Yet another generalization of the Kruskal–Katona theorem

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Received 21 March 1996; received in revised form 2 April 1997; accepted 14 April 1997

Abstract

For an n-tuple \( t = (t_1, t_2, \ldots, t_n) \) of integers satisfying \( 1 \leq t_1 \leq t_2 \leq \cdots \leq t_n \), \( T(t) = T \) denotes the ranked partially ordered set consisting of n-tuples \( a = (a_1, a_2, \ldots, a_n) \) of integers satisfying \( t_i - t_{i+1} \leq a_i \leq t_i \), \( i = 1, 2, \ldots, n \), partially ordered by defining \( a \) to precede \( c \) if \( a_i = c_i \) or \( c_i - t_i \) for \( i = 1, 2, \ldots, n \). The rank \( r(a) \) of \( a \) is \( \{ i : a_i = t_i \} \). For \( 0 \leq l \leq n \), the set consisting of all elements of rank \( l \) is called the \( l \)-th rank and is denoted \( T_l \). Let \( b, l, m \) denote positive integers satisfying \( b \leq l \leq m \leq n \). For a subset \( S \) of \( T_l \), \( A^b.S \) denotes the elements of \( T_{l-b} \) which precede at least one element of \( S \). An algorithm is given for calculating \( \min \{|A^b.S|\} \), where the minimum is taken over all \( m \)-element subsets \( S \) of \( T_l \). If \( t_1 = t_2 = \cdots = t_n = 1 \), it reduces to the Kruskal–Katona algorithm.

1. Introduction

Let \( B_n \) denote the partially ordered set consisting of the \( 2^n \) subsets of \( I = \{1, 2, \ldots, n\} \), partially ordered by setwise inclusion. For positive integers \( b, l, m \) where \( b \leq l \leq m \) and \( m \leq \binom{n}{l} \), the Kruskal–Katona theorem [7,8] gives an algorithm for calculating \( \min \{|A^b.A|\} \), where here and below the minimum is understood to be taken over all \( m \)-element families of \( l \)-element subsets of \( I \), and \( A^b.A \) denotes the set of \( (l-b) \)-element subsets of \( I \) which are contained in at least one element of \( A \).

The algorithm is as follows. Let \( m_l \) be the largest integer such that \( m \geq \binom{n}{l} \), let \( m_{l-1} \) be the largest integer such that \( m - \binom{m_l}{l} \geq \binom{m_{l-1}}{l-1} \), etc. until equality is attained. This gives the so-called \( l \)-binomial representation of \( m \):

\[
m = \binom{m_l}{l} + \binom{m_{l-1}}{l-1} + \cdots + \binom{m_j}{j},
\]

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1 The author is indebted to the University of Colorado for the sabbatical during which this paper was written.
where \( m_1 > m_{i-1} > \cdots > m_j \geq j \geq 1 \). Then

\[
\min |A^b \mathcal{A}| = \left( \begin{array}{c} m_l \\ l - b \end{array} \right) + \left( \begin{array}{c} m_{l-1} \\ l - 1 - b \end{array} \right) + \cdots + \left( \begin{array}{c} m_j \\ j - b \end{array} \right).
\]

The purpose of this paper is to generalize this algorithm from \( B_n \) to the posets \( T(t) \). For an \( n \)-tuple \( t = (t_1, t_2, \ldots, t_n) \) of positive integers satisfying \( t_1 \leq t_2 \leq \cdots \leq t_n \), \( T(t) \) denotes the partially ordered set (poset) consisting of \( n \)-tuples \( a = (a_1, a_2, \ldots, a_n) \) of integers satisfying \( t_n - t_i \leq a_i \leq t_i \), \( i = 1, 2, \ldots, n \), partially ordered by defining \( a \leq c \) if \( a_i = c_i \) or \( c_i = t_i \), \( i = 1, 2, \ldots, n \).

\( T = T((1, 1, \ldots, 1)) \) (\( n \) 1's) is isomorphic to \( B_n \), \( a \in T \) corresponding to \( \{i \mid a_i = 1\} \), and \( T = T((2, 2, \ldots, 2)) \) (\( n \) 2's) is isomorphic to the cubical poset, which consists of the faces of the \( n \)-dimensional cube, partially ordered by setwise inclusion, \( a \in T \) corresponding to the face \( \{z = (z_1, z_2, \ldots, z_n) \mid z_i = a_i \text{ if } a_i \neq 2, 0 \leq z_i \leq 1 \text{ if } a_i = 2\} \).

\( T \) is a ranked poset, the rank \( r(a) \) of \( a \in T \) being \( \sum a_i \). For the basic facts about ranked posets see, e.g., [1]. For any ranked poset \( P \), the set of elements of rank \( l \) is called the \( l \)th rank and is denoted \( P_l \). Evidently

\[
|T_l(t)| = \sum t_1, t_2, \ldots, t_{n-l},
\]

where the sum is taken over the \( \binom{n}{n-l} \) combinations \( i_1, i_2, \ldots, i_{n-l} \) of the first \( n \) positive integers taken \( (n - l) \) at a time. The shadow \( \Delta a \) of an element \( a \) of \( T \) is \( \{c \mid c \leq a, \ r(c) = r(a) - 1\} \). The shadow \( \Delta \mathcal{A} \) of a subset \( \mathcal{A} \) of \( T \) is \( \bigcup_{a \in \mathcal{A}} \Delta a \), and \( \Delta^2(\mathcal{A}) = \Delta(\Delta \mathcal{A}) \), etc.

In [4], the Kruskal–Katona algorithm is generalized from \( B_n = T((1, 1, \ldots, 1)) \) to \( T((t, t, \ldots, t)) \) for any integer \( t > 1 \). The purpose of this paper is to generalize it to \( T(t) \) where \( 1 \leq t_1 \leq t_2 \leq \cdots \leq t_n \).

A generalization in a different direction is given in [3]. There the generalization is from \( B_n \) to \( S(t) \), the poset consisting of all \( n \)-tuples \( a = (a_1, a_2, \ldots, a_n) \) of integers satisfying \( 0 \leq a_i \leq t_i \), \( i = 1, 2, \ldots, n \) partially ordered by defining \( a \leq c \) if \( a_i \leq c_i \) for \( i = 1, 2, \ldots, n \). The rank \( r(a) \) of \( a \in S(t) \) is \( a_1 + a_2 + \cdots + a_n \). \( S(t) \) and \( T(t) \) are isomorphic if and only if \( t_1 = t_2 = \cdots = t_n = 1 \), (in which case both are isomorphic to \( B_n \)).

Our algorithm involves the fact that \( T(t) \) is a Macaulay poset. A ranked poset \( P \) is a Macaulay poset [6] if there is a linear order \( \prec \) for \( P \), called the Macaulay order, such that for fixed \( l \) and \( m \leq |P_l| \),

\[
\min |A \mathcal{A}| = |AF(m, P_l)|
\]

and

\[
AF(m, P_l) = F(|AF(m, P_l)|, P_{l-1}),
\]

where for any subset \( \mathcal{A} \) of \( P, F(m, \mathcal{A}) \) denotes the first \( m \) elements of \( \mathcal{A} \) in the Macaulay order \( \prec \) and, as usual, the minimum is taken over all \( m \)-element subsets \( \mathcal{A} \) of \( P_l \).
Thus in Macaulay posets, evaluating \( \min |AF| \) is the same as evaluating \( |AF(m, P_i)| \). Sets of the form \( F(m, P_i) \) are called initial segments, so (4) means that shadows of initial segments are initial segments.

We define \( P \) to be weakly Macaulay [4] if there is a linear order for which (3) holds. The ranked poset \( L(2,3) \), where \( L(m,n) \) consists of all \( n \)-tuples \( a = (a_1, a_2, \ldots, a_n) \) of non-negative integers satisfying \( a_1 \leq a_2 \leq \cdots \leq a_n \leq m \) partially ordered by defining \( a \leq c \) if \( a_i \leq c_i, \ i = 1, 2, \ldots, n \) and \( r(a) = |\{i | a_i = m\}| \), is weakly Macaulay but not Macaulay. \( L(5,3) \) is not even weakly Macaulay, there being no linear order for which (3) holds for all admissible \( m \) when \( l = 8 \).

Macaulay [13] showed that \( S(\infty,\infty,\ldots,\infty) \), the set of all \( n \)-tuples of non-negative integers, partially ordered by defining \( a \leq c \) if \( a_i \leq c_i, \ i = 1, 2, \ldots, n \), is what is now called a Macaulay poset, the Macaulay order being lexicographic order: \( a < c \) if \( a_i < c_i \) for the smallest integer \( i \) for which \( a_i \neq c_i \). More generally, Clements and Lindström [5] showed that lexicographic order is also a Macaulay order for \( S(t) \). Kruskal [9] noted similarities between \( T((1,1,\ldots,1)) \) and \( T((2,2,\ldots,2)) \) and in effect asked if the latter were a Macaulay poset. Lindstrøm [12] found that it was. Leeb [11] found that \( T(t,t,\ldots,t) \) is Macaulay and stated that \( T(t_1,t_2,\ldots,t_n) \) is Macaulay.

Bezrukov [2] showed, independently of Leeb, that \( T((t,t,\ldots,t)) \) is Macaulay and Leck [10] showed that \( T(t_1,t_2,\ldots,t_n) \) is Macaulay. Both authors used the Clements–Lindstrøm method of proof. Engel [6] has been able to simplify these proofs in several places by means of a new description of Leeb’s order.

Clements [5], in the course of extending the Kruskal–Katona algorithm to \( T(t_1,t_2,\ldots,t_n) \) rediscovered Bezrukov’s order and showed it was the same as Leeb’s. Leck [10] has also given a Kruskal–Katona algorithm for \( T(t_1,t_2,\ldots,t_n) \).

In the next section we describe a Macaulay order for \( T(t) \) (Engel’s) and formulate our algorithm. The final section is devoted to its proof.

### 2. A Macaulay order for \( T(t) \)

For \( 0 \leq l \leq t_n \) we define \( a(l) \) to be the \( 0 \)-\( 1 \)-\( n \)-tuple with \( i \)-th coordinate equal to \( 1 \) if and only if \( a_i = l \), and we associate with each element \( a \in T(t) \) the \( (t_n + 1) \times n \) \( 0 \)-\( 1 \) matrix \( M(a) \) with rows \( a(0), a(1), \ldots, a(t_n) \). It is convenient to refer to the top row as the 0th row, etc. We define \( M(a) < R M(c) \) if \( a(l) < R c(l) \) for the smallest integer \( l \) for which \( a(l) \neq c(l) \). The order \( <_t \) defined as follows is a Macaulay order for \( T(t) \) [6].

**Definition 1.** For distinct elements \( a, c \) of \( T(t) \), \( a <_t c \) if and only if \( M(c) < R M(a) \).
Many properties of the elements of $T$ and the relations between them are simply reflected in their matrices. For example, if $c$ is in the shadow of $a$, its matrix can be obtained from $M(a)$ by moving a 1 in the last row of $M(a)$ up one or more rows, keeping it in the same column. Also $r(a)$ is the number of 1’s in the $t_n$th (last) row of $M(a)$. We will see that $a(0)$, the top (0th) row of $M(a)$, has special significance. Let $k = k(t)$ denote the largest integer such that $t_{n+1-k} = t_{n+2-k} = \ldots = t_n$. For $a \in T(t)$, $a_i = t_n - t_i \geq t_{n-1} - t_{n-k} \geq 1$ for $i = 1, 2, \ldots, n - k$, so the possible 0th rows of $M(a)$ are the $2^k$ 0-1 $n$-tuples with first $n - k$ components equal to 0. We will use $d_0, d_1, \ldots, d_{2^k-1}$ to denote these possible first rows arranged in decreasing reverse lexicographic order.

If $d = (d_1, d_2, \ldots, d_n)$ denotes any one of $d_0, d_1, \ldots, d_{2^k-1}$, we define $t(d)$ to be the result of deleting from $t$ those coordinates $t_i$ for which $d_i = 1$ and reducing by 1 those coordinates for which $i > n - k$ and $a_{i-k} = 0$. Thus the first $n - k$ coordinates of $t$ and $t(d)$ are the same. We will abbreviate $t(a(0))$ to $t(a)$. In the following figure the elements $a$ of $T((1, 3, 3))$ are arrayed in increasing Macaulay order from left to right, top to bottom, always writing elements of rank $r$ in column $r$. The superscript appearing with $a$ is $a(0)$, the top row of $M(a)$. Here and below we omit commas and parentheses from $n$-tuples if there is no danger of confusion. We also exhibit the posets $T(t(d_i))$ for $i = 0, 1, \ldots, 2^k-1$ where $k = k(1, 3, 3) = 2$.

Fig. 1 suggests thinking of $T(t)$ as $\bigcup_{i=0}^{2^k-1} T(t, d_i)$, where $T(t, d) = \{a \mid a \in T(t), a(0) = d\}$ is somehow isomorphic to $T(t(d))$. This will be clarified in Lemma 2 below.

Also note that if $a$ is the last element of an initial segment of rank $l$, then the shadow of that segment is the initial segment of rank $l - 1$ consisting of all elements that appear not lower than $a$ in the diagram, e.g., $\Delta F(3, T_3(1, 3, 3)) = F(10, T_2(1, 3, 3))$ and $\Delta^2 F(3, T_3(1, 3, 3)) = F(7, T_1(1, 3, 3))$. The foregoing observations suggest our algorithm.

**Theorem.** Let $t = (t_1, t_2, \ldots, t_n)$ denote an $n$-tuple of integers satisfying $1 \leq t_1 \leq \cdots \leq t_n$ and let $b, l, m$ denote integers satisfying $1 \leq b \leq l \leq n$, $m \leq |T_l(t)|$. Let $k$ denote the largest integer such that $t_{n-k+1} = t_{n-k+2} = \cdots = t_n$ and let $d_0, d_1, \ldots, d_{2^k-1}$ denote the 0-1 $n$-tuples with first $n - k$ coordinates equal to 0, arranged in decreasing reverse lexicographic order. If $j$ is the largest integer such that $\sum_{i=0}^j |T_l(t(d_i))| = \Sigma \leq m$ and $r = m - \Sigma$, then

$$|\Delta^0 F(m, T_l(t))| = \sum_{i=0}^j |T_{l-b}(t(d_i))| + |\Delta^0 F(r, T_l(t(d_{j+1})))|.$$  \hfill (5)

The sum in (5) can be evaluated using (2). The maximum coordinate in $t(d_{j+1})$ is always strictly less than the maximum coordinate in $t$, so after a finite number of applications of (5) one is left to evaluate $|\Delta^0 F(r', T_l(t'))|$ where each coordinate of $t'$ is 1. This can be done using the Kruskal–Katona algorithm (1).
**Example.** We calculate $|A^2F(3, T_2(1,3,3))|$ by means of the theorem. (In view of Fig. 1, the answer will be 7.) We have $k(1,3,3)=2$ and

\[ d_0 = 011, \quad |T_2(t(d_0))| = |T_2(1)| = 0, \quad |T_0(1)| = 1, \]

\[ d_1 = 001, \quad |T_2(t(d_1))| = |T_2(1,2)| = 1, \quad |T_0(1,2)| = 1 \cdot 2, \]

\[ d_2 = 010, \quad |T_2(t(d_2))| = |T_2(1,2)| = 1, \quad |T_0(1,2)| = 1 \cdot 2, \]

where $|T(t)| = T((1,3,3))$ and $T(t(d)) = T(t(0,0,0)) = T((1,2,3))$.

\[
\begin{array}{cccccccc}
\text{Rank} & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\
200^{011} & 300^{011} & 0 & 1 \\
210^{001} & 310^{001} & 10 & 20 \\
220^{001} & 320^{001} & 11 & 21 \\
230^{001} & 330^{001} & 12 & 22 \\
201^{010} & 301^{010} & 10 & 20 \\
202^{010} & 302^{010} & 11 & 21 \\
203^{010} & 303^{010} & 12 & 22 \\
211^{000} & 311^{000} & 100 & 200 \\
221^{000} & 321^{000} & 110 & 210 \\
231^{000} & 331^{000} & 120 & 220 \\
212^{000} & 312^{000} & 101 & 201 \\
213^{000} & 313^{000} & 102 & 202 \\
223^{000} & 323^{000} & 112 & 212 \\
233^{000} & 333^{000} & 122 & 222 \\
\end{array}
\]

**Fig. 1.**

Example. We calculate $|A^2F(3, T_2(1,3,3))|$ by means of the theorem. (In view of Fig. 1, the answer will be 7.) We have $k(1,3,3)=2$ and $\ldots$
\[ d_3 = 000, |T_2(t(d_3))| = |T_2(1,2,2)| = 1 + 2 + 2 = 5, \quad |T_0(1,2,2)| = 1 \cdot 2 \cdot 2 = 4. \]

Thus \(3 = \sum_{i=0}^{2} |T_2(t(d_i))| + 1\) and
\[
|A^2F(3, T_2(1,3,3))| = \sum_{i=0}^{2} |T_0(t(d_i))| + |A^2F(1, T_2(t(d_1)))| = 5 + |A^2F(1, T_2(1,2,2))|.
\]

To evaluate the last term, we apply the theorem with \(t = (1,2,2)\). Since \(k(1,2,2) = 2\) as before, \(d_0, d_1, d_2, d_3\) are as above and
\[
|T_2(t(d_0))| = |T_2(1)| = 0, \quad |T_0(1)| = 1, \quad |T_2(t(d_1))| = |T_2(1,1)| = 1, \quad |T_0(1,1)| = 1 \cdot 1.
\]

Thus
\[
1 = \sum_{i=0}^{1} |T_2(t(d_i))| + 0,
\]
\[
|A^2F(1, T_2(1,2,2))| = \sum_{i=0}^{1} |T_0(t(d_i))| + |A^2F(0, T_2(t(d_2)))| = 2 + 0 = 2
\]
and
\[
|A^2F(3, T_2(1,3,3))| = 5 + 2 = 7 \quad \text{(as anticipated).}
\]

3. Proof of the theorem

We begin by giving an inductive formulation of Engel’s order. Recall that \(k(t)\) is the number of final coordinates in \(t\) that are equal and that \(t(a(0)) = t(a)\) is obtained from \(t\) by altering its last \(k\) coordinates by deletion or reduction by 1 according as the corresponding coordinate of \(a(0)\) is 1 or 0.

We now define \(a(t)\) to be the result of deleting from \(a\) 0 coordinates and reducing non-zero coordinates by \((t_n - t_{n-k})\) or 1 according as the last \(k\) coordinates of \(a\) are all 0 or not. Note that \(a(t) \in T(t(a))\). For example, if the last \(k\) coordinates of \(a\) are 0’s, then the coordinates of \(a(t)\) are \(a_i - (t_n - t_{n-k})\), \(i = 1,2,\ldots,n-k\). Since \(a \in T(t)\), \(t_n - t_i \leq a_i \leq t_n\) for \(i = 1,2,\ldots,n-k\), so \(t_{n-k} - t_i \leq a_i - (t_n - t_{n-k}) \leq t_{n-k}\) for \(i = 1,2,\ldots,n-k\), and \(a(t) \in T(t_1,t_2,\ldots,t_{n-k}) = T(t(a))\).

**Definition 2.** For distinct elements \(a, c\) of \(T(t), a \prec_t c\) if and only if
(i) \(c(0) <_R a(0)\)
or
(ii) \(c(0) = a(0)\) and \(a(t) \prec_{t(a)} c(t)\).
As already noted, the maximum coordinate of $t(a)$ is always strictly less than the maximum coordinate of $t$, so deciding $a \prec_t c$ eventually comes down to deciding $a' \prec_{t'} c'$ for distinct elements $a', c'$ of $T(t')$, where all coordinates of $t'$ are 1. But then $a'(0)$ and $c'(0)$ are distinct and $a' \prec_{t'} c'$ is equivalent to $c(0) \prec_R a(0)$. Thus the order $\prec_{t'}$ is well defined. We now show that it is actually the same as $\prec_t$.

**Lemma 1.** For distinct elements $a, c$ of $T(t), a \prec_t c$ if and only if $a \prec_{t'} c$.

**Proof.** The proof is a double induction, first on the value of $t_n$ and then on $n$.

If $t_n=1$, and therefore, $t_1 = t_2 = \cdots = t_n = 1$, then both $a \prec_t c$ and $a \prec_{t'} c$ are equivalent to $c(0) \prec_R a(0)$. Now assuming the lemma for $t = (t_1, t_2, \ldots, t_n)$ where $1 \leq t_n \leq m$ and $n$ is any positive integer, we show that it holds for $t = (t_1, t_2, \ldots, t_n)$ where $t_n = m + 1$ and $n$ is any positive integer by induction on $n$.

If $n = 1$ and $a = (0), c = (1)$ and either $i$ or $j$ is 0, then it must be $i$ since we would otherwise have the contradiction $c(0) = 1 \prec_R a(0) = 0$. If neither is 0, then with $t = (m + 1)$ we have $t(a) = (m)$ and $a(t) = (i - 1) - m, (j - 1) - m$ which implies the contradiction $(i - 1)(0) = 1 \prec_R (j - 1)(0) = 0$. Thus $i < j$. Conversely one can check that if $i < j$, then $(i) \prec_{m+1} (j)$. Thus $(i) \prec_{m+1} (j)$ is equivalent to $i < j$. It is also simple to check that $(i) \prec_{m+1} (j)$ is equivalent to $i < j$, so our induction on $n$ is anchored.

Now assuming the lemma holds for $t = (t_1, t_2, \ldots, t_n)$ for any integer $i$ if $t_i \leq m$ and that it holds for $1 \leq i < n$ if $t_i = m + 1$, we prove it for $t = (t_1, t_2, \ldots, t_n)$ where $t_n = m + 1$.

First suppose $a \prec_t c$. Then $c(l) \prec_R a(l)$ where $l$ is the smallest integer such that $a(l) = c(l)$. If $l = 0$, $a \prec_t c$ is immediate so we henceforth assume $l > 0$.

If the last $k(t)$ components of $a$ are 0 (and therefore the last $k$ components of $c$ are also 0 since $l > 0$), then we must show that, with $t_{n-k} = e$,

$$a' = a(t') = (a_1 - e, \ldots, a_{n-k} - e) \prec_{t'} (c_1 - e, \ldots, c_{n-k} - e) = c(t) = c',$$

where $t' = t(a) = (t_1, t_2, \ldots, t_{n-k})$. (Note that $k < n$ since $k = n$ would imply $a = c = 0$ contradicting that $a$ and $c$ are distinct.)

$M(a')$ is obtained from $M(a)$ by deleting the last $k$ columns and the first $e$ rows. If $e = 1$, only the $l$th row is deleted — so the $l$th row is not deleted since $l > 0$ by hypothesis.

If $e > 1$, then rows $1, 2, \ldots, e - 1$ of $M(a)$ must be 0 rows since $a_i = 0$ for $i > n - k$ by hypothesis and $a_i \geq t_0 - t_i \geq t_n - t_{n-k} = e$ for $i = 1, 2, \ldots, n - k$. The same discussion applies to $M(c')$. Since $M(a)$ and $M(c)$ differ for the first time at row $l$, it follows that $l \geq e$ and that matrices $M(a')$ and $M(c')$ differ for the first time at row $l - e$ ($\geq 0$). These rows are obtained from rows $l$ of $M(a)$ and $M(c)$ by deleting the final $k$ entries, all of which are 0. Thus reverse lexicographic order between these rows is preserved, so $a' \prec_{t'} c'$ where $t' = (t_1, t_2, \ldots, t_{n-k})$. Since $t_{n-k} < t_n = m + 1$, the induction hypothesis allows us to conclude that (6) does indeed hold.

If $l > 0$ and the last $k$ components of $a$ are not all 0’s similar (actually somewhat simpler) arguments show that $a \prec_t c$. 

Conversely, suppose \( a \prec_t c \). If \( l \) is the smallest integer for which \( a(l) \neq c(l) \), we may again assume \( l > 0 \). This time we will provide the details for the case in which the last \( k \) components of \( a \) (and therefore \( c \)) are not all 0's. Let \( i_1, i_2, \ldots, i_j \), where \( n - k < i_1 < i_2 < \cdots < i_j \leq n \), be the integers for which the corresponding components of \( a \) are not zero. By definition of \( a \prec_t c \),

\[
\begin{align*}
a' &= a(t) = (a_1 - 1, \ldots, a_{n-k} - 1, a_{i_1}, \ldots, a_{i_j} - 1) \\
c(t) &= (c_1 - 1, \ldots, c_{n-k} - 1, c_{i_1}, \ldots, c_{i_j} - 1) = c'
\end{align*}
\]

where \( t' = t(a) = (t_1, \ldots, t_{n-k}, t_{i_1} - 1, \ldots, t_{i_j} - 1) \). Since \( t_{i_j} - 1 = t_n - 1 = m \), we conclude by means of the induction hypothesis that \( a' \prec_t c' \). Thus, if \( l' \) is the smallest integer such that \( a'(l') \neq c'(l') \), then \( c'(l') <_R a'(l') \). \( M(a) \) is obtained from \( M(a') \) by inserting \( n - k - j \) columns of 0's so that they become columns \( i_s \), \( i_s > n-k \), \( i_s \neq i_s \) for \( s = 1, 2, \ldots, j \) in the resulting matrix, and then adding \( a(0) \) as top row. \( M(c) \) is obtained from \( M(c') \) in exactly the same way. Since \( a(0) = c(0) \) (because \( l > 0 \)), it follows that \( l = l' + 1 \). Since the \( l \)th rows of \( M(a) \) and \( M(c) \) are obtained by inserting 0's into the \( l' \)th rows of \( M(a') \) and \( M(c') \) in exactly the same way and \( c'(l') \prec_R a'(l') \) it follows that \( c(l) \prec_R a(l) \) and therefore \( a \prec_t c \). The remaining case, in which the final \( k \) components of \( a \) are all 0's can be handled similarly. This completes the proof of Lemma 1. We are henceforth free to regard either Definition 1 or Definition 2 as giving the Macaulay order on \( T(t) \).

Recall that for \( a \in T(t) \), \( a(t) \) denotes the result of deleting from \( a \) the zero coordinates and reducing the non-zero coordinates by \( t_n - t_{n-k} \) or 1 according as the last \( k \) coordinates of \( a \) are all 0 or not.

If \( d \) denotes any one of \( d_0, d_1, \ldots, d_{2^k-1} \) — i.e., \( d \) is a 0–1 \( n \)-tuple the first \( n - k \) coordinates of which are 0's, let \( T(t, d) = \{ a \mid a \in T(t); a(0) = d \} \).

**Lemma 2.** The mapping \( a \rightarrow a(t) \) from \( T(t, d) \) to \( T(t(d)) \) is 1–1, onto and preserves both poset and Macaulay order.

**Proof.** We have already remarked that \( a(t) \in T(t(a(0))) \), so if \( a \in T(t, d) \), \( a(t) \in T(t(d)) \) and it follows from the definition of \( a(t) \) that the mapping is 1–1. Hence to check that the mapping is onto, it suffices to check that \( |T(t, d)| = |T(t(d))| \). If \( a_{n+k+1} = \cdots = a_n = 0 \), \( |T(t, d)| = \prod_{s=1}^{n-k}(t_i + 1) = |T(t, \ldots, t_{n-k})| = |T(t(d))| \); if \( a_i \neq 0 \) for integers \( i_1, \ldots, i_j \) satisfying \( n - k + 1 \leq i_1 < i_2 < \cdots < i_j \leq n \), then \( |T(t, d)| = \prod_{s=1}^{n-k} (t + 1) \prod_{i=1}^{n-k} t_i = |T(t(d))| \).

We now check that our mapping preserves both poset and Macaulay order. Let \( a \) and \( c \) denote distinct elements of \( T(t, d) \). It follows from the proof of Lemma 1 that \( a \prec_t c \) implies \( a(t) <_{t(a)} c(t) \) — i.e., that Macaulay order is preserved so we only need check that poset order is preserved.

If the last \( k \) coordinates of \( a \) and \( c \) are all 0's, then with \( t_n - t_{n-k} = e \) we have

\[
\begin{align*}
a' &= a(t) = (a_1 - e, a_2 - e, \ldots, a_{n-k} - e) \\
c' &= c(t) = (c_1 - e, c_2 - e, \ldots, c_{n-k} - e)
\end{align*}
\]
and
\[ t' = t(d) = t(a) = t(c) = (t_1, t_2, \ldots, t_{n-k}). \]

If \( a \subseteq c \), then \( a_i = c_i \) or \( c_i = t_n \) for \( i = 1, 2, \ldots, n \) and \( a_i - e = c_i - e \) or \( c_i - e = t_{n-k} \) for \( i = 1, 2, \ldots, n-k \), so \( a' \subseteq c' \) follows.

If there are integers \( i_1, i_2, \ldots, i_j \) satisfying \( n-k+1 \leq i_1 < i_2 < \cdots < i_j \leq n \) for which the corresponding coordinates of \( a \) and \( c \) are not zero, then
\[ a' = a(t) = (a_1 - 1, \ldots, a_{n-k} - 1, a_i - 1, \ldots, a_{i_j - 1}), \]
\[ c' = c(t) = (c_1 - 1, \ldots, c_{n-k} - 1, c_i - 1, \ldots, c_{i_j - 1}) \]
and
\[ t' = t(d) = t(a) = t(c) = (t_1, \ldots, t_{n-k}, t_{i_1 - 1}, \ldots, t_{i_j - 1}). \]

If \( a \subseteq c \), then \( a_i = c_i \) or \( c_i = t_n \) for \( i = 1, 2, \ldots, n \). If \( a_i = c_i, a_i - 1 = c_i - 1 \); if \( c_i = t_n \), then \( c_i - 1 = t_n - 1 = t_{i_j - 1} \), so \( a' \subseteq c' \) follows. This completes the proof of Lemma 2.

We are now ready to prove the theorem. Let an \( n \)-tuple \( t = (t_1, t_2, \ldots, t_n) \) of integers satisfying \( 1 \leq t_1 \leq t_2 \leq \cdots \leq t_n \) be given and let \( b, l, m \) denote positive integers satisfying \( b \leq l \leq n \) and \( m \leq |T(t)| \). Let \( k = k(t) \) denote the largest integer such that \( t_{n+k+1} = t_{n+k+2} = \cdots = t_n \) and for any \( 0-1 \) \( n \)-tuple having first \( n-k \) coordinates equal to zero, let \( T_i(t, d) \) denote \( \{a | a \in T(t); a(0) = d\} \). In view of Definition 2 of our Macaulay order,
\[ F(m, T_i(t)) = \left( \bigcup_{i=0}^{j} T_j(t, d_i) \right) \cup F(r, T_i(t, d_{j+1})), \]
where \( j \geq 0 \) is the largest integer such that
\[ m \geq \sum_{i=0}^{j} |T_i(t, d_i)| = \Sigma \quad \text{and} \quad r = m - \Sigma \geq 0. \]

Then
\[ A^b F(m, T_i(t)) = R \cup S, \]
where \( R = A^b \left( \bigcup_{i=0}^{j} T_i(t, d_i) \right) \) and \( S = A^b F(r, T_i(t, d_{j+1}) \). We claim that \( R \) is the same as \( R' = \bigcup_{i=0}^{j} T_{i-b}(t, d_i) \). In view of Definition 2, \( R' \) is an initial segment of \( T_{i-b}(t) \) and since shadows of initial segments are initial segments, \( R \) is also an initial segment of \( T_{i-b} \). Consider the largest element \( a \) of \( T_i(t, d_i) \). In view of Definition 1, one can form \( M(a) \) by starting with the \((t_n + 1) \times n \) 0-matrix, replacing the top and bottom rows by \( d_i \) and the complement of \( d_i \), respectively, and then, going from left to right, raising 1’s in the last row by one row (not changing columns) until the last row contains exactly \( l \) 1’s. If \( a' \) is the largest element in \( R' \), then it is the largest element in \( T_{i-b}(t, d_i) \) and \( M(a') \) can be formed exactly as \( M(a) \) was formed, except that \( b \) more 1’s are raised from the bottom row. It follows that \( a' \in A^b a \) and therefore \( R \supseteq R' \). If \( R \) properly contains \( R' \), it contains the next element in \( T_{i-b}(t) \) after \( c' \), call it \( c'' \). But then \( c''(0) = d_i \) with \( s > j \) while for \( a \in R, a(0) = d_i \) with \( i \leq j \). Thus \( R \) is indeed \( R' \).
If we partition $S$ into the disjoint union $S^m \cup S^c$ where $S^m = \{a \mid a \in S, a(0) = d_{j+1}\}$, then $S^c \subseteq R = R'$ and

$$\Delta^b F(m, T_i(t)) = R' \cup S^c,$$

the union being disjoint. Thus

$$|\Delta^b F(m, T_i(t))| = |R'| + |S^c| = \sum_{i=0}^{j} |T_i-b(t, d_i)| + |S^c|.$$

In view of the isomorphism between $T_i(t, d_i)$ and $T_i(t(d_i)), i = 1, 2, \ldots, j + 1$ (Lemma 2), this can be written

$$|\Delta^b F(m, T_i(t))| = \sum_{i=0}^{j} |T_i-b(t(d_i))| + |\Delta^b F(r, T(t(d_{j+1})))|,$$

completing the proof of the theorem.

References