 2.1 (Riemann's Existence Theorem). *The group G acts on the surface S, of genus σ, with branching data $(\varrho : m_1, \dots, m_r)$ if and only if the Riemann–Hurwitz equation* (2.4) *above is satisfied, and G has a generating $(\varrho : m_1, \dots, m_r)$-vector.* □

The relation of equivalence of actions induces an equivalence relation on generating vectors. Let $\varepsilon, \varepsilon', \omega, h$ be as in (2.1), $\eta, \eta' : G^* \to G$ the corresponding maps determined as above. The map h lifts to an orientation-preserving homeomorphism h^* of $\mathbb{H}$ such that $h^* G^* (h^*)^{-1} = G^*$, giving rise to an automorphism $\phi : G^* \to G^*$ defined by

$$\phi(g) = h^* g (h^*)^{-1}. \tag{2.8}$$

The following commutative diagram,

$$\begin{array}{ccc} G^* & \xrightarrow{\ \eta\ } & G \\ \phi \downarrow & & \downarrow \omega \\ G^* & \xrightarrow{\ \eta'\ } & G \end{array}$$

results, from which we get

$$\eta' = \omega \circ \eta \circ \phi^{-1}. \tag{2.9}$$

Let $\mathscr{B}$ be the subgroup of $\mathrm{Aut}(G^*)$ induced by orientation-preserving homeomorphisms as in (2.8). The group $\mathrm{Aut}(G) \times \mathscr{B}$ acts on the set $\{\eta : G^* \to G,\ \eta \text{ surjective, } \ker(\eta) \text{ torsion free}\}$ by the formula (2.9) and hence acts on the generating $(\varrho: m_1, \dots, m_r)$-vectors of G. By Zieschang's improvement of Nielsen's original theorem on automorphisms of surfaces (cf., e.g., Theorem 5.8.3 of [70]) each automorphism ϕ of G^* is induced by an h^*, as in (2.8), though h^* may be orientation-reversing. We have the following:

Proposition 2.2. *Two generating* $(\varrho: m_1, \ldots, m_r)$*-vectors of the finite group G define the same equivalence class of G-actions if and only if the generating vectors lie in the same* $\mathrm{Aut}(G) \times \mathscr{B}$*-class.* □

The non-trivial part of the proof, that $\mathrm{Aut}(G) \times \mathscr{B}$ equivalent generating vectors give equivalent G-actions, depends on the theory of quasi-conformal mappings and Teichmüller's uniqueness theorem. Proofs are given in [27] and [50].

In order to classify actions of a specific group we need an algebraic characterization of $\mathscr{B}$, and ways of producing elements of $\mathscr{B}$. One may construct automorphisms of G^* by constructing them in a certain one-relator group lying over G^*. Let F be the free group on the $\alpha_i, \beta_i, \gamma_j$ and Π be the group

$$\Pi = \left\langle \alpha_1, \ldots, \alpha_\varrho, \beta_1, \ldots, \beta_\varrho, \gamma_1, \ldots, \gamma_r : \prod_{i=1}^{\varrho} [\alpha_i, \beta_i] \prod_{j=1}^{r} \gamma_j = 1 \right\rangle. \tag{2.10}$$

There are obvious projections $\Pi \to G^*$, $F \to \Pi$ defined by the presentations (2.2), (2.10). By using a topological interpretation of Π as a fundamental group ($\pi_1(R^\circ)$, R° introduced below) and (2.8) the following can be shown:

Proposition 2.3. *Let* Π *be as above and* ϕ *an automorphism of* G^**. Then,* ϕ *may be lifted to an automorphism* $\tilde{\phi}$ *of* Π:

$$\begin{array}{ccc} \Pi & \xrightarrow{\tilde{\phi}} & \Pi \\ \downarrow & & \downarrow \\ G^* & \xrightarrow{\phi} & G^* \end{array}$$

such that for each j, $\tilde{\phi}(\gamma_j)$ *is conjugate to some* $\gamma_{j'}$ *or* $\gamma_{j'}^{-1}$*, where* $\gamma_j, \gamma_{j'}$ *are identified with their images in* Π*. The induced permutation representation* $\mathscr{B} \to \Sigma_r$ *preserves the branching orders, i.e.,* $m_j = m_{j'}$*.* □

Using some facts about one-relator groups (cf. [70], p. 54]), and some elementary surface topology it is easy to establish the following:

Proposition 2.4. *Let* $\Pi, F, \phi, \tilde{\phi}$ *be as above and* $\phi^{\#}$ *a lift of* $\tilde{\phi}$ *to* F:

$$\begin{array}{ccc} F & \xrightarrow{\phi^{\#}} & F \\ \downarrow & & \downarrow \\ \Pi & \xrightarrow{\tilde{\phi}} & \Pi \end{array}$$

Let $c=\prod_{i=1}^{\varrho}[\alpha_i,\beta_i]\prod_{j=1}^{r}\gamma_j$, *considered as a word in F. Then, either:*

(1) $\phi^{\#}(c)$ *is conjugate to* c *and* ϕ *is orientation-preserving and induces an element of* $\mathscr{B}$ *or*

(2) $\phi^{\#}(c)$ *is conjugate to* c^{-1} *and* ϕ *is orientation-reversing.* □

Let R° be the punctured surface obtained by removing the r branch points from $R=S/G$. The elements of $\mathscr{B}$ may be identified with a certain subgroup of finite index of the mapping class group of R°. This group is known to be finitely generated and generating sets of various sorts have been determined by various authors, e.g. [1,12,28,66]. A complete description of the action for abelian G is given in [27]. These results could be used to obtain a finite generating set for the action of $\mathscr{B}$ on generating vectors. However, we shall not need to go this far for the following reason. To prove two generating vectors equivalent one only need produce an appropriate element of $\mathrm{Aut}(G)\times\mathscr{B}$ effecting the equivalence. In most of our calculations we can produce the automorphisms to achieve this in an ad hoc fashion. Proposition 2.4 can be used as a test to verify the proposed automorphism really is an element of $\mathscr{B}$. A collection of such automorphisms, suitable for our needs, is given in Proposition 2.5 below. The proposition may be verified as suggested or the automorphisms directly constructed by using Dehn twists on R° (cf. [1]). Examples of the use of these automorphisms are given in case 3.m in Section 4.

Proposition 2.5. *Suppose that* $\varrho=0$ *or* 1, *let* $\alpha=\alpha_1,\beta=\beta_1$ *and let other notation be as above. Consider the automorphisms of F defined by the formulae below, where the action of an automorphism on a generator of F is written down only if it actually moves the generator. A product of these automorphisms defines an element of* $\mathscr{B}$, *as in Proposition* 2.4, *if and only if the induced permutation described in Proposition* 2.3 *preserves branching orders.*

Type I.a	$\alpha\to\alpha\beta^n,\quad n\in\mathbb{Z},$
Type I.b	$\beta\to\beta\alpha^n,\quad n\in\mathbb{Z},$
Type II(j)	$\gamma_j\to\gamma_{j+1},\quad \gamma_{j+1}\to\gamma_{j+1}^{-1}\gamma_j\gamma_{j+1},\qquad 1\le j\le r-1,$
Type III.a(j)	$\alpha\to x\alpha,\qquad \gamma_j\to y\gamma_j y^{-1},\quad 1\le j\le r,$
	$x=\beta^{-1}wz,\quad y=z\beta^{-1}w,\quad w=\gamma_1\cdot\dots\cdot\gamma_{j-1},\quad z=\gamma_{j+1}\cdot\dots\cdot\gamma_r,$
Type III.b(j)	$\beta\to x\beta,\qquad \gamma_j\to y\gamma_j y^{-1},\quad 1\le j\le r,$
	$x=wz\alpha,\quad y=z\alpha w,\quad w,z$ *as above.* □

The full knowledge of the action of $\mathscr{B}$ is only required to show that two generating vectors are not equivalent. In our calculations there will only be a few cases or types of cases where this situation occurs. In these cases the action of $\mathscr{B}$ is quite simple so it is worth recording it in the following propositions.

Proposition 2.6. *Suppose that G is abelian and that S/G is a sphere, so* $\varrho=0$. *Then,* $\mathscr{B}$ *acts on the generating vectors by permuting the* c_j*'s so that orders of elements are preserved.* □

Proposition 2.7. *If G^* is a triangle group ($\varrho=0, r=3$), then $\mathscr{B}$ is the subgroup, preserving branching orders as in Proposition 2.4, of the group generated by the two automorphisms $(\gamma_1,\gamma_2,\gamma_3)\to(\gamma_2,\gamma_3,\gamma_1)$ and $(\gamma_1,\gamma_2,\gamma_3)\to(\gamma_1,\gamma_3,\gamma_3^{-1}\gamma_2\gamma_3)$. If the induced permutation of the indices under an automorphism is trivial, then the automorphism is inner. Thus, if the m_j are distinct, then $\mathscr{B}$ acts trivially modulo the inner action of G.* □

Proposition 2.6 follows from Propositions 2.3 and 2.4, Proposition 2.7 follows from the presentation of the mapping class group of R^0 given in [1].

Relative projections. (See [58, 59, 61] for additional details.) Let G act on S and $H \subseteq G$. Let $(\tau: n_1,\dots,n_t)$ be the branching data of the restricted action of H on S. If $\eta: G^*\to G$ defines the G-action, then $H^*=\eta^{-1}(H)$ is a Fuchsian group and all the above discussion holds for $\eta: H^*\to H$ and the branching data $(\tau: m_1,\dots,m_t)$. Also, the map η induces a bijective correspondence between the coset spaces G^*/H^* and G/H which intertwines the G^*-action and the G-action. Let ν_l^j denote the number of $\langle\gamma_j\rangle$-orbits of size l in the right coset space G^*/H^*. We symbolize the $\langle\gamma_j\rangle$-orbit structure by $1^{\nu_1^j}\cdot 2^{\nu_2^j}\cdot\cdots$, leaving out those $l^{\nu_l^j}$ for which $\nu_l^j=0$. Observe that $\nu_l^j=0$ if l does not divide m_j. We call this symbol the *fibre type* associated to γ_j and denote it by f_j. The terminology fibre type is used since f_j gives the ramification numbers of the relative projection $\pi: \mathbb{H}/H^*\to\mathbb{H}/G^*$ over the branch point $Q_j\in\mathbb{H}/G^*\simeq S/G$, associated to γ_j. Let $\gamma_1',\dots,\gamma_t'$ denote the elements in the presentation of H^* as in (2.2). It can be shown, using the projection π, that for each γ_j and for each $\langle\gamma_j\rangle$-orbit, of size strictly smaller than m_j, the stabilizer of this orbit is conjugate in H^* to a unique $\langle\gamma_k'\rangle$ and each γ_k' occurs this way. In the correspondence of the γ_k' with stabilizers exactly ν_l^j of them will have order m_j/l when l strictly divides m_j. Therefore, we may reorder the n_k as $n_1^1,\dots,n_{s_1}^1;\dots;n_1^r,\dots,n_{s_r}^r$ so that the first ν_1^j orders in the sequence $n_1^j,\dots,n_{s_j}^j$ equal m_j, the next ν_2^j orders in the sequence $n_1^j,\dots,n_{s_j}^j$ equal $m_j/2$ etc. We write

$$[f_1,\dots,f_r]:[\tau: n_1^1,\dots,n_{s_1}^1;\dots;n_1^r,\dots,n_{s_r}^r]\to[\varrho: m_1,\dots,m_r] \tag{2.11}$$

to symbolize the projection $\pi: S/H\to S/G$ which may be identified with the projection $\pi:\mathbb{H}/H^*\to\mathbb{H}/G^*$. Some of the sequences $n_1^j,\dots,n_{s_j}^j$ may be empty, we permute the f_i and the m_j so that these sequences occur last in the list. The square brackets are used to indicate that the branching data may not be in the usual non-decreasing order. As an example, consider $\mathrm{GL}_2(3)$ acting on an S_2 with branching data $(2,3,8)$. If H is the 2-Sylow subgroup of G, then H acts with branching data $(2,4,8)$. The relative projection is of the form $[1\cdot 2, 1\cdot 2, 3]:[2;8,4]\to[2,8,3]$. We call the expression (2.11), and also the symbol $[f_1,\dots,f_r]$, the *symbol of the projection $S/G\to S/H$.*

Let n be the degree of π which in turn equals the index of H^* in G^* and also H in G. Then, from the definition it follows that

$$n=\sum_l l\nu_l^j,\quad 1\le j\le r. \tag{2.12}$$

The relative Hurwitz formula applied to $S/H \to S/G$ can easily be transformed into

$$n(2\varrho-2+r)+2-2\tau = \sum_{j,l} v_l^j, \tag{2.13}$$

since the right-hand side is the total number of points in the fibres of π lying over the branch points on S/G. Any expression of the form (2.11) satisfying (2.12), (2.13) we will call a *numerical projection* between branching data even though it may not necessarily correspond to an actual projection $S/H \to S/G$.

If H is normal in G, then, since all $\langle \gamma_j \rangle$-orbits in G^*/H^* have the same size, the fibre type f_i has the form e_j^{n/e_j}, where e_j is the order of the image of γ_j in the quotient group $G^*/H^* \simeq G/H$. We say that an arbitrary numerical projection is *uniformly branched* if all fibre types are of the above form. A uniformly branched numerical projection need not arise from a normal inclusion of groups $H \subset G$ but we do have the following:

Proposition 2.8. *Let G act on S and $H \subset G$ be such that S/H is a sphere* ($\tau=0$). *Then, H is normal in G if and only if the corresponding numerical projection is uniformly branched.* □

In [25], Greenberg proves a similar theorem for branched covers $\phi: D \to S$, where S is any Riemann surface and D is the unit disc. Using the simple connectivity of the sphere, his proof may be easily modified to show that the map $S/H \to S/G$ is regular, from which the proposition follows. We list the possibilities that arise from the proposition in Table 1.

Now suppose H is not necessarily normal and let $N = \mathrm{core}_G(H)$, let $(\lambda: p_1, \ldots, p_s)$ be the branching data for N acting on S, let $g \to \bar{g}$ be the quotient map $G \to G/N$, let $(a_1, \ldots, a_\varrho, b_1, \ldots, b_\varrho, c_1, \ldots, c_r)$ be a generating vector for the G-action and let $e_1, \ldots, e_r$ be as above. The order e_j is the least common multiple of the sizes of the $\langle \gamma_j \rangle$-orbits on G^*/H^* and may be read off from f_j, the fibre type. Now G/N acts on the surface S/N. A simple topological argument shows that $(\bar{a}_1, \ldots, \bar{a}_\varrho, \bar{b}_1, \ldots, \bar{b}_\varrho, \bar{c}_1, \ldots, \bar{c}_r)$ is a generating $(\varrho: e_1, \ldots, e_r)$-vector for the action of G/N on S/N (for this discussion we allow some of the e_j to equal 1 and do not assume that they are in non-decreasing order). The genus λ of S/N and $|N|$ are not

Table 1

Symbol of π[a]	$\lvert G/H \rvert = \deg(\pi)$	G/H[b]
$[n, n]$	n	$\mathbb{Z}_n$
$[2^n, 2^n, 2^n]$	$2n$	D_n
$[2^8, 3^4, 3^4]$	12	A_4
$[2^{12}, 3^8, 4^6]$	24	Σ_4
$[2^{30}, 3^{20}, 5^{12}]$	60	A_5

[a] One may have to add some 'unramified' fibres 1^n, $n = |G/H|$.
[b] See statement (4.1) for group notation.

uniquely determined but must satisfy $\tau \le \lambda \le \sigma$ ($\lambda \le (\sigma+1)/2$, if N is not trivial), $|N|(2\lambda-2) = |G|/(2\varrho-2+\sum_{j=1}^{r}(1-1/e_j))$, and there must exist a $(\varrho: e_1, \dots, e_r)$ action on a surface of genus λ. Thus, if N is not trivial, the possible structures of G/N can be determined from the classification of actions on surfaces of lower genus. The only cases we shall require have $\lambda = 0$ and already occur in Table 1. If $\lambda = 1$, then the classification of these groups may be found in [69], the branching data that occur are (1: -), (2, 4, 4), (2, 3, 6), (3, 3, 3) and (2, 2, 2, 2). Another approach for non-normal H is to work out the structure of $G^*/\mathrm{core}_{G^*}(H^*)$ directly from the permutation representation of G^* on G^*/H^*. This handles all pairs $H \subset G$ with the same relative projection at once. Singerman [59] has done this for all inclusions of triangle groups, i.e., $\varrho = \tau = 0$, $r = t = 3$.

Homology representation and fixed points. The group G acts on the homology group $H_1(S;\mathbb{C})$, this representation may be used to prove that certain actions on S cannot occur. Let the branching data and generating vector for the G-action be as above, let η, ϱ and χ_0 denote, respectively, the homology, regular and trivial characters of G and let ϱ_j for $1 \le j \le r$ denote permutation character of G on the coset space $G/\langle c_j \rangle$. In [2] (cf. also [64]) it is shown that

$$\eta = 2\chi_0 + (2\varrho - 2 + r)\varrho - \sum_{j=1}^{r} \varrho_j. \tag{2.14}$$

This can be reinterpreted for a irreducible character χ of G as follows (cf. [2]). Let χ be a non-trivial, irreducible character of G, let $g \in G$ and define

$$l_g(\chi) = \frac{1}{\mathrm{o}(g)} \sum_{i=1}^{\mathrm{o}(g)} \chi(g^i).$$

This number is the multiplicity of the trivial character in the restriction of χ to $\langle g \rangle$, so it follows from Frobenius reciprocity that for non-trivial χ the multiplicity of χ in the homology representation equals $(2\varrho-2+r)\chi(1) - \sum_{j=1}^{r} l_{c_j}(\chi)$. Since multiplicities are non-negative we have

$$(2\varrho - 2 + r)\chi(1) \ge \sum_{j=1}^{r} l_{c_j}(\chi). \tag{2.15}$$

For $1 \ne g \in G$ let S^g be the set of points on S fixed by g. The number of fixed points $|S^g|$ is finite and is non-zero if and only if g is conjugate to a power of some c_j. The Lefschetz fixed point theorem [18] states that

$$\eta(g) = 2 - |S^g|. \tag{2.16}$$

Since ϱ_j is the permutation character the coset of a *cyclic* group, then $\varrho_j(g) = |N_G(\langle g \rangle)|\,\delta_j(g)/m_j$ where $\delta_j(g)$ equals 1 if g is conjugate to a power of c_j and 0 otherwise. From (2.14) and (2.16) we get

$$|S^g| = |N_G(\langle g \rangle)| \sum_{j=1}^{r} \delta_j(g)/m_j. \tag{2.17}$$

Remark. If $G=\langle g\rangle$, then the number of fixed points of g is the number of times $|G|$ occurs among the m_j.

3. Some examples and techniques

Example 3.1. *Simple Hurwitz groups of low order.* By Hurwitz' Theorem, if G acts on S_σ, then $|G|\le 84(\sigma-1)$. A group for which equality is obtained is called a Hurwitz group. In this example we show how character theory may be used to determine all possible Hurwitz actions for simple groups of low order. To illustrate the method let us first show that $G=\mathrm{PSL}_2(7)$ is the only Hurwitz group acting on S_3 and that the action is unique up to equivalence. For such a Hurwitz group, the branching data must be $(2,3,7)$ and $|G|=168$. If G is not isomorphic to $\mathrm{PSL}_2(7)$, then G is solvable since $\mathrm{PSL}_2(7)$ is the only simple group whose order divides 168. Since an abelian group generated by x,y,z satisfying $x^2=y^3=z^7=xyz=1$ is necessarily trivial, a Hurwitz group cannot have a non-trivial abelian quotient. It follows that our G is isomorphic to $\mathrm{PSL}_2(7)$.

Let X be the set of $(2,3,7)$-vectors in $\mathrm{PSL}_2(7)$. Since all the proper subgroups of $\mathrm{PSL}_2(7)$ are solvable every vector in X must generate $\mathrm{PSL}_2(7)$ by the above argument. We may compute the number of elements in X from the character table of G, using formula (3.1) below. Let $K_1,\dots,K_s$ be s conjugacy classes in a finite group G, let $x_i\in K_i$ and let $X(K_1,\dots,K_s)=\{y_1,\dots,y_s\colon y_i\in K_i,\ y_1\cdot y_2\cdot\dots\cdot y_s=1\}$. We shall call an element of $X(K_1,\dots,K_s)$ a $(K_1,\dots,K_s)$-*vector*. It is well known that

$$|X(K_1,\dots,K_s)|=\frac{|G|^{s-1}}{|\mathrm{Cent}(x_1)|\cdot\dots\cdot|\mathrm{Cent}(x_s)|}\sum_\chi\frac{\chi(x_1)\cdot\dots\cdot\chi(x_s)}{\chi(1)^{s-2}},\qquad(3.1)$$

where the sum is over the irreducible characters of G. In $\mathrm{PSL}_2(7)$ there is one conjugacy class of elements of order 2, K_2, one of elements of order 3, K_3, and two of elements of order 7, K_7^+,K_7^-, where $K_7^-=\{x^{-1}\colon x\in K_7^+\}$. Thus $X=X(K_2,K_3,K_7^+)\cup X(K_2,K_3,K_7^-)$ (disjoint). Applying the above formula to $\mathrm{PSL}_2(7)$ (character table: [52, p. 1215] or [30, p. 289]) we determine that $|X|=2\cdot 168$. The action of $\mathrm{Aut}(\mathrm{PSL}_2(7))$ on X is fixed point free since an automorphism of a group fixing a generating set is trivial. As $\mathrm{Aut}(\mathrm{PSL}_2(7))=\mathrm{PGL}_2(7)$, acting by conjugation, then $|\mathrm{Aut}(\mathrm{PSL}_2(7))|=2\cdot 168$. All $(2,3,7)$-vectors are therefore equivalent and $\mathrm{PSL}_2(7)$ has only one equivalence of actions on S_3.

Now let us apply this method, formula (2.15) and previously known results to determine all the Hurwitz actions of all the simple groups, G, whose character tables appear in McKay's paper [52], namely, $\mathrm{PSL}_2(p^n)$ or $|G|<10^6$. The results are given in Table 2.

In our proof of these results the unreferenced group-theoretic facts we use may be extracted from the tables in McKay's paper on character tables [52] or Fischer and McKay's paper on maximal subgroups [20]. The results for $\mathrm{PSL}_2(p^n)$ are proven in [51, Theorem 8], using a method different from ours. The seven inequivalent actions of $J(1)$ were determined by Sah [55, Proposition 2.7], using our method. Now let

Table 2

G	No. of actions
$\mathrm{PSL}_2(7)$	1
$\mathrm{PSL}_2(p)$, $p^2 \equiv 1 \bmod 7$	3
$\mathrm{PSL}_2(p^3)$, $p^2 \not\equiv 1 \bmod 7$, $p \neq 7$	1
$J(1)$[a]	7
$J(2)$[a]	10

[a] $J(1)$ and $J(2)$ are Janko's first and second groups.

G be any group in McKay's tables not isomorphic to $\mathrm{PSL}_2(p^n)$ or $J(1)$ and suppose it is a Hurwitz group. Since $|G| = 84(\sigma - 1)$ the order of G is divisible by 84, so, taking isomorphisms into account, G can only be one of: Alt(7), Alt(8), Alt(9), $L(3,4)$, $U(3,3)$, $U(3,5)$, $M(22)$ or $J(2)$. The first seven of these cannot be Hurwitz groups since for every potential generating $(2,3,7)$-vector, (c_1, c_2, c_3), formula (2.15) is violated for some character. If χ_n indicates the nth character in the table of a group in [52], then the characters which yield the desired contradiction for these groups are: Alt(7): χ_5, χ_6, Alt(8): χ_2, χ_3, Alt(9): χ_2, χ_{12}, $L(3,4)$: χ_2, $U(3,3)$: χ_4, χ_6, $U(3,5)$: χ_3 and $M(22)$: χ_2. The alternating groups could also have been eliminated by appealing to Conder's results [10].

From the character table of $J(2)$ we see that there are two classes of involutions, $2A, 2B$, two classes of elements of order 3, $3A, 3B$, and one class of elements of order 7, $7A$. There are no $(2A, 3A, 7A)$, $(2A, 3B, 7A)$ or $(2B, 3A, 7A)$-vectors, since (2.15) fails for each such vector and one of the characters χ_2 or χ_4. Using formula (3.1) we calculate that the number of $(2B, 3B, 7A)$-vectors is $10|J(2)|$. If we can show that each one of these vectors is a generating vector, then there will be 10 inequivalent actions since $|\mathrm{Aut}(J(2))| = |(J(2))|$, (cf., e.g., [24] or [31]).

Suppose that one of the vectors is not a generating vector and let H denote a Hurwitz subgroup generated by the vector. From previous arguments H has no abelian quotients. Using the classification of maximal subgroups of $J(2)$, given in [20], the subgroup must lie in either a maximal $U(3,3)$ or in the normalizer of a $\mathrm{PSL}_2(7)$, of order $336 = 2|\mathrm{PSL}_2(7)|$. The subgroup H cannot be a $U(3,3)$ since this was eliminated earlier. If H lies in a $U(3,3)$, then, using [20] one more time for $U(3,3)$, H must be isomorphic to $\mathrm{PSL}_3(2) \simeq \mathrm{PSL}_2(7)$. If H lies in the normalizer of $\mathrm{PSL}_2(7)$, then it must equal $\mathrm{PSL}_2(7)$ since H has no abelian quotients. Thus in all cases $H \simeq \mathrm{PSL}_2(7)$. Now consider an element g of order 4 in H. It must belong to the conjugacy class $4A$ of $J(2)$ and hence g^2 must belong to $2A$, again using McKay's tables. But all involutions in H are conjugate and they must belong to $2B$ by construction. This contradiction shows that $H = J(2)$.

Example 3.2. *Actions of* $\mathbb{Z}_p$. This is probably the most extensively studied case of groups acting on surfaces. We will just recall enough facts here for later use in our calculations in Section 4. If $\mathbb{Z}_p$ acts on S, then its branching data must be $(\varrho: p, \ldots, p)$

with r branch points. The corresponding surface has genus

$$\sigma = (\varrho - 1)p + r(p-1)/2 + 1.$$

The case we shall most frequently encounter is $r=3$, $\varrho=0$. Let X be the set of (p,p,p)-vectors, its cardinality is easily seen to be $(p-1)(p-2)$ and all such vectors are generating vectors. By Proposition 2.6, $\mathscr{B}$ acts as Σ_3, permuting entries of the generating vector, and $\mathrm{Aut}(\mathbb{Z}_p)$ is $\mathbb{Z}_p^*$, acting by multiplication. The only automorphisms in $\Sigma_3 \times \mathbb{Z}_p^*$ which have fixed points on X are conjugate to $(c_1, c_2, c_3) \to (c_2, c_1, c_3)$ which fixes $(c_1, c_1, (p-2)c_1)$ and $(c_1, c_2, c_3) \to (ac_3, ac_1, ac_2)$, where $p \equiv 1 \bmod 3$, $a^3 \equiv 1 \bmod p$, which fixes (c_1, ac_1, a^2c_1). We list the equivalence classes of vectors in Table 3.

Example 3.3. *Cyclic and abelian groups.* Here we record from [26] and [27] necessary and sufficient conditions on branching data in order for S_σ, $\sigma \geq 2$, to admit an automorphism of order n. The cyclic group $\mathbb{Z}_n$ acts on S_σ with branching data $(\varrho: m_1, \dots, m_r)$ if and only if the Riemann-Hurwitz equation (2.4) with $|G| = n$ holds and all the conditions below hold. Let $m = \mathrm{l.c.m.}(m_1, \dots, m_r)$.

$$\mathrm{l.c.m.}(m_1, \dots, m_{j-1}, m_{j+1}, \dots, m_r) = m \quad \text{for all } j, \tag{3.2}$$

$$m \mid n, \quad m = n, \quad \text{if } \varrho = 0, \tag{3.3}$$

$$r \neq 1 \text{ and } r \geq 3, \quad \text{if } \varrho = 0 \tag{3.4}$$

and

if m is even, then the number of m_j divisible by the maximal power of 2 dividing m is even. (3.5)

Now let $\mathbb{Z}_n = \langle x \rangle$ and let $(a_1, \dots, a_\varrho, b_1, \dots, b_\varrho, c_1, \dots, c_r)$ be a $(2\varrho + r)$-tuple of elements of $\mathbb{Z}_n$, set $c_j = x^{s_j}$. Then, it is easily shown that $(a_1, \dots, a_\varrho, b_1, \dots, b_\varrho, c_1, \dots, c_r)$ is a generating $(\varrho: m_1, \dots, m_r)$-vector if and only if the following hold:

$$\sum_{j=1}^{r} s_j \equiv 0 \bmod n, \tag{3.6}$$

$$\mathrm{g.c.d.}(s_j, n) = n/m_j \tag{3.7}$$

and

$$a_1, \dots, a_\varrho, b_1, \dots, b_\varrho \text{ generate } \mathbb{Z}_n/\mathbb{Z}_m. \tag{3.8}$$

Table 3

$p \equiv 1 \bmod 3$	$p \not\equiv 1 \bmod 3$
$(1, 1, p-2)$-class	$(1, 1, p-2)$-class
$(1, a, a^2)$-class, $a^3 \equiv 1 \bmod p$, $a \leq p/2$	
$(p-7)/6$ other classes	$(p-5)/6$ other classes

Finally from Theorem 14, p. 396 of [27] we have:

each equivalence class of generating $(\varrho: m_1, \ldots, m_r)$-vectors contains a vector with $a_2 = \cdots = a_\varrho = b_1 = \cdots b_\varrho = 0$ and a_1 generates $\mathbb{Z}_n/\mathbb{Z}_m$. (3.9)

If G is abelian, then (3.2) and (3.4) still hold, we leave the proof to the reader. There are analogues of (3.8) and (3.9) which we illustrate with an example. Suppose that $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts on a surface of genus 3 with branching data (1: 2, 2). Write $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle x, y: x^2 = y^2 = [x, y] = 1\rangle$. Suppose (a, b, c_1, c_2) is a generating vector. Since $[a, b]c_1c_2 = 1$, then $c_1 = c_2$. Transforming by an automorphism, we may assume that $c_1 = c_2 = x$. Now consider the effect of the two $\mathscr{B}$-transformations of Type I, $(\alpha, \beta, \gamma_1, \gamma_2) \to (\alpha\beta, \beta, \gamma_1, \gamma_2)$, $(\alpha, \beta, \gamma_1, \gamma_2) \to (\alpha, \beta\alpha, \gamma_1, \gamma_2)$ and the two $\mathscr{B}$-transformations of Type III, $(\alpha, \beta, \gamma_1, \gamma_2) \to (\beta^{-1}\gamma_2\alpha, \beta, \gamma_2\beta^{-1}\gamma_1\beta\gamma_2^{-1}, \gamma_2)$, $(\alpha, \beta, \gamma_1, \gamma_2) \to (\alpha, \gamma_2\alpha\beta, \gamma_2\alpha\gamma_1\alpha^{-1}\gamma_2^{-1}, \gamma_2)$, on generating vectors (see Proposition 2.5). Using combinations of the transformations we obtain the equivalent vectors (xa, b, x, x), $(a, xb, x, x)(xa, xb, x, x)$. Thus we may assume that $\langle a, b\rangle = \langle y\rangle$. Again using combinations of the transformations we get the equivalent vectors (a, ab, x, x), (b, a, x, x). It follows, then, that every generating vector is equivalent to $(y, 1, xx)$ and that there is only one equivalence class of actions. Harvey has given generators for the $\mathscr{B}$-action on abelian G in [27].

Example 3.4. *Split metacyclic groups.* Frequently we have the situation $|G| = pq$ and S/G has branching data (p, q, r). For the purposes of this example we do not necessarily assume that $p \le q \le r$. Let (c_1, c_2, c_3) be a generating vector. Suppose by Sylow theorems or $p = 2$ that we are able to assume that every cyclic subgroup of order q is normal; thus $\langle c_2\rangle \lhd G$. We allow ourselves to reverse the roles of p and q to achieve this. Now $\langle c_1\rangle \cap \langle c_2\rangle = \langle 1\rangle$, otherwise c_1, c_2 generate a group of order less than pq. Therefore, $G = \langle c_1\rangle \ltimes \langle c_2\rangle$, and, consequently, $c_1c_2c_1^{-1} = c_2^j$ for a j satisfying

$$j^p \equiv 1 \bmod q. \tag{3.10}$$

Taking $x = c_1, y = c_2$, we get a presentation: $G = \langle x, y: x^p = y^q = 1, xyx^{-1} = y^j\rangle$. We denote this group by $D_{p,q,j}$ since it is an analogue of a dihedral group. If we have:

$$k = j^s \quad \text{where } (s, p) = 1, \tag{3.11}$$

then $D_{p,q,j} \simeq D_{p,q,k}$, since $z = x^s$ and y generate $D_{p,q,j}$ and $zyz^{-1} = y^k$. Note that (3.11) is equivalent to saying that k and j generate the same subgroup of $\mathrm{Aut}(\mathbb{Z}_q)$. If (3.11) is not satisfied, then we can usually establish that $D_{p,q,j} \not\simeq D_{p,q,k}$. We suggest some ways to do this later in this example.

Let $wz, z = c_1^s, w = c_2^l$ be an element of G and let $k = j^s, a = \mathrm{o}(z)$, we do not asume $(s, p) = 1$. For any integer t we get

$$(wz)^t = ww^k \cdot \cdots \cdot w^{k^{t-1}} z^t.$$

It follows that $a \mid \mathrm{o}(wz)$, $(wz)^a = w^{1 + k + \cdots + k^{a-1}}$ and hence that $\mathrm{o}(wz) = ab$, where $b = q/(q, 1 + k + \cdots + k^{a-1})$. Now $c_1c_2 = c_3^{-1}$ has order r so, taking $s = l = 1$ in the above, we get $a = p$,

$$r = pb \quad \text{for some } b \mid q \tag{3.12.i}$$

and

$$(q, 1 + j + \cdots + j^{p-1}) = q/b. \tag{3.12.ii}$$

This limits the values of r and j that can occur. Working backwards, if p, q, j and r satisfy these two equations, then $D_{p,q,j}$ has a generating (p, q, pb)-vector. By Proposition 2.1, $D_{p,q,j}$ must act on a surface of genus $((p-1)(q-1)+1-q/b)/2$, as long as this number is an integer.

Now suppose that (d_1, d_2, d_3) is any other generating (p, q, r)-vector of $G = D_{p,q,j}$. By the assumption on cyclic subgroups of order q, $\langle d_2 \rangle \lhd G$, so then $d_1 d_2 d_1^{-1} = d_2^k$ for some k. The integer k must satisfy (3.12.ii) with k replacing j. Clearly the two vectors (c_1, c_2, c_3) and (d_1, d_2, d_3) are Aut(G)-equivalent if and only if $j \equiv k \bmod q$. Let $(p, s) = 1$ and $k = j^s \bmod q$. Since $\{s, 2s, \ldots, (p-1)s\}$ is simply a permutation of $\{1, 2, \ldots, (p-1)\}$, it follows from (3.12.ii) that $(d_1, d_2, d_3) = (c_1^s, c_2, (c_1^s c_2)^{-1})$, is a generating (p, q, r)-vector. Thus, if we also know that $D_{p,q,j}$ and $D_{p,q,k}$ are isomorphic if and only if (3.11) holds, then the Aut(G)-equivalence classes of generating (p, q, r)-vectors are in 1–1 correspondence with the set $(\bmod q)$ $\{j^s : (s, p) = 1\}$. If any two of p, q or r are equal, then it is necessary to work out the action of $\mathscr{B}$, given in Proposition 2.7, on these vectors. Also, even if it is not true that every subgroup of order q is normal we still get all of the above generating vectors of $D_{p,q,j}$, though there may be more.

Here are some ways to establish $D_{p,q,j} \not\simeq D_{p,q,k}$. If x, y are as in the presentation of $G = D_{p,q,j}$ above, then $[x, y] = y^{j-1}$ and it follows that the order of the derived subgroup $[G, G]$ is $q/(j-1, q)$. Thus if $D_{p,q,j} \simeq D_{p,q,k}$, then

$$(j-1, q) = (k-1, q). \tag{3.13}$$

If $D_{p,q,j}$ has a unique subgroup N of order q, then a unique cyclic subgroup of Aut(N) is determined by the action of G on N. It follows from the comment immediately following (3.11) that (3.11) is satisfied when this condition holds. If p is a prime, then $D_{p,q,j} \simeq D_{p,q,k}$ if and only if (3.11) holds. To prove this we may assume that $D_{p,q,j}$ does not have a unique cyclic subgroup of order q, according to the last argument. Using the notation, wz, a, b, k, s above, let wz be an element of order q not lying in $\langle y \rangle$. It then follows that $a = p$, $(s, p) = 1$, $(q, 1 + k + \cdots + k^{p-1}) = p$. Since $(k-1)(1 + k + \cdots + k^{p-1}) = k^p - 1 \equiv 0 \bmod q$, then the last formula of the last sentence yields $k - 1 \equiv 0 \bmod q$. We get $k \equiv 1 + eq/p \bmod q$ for some e, where $0 < e < p$. If t is chosen such that $st \equiv 1 \bmod p$, then by applying the binomial theorem we see that q/p divides $(1 + eq/p)^t - 1$ and hence $j \equiv k^{st} \equiv 1 + fq/p \bmod q$. Thus the number of pth roots of unity in $\mathbb{Z}_q$ is at most p and so (3.11) holds.

If a group of order 16 acts on a surface of genus 3 with branching data $(2, 8, 8)$, then $G = \mathbb{Z}_2 \times \mathbb{Z}_8$ or $D_{2,8,5}$, with one Aut(G)-class of generating vectors each. If a group of order 21 acts on a surface of genus 3 with branching data $(3, 3, 7)$, then $G = D_{3,7,2}$ with two Aut(G)-classes of generating vectors. The two classes are $\mathscr{B}$-equivalent using the automorphism in Proposition 2.7 which interchanges c_1 and c_2.

Example 3.5. *Inductive use of classification.* Here we give an example of the inductive method of classification outlined in step 1.4. Using the method of relative projections, we determine all actions of a group of order 16 on a surface of genus 3, with branching data (4, 4, 4), assuming we know the actions of groups of order 4 and 8. First let us assume that there is normal cyclic subgroup $H \subset G$ of order 4, generated by an element fixing at least one point. Let (c_1, c_2, c_3) be a generating (4, 4, 4)-vector. Since H is conjugate to one of the $\langle c_j \rangle$'s and is normal, then H is generated by one of the c_j's. Using $\mathscr{B}$-transformations of Type II if necessary, we may assume that $H = \langle c_3 \rangle$. The action of H is given by one of the cases 3.f, 3.g or 3.i.1 of Table 5. According to the remark following formula (2.17) a generator of H fixes four, two or no points, respectively, in these three cases. Since H is normal and a generator of H fixes at least one point, we may apply the fixed point formula (2.17) to conclude that H fixes at least four points. Therefore, the branching data for H acting on S is (4, 4, 4, 4) and the symbol of the projection $S/H \to S/G$ is $[1^4, 4, 4] : [4, 4, 4, 4] \to [4, 4, 4]$. From this symbol and description of uniformly branched projections immediately preceding Proposition 2.8, we see that both $\langle c_1 \rangle$ and $\langle c_2 \rangle$ are subgroups complementary to H. As in Example 3.4, $G \simeq \mathbb{Z}_4 \times \mathbb{Z}_4$ or $D_{4,4,-1}$ has a presentation

$$G = \langle x, y : x^4 = y^4 = 1, xyx^{-1} = y^j \rangle, \tag{3.14}$$

with $j = 1$ for $\mathbb{Z}_4 \times \mathbb{Z}_4$ and $j = -1$ for $D_{4,4-1}$. In both cases $(x, (yx)^{-1}, y)$ is generating a (4, 4, 4)-vector.

Now let (d_1, d_2, d_3) be any generating (4, 4, 4)-vector. At least one of $\langle d_1 \rangle$, $\langle d_2 \rangle$ or $\langle d_3 \rangle$ is normal. This is trivial for $\mathbb{Z}_4 \times \mathbb{Z}_4$. For $D_{4,4,-1}$ we argue as follows. There are eight elements of order 4 in $D_{4,4,-1}$ which do not generate normal subgroups, namely: $x, xy, xy^2, xy^3, x^{-1}, x^{-1}y, x^{-1}y^2$ and $x^{-1}y^3$. The product of any two of these elements has the form y^s or x^2y^s and such elements generate normal subgroups. Using $\mathscr{B}$-transformations we may now assume $\langle d_3 \rangle$ is normal. Since $d_1 \notin \langle d_3 \rangle$, then d_1, d_3 satisfy $d_1^4 = d_3^4 = 1$, $d_1 d_2 d_1^{-1} = d_3^j$. Since (3.14) is a presentation of G, then there is an automorphism of G carrying (d_1, d_3) to (x, y) and hence there is one equivalence class of generating (4, 4, 4)-vectors.

It remains to show that for any group of order 16 with a generating (4, 4, 4)-vector (c_1, c_2, c_3) at least one $\langle c_j \rangle$ is normal. Assume that $H = \langle c_3 \rangle$ is not normal. Since G is a 2-group K, the normalizer of H, must properly contain H, so $|K| = 8$. By Table 5, S/H must be a sphere hence S/K is also a sphere. Therefore, the symbol of the relative projection $S/K \to S/G$ is $[1^2, 2, 2]$. From the classification of groups of order 8, K has branching data $(2^2, 4^2)$ and the symbol of $S/K \to S/G$ is $[1^2, 2, 2] : [4, 4; 2; 2] \to [4, 4, 4]$. By Table 5, $K = \mathbb{Z}_2 \times \mathbb{Z}_4$ or D_4. The case $K = D_4$ can be eliminated, since H would be characteristic in K, hence normal in G. Let $y = c_3$ and write $K = \langle y, x : y^4 = z^2 = [y, z] = 1 \rangle$. Let $x = c_1$, since c_1 and c_3 generate, then $x \notin K$, but $x^2 \in K$. Let θ be the automorphism of K induced by conjugation by x; since $x^2 \in K$, then θ has order 2. The subgroup K has two cyclic subgroups of order 4, namely $\langle y \rangle$ and $\langle yz \rangle$. Since $\langle y \rangle$ is not normal, then θ interchanges these subgroups and θ either interchanges y and yz or y and $y^{-1}z$. The automorphism $y \to y$, $z \to zy^2$ conjugates each of these possible automorphisms into the other, so we may assume that $\theta(y)=yz$. It is easy to check that θ fixes z, y^2 or y^2z, the three elements of order 2 in K. The element x^2 is an element of order 2 in K, so it must equal one of z, y^2 or y^2z. Since $yx=c_2^{-1}$ has order 4 and $(yx)^2=yxyx^{-1}x^2=y^2zx^2$, x^2 cannot equal y^2z. If $x^2=y^2$, then $(yx)^2=y^2zx^2=z$ and $y^{-1}(yx)y=xyx^{-1}x=yzx^{-1}=(yx)^3$. The elements y and yx generate G and normalize $\langle yx\rangle$, so $\langle c_2\rangle$ is normal. If $x^2=z$, a similar calculation shows that $\langle c_1\rangle$ is normal.

Example 3.6. *Split extensions.* In this example we show how relative projections $S/N\to S/G$ may be used to show that $N\to G\to G/N$ is split for certain $N\lhd G$. We determine all actions of a group G of order 96 acting on a surface S of genus 3, with branching data $(2,3,8)$, assuming the classification of actions of groups of lower order.

Let H be the 2-Sylow subgroup of G, then $S/H\to S/G$ is $[1\cdot 2, 1\cdot 2, 3]:[2;4,8]\to[2,8,3]$. Let $N=\mathrm{core}_G(H)$, then from Table 1 we have $G/N=\Sigma_3$. Let K be such $K/N\simeq A_3\subseteq G/N$. The symbol of $S/K\to S/G$ must be $[1^2,2,2]:[3,3;4]\to[3,8,2]$ and the branching data for K acting on S is $(3,3,4)$. By 3.aq of Table 5, $N\simeq\mathbb{Z}_4\times\mathbb{Z}_4$ since N is a normal 2-Sylow subgroup of K. By standard group cohomology arguments the sequence $N\to G\to\Sigma_3$ is split if and only if the sequence

$$N\to G_p\to(\Sigma_3)_p \tag{3.15}$$

is split for the primes $p=2,3$, where $(\Sigma_3)_p$ is the p-Sylow subgroup of Σ_1 and G_p is inverse image of $(\Sigma_3)_p$ under the map $G\to\Sigma_3$. The sequence splits for $p=3$ since $|N|$ and 3 are coprime. For $p=2$ observe that the symbol of $S/N\to S/G$ is $[2^3,2^3,3^2]:[4,4,4]\to[8,3,2]$, thus, there is an $x\in G$ of order 2 such that x maps to a transposition in Σ_3. Since $(\Sigma_3)_2\simeq\mathbb{Z}_2$, then (3.15) is split for $p=2$ and $G\simeq\Sigma_3\ltimes(\mathbb{Z}_4\times\mathbb{Z}_4)$. Let $x,y\in G$ generate a complementary subgroup to N, say $x^2=y^3=1$, $xyx^{-1}=y^{-1}$.

The element y acts without non-trivial fixed points on N. To see this, note that the number of fixed points of y, $|S^y|$, is 2 or 5, according to the remark following (2.17) and cases 3.d–3.e. But, by (2.17), $|N_G(\langle y\rangle)|=3|S^y|$, so it follows that $|N_G(\langle y\rangle)|=6$, and hence, $N_G(\langle y\rangle)=\langle x,y\rangle$. Therefore, y cannot commute with any elements of N. Let $z\in N$ have order 4 and set $w=yzy^{-1}$. Since $z(yzy^{-1})(y^2zy^{-2})$ is y-invariant, then it equals 1, so $y^2zy^{-1}=(zw)^{-1}$ and $\langle z,w\rangle$ is a y-invariant subgroup of order 4, 8 or 16. If the y-action on this subgroup is to be fixed point free, then $|\langle z,w\rangle|\equiv 1 \bmod 3$. The order of $\langle z,w\rangle$ cannot be 4 since $\langle z,w\rangle$ will be cyclic and y will act trivially, therefore, z, w generate N. Let $N_2=\{g^2: g\in N\}$, N_2 and N/N_2 are both isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2$ and Σ_3 must act faithfully on N_2 and N/N_2. By changing x to another transposition in Σ_3 if necessary we may assume that $xzx^{-1}\equiv w \bmod N_2$, $xwx^{-1}\equiv z \bmod N_2$. Thus $xzx^{-1}=w\beta(z)$, $xwx^{-1}=z\beta(w)$, $\beta(z),\beta(w)\in N_2$. From $z=x(xzx^{-1})x^{-1}$, we get $\beta(w)=x\beta(z)x^{-1}$. From $xyzy^{-1}x^{-1}=y^{-1}xzx^{-1}y$, we get $\beta(w)=y^{-1}\beta(z)y$. Thus, $yx\beta(z)(yx)^{-1}=\beta(z)$, i.e., $\beta(z)=1,\beta(w)=1$ or $\beta(z)=w^2,\beta(w)=z^2$. For those two cases we get $xzx^{-1}=w, xwx^{-1}=z$ and $xzx^{-1}=w^{-1}, xwz^{-1}=z^{-1}$, re-

Table 4
Finite group actions on a genus 2 surface[a]

Case	G	$\|G\|$	Branching data	Presentation	Generating vectors
2.a	$\mathbb{Z}_2$	2	(2^6)	$\langle x: x^2=1\rangle$	(x,x,x,x,x,x)
2.b	$\mathbb{Z}_2$	2	$(1:2^2)$	$\langle x: x^2=1\rangle$	$(1,1,x,x)$
2.c	$\mathbb{Z}_3$	3	(3^4)	$\langle x: x^3=1\rangle$	(x,x,x^{-1},x^{-1})
2.e	$\mathbb{Z}_4$	4	$(2^2,4^2)$	$\langle x: x^4=1\rangle$	(x^2,x^2,x,x^{-1})
2.f	$\mathbb{Z}_2\times\mathbb{Z}_2$	4	(2^5)	$\langle x,y: x^2=y^2=[x,y]=1\rangle$	(x,x,x,y,xy)
2.h	$\mathbb{Z}_5$	5	$(5,5,5)$	$\langle x: x^5=1\rangle$	(x,x,x^3)
2.i	$\mathbb{Z}_6$	6	$(3,6,6)$	$\langle x: x^6=1\rangle$	(x^4,x,x)
2.k.1	$\mathbb{Z}_6$	6	$(2^2,3^2)$	$\langle x: x^6=1\rangle$	(x^3,x^3,x^2,x^4)
2.k.2	D_3	6	$(2^2,3^2)$	$\langle x,y: x^2=y^3=1, xyx^{-1}=y^{-1}\rangle$	(x,x,y,y^{-1})
2.l	$\mathbb{Z}_8$	8	$(2,8,8)$	$\langle x: x^8=1\rangle$	(x^4,x^3,x)
2.m	$\tilde{D}_2$	8	$(4,4,4)$	$\langle x,y: x^4=y^4=1, x^2=y^2, xyx^{-1}=y^{-1}\rangle$	(x,y,yx)
2.n	D_4	8	$(2^3,4)$	$\langle x,y: x^2=y^4=1, xyx^{-1}=y^{-1}\rangle$	(x,xy,y^2,y)
2.o	$\mathbb{Z}_{10}$	10	$(2,5,10)$	$\langle x: x^{10}=1\rangle$	(x^5,x^4,x)
2.p	$\mathbb{Z}_2\times\mathbb{Z}_6$	12	$(2,6,6)$	$\langle x,y: x^2=y^6=[x,y]=1\rangle$	(x,xy,y^{-1})
2.r	$D_{4,3,-1}$	12	$(3,4,4)$	$\langle x,y: x^4=y^3=1, xyx^{-1}=y^{-1}\rangle$	$(y,(xy)^{-1},x)$
2.s	D_6	12	$(2^3,3)$	$\langle x,y: x^2=y^6=1, xyx^{-1}=y^{-1}\rangle$	(x,xy,y^3,y^2)
2.u	$D_{2,8,3}$	16	$(2,4,8)$	$\langle x,y: x^2=y^8=1, xyx^{-1}=y^3\rangle$	$(x,(yx)^{-1},y)$
2.w	$\mathbb{Z}_2\ltimes(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)$	24	$(2,4,6)$	$\langle x,y,z,w: x^2=y^2=z^2=w^3=[y,z]=[y,w]=[z,w]=1,$ $xyx^{-1}=y, xzx^{-1}=zy, xwx^{-1}=w^{-1}\rangle$	$(x,(zwx)^{-1},zw)$
2.x	$SL_2(3)$	24	$(3,3,4)$	$\langle x,y: x=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), y=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)\rangle$	$(x,(yx)^{-1},y)$
2.aa	$GL_2(3)$	48	$(2,3,8)$	$\langle x,y: x=\left(\begin{smallmatrix}1&1\\0&-1\end{smallmatrix}\right), y=\left(\begin{smallmatrix}0&-1\\-1&-1\end{smallmatrix}\right)\rangle$	$(x,y,(xy)^{-1})$

[a] See notes (4.1)–(4.6) for group notation.

spectively. Viewing the two actions as $\mathbb{Z}_4$-representations $\xi_1, \xi_2 : \Sigma_3 \to \mathrm{GL}_2(\mathbb{Z}_4)$ we get, with respect to the basis, $\{z, w\}$: $\xi_1 : y \to \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, $x \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\xi_2 : y \to \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, $x \to \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. The matrix $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_4)$ intertwines these two representations so we only get one semi-direct product $\Sigma_3 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)$.

Now let (c_1, c_2, c_3) be a generating $(2,3,8)$-vector. The corresponding image vector $(\bar{c}_1, \bar{c}_2, \bar{c}_3)$, under the quotient map $G \to \Sigma_3$, $c \to \bar{c}$, is a generating $(2,3,2)$-vector. All $(2,3,2)$-vectors for Σ_3 are equivalent by Σ_3-conjugation or a $\mathscr{B}$-transformation as given in Proposition 2.7, so we may assume $(c_1, c_2, c_3) = (xy^{-1}g_1, yg_2, xg_3)$, where $g_1, g_2, g_3 \in N$. Since c_1 has order 2, then g_1 is one of 1 or w^2. A simple calculation shows that $xy^{-1}g_1 = gxy^{-1}g^{-1}$ for some $g \in N$, so we may assume $g_1 = 1$. From $c_1c_2c_3 = 1$ it follows that $g_2 = xg_3^{-1}x^{-1}$. If $g_3 = z^r w^s$, then xg_3 has order 8 if and only if $(xg_3)^2 = (wz)^{r+s}$ has order 4, i.e., $g_3 = z, w, z^{-1}, w^{-1}, zw^2, z^{-1}w^2, wz^2$ or wz^{-2}. Conjugating xg_3 by elements of $\mathrm{Cent}_N(xy^{-1})$ we transform xg_3 into $xg_3(zw^{-1})^r$, $r = 0, 1, 2, 3$, so we may assyme that xg_3 is either xz or xz^{-1}. But $(g,h) \to (g, h^{-1})$ is an automorphism of G fixing Σ_3 and interchanging xz and xz^{-1}, so we arrive at a unique representative (xy^{-1}, yw, xz^{-1}) for (c_1, c_2, c_3). To see that (c_1, c_2, c_3) generates, observe that $c_3c_2c_3^{-1}c_2 = w^{-1}$. By conjugating w^{-1} by c_1, c_2, c_3 repeatedly we generate N and then all of G.

4. Classification of actions for genus 2 and 3

Here we carry out the program outlined in Section 1 for finite group actions on surfaces of genus 2 and 3. We state this as the following:

Theorem 4.1. *Let G be a finite group acting on a surface of genus $\sigma = 2$ or 3. Then, G is isomorphic to one of the groups listed in* Table 4 ($\sigma = 2$) *or* Table 5 ($\sigma = 3$). *The action of G is determined, up to topological equivalence, by the branching data and an appropriate generating vector listed in the table.*

Notes for Theorem 4.1 and Tables 4 and 5.

(4.1) The symbols $\mathbb{Z}_n$, D_n, A_n and Σ_n denote, respectively, the cyclic group of order n, the dihedral group of order $2n$, the alternating group on n letters and the symmetric group on n letters.

(4.2) The notation $D_{p,q,r}$ for split metacyclic groups is explained in Example 3.4. $D_{2,n,-1}$ and D_n are the same group.

(4.3) For the subgroups D_n, A_4, Σ_4 and A_5 of SO(3) we denote their double covers in SU(2) by $\tilde{D}_n$, $\tilde{A}_4$, $\tilde{\Sigma}_4$ and $\tilde{A}_5$ (binary polyhedral groups). These groups often occur as symmetry groups of hyperelliptic curves.

(4.4) The entries in the tables are ordered by lexicographically ordering the vectors $(|G|, \varrho, r, m_1, \ldots, m_r)$ derived from the branching data vectors. Recall that $|G| = (2\sigma - 2)/(2\varrho - 2 + \sum_{j=1}^{r} (1 - 1/m_j))$ by (2.4).

Table 5
Finite group actions on a genus 3 surface[a]

Case	G	$\lvert G\rvert$	Branching data	Presentation	Generating vectors
3.a	$\mathbb{Z}_2$	2	(2^8)	$\langle x: x^2=1\rangle$	(x,x,x,x,x,x,x,x)
3.b	$\mathbb{Z}_2$	2	$(1:2^4)$	$\langle x: x^2=1\rangle$	$(1,1,x,x,x,x)$
3.c	$\mathbb{Z}_2$	2	$(2: -)$	$\langle x: x^2=1\rangle$	$(x,1,1,1)$
3.d	$\mathbb{Z}_3$	3	(3^5)	$\langle x: x^3=1\rangle$	(x,x,x,x,x^{-1})
3.e	$\mathbb{Z}_3$	3	$(1:3^2)$	$\langle x: x^3=1\rangle$	$(1,1,x,x^{-1})$
3.f	$\mathbb{Z}_4$	4	(4^4)	$\langle x: x^4=1\rangle$	(x,x,x,x), (x,x,x^{-1},x^{-1})
3.g	$\mathbb{Z}_4$	4	$(2^3,4^2)$	$\langle x: x^4=1\rangle$	(x^2,x^2,x^2,x,x)
3.h	$\mathbb{Z}_2\times\mathbb{Z}_2$	4	(2^6)	$\langle x,y: x^2=y^2=[x,y]=1\rangle$	(x,x,y,y,xy,xy), (x,x,y,y,y,y)
3.i.1	$\mathbb{Z}_4$	4	$(1:2^2)$	$\langle x: x^4=1\rangle$	$(x,1,x^2,x^2)$
3.i.2	$\mathbb{Z}_2\times\mathbb{Z}_2$	4	$(1:2^2)$	$\langle x,y: x^2=y^2=[x,y]=1\rangle$	$(y,1,x,x)$
3.j	$\mathbb{Z}_6$	6	$(2^2,6^2)$	$\langle x: x^6=1\rangle$	(x^3,x^3,x,x^{-1})
3.k	$\mathbb{Z}_6$	6	$(2,3^2,6)$	$\langle x: x^6=1\rangle$	(x^3,x^4,x^4,x)
3.m	D_3	6	$(2^4,3)$	$\langle x,y: x^2=y^3=1, xyx^{-1}=y^{-1}\rangle$	(x,x,x,xy^{-1},y)
3.n	D_3	6	$(1:3)$	$\langle x,y: x^2=y^3=1, xyx^{-1}=y^{-1}\rangle$	(x,xy,y)
3.o	$\mathbb{Z}_7$	7	$(7,7,7)$	$\langle x: x^7=1\rangle$	$(x,x,x^5),(x,x^2,x^4)$
3.p	$\mathbb{Z}_8$	8	$(4,8,8)$	$\langle x: x^8=1\rangle$	$(x^6,x,x),(x^2,x,x^5)$
3.q.1	$\mathbb{Z}_2\times\mathbb{Z}_4$	8	$(2^2,4^2)$	$\langle x,y: x^2=y^4=[x,y]=1\rangle$	(x,x,y^{-1},y), (x,xy^2,y,y), (x,y^2,xy,y)
3.q.2	D_4	8	$(2^2,4^2)$	$\langle x,y: x^2=y^4=1, xyz^{-1}=y^{-1}\rangle$	(x,x,y^{-1},y)
3.r.1	$(\mathbb{Z}_2)^3$	8	(2^5)	$\langle x: x^2=1\rangle\times\langle y: y^2=1\rangle\times\langle z: z^2=1\rangle$	(x,y,z,yz)
3.r.2	D_4	8	(2^5)	$\langle x,y: x^2=y^4=1, xyx^{-1}=y^{-1}\rangle$	(x,x,xy,xy^3,y^2)
3.s.1	$\tilde{D}_2$	8	$(1:2)$	$\langle x,y: x^4=y^4=1, x^2=y^2, xyx^{-1}=y^{-1}\rangle$	(x,y,x^2)
3.s.2	D_4	8	$(1:2)$	$\langle x,y: x^2=y^4=1, xyx^{-1}=y^{-1}\rangle$	(x,xy,y^2)
3.t	$\mathbb{Z}_9$	9	$(3,9,9)$	$\langle x: x^9=1\rangle$	(x^3,x^5,x)

Table 5 (continued)

Case	G	$\|G\|$	Branching data	Presentation	Generating vectors
3.u	$\mathbb{Z}_{12}$	12	$(2,12,12)$	$\langle x: x^{12}=1\rangle$	(x^6,x^5,x)
3.v	$\mathbb{Z}_{12}$	12	$(3,4,12)$	$\langle x: x^{12}=1\rangle$	(x^8,x^3,x)
3.x	$D_{4,3,-1}$	12	$(4,4,6)$	$\langle x,y: x^4=y^3=1, xyx^{-1}=y^{-1}\rangle$	(x,xy^{-1},x^2y)
3.y	D_6	12	$(2^3,6)$	$\langle x,y: x^2=y^6=1, xyx^{-1}=y^{-1}\rangle$	(x,xy^2,y^3,y)
3.z	A_4	12	$(2^2,3^2)$	$\langle x,y: x=(1,2)(3,4), y=(1,2,3)\rangle$	(x,x,y,y^{-1})
3.aa	$\mathbb{Z}_{14}$	14	$(2,7,14)$	$\langle x: x^{14}=1\rangle$	(x^7,x^6,x)
3.ab.1	$\mathbb{Z}_2\times\mathbb{Z}_8$	16	$(2,8,8)$	$\langle x,y: x^2=y^8=[x,y]=1\rangle$	(x,xy^{-1},y)
3.ab.2	$D_{2,8,5}$	16	$(2,8,8)$	$\langle x,y: x^2=y^8=1, xyx^{-1}=y^5\rangle$	(x,xy^{-1},y)
3.ac.1	$\mathbb{Z}_4\times\mathbb{Z}_4$	16	$(4,4,4)$	$\langle x,y: x^4=y^4=[x,y]=1\rangle$	$(x,y,(xy)^{-1})$
3.ac.2	$D_{4,4,-1}$	16	$(4,4,4)$	$\langle x,y: x^4=y^4=1, xyx^{-1}=y^{-1}\rangle$	$(x,(yx)^{-1},y)$
3.ad.1	$\mathbb{Z}_2\times D_4$	16	$(2^3,4)$	$\langle x: x^2=1\rangle\times\langle y,z: y^2=z^4=1, yzy^{-1}=z^{-1}\rangle$	(x,y,yxz^{-1},z)
3.ad.2	$\mathbb{Z}_2\ltimes(\mathbb{Z}_2\times\mathbb{Z}_4)$	16	$(2^3,4)$	$\langle x,y,z: x^2=y^2=z^4=[y,z]=1, [x,z]=1, xyx^{-1}=yz^2\rangle$	(x,xzy,y,z^{-1}), (x,xyz,z^2,xy)
3.ag	$D_{3,7,2}$	21	$(3,7,7)$	$\langle x,y: x^3=y^7=1, xyx^{-1}=y^2\rangle$	$(x,(yx)^{-1},y)$
3.ah	$D_{2,12,5}$	24	$(2,4,12)$	$\langle x,y: x^2=y^{12}=1, xyx^{-1}=y^5\rangle$	$(x,(yx)^{-1},y)$
3.ai	$\mathbb{Z}_2\times A_4$	24	$(2,6,6)$	$\langle x: x^2=1\rangle\times\langle y,z: y=(1,2)(3,4), z=(1,2,3)\rangle$	$(y,xz,x(yz)^{-1})$
3.aj	$\mathrm{SL}_2(3)$	24	$(3,3,6)$	$\langle x,y: x=\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right), y=\left(\begin{smallmatrix}1&-1\\0&1\end{smallmatrix}\right)\rangle$	$(x,y,(xy)^{-1})$
3.ak	Σ_4	24	$(3,4,4)$	$\langle x,y: x=(1,2,3,4), y=(1,4,3,2)\rangle$	$((xy)^{-1},x,y)$
3.al	Σ_4	24	$(2^3,3)$	$\langle x,y,z: x=(1,2), y=(2,3), z=(3,4)\rangle$	$(x,y,yxzy,yz)$
3.am.1	$\mathbb{Z}_2\ltimes(\mathbb{Z}_2\times\mathbb{Z}_8)$	32	$(2,4,8)$	$\langle x,y,z: x^2=y^2=z^8=[y,z]=1, [x,y]=1, xzx^{-1}=yz^3\rangle$	(x,xz,z^{-1})
3.am.2	$\mathbb{Z}_2\ltimes D_{2,8,5}$	32	$(2,4,8)$	$\langle x,y,z: x^2=y^2=z^8, yzy^{-1}=z^5, xyx^{-1}=yz^4, xzx^{-1}=yz^3\rangle$	(x,xz,z^{-1})
3.ap	$\mathbb{Z}_2\times\Sigma_4$	48	$(2,4,6)$	$\langle x: x^2=1\rangle\times\langle y,z\ \ y=(1,2), z=(2,3,4)\rangle$	$(xy,(zy)^{-1},xz)$
3.aq	$\mathbb{Z}_3\ltimes(\mathbb{Z}_4\times\mathbb{Z}_4)$	48	$(3,3,4)$	$\langle x,y,z: x^3=y^4=z^4=[y,z]=1, xyx^{-1}=z, xzx^{-1}=(yz)^{-1}\rangle$	$(x,(yx)^{-1},y)$
3.as	$\Sigma_3\ltimes(\mathbb{Z}_4\times\mathbb{Z}_4)$	96	$(2,3,8)$	$\langle x,y,z,w: x^3=y^4=z^4=w^4=1, [y,z]=1, xyx^{-1}=y^{-1},$ $xzx^{-1}=w, xwx^{-1}=z, yzy^{-1}=w, ywy^{-1}=(zw)^{-1}\rangle$	(xy^{-1},yw,xz^{-1})
3.at	$\mathrm{PSL}_2(7)$	168	$(2,3,7)$	$\langle x,y: x=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right), y=\left(\begin{smallmatrix}1&1\\-1&2\end{smallmatrix}\right)\rangle$	$(x,y,(xy)^{-1})$

[a] See Notes (4.1)–(4.6) for group notation.

(4.5) The abbreviated notation for the branching data is explained near the beginning of Section 2.

(4.6) The 'missing' entries 2.d, 2.g, ..., 3.l, 3.w, ..., correspond to $(r+1)$-tuples found in step 1.1 of Section 1 for which there was no corresponding group action.

Proof outline. To follow the program outlined in Section 1 we first need to determine the possible orders of automorphisms of surfaces of genus 2 and 3. It is well known (e.g. [67]) that the orders of automorphisms are 2, 3, 4, 5, 6, 8, 10 for $\sigma=2$ and 2, 3, 4, 6, 7, 8, 9, 12, 14 for $\sigma=3$. Harvey's results [26], restated in Example 3.3, would allow us to determine exactly which orders occur for any genus. We can cut down on the number of cases to check using Harvey's results by considering only those orders n which satisfy the following:

$$n \geq 4\sigma+2, \tag{4.7.i}$$

$$\text{if } n \text{ is prime, then } n=2\sigma+1 \text{ or } n \leq \sigma+1, \tag{4.7.ii}$$

$$\phi(n) \leq 2\sigma \ (\phi = \text{Euler function}). \tag{4.7.iii}$$

Note that (4.7.ii) applies to all prime factors of the order of an automorphism of S_σ. The inequality (4.7.i) was originally proven by Wiman [67], and both (4.7.i) and (4.7.ii) are proven in [26, Corollary to Theorem 6] and [27, Corollary 11], respectively. Statement (4.7.iii) follows from applying (2.14) to the homology representation of $\mathbb{Z}_n$. The homology representation is integral and (2.14) implies that a primitive nth root of unity is an eigenvalue of the representation. The result on possible orders for $\sigma=2,3$ follows directly from (4.7.i)–(4.7.iii).

Next a list of branching data must be prepared, exactly as suggested in 1.1. The lists, except for excluded cases, are given as part of the data in Tables 4 and 5. This calculation is straightforward though tedious. After discovering how long calculations were by hand, the author wrote a couple of computer programs to calculate the orders of automorphisms and possible branching data for low genus ($\sigma \leq 50$). The programs verified the hand calculations for $\sigma=2,3$ in a few seconds on a microcomputer.

There only remains the lengthy case by case analysis outlined in steps 1.2–1.4. We break this into two parts, one for each genus. If $|G|$ is prime or some $m_j=|G|$, then the actions can be classified by direct application of Harvey's results given in Example 3.3 and also the results described in Example 3.2. We omit the proofs of these cases and those cases already proven in the other examples in Section 3. For each other case in Tables 4 and 5 we will write down an analysis of the form: case, $|G|$, branching data, group (possibly with a presentation), followed by one or more paragraphs of proof. The presentation is written down if the proof requires an explicit description of the group in order to classify the generating vectors. We will also have additional analyses of the form: case, $|G|$, branching data, no group exists. These are the justifications for the 'missing' entries described in (4.6) above.

Case by case analysis for genus 2

$$3,\ (1{:}\,3),\ \text{no group exists.} \tag{2.d}$$

Statement (3.4) fails.

$$4,\ (2^5),\ \mathbb{Z}_2\times\mathbb{Z}_2 = \langle x, y : x^2 = y^2 = [x,y] = 1\rangle. \tag{2.f}$$

By statement (3.3), G can only be $\mathbb{Z}_2\times\mathbb{Z}_2$. The non-identity elements of G are x, y and $z = xy$. Aut(G) acts as the full symmetric group on $\{x, y, z\}$ and $\mathscr{B}$ acts as the full symmetric group on $\{c_1, \dots, c_5\}$. If only one or two of x, y and z occur in $\{c_1, \dots, c_5\}$, then either the c_j's do not generate G or their product cannot equal 1. At least one x, y and z must occur three times, otherwise the product of the c_j's cannot equal 1. It now follows that every generating vector is equivalent to (x, x, x, y, xy).

$$4,\ (1{:}\,2),\ \text{no group exists.} \tag{2.g}$$

Statement (3.4) fails.

$$6,\ (2^3, 6),\ \text{no group exists.} \tag{2.j}$$

Statement (3.2) fails.

$$6,\ (2^2, 3^2),\ \mathbb{Z}_6. \tag{2.k.1}$$

$$6,\ (2^2, 3^2),\ D_3 = \langle x, y : x^2 = y^3 = 1, xyx^{-1} = y^{-1}\rangle. \tag{2.k.2}$$

If G is abelian, then $G = \mathbb{Z}_6$ and we get case 2.k.1. Assume $G = D_3$ and (c_1, c_2, c_3, c_4) is a generating vector. There is $\omega \in \text{Aut}(G)$ with $\omega(c_1, c_4) = (x, y^{-1})$. Since $c_3 = y$ or y^{-1}, then every vector is equivalent to (x, x, y, y^{-1}) or $(x, xy^{-1}, y^{-1}, y^{-1})$. Now $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \to (\gamma_1, \gamma_3^{-1}\gamma_2\gamma_3, \gamma_3^{-1}\gamma_2^{-1}\gamma_3\gamma_2\gamma_3, \gamma_4)$ is a $\mathscr{B}$-transformation (Proposition 2.4) and it maps the second vector to the first.

$$8,\ (4,4,4),\ \tilde{D}_2 = \langle x, y : x^4 = y^4 = 1, x^2 = y^2, xyx^{-1} = y^{-1}\rangle. \tag{2.m}$$

The groups of order 8 containing an element of order 4 are $\mathbb{Z}_8$, $\mathbb{Z}_2\times\mathbb{Z}_4$, D_4 and $\tilde{D}_2$ (quaternions). The elements of order 4 in $\mathbb{Z}_8$ and D_4 form proper subgroups, so these cases are excluded. Let (c_1, c_2, c_3) be a generating vector. Now, in $\mathbb{Z}_2\times\mathbb{Z}_4$ there are two cyclic subgroups of order 4, so exactly two of $\langle c_1\rangle$, $\langle c_2\rangle$, $\langle c_3\rangle$ are equal, and two of c_1, c_2, c_3 have four fixed points by (2.17). But, from 2.e of Table 4 and (2.17) an element of order 4 can have only 2 fixed points, thus $G \simeq \tilde{D}_2$. If $c_1, c_2 \in \tilde{D}_2$ have order 4 and generate $\tilde{D}_2$, then $c_1^2 = c_2^2$ and $c_1c_2c_1^{-1} = c_2^{-1}$. From the presentation of $\tilde{D}_2$ above, we see that there is $\omega \in \text{Aut}(G)$ such that $\omega(x, y) = (c_1, c_2)$ and, hence, there is one class of generating vectors, represented by (x, y, yx).

$$8,\ (2^3, 4),\ D_4 = \langle x, y : x^2 = y^4 = 1, xyx^{-1} = y^{-1}\rangle. \tag{2.n}$$

By considering images in $G/\langle c_4\rangle$, exactly two of c_1, c_2, c_3 do not lie in $\langle c_4\rangle$. Using $\mathscr{B}$-transformations, if necessary, we may assume that $c_1, c_2 \notin \langle c_4\rangle$. Since the normal subgroup $\langle c_4\rangle$ of order 4 is complemented by $\langle c_1\rangle$ and G is non-abelian, then G must be isomorphic to D_4. There is an element of $\mathrm{Aut}(G)\simeq D_8$ taking the pair (c_1, c_4) to (x, y). Since we assume $c_3\in\langle y\rangle$, then $c_3=y^2$ and $c_2=xy$. Thus, all generating vectors are equivalent to (x, xy, y^2, y).

$$12,\ (2,6,6),\ \mathbb{Z}_2\times\mathbb{Z}_6. \tag{2.p}$$

Apply the method of Example 3.4.

$$12,\ (3,3,6),\ \text{no group exists.} \tag{2.q}$$

If (c_1, c_2, c_3) is a generating vector, then $\langle c_3\rangle\lhd G$. The subgroup $H=\langle c_3^2\rangle$ is characteristic in $\langle c_3\rangle$ so $H\lhd G$. By Sylow theorems K is the unique cyclic subgroup of order 3, so $H\lhd\langle c_1\rangle$, $\langle c_2\rangle$ and $\langle c_3\rangle$. Apply (2.17) to conclude that non-trivial elements of H must have ten fixed points. But, by the remark following (2.17) and the classification of automorphisms of order 3, given by case 2.c, such an automorphism always has four fixed points.

$$12,\ (3,4,4),\ D_{4,3,-1}. \tag{2.r}$$

If $H=\langle c_1\rangle$, then the only possible numerical projection for $S/H\to S/G$ is $[1^4,4,4]:[3,3,3,3]\to[3,4,4]$, so H is normal by Proposition 2.8. By Sylow theorems, H is the unique subgroup of order 3. Now apply the method of Example 3.4.

$$12,\ (2^3,3),\ D_6=\langle x,y\colon x^2=y^6=1, xyx^{-1}=y^{-1}\rangle. \tag{2.s}$$

For $H=\langle c_4\rangle$ the only possible projection $S/H\to S/G$ is: $[1^4,2^2,2^2,2^2]:[3,3,3,3]\to[3,2,2,2]$. Thus $H\lhd G$ and $G/H\simeq\mathbb{Z}_2\times\mathbb{Z}_2$ by Proposition 2.8 and Table 1. Since some element of G/H acts trivially on H there is $K\simeq\mathbb{Z}_6$ with $H\lhd K\lhd G$. Let (c_1, c_2, c_3, c_4) be a generating vector. Not all of c_1, c_2 and c_3 can lie in K, so K is complemented in G and $G\simeq\mathbb{Z}_2\times\mathbb{Z}_6$ or D_6. By statement (3.2), $G\not\simeq\mathbb{Z}_2\times\mathbb{Z}_6$. By considering images in G/K, we see that exactly one of c_1, c_2 and c_3 must lie in K, using $\mathscr{B}$-transformations we may assume the element is c_3. Since G has a unique subgroup of order 6, then $K=\langle y\rangle$ and we may use $\mathrm{Aut}(G)$ to force $c_1=x$ and $c_4=y^2$. The unique class of generating vectors is thus represented by (x, xy, y^3, y^2).

$$15,\ (3,3,5),\ \text{no group exists.} \tag{2.t}$$

G must be $\mathbb{Z}_{15}$, but then condition (3.2) fails.

$$16,\ (2,4,8),\ D_{2,8,3}. \tag{2.u}$$

Apply the method of Example 3.4.

$$20,\ (2,5,5),\ \text{no group exists.} \tag{2.v}$$

There is a unique 5-Sylow subgroup, now apply the fixed point argument in case 2.q.

$$\begin{aligned}&24,\ (2,4,6),\ \mathbb{Z}_2\ltimes(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)\\ &\quad=\langle x,y,z,w\colon x^2=y^2=z^2=w^3=[y,z]=[y,w]=[z,w]=1,\\ &\qquad xyx^{-1}=y, xzx^{-1}=zy, xwx^{-1}=w^{-1}\rangle.\end{aligned} \tag{2.w}$$

Let $H=\langle c_3\rangle$, $S/H\to S/G$ must be $[1^2\cdot 2,2^2,4]:[6,6,3]\to[6,2,4]$. Let $N=\mathrm{core}_G(H)$. From Table 1 and the discussion below Proposition 2.8, $|N|=3$, the branching data for G/N acting on S/N is $(2,2,4)$ and $G/N=D_4$. Since $N_G(H)/H\simeq N_{G/N}(H/N)/(H/N)$, then $|N_G(H)|=12$ or 24; since $|N_{D_4}(J)/J|=2$ or 4 for every proper subgroup J of D_4. Since H is not normal in G, by a fixed point argument, then $|N_G(H)|=12$. Let $K=N_G(H)$, then K acts on S with branching data $(2,6,6)$ and $K\simeq\mathbb{Z}_2\times\mathbb{Z}_6\simeq\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3$ (2.p–2.s of Table 4). Since $S/K\to S/G$ is $[1^2,2,2]:[6,6;2]\to[6,4,2]$, then $\langle c_1\rangle$ must be a subgroup complementary to K. Write $K=\langle y,z,w: y^2=z^2=w^3=[y,z]=[y,w]=[z,w]\rangle$. The action of c_1 on $\langle y,z\rangle$, induced by conjugation, must fix a point, by picking new generators we may assume $c_1yc_1^{-1}=y$, $c_1zc_1^{-1}=yz$. Set $J=\langle y,z\rangle$, then J has branching data (2^5) and the projection $S/J\to S/G$ is $[2^3,3^2,2^3]:[2,2,2;2,2]\to[4,6,2]$. Therefore, $S/J\simeq\Sigma_3$, so that c_1 acts non-trivially on elements of order 3 and $c_1wc_1^{-1}=w^{-1}$. Setting $x=c_1$, $G=\langle x,y,z,w: x^2=y^2=z^2=w^3=[y,z]=[y,w]=[z,w]=1, xyx^{-1}=y, xzx^{-1}=zy, xwx^{-1}=w^{-1}\rangle$.

A generating vector is $(x,(zwx)^{-1},zw)$. Let (c_1,c_2,c_3) be any other generating $(2,4,6)$-vector. Redefine K to be $\langle y,z,w\rangle$. Every element of order 6 lies in K, so $\langle c_4\rangle\subset K$. Since c_1 and c_4 generate G, then $\langle c_1\rangle$ is a complement to K. Furthermore, since c_1 has order 2, then $c_1=xh$, for some $h\in K$, $xhx^{-1}=h^{-1}$. A straightforward calculation shows that $h=x^{-1}gxg^{-1}$, for some $g\in K$, so $c_1=gxg^{-1}$, and hence we may assume $c_1=x$. Since $\langle c_3\rangle$ is not normal, then $c_3=zw, zw^{-1}, zyw$ or zyw^{-1}. Conjugating by x fixes c_1 and reduces the choices for c_3 to zw and zw^{-1}. The map $x\to x, y\to y, z\to z, w\to w^{-1}$ is easily seen to be an automorphism, from the presentation of G. Thus, there is a single equivalence class of generating vectors, represented by $(x,(zwx)^{-1},zw)$.

$$24,\ (3,3,4),\ \mathrm{SL}_2(3)=\left\langle x,y: x=\begin{pmatrix}1&1\\0&1\end{pmatrix}, y=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right\rangle. \tag{2.x}$$

If $H\subseteq G$ is the 2-Sylow subgroup, then the projection can only be $[1^3,3,3]:[4,4,4]\to[4,3,3]$, thus $H\lhd G$ and $H\simeq\tilde{D}_2$ (quaternions). Since H is complemented by $\langle c_1\rangle$, $G\simeq\mathbb{Z}_3\ltimes\tilde{D}_2$. Let J be the centre of H, then J fixes six points and $S/J\to S/G$ must be $[2^6,3^2,3^2]:[2,2,2,2,2,2]\to[4,3,3]$, so $G/J\simeq A_4$, by Table 1. It follows that c_1 acts non-trivially on H. Since all automorphisms of $\tilde{D}_2$ of order 2 are conjugate in $\mathrm{Aut}(\tilde{D}_2)$, then there is a unique non-trivial semi-direct product $\mathbb{Z}_3\ltimes\tilde{D}_2$ isomorphic to $\mathrm{SL}_2(3)$. By a character table calculation (cf. [30] for table) there are 24 $(3,3,4)$-triples in G and $\mathrm{Aut}(G)\simeq\mathrm{PGL}_2(3)$ acts transitively on them. There is a single class of vectors all equivalent to $(x,(yx)^{-1},y)$.

30, (2, 3, 10), no group exists. (2.y)

Let H be a 5-Sylow subgroup. If there are 5-Sylow subgroups other than H, then there are six of them, each of which is its own normalizer. However, this contradicts the existence of an element of order 10. Any element of order 3 must centralize H, therefore there is an element of order fifteen. This contradicts case 2.t.

40, (2, 4, 5), no group exists. (2.z)

Argue as in case 2.v.

$$48,\ (2,3,8),\ \mathrm{GL}_2(3) = \left\langle x, y\colon x = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\rangle. \tag{2.aa}$$

Let H be a 2-Sylow subgroup of G and $N = \mathrm{core}_G(H)$. The symbol of the projection $S/H \to S/G$ is $[1\cdot 2, 1\cdot 2, 3] : [2;4,8] \to [2,8,3]$. From Table 1 it follows that $|N| = 8$ and the symbol of $S/N \to S/G$ is $[2^3, 2^3, 3^2] : [4,4,4] \to [8,2,3]$. Again from the table $N = \tilde{D}_2$, let $J = \mathrm{Cent}(N) \simeq \mathbb{Z}_2$. From formula (2.17), J fixes six points and the symbol of the projection $S/J \to S/G$ is $[4^6, 2^{12}, 3^8] : [2,2,2,2,2,2] \to [8,3,2]$. Therefore G/J is Σ_4 by Table 1. The classification of central extensions of symmetric groups may be found in [57], the only possibilities for G are $\mathbb{Z}_2 \times \Sigma_4$, $\tilde{\Sigma}_4$ and $\mathrm{GL}_2(3)$. Now $G \neq \mathbb{Z}_2 \times \Sigma_4$ since $\mathbb{Z}_2 \times \Sigma_4$ has no elements of order 8. Also $G \neq \tilde{\Sigma}_4$ since $\tilde{\Sigma}_4$ has unique element of order 2 which generates the centre. But, in this case c_1 would be central and c_1, c_2, c_3 would generate an abelian group, a contradiction. By a character table calculation $\mathrm{GL}_2(3)$ has 48 (2, 3, 8)-vectors. Now $\mathrm{Aut}(\mathrm{GL}_2(3)) \simeq \Sigma_4 \times \mathbb{Z}_2$, Σ_4 corresponding to inner automorphisms and $\mathbb{Z}_2$ generated by the central automorphism $g \to g \cdot \det(g)$. Since there is a generating (2, 3, 8)-vector all the vectors are equivalent generating vectors, $(x, y, (xy)^{-1})$ is a representative.

Case by case analysis for genus 3

$$4,\ (2^6),\ \mathbb{Z}_2 \times \mathbb{Z}_2. \tag{3.h}$$

This is analogous to case 2.f.

$$4,\ (1\colon 2^2),\ \mathbb{Z}_4 \text{ or } \mathbb{Z}_2 \times \mathbb{Z}_2. \tag{3.i}$$

If G is cyclic, then apply Harvey's results to get case 3.i.1. The non-cyclic case was done in Example 3.3.

$$6,\ (3^4),\ \text{no group exists.} \tag{3.l}$$

In any group of order 6 the elements of order 3 generate a proper subgroup.

$$6,\ (2^4, 3),\ D_3 = \langle x, y\colon x^2 = y^3 = xyx^{-1} = y^{-1} \rangle. \tag{3.m}$$

The group G cannot be abelian, by statement (3.2), so $G \simeq D_3$. Let $z = xy$ and $w = xy^{-1}$ be the other two reflections in D_3. We may assume, via $\mathrm{Aut}(G)$-action, that $c_5 = y$ and that at least two of the reflections in $\{c_1, c_2, c_3, c_4\}$ equal x. Let (c_j, c_{j+1}), $1 \le j \le 3$ be an adjacent pair of reflections and consider the image of the pair under the action of a $\mathcal{B}$-transformation of Type II(j). The transformation maps the pairs $(z,x), (w,x), (z,w)$ and (w,z) to $(x,w), (x,z), (w,x)$ and (z,x), respectively. By repeated application of these transformations, we may assume that (c_1, c_2, c_3, c_4) is one of $(x,x,x,w), (x,x,x,z), (x,x,w,w)$ or (x,x,z,z). However, since $c_5 = y$ only the first of these yields a generating vector namely (x,x,x,xy^{-1},y).

$$6,\ (1\colon 3),\ D_3 = \langle x, y\colon x^2 = y^3 = xyx^{-1} = y^{-1} \rangle. \tag{3.n}$$

The group G cannot be abelian by (3.4), so $G \simeq D_3$. If (a,b,c) is a generating vector at least one of a,b must be a reflection. If a is a reflection, then (a,b,c) must be $\mathrm{Aut}(G)$-equivalent to (x,xy,y). If a has order 3, then it can be transformed into a reflection by a transformation of Type I.a.

$$8,\ (2^2,4^2),\ \mathbb{Z}_2\times\mathbb{Z}_4 = \langle x,y\colon x^2=y^4=[x,y]=1\rangle. \tag{3.q.1}$$

$$8,\ (2^2,4^2),\ D_4 = \langle xy\colon x^2=y^4=1, xyx^{-1}=y^{-1}\rangle. \tag{3.q.2}$$

As argued in case 2.n, $G=\langle c_1\rangle \ltimes \langle c_4\rangle$. Therefore, $G=\mathbb{Z}_2\times\mathbb{Z}_4$ or D_4, and there is an automorphism of G carrying (c_1,c_4) to (x,y). Now consider the equation $c_2c_3=xy^{-1}$. In D_4 there are two solutions: $(c_2,c_3)=(x,y^{-1})$ or (xy^2,y). For $G\simeq\mathbb{Z}_2\times\mathbb{Z}_4$, we get these two solutions and the additional solution: (y^2,xy). For D_4, the square of a transformation of Type II(2) transforms (x,xy^2,y,y) into (x,x,y^{-1},y), and we get one class of vectors. Each of the vectors $(x,x,y^{-1},y),(x,xy^2,y,y)$, and (x,y^2,xy,y) defines a distinct equivalence class of actions of $\mathbb{Z}_2\times\mathbb{Z}_4$. We can see this by observing that these vectors are characterized, in the same order as above, by: $c_1=c_2$, $c_3=c_4$ and $G=\langle c_3,c_4\rangle$. Since each of these characterizations is invariant under $\mathrm{Aut}(G)\times\mathscr{B}$ we get three equivalence classes of actions.

$$8,\ (2^5),\ \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2 = \langle x,y,z\colon x^2=y^2=z^2=[y,z]=[x,z]=[x,y]=1\rangle. \tag{3.r.1}$$

$$8,\ (2^5),\ D_4 = \langle x,y\colon x^2=y^4=1, xyx^{-1}=y^{-1}\rangle. \tag{3.r.2}$$

If G is abelian, then $G\simeq\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$. By using $\mathscr{B}$-transformations of Type II, we can assume that c_1,c_2,c_3 generate G, and transforming by $\mathrm{Aut}(G)$, we can assume $c_1=x$, $c_2=y$, $c_3=z$. Since $c_4c_5=xyz$ and c_4,c_5 both have order 2, c_4 is one of x,y,z,xy,xz,yz. Transforming by $\mathrm{Aut}(G)$, we may assume that $c_4=x$ or yz and (c_1,c_2,c_3) is a permutation of x,y,z. Using a $\mathscr{B}$-transformation, we can permute (c_1,c_2,c_3) to obtain (x,y,z,x,yz) or (x,y,z,yz,x). These two are equivalent by interchange of c_4 and c_5.

If G is non-abelian, then $G\simeq D_4$, since this is the only non-abelian group of order 8 generated by elements of order 2. An even number of the c_j's must be reflections and the remaining c_j's must be the central element y^2. If only two of these elements are reflections, then the c_j's cannot generate G. Thus, we may assume that $c_5=y^2$. By considering images of the remaining c_j's in $G/\langle y^2\rangle$ we see that there must be two pairs of commuting reflections. Now we can use $\mathrm{Aut}(G)$ and the transformations used in case 3.m to transform any generating vector into (x,x,xy,xy^3,y^2).

$$8,\ (1\colon 2),\ \tilde{D}_2 = \langle x,y\colon x^4=y^4=1, x^2=y^2, xyx^{-1}=y^{-1}\rangle. \tag{3.s.1}$$

$$8,\ (1\colon 2),\ D_4 = \langle x,y\colon x^2=y^4=1, xyx^{-1}=y\rangle. \tag{3.s.2}$$

By (3.4), G cannot be abelian, so it must be the quaternion group $\tilde{D}_2$ or the dihedral group D_4. If (a,b,c) is a generating vector, then $[a,b]=c^{-1}$, so a,b do not commute. For $a, b \in \tilde{D}_2$ with $[a, b] \neq 1$, both a, b have order 4 and all such pairs are $\mathrm{Aut}(\tilde{D}_2)$-equivalent, by a presentation argument. We may pick (x, y, y^2) as a representative. The case $G \simeq D_4$ can be handled as in case 3.n, yielding a single class of generating vectors represented by (x, xy, y^2).

12, $(3, 6, 6)$, no group exists. (3.w)

The proof is similar to that in case 2.q except that an automorphism of order 3 of a surface of genus 3 has either 2 or 5 fixed points.

12, $(4, 4, 6)$, $D_{4,3,-1} = \langle x, y : x^4 = y^3 = 1, xyx^{-1} = y^{-1} \rangle$. (3.x)

As in case 2.q the 3-Sylow subgroup is normal, so G must be the non-abelian split metacyclic group $D_{4,3-1}$. In this group if z, w are any elements of orders 4 and 4 respectively, then $zwz^{-1} = w^{-1}$. This implies that any such pair (z, w) is $\mathrm{Aut}(G)$-equivalent to (x, y). From this it follows that all generating vectors are equivalent to (x, xy^{-1}, x^2y).

12, $(2^3, 6)$, $D_6 = \langle x, y : x^2 = y^6 = 1, xyx^{-1} = y^{-1} \rangle$. (3.y)

Apply the argument of case 2.n to conclude that $G \simeq D_6$ and that every generating vector is equivalent to (x, xy^2, y^3, y).

12, $(2^2, 3^2)$, $A_4 = \langle x, y : x = (1,2)(3,4), y = (1,2,3) \rangle$. (3.z)

If H is the 2-Sylow subgroup, then $S/H \to S/G$ must be $[1^3, 1^3, 3, 3] : [2,2,2;2,2,2] \to [2,2,3,3]$ or $[1 \cdot 2, 1 \cdot 2, 3, 3] : [1:2;2] \to [2,2,3,3]$. In the latter case H is not normal, which forces the 3-Sylow subgroup to be normal. But, a fixed point argument, as in case 2.q, yields a contradiction. Thus, we may assume that the first projection is correct, that $H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $H \lhd G$, by Table 1. Since H is complemented, G is a semi-direct product $\mathbb{Z}_3 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$. Now G cannot be abelian, for then the 3-Sylow subgroup would be normal and this was eliminated above, therefore $G \simeq A_4$. There are exactly 24 pairs of elements (z, w) in A_4 with $o(z) = 2$, $o(w) = 3$ and Σ_4 acts on them simply transitively by conjugation. Thus we may assume that $(c_1, c_3) = (x, y)$. If $c_1 = c_2$, then the generating vector is (x, x, y, y^{-1}). Now suppose that $c_1 \neq c_2$. If c is an element of order 3, then one of $cc_2c^{-1}, c^2c_2c^{-2}$ equals c_1 and if d is conjugate to c in A_4, then $dcc_2c^{-1}d^{-1} = c^2c_2c^{-2}$. The square of a transformation of Type II(3) is $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \to (\gamma_1, \gamma_3^{-1}\gamma_2\gamma_3, \gamma_3^{-1}\gamma_2^{-1}\gamma_3\gamma_2\gamma_3, \gamma_4)$ which is a $\mathscr{B}$-transformation by Proposition 2.4. By applying this transformation once or twice we can transform c_2 into cc_2c^{-1} or $dcc_2c^{-1}d^{-1}$, respectively, where c and d are conjugate elements of order 3. Thus, the generating vector is equivalent to a vector with $c_1 = c_2$.

Done in Example 3.4. (3.ab.1)–(3.ab.2)

Done as Example 3.5. (3.ac.1)–(3.ac.2)

16, $(2, 2, 2, 4)$, $\mathbb{Z}_2 \ltimes D_4 = \langle x : x^2 = 1 \rangle \times \langle y, z : y^2 = z^4 = 1, yzy^{-1} = z^{-1} \rangle$. (3.ad.1)

16, (2, 2, 2, 4), $\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_4)$

$$= \langle x, y, z \colon x^2 = y^2 = z^4 = [y, z] = 1, [x, z] = 1, xyx^{-1} = yz^2 \rangle. \qquad (3.\text{ad}.2)$$

We first show that $H = \langle c_4 \rangle$ is normal. Let K be a subgroup of order 8 containing H. By arguments similar to those in Example 3.5, $K \simeq D_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_4$. If $K \simeq D_4$, then H is characteristic in K and H is normal, therefore, we may suppose $K \simeq \mathbb{Z}_2 \times \mathbb{Z}_4$. The branching data for K is $(2^2, 4^2)$ and the relative projection is $[1^2, 1^2, 2, 2] : [4, 4; 2, 2] \to [4, 2, 2, 2]$, so we conclude that exactly two of $\langle c_1 \rangle$, $\langle c_2 \rangle$ and $\langle c_3 \rangle$ are complements to K. Using $\mathscr{B}$-transformations we may assume that $c_3 \in K$ and $c_1, c_2 \notin K$. Let $w = c_3 c_4$, w lies in K, has order 4 and $c_2 = wc_1$. Since c_1 and c_2 have order 2, then $c_1 w c_1^{-1} = w^{-1}$. Since K has exactly two subgroups of order 4, then c_1 normalizes both of them and hence $H \lhd G$.

As in Example 3.5, the branching data for H is (4^4) since H is normal. The associated relative projection has the symbol $[1^4, 2^2, 2^2, 2^2] : [4, 4, 4, 4] \to [4, 2, 2, 2]$. Therefore, none of the subgroups $\langle c_1 \rangle$, $\langle c_2 \rangle$ and $\langle c_3 \rangle$ lie in H, and $G/H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. One of $\langle c_1 \rangle$, $\langle c_2 \rangle$ or $\langle c_3 \rangle$ acts trivially on H so, as above, we may assume that $K = \langle c_3, c_4 \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_4$. Let $x = c_1$, $y = c_3$ and $z = c_4$. From previous arguments, the automorphism of K induced by conjugation by x must be one of $y \to y, z \to z^{-1}$; $y \to yz^2, z \to z^{-1}$ or $y \to yz^2, z \to z$. In the first case x, z generate a subgroup isomorphic to D_4 and y generates a commuting complementary subgroup, so $G \simeq \mathbb{Z}_2 \times D_4$. Interchanging the roles of x and y we arrive at the first presentation above. The last two automorphisms are conjugate in $\text{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ by $y \to yz$, $z \to z$, so we get $\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_4)$, with the second presentation above. The two groups are not isomorphic since their centres are not isomorphic.

Suppose (c_1, c_2, c_3, c_4) is an arbitrary generating vector for $G = \mathbb{Z}_2 \times D_4$. The group $\text{Aut}(G)$ is generated by $\text{Aut}(D_4)$ and the central automorphisms: $g \to g\lambda(g)$, where $\lambda : G \to Z(G)$ is a suitable homomorphism. From this description we see that any two elements of order 4 are $\text{Aut}(G)$-equivalent, thus, we may assume that $c_4 = z$. None of the elements c_1, c_2 or c_3 lie in $\langle c_4 \rangle$ and exactly one of these elements centralizes $\langle c_4 \rangle$, and hence all of G. Therefore, by a combination of central automorphisms and $\mathscr{B}$-transformations we may assume that $c_1 = x$. Now c_2 has the form $yz^s x^r$, by using suitable combinations of the automorphisms $\text{Ad}_z : g \to zgz^{-1}$ and central automorphisms, we can transform c_2 into y. Thus, there is a unique generating vector represented by (x, y, yxz^{-1}, z).

Now let (c_1, c_2, c_3, c_4) be an arbitrary generating vector for $G = \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_4)$. As above, we may assume that $K = \langle c_3, c_4 \rangle = \langle y, z \rangle$ and the action of c_1 and c_2 is given by the x-action in the presentation. The subgroup K has two cyclic subgroups of order 4, namely $\langle z \rangle$ and $\langle yz \rangle$. The element x normalizes both of these subgroups, fixing z and conjugating yz to its inverse. By calculation above, $c_3 c_4$ is conjugated to its inverse by c_1. Thus, the pair (c_3, c_4) must be one of $(y, z), (y, z^{-1}), (yz^2, z)$, $(yz^2, z^{-1}), (z^2, yz)$ or (z^2, yz^{-1}). Transforming by combinations of the automorphisms Ad_x and $\mu : x \to x, y \to y, z \to z^{-1}$, we see that (c_3, c_4) is equivalent to one of (y, z^{-1}) or (z^2, yz). Since $c_1 \notin K$ and has order 2, then $c_1 = xw$, where $w \in K$ satisfies $xwx^{-1} =$

w^{-1} and c_1 is one of $x, xzy, xz^{-1}y$ or xz^2. Transforming by the automorphisms $\mathrm{Ad}_g, g \in K$, we can cut down this list to x and xzy without changing c_3 or c_4. Any vector must now be equivalent to one of the following vectors: (x, xzy, y, z^{-1}), (xzy, xz^2, y, z^{-1}), (x, xzy, z^2, yz), (xzy, xz^2, z^2, yz). A $\mathscr{B}$-transformation of Type II(1) interchanges the first and second vector and the third and fourth vectors. The first and third vectors are not equivalent. In the third vector, c_4 generates the centre of G but this fails for the first vector. This characterization is invariant under the $\mathrm{Aut}(G) \times \mathscr{B}$ action.

18, (2, 6, 9), no group exists. (3.ae)

By Example 3.4, $G = D_{2,9,j}$ where j satisfies equation (3.10) with $p = 2$ and $q = 9$. But, the solutions of this equation, $1, -1$, do not satisfy equations (3.12.i)–(3.12.ii) with $r = 6$.

18, (3, 3, 9), no group exists. (3.af)

The 3-Sylow subgroup must be normal, but then G cannot be generated by elements of order 3 and 9.

Done in Example 3.4. (3.ag)

24, (2, 4, 12), $D_{2,12,5}$. (3.ah)

Apply the method of Example 3.4.

24, (2, 6, 6), $\mathbb{Z}_2 \times A_4 = \langle x : x^2 = 1 \rangle \times \langle y, z : y = (1,2)(3,4), z = (1,2,3) \rangle$. (3.ai)

If H is the 2-Sylow subgroup, then $S/H \to S/G$ must be $[1^3, 3, 3] : [2, 2, 2; 2; 2] \to [2, 6, 6]$. Thus, $H = D_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and is normal. Let g be an element of order 3, $K = \langle g \rangle$. Using a fixed point argument as in cases 2.q and 3.w, we see that K is not a normal subgroup. Thus, g acts non-trivially on H. As $|\mathrm{Aut}(D_4)| = 8$, we must have $H = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Considering g as a matrix in $\mathrm{GL}_3(2)$, we get $H = H_1 \times H_2$, where $|H_1| = 2$, $|H_2| = 4$, $g = \mathrm{id}$ on H_1 and g has no non-trivial invariants in H_2. The subgroup $\langle g \rangle \ltimes H_2$ is isomorphic to A_4 and we get a direct product decomposition: $G = H_1 \times (\langle g \rangle \ltimes H_2) \simeq \mathbb{Z}_2 \times A_4$. The projection of a generating (2, 6, 6)-vector to $\mathbb{Z}_2$ or A_4 must be a generating (1, 2, 2)-vector or (2, 3, 3)-vector respectively. There are 8 elements of order 3 in A_4 and hence there are 48 (2, 3, 3)-vectors in A_4. As $|\mathrm{Aut}(A_4)| = |\Sigma_4| = 24$ there are two $\mathrm{Aut}(A_4)$-classes of (2, 3, 3)-vectors. These two vectors are transformed into each other by the second transformation listed in Proposition 2.7. We may combine the (1, 2, 2)-vector $(1, x, x)$ of $\mathbb{Z}_2$ and the (2, 3, 3)-vector $(y, z, (yz)^{-1})$ of A_4 to get the (2, 6, 6)-vector $(y, xz, x(yz)^{-1})$ to which all the other generating vectors are equivalent.

24, (3, 3, 6), $\mathrm{SL}_2(3) = \left\langle x, y : x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, y = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right\rangle$. (3.aj)

If H is a 2-Sylow subgroup, then the only possible projection $S/H \to S/G$ is $[3,3,3]:[1:2] \to [6,3,3]$, and $H = D_4$ or the quaternions $\tilde{D}_2$. As in the last case a 3-Sylow subgroup, K, cannot be normal. Thus, there are four 3-Sylow subgroups, each of index 2 in its normalizer. Therefore, there are exactly eight elements of order 3 and eight elements of order 6. It follows that $H \lhd G$ and $G = K \ltimes H$. Now just modify the argument of 2.x to show that $G \simeq \mathrm{SL}_2(3)$ and that all vectors are equivalent to $(x, y, (xy)^{-1})$.

$$24,\ (3,4,4),\ \Sigma_4 = \langle x, y : x = (1,2,3,4), y = (1,3,4,2)\rangle. \tag{3.ak}$$

Let H be a 2-Sylow subgroup, then $S/H \to S/G$ is $[1\cdot 2, 1\cdot 2, 3]:[4,2;4,2] \to [4,4,3]$. Let $N = \mathrm{core}_G(H)$, then $G/N \simeq \Sigma_3$, the projection $S/N \to S/G$ is $[2^3, 2^3, 3]:[2,2,2;2,2,2] \to [4,4,3]$ and $N \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. It follows that G has a subgroup K of index 2 and $S/K \to S/G$ is $[1^2, 2, 2]:[3,3;2,2] \to [3,4,4]$, so $K \simeq A_4$, by case 3.z. Now $H \simeq D_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_4$, by cases 3.q.1–3.q.2. If $H \simeq D_4$, then $N = K \cap H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and is normal. The set $H - N$ consists of 2 reflections and two elements of order 4, let x be a reflection. Then, $G = \langle x \rangle \ltimes A_4$. Since x induces an involuntary automorphism of A_4, acting non-trivially on $H \cap A_4$, then x acts as a transposition from Σ_4 and hence $G \simeq \Sigma_4$. By a character table calculation (see [30, p. 287] for table), there is a unique equivalence class of generating $(3,4,4)$-vectors with representative $((xy)^{-1}, x, y)$. The case $H \simeq \mathbb{Z}_2 \times \mathbb{Z}_4$ cannot occur. Every involution of $\Sigma_3 \simeq G/N$ is the image of an element of some conjugate of H. Since H is abelian, the involutions of Σ_3, and hence all elements of Σ_3 act trivially on $A_4 \cap H$. Thus, elements of order 3 in A_4 act trivially on $H \cap A_4$, a contradiction.

$$24,\ (2^3, 3),\ \Sigma_4 = \langle x, y, z : x = (1,2), y = (2,3), z = (3,4)\rangle. \tag{3.al}$$

Mimic the proof of case 3.ak to show that $G = \Sigma_4$. If (c_1, c_2, c_3, c_4) is a generating $(2,2,2,3)$-vector, exactly two of c_1, c_2, c_3 are transpositions, we may assume they are c_1, c_2, by using $\mathscr{B}$-transformations. The elements c_1, c_2 cannot commute, for then $c_1 c_2 c_3 = c_4^{-1}$ lies in the Klein 4-group, a contradiction. There are 24 pairs of non-commuting transpositions in Σ_4, Σ_4 acts transitively on them so we may assume $c_1 = (1,2)$, $c_2 = (3,4)$. By using transformations as in case 3.z we may assume $c_3 = (1,3)(2,4)$ and that the generating vector is $(x, y, yxzy, yz)$.

$$\begin{aligned} &32,\ (2,4,8),\ \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8) \\ &\quad = \langle x, y, z : x^2 = y^2 = z^8 = [x,y] = [y,z] = 1, xzx^{-1} = yz^3\rangle. \end{aligned} \tag{3.am.1}$$

$$\begin{aligned} &32,\ (2,4,8),\ \mathbb{Z}_2 \ltimes D_{2,8,5} \\ &\quad = \langle x, y, z : x^2 = y^2 = z^8 = 1, yzy^{-1} = z^5, xyx^{-1} = yz^4, xzx^{-1} = yz^3\rangle. \end{aligned} \tag{3.am.2}$$

Let H be a subgroup of index 2 containing $K = \langle c_3 \rangle$. From cases 3.ab.1–3.ab.2, $H = \mathbb{Z}_2 \times \mathbb{Z}_8$ or $D_{2,8,5}$ with presentations: $\langle y, z : y^2 = z^8 = 1, yzy^{-1} = z^j\rangle$, $j = 1$ or 5 respectively. Since $(yz^s)^2 = z^{2s}$ or z^{6s} as $j = 1$ or 5, respectively, then in both cases H has 2 cyclic subgroups of order 8, $\langle z \rangle$ and $\langle yz \rangle$, and three elements of order 2, namely y, yz^4 and z^4, of which only z^4 is a square. Let $x=c_1$, since c_1 and c_3 generate G, then $c_1 \notin H$, so $G=\langle x\rangle \ltimes H$. By a fixed point argument $\langle c_3\rangle$ cannot be normal. Thus, x conjugates $\langle z\rangle$ and $\langle zy\rangle$ into each other, and $xyx^{-1}=y$ or yz^4. Any automorphism satisfying these properties, involutary or not, must have the form $y\to yz^m, z\to yz^k$ where $m=0,4$, $k=1,3,5,7$. We may assume $k=1$ or 3 since, for both groups, $y\to yz^4, z\to z$ defines a central automorphism which conjugates the automorphisms for $k=1,3$ to the automorphisms for $k=5,7$ respectively. It is straightforward to check that among these automorphisms we only get involutary automorphisms for the cases $(j,m,k)=(1,0,1),(1,0,3),(5,0,1),(5,4,3)$. We may eliminate $(j,m,k)=(1,0,1)$ as follows. The subgroup $N=\langle x,y\rangle$ is a normal, isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2$ and has quotient $G/N\simeq\langle z\rangle\simeq\mathbb{Z}_8$. But then, a generating $(2,4,8)$-vector for G cannot project to a generating vector for $\mathbb{Z}_8$, a contradiction. We may also eliminate $(j,m,k)=(5,0,1)$ as follows. In this case, $y(yz)y^{-1}=(yz)^5$ and $x(yz)x^{-1}=(yz)^k$, hence, $x\to x, y\to y, z\to yz$ is an automorphism of G. Since $x=c_1$ and $\langle c_3\rangle=\langle z\rangle$ or $\langle yz\rangle$ we may use this automorphism to force $c_3=\langle z\rangle$, but not move c_1. Thus, $c_2=xz^s$ for some s relatively prime to 8, but $c_2^2=yz^2$ and so c_2 has order 8, a contradiction. Thus, our groups must have the presentations above and (x, xz, z^{-1}) is a generating $(2,4,8)$-vector for both cases. The groups are non-isomorphic since the centres of the groups are $\mathbb{Z}_2\times\mathbb{Z}_2$ and $\mathbb{Z}_4$ in cases 3.m.1 and 3.m.2., respectively.

Now suppose (c_1,c_2,c_3) is an arbitrary $(2,4,8)$-vector. For every element $w\in H$, $xwxw$ has order 1 or 2, thus the elements of order 8 all lie in H. These elements are z, z^3, z^5, z^7, lying in $\langle z\rangle$, and yz, yz^3, yz^5, yz^7, lying in $\langle yz\rangle$. Since conjugation by x interchanges $\langle z\rangle$ and $\langle yz\rangle$, we may assume that c_3 is one of z, z^3, z^5, z^7. For $H=\mathbb{Z}_2\times\mathbb{Z}_8$ the maps $x\to x, y\to y, z\to z^k$, $k=1,3,5,7$ are automorphisms and for $H=D_{2,8,5}$ the maps $x\to x, y\to y, z\to z^k$, $k=1,5$ and $x\to x, y\to yz^4, z\to z^k$, $k=3,7$ are also automorphisms. Thus, we may assume that $c_3=z$. Now $c_1=xy^rz^s$, otherwise $c_1,c_2,c_3\in\langle y,z\rangle$. Since c_1 has order 2, then $c_1\in\{x, xz^2, xz^4, xz^6, xy, xyz^2, xyz^4, xyz^6\}$ for $H=\mathbb{Z}_2\times\mathbb{Z}_8$ and $c_1\in\{x, xz^4, xyz^2, xyz^6\}$ for $D_{2,8,5}$. By conjugating by powers of z, we fix z and reduce these lists to $\{x, xy\}$ and $\{x\}$, respectively. In the case of $H=\mathbb{Z}_2\times\mathbb{Z}_8$, $x\to xy, y\to y, z\to z$ gives an automorphism of G so we may assume $c_1=x$ in both cases.

42, (2, 3, 14), no group exists. (3.an)

The 7-Sylow subgroup, H, is normal. A generating $(2,3,14)$-vector projects to a $(2,3,2)$-vector in G/H so G/H must be Σ_3. Since there is an element of order 14 in G there must be an element of order 2 in G/H which acts trivially on H. Since G/H has no non-trivial normal subgroup of even order, then all of G/H acts trivially on H. Hence, there is an element of order 21 in G, but this has been previously excluded.

48, (2, 3, 12), no group exists. (3.ao)

If $H=\langle c_3\rangle$, then the only possible numerical projection for $S/H\to S/G$ is

$[1\cdot 3, 1\cdot 3, 2^2]:[3;12,4]\to[3,12,2]$. From Table 1 we conclude that $N=\mathrm{core}_G(H)\simeq\mathbb{Z}_4$ and $G/N\simeq A_4$. Since c_3 has order 12 and A_4 has no non-trivial normal subgroups with order divisible by 3, we conclude, as in the case 3.an above, that G/N acts trivially on N. Therefore, there is a normal abelian 2-Sylow subgroup, K, of order 16. From Table 5 the branching data K must be $(2,8,8)$ or $(4,4,4)$, but then the symbol of the projection $S/K\to S/G$ does not exist.

$$48,\ (2,4,6),\ \mathbb{Z}_2\times\Sigma_4=\langle x\colon x^2=1\rangle\times\langle y,z\colon y=(1,2), z=(2,3,4)\rangle. \qquad (3.\mathrm{ap})$$

Let H be a 2-Sylow subgroup, then $S/H\to S/G$ is $[1\cdot 2,1\cdot 2,3]:[2;4,2;2]\to[2,4,6]$. Let $N=\mathrm{core}_G(H)$. By Table 1 the symbol of the projection $S/N\to S/G$ is $[2^3,3^2,2^3]:[2,2,2;2,2]\to[4,6,2]$, so $N=\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ or D_4 and $G/N\simeq\Sigma_3$. Let g be an element of order 6, fixing a point. The number of fixed points of g^2 is $N_G(\langle g^2\rangle)/6$. If g^2 acts trivially on N, then the number of fixed points of g^2 is a multiple of 4, but an element of order 3 has only 2 or 5 fixed points by inspection of cases 3.d–3.e. Therefore, g^2 acts non-trivially on N. The group D_4 has no automorphisms of order 3, so $N=\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$. Thus, N affords a 3-dimensional, non-trivial $\mathbb{F}_2$-representation of Σ_3. Restricted to $\mathbb{Z}_3\simeq A_3\lhd\Sigma_3$, N splits into two irreducible submodules of dimensions 1 and 2, moreover these submodules are invariant all of Σ_3. We may argue as in case 3.ai that $G\simeq N_1\times(\Sigma_3\ltimes N_2)$, where $N=N_1\times N_2$, $|N_1|=2$, $|N_2|=4$ and $\Sigma_3\ltimes N_2=\Sigma_4$. Thus $G\simeq\mathbb{Z}_2\times\Sigma_4$.

Let (c_1,c_2,c_3) be a generating $(2,4,6)$-vector. The element c_3^3 commutes with c_3^2, an element of order 3, therefore, $c_3^3=x$. By (2.17) it follows that x has eight fixed points and that $x\notin\langle c_1\rangle,\langle c_2\rangle$. Using this fact and using the projections of a generating vector to the factor groups as in case 3.ai, we may calculate that every vector is equivalent $(xy,(zy)^{-1},xz)$ or $(y,x(zy)^{-1},xz)$. There is a homomorphism $\delta: G\to\langle x\rangle=Z(G)$ such that $\delta(x)=\delta(z)=1, \delta(y)=x$, the central automorphism $g\to g\delta(g)$ interchanges the above two vectors. In the calculation one needs to show that all $(2,4,3)$-vectors in Σ_4 are equivalent, this may be done by a character table calculation.

$$48,\ (3,3,4),\ \mathbb{Z}_3\ltimes(\mathbb{Z}_4\times\mathbb{Z}_4)$$
$$=\langle x,y,z\colon x^3=y^4=z^4=[y,z]=1, xyx^{-1}=z, xzx^{-1}=(yz)^{-1}\rangle. \qquad (3.\mathrm{aq})$$

Let x be an element of order 3. By a Sylow theorem all elements of order 3 generate conjugate subgroups, so the number of fixed points of x, acting on S, is $2N_G(\langle x\rangle)/3$, by (2.17). As previously mentioned, x can have only 2 or 5 fixed points, so $N_G(\langle x\rangle)=3$. If H is a 2-Sylow subgroup, then $S/H\to S/G$ is $[1^3,3,3]:[4,4,4]\to[3,3,4]$ and $H\lhd G$. Since $\langle x\rangle$ is self-normalizing, the adjoint action of x on H is fixed point free. By case 3.ac, $H\simeq\mathbb{Z}_4\times\mathbb{Z}_4$ or $D_{4,4,-1}$. In $D_{4,4,-1}$ the intersection of the normal subgroups of order 4 is a characteristic subgroup of order 2, so x must fix this element, a contradiction. Let y be any element of order 4 in $\mathbb{Z}_4\times\mathbb{Z}_4$, $yxyx^{-1}x^2yx^{-2}$ is x-invariant, hence trivial. Set $z=xyx^{-1}$, then $xzx^{-1}=(yz)^{-1}$. If y,z do not generate $\mathbb{Z}_4\times\mathbb{Z}_4$, then $\langle y,z\rangle=\langle y\rangle$ is an x-invariant subgroup of order 4, on which x must act trivially, yielding a contradiction. Thus, we get the presentation above.

Suppose (c_1,c_2,c_3) is a $(3,3,4)$-vector. If u and v are any elements of G of orders 3 and 4, respectively, and $w=uvu^{-1}$, then the relations in the presentation are satisfied when x, y and z are replaced by u, v and w, respectively. This follows from the argument immediately above. Thus, there is an automorphism of G carrying the pair (c_1, c_3) onto the pair (x, y) and, hence, all generating vectors are equivalent to $(x, (yx)^{-1}, y)$.

72, (2, 3, 9), no group exists. (3.ar)

The element c_3 has exactly two fixed points by the remark following (2.17) and case 3.t. From (2.17) $N_G(\langle c_3 \rangle)$ is a subgroup of order 18, but no group of order 18 acts on a surface of genus 3.

Done in Example 3.6. (3.as)

Done in Example 3.1. (3.at)

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