Bifurcations Caused by Perturbing the Domain in an Elliptic Equation

JOSÉ M. VEGAS∗†

Lefschetz Center for Dynamical Systems, Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912

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We consider the Neumann problem \(-Au = \lambda u - u^p\) on a continuous family of bounded domains \(\Omega\), which approach (as \(\epsilon \to 0\)) a set \(\Omega_0\) with two connected components, and analyze it as a bifurcation problem with two parameters, \(\lambda\) and \(\epsilon\). The bifurcation diagrams and the qualitative properties of the bifurcation sets for \(p\) odd and \(p\) even are obtained, and the relations between them are studied by considering the problem \(-Au = \lambda u - au^2 - u^3\) for different values of \(a\).

1. INTRODUCTION

In the study of the asymptotic behavior of the solutions of the scalar reaction–diffusion equation

\[
\begin{align*}
\frac{\partial u}{\partial t} &= Au + f(u) \quad \text{in } \Omega \times (0, \infty), \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega \times (0, \infty)
\end{align*}
\]  

(1.1)

the analysis of its stationary solutions, i.e., solutions of

\[
\begin{align*}
Au + f(u) &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega
\end{align*}
\]  

(1.2)

is essential: the \(\omega\)-limit set of any bounded solution of (1.1) consists exclusively of solutions of (1.2). Aside from the constants (spatially

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†Present address: Departamento de Educaciones Functionales, Facultad de Matemáticas, Universidad Complutense, Madrid-3, Spain.
homogeneous functions: the zeros of $f$), the existence of nonconstant solutions of (1.2) reflects the fact that a very special balance between the reaction and the diffusion mechanisms is taking place. The purpose of this paper is to analyze how small changes in the shape of the domain $\Omega$ may lead to changes in the stability properties or the number of solutions of (1.2). Thus, the emphasis is on bifurcation phenomena.

If $\Omega$ is a convex domain, it is known (Chafee [4], Casten and Holland [3], Matano [10]) that all stable stationary solutions of (1.1) are constant; on the other hand, Matano [10] exhibited examples of connected domains $\Omega$ and functions $f$ for which there are stable nonconstant solutions. How does the transition between so different types of behavior take place? In order to study it, we consider a "continuous" family of domains $\Omega_\epsilon$ such that, for $\epsilon = 0$, $\Omega_0$ has two connected components, $\Omega_0^R$ and $\Omega_0^L$, and, for $\epsilon > 0$, $\Omega_\epsilon$ is just $\Omega_0 \cup R_\epsilon$, where $R_\epsilon$ looks like a small channel connecting $\Omega_0^R$ and $\Omega_0^L$. If $a, b$ are zeros of $f$, we denote by $[a, b]_\epsilon$ the function $[a, b]_\epsilon \equiv a$ on $\Omega_\epsilon^R$, $[a, b]_\epsilon \equiv b$ on $\Omega_\epsilon^L$, which is a solution of (1.2) for $\Omega = \Omega_\epsilon$. If $\epsilon$ is very small, it is then expected that there will exist a solution $u_\epsilon$ of (1.2) for $\Omega = \Omega_\epsilon$, which is very close to $[a, b]_0$ on $\Omega_0$.

Hale and Vegas [9] proved that this is indeed the case if all the zeros of $f$ are simple. Moreover, the solution $u_\epsilon$, denoted $[a, b]_\epsilon$, is unique and has the stability properties of $[a, b]_0$.

In this paper, we analyze the opposite or critical case. As a first example, we consider the function $f(u) = \lambda u - u^p$, where $\lambda$ is a small parameter. If $p$ is odd and $\lambda > 0$, $f$ has three zeros, namely, $u = 0$, $u = \pm \lambda^{1/p-1}$. Thus, for each fixed $\lambda > 0$, there exists $\epsilon = \epsilon(\lambda)$ such that (1.2) has nine solutions for $0 \leq \epsilon < \epsilon(\lambda)$. However, if we fix $\epsilon$, for $\lambda$ sufficiently small only the three constant solutions exist. Therefore, some bifurcation phenomena must occur when $\lambda$ and $\epsilon$ vary independently, with the possibility of existence of secondary bifurcations, as indicated by Hale [8]. It turns out that, for $\lambda$ fixed, if we call $a = \lambda^{1/p-1}$, the solutions $[a, 0]_\epsilon$, $[a, -a]_\epsilon$ and $[0, -a]_\epsilon$ "merge" at a certain value of $\epsilon$, and so the other three nonconstant solutions $[0, a]_\epsilon$, $[-a, a]_\epsilon$ and $[-a, 0]_\epsilon$ at the same value of $\epsilon$. Finally, these two new solutions coalesce with the zero solution at a larger value of $\epsilon$. If we fix $\epsilon > 0$, a more familiar bifurcation pattern appears in which two simultaneous bifurcations occur (Fig. 4). If $p$ is even, the bifurcation set has a completely different structure (Fig. 6). In order to clarify the relationship between these two pictures, we make them appear as the extreme points in a continuum by considering the function $f(u) = \lambda u - au^2 - u^3$ and letting $a$ vary. In this way, we obtain a variety of bifurcation sets, some of which present very curious features (Figs. 9, 11 and 13).

Concerning the method of analysis, the first difficulty lies precisely in having to deal with a variable domain. By expressing $u$ in terms of the eigenfunctions of $A$ in $\Omega_\epsilon$, we reduce the problem (via the Liapunov–Schmidt
method) to a system of two bifurcation equations whose dependence on \( \epsilon \) can be shown to be sufficiently regular by a careful analysis of the properties of the domains \( \Omega_\epsilon \) and the corresponding eigenfunctions and eigenvalues. Once this question is settled, the analysis of the equations is based on the scaling techniques discussed in Chow, Hale and Mallet-Paret [5].

Section 2 contains the precise description of the domains \( \Omega_\epsilon \), the application of the Liapunov–Schmidt method and the statements of the results. The proofs of these appear in Sections 3 and 4. An application to a selection–migration model is included in Section 5. Section 6 consists of a few remarks concerning generalizations and open questions. Finally, Section 7 deals with the very technical subject of the regularity of the Bifurcation Equations and is completely independent of the rest.

Many of the proofs are very sketchy, and some have been omitted since they are not considered to be essential or hard to reconstruct. Their complete version appears in the author’s thesis at Brown University [11], which was done under the direction of Jack K. Hale, to whom the author wishes to express his acknowledgment.

2. Statement of the Problem and Summary of Results

Consider the reaction–diffusion equation

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u + f(\lambda, u) \quad \text{in } \Omega_\epsilon \times (0, \infty), \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega_\epsilon \times (0, \infty),
\end{align*}
\] (2.1)

where \( f \) is a given smooth function of \( \lambda, u \) (\( \lambda \) in a Banach space \( A \)), \( u \in \mathbb{R} \), and, for \( \epsilon \in (0, 1] \), \( \Omega_\epsilon \) is a bounded, smooth, connected domain in \( \mathbb{R}^2 \). For \( \epsilon = 0, \Omega_0 \) is the union of two bounded, connected domains \( \Omega_0^1 \) and \( \Omega_0^2 \) whose closures are disjoint and the family \( \{ \Omega_\epsilon \} \) is "continuous" in the sense that \( \Omega_\epsilon \subset \Omega_\delta \) if \( \epsilon \leq \delta \) and the function \( \epsilon \mapsto |\Omega_\epsilon| \) is continuous on \([0, 1]\) (\( |\cdot| \) represents Lebesgue measure in \( \mathbb{R}^2 \)).

Our objective is to study the stationary solutions of (2.1) for \( |\lambda|, \epsilon \) small. The particular form of the domains \( \Omega_\epsilon \) will be given in Definition 2.1. At this point, we only need two basic spectral properties which will enable us to apply the Liapunov–Schmidt method.

Let us first introduce some notation: \( \mathcal{A}_\epsilon \) represents the Laplacian on \( \Omega_\epsilon \) with homogeneous Neumann boundary condition, i.e., \( \mathcal{D}(\mathcal{A}_\epsilon) = \{ u \in H^2(\Omega_\epsilon) : \partial u/\partial n = 0 \text{ on } \partial \Omega_\epsilon \} \). If \( D \) is a smooth domain in \( \mathbb{R}^2 \), \( \lambda^{(k)}(D) \) represents the sequence of eigenvalues of \( -\Delta \) on \( D \) with homogeneous Neumann boundary conditions, where each eigenvalue is repeated according
to its multiplicity. If \( w^{(k)} \in H^1(D) \) represents the corresponding eigenfunction, we have the variational characterization:

\[
\lambda^{(k)}(D) = \min \left\{ \int_D |\nabla v|^2 \, dv : v \in H^1(D), \int_D v w^{(j)} = 0, \ 1 \leq j \leq k-1 \right\}.
\]

The two basic properties of \( \Omega_\epsilon \) mentioned before are

\[
\lambda_\epsilon \overset{\text{def}}{=} \lambda^{(3)}(\Omega_\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0, \tag{2.2}
\]

\( \lambda^{(3)}(\Omega_\epsilon) \) is bounded away from zero as \( \epsilon \to 0. \tag{2.3} \)

These statements imply that \( \lambda_\epsilon \) is a simple eigenvalue for \( \epsilon \) small. Let \( w_\epsilon \) denote the eigenfunction corresponding to \( \lambda_\epsilon \) satisfying \( \int_{\Omega_\epsilon} w_\epsilon^2 = 1 \) and \( \int_{\Omega_\epsilon^c} w_\epsilon > 0 \). It can be shown that \( \int_{\Omega_\epsilon^c} w_\epsilon \neq 0 \) for \( \epsilon \) small and, thus, \( w_\epsilon \) is well-defined.

Let

\[
U_\epsilon = \text{span}\{1, w_\epsilon\} \subset L^2(\Omega_\epsilon),
\]

\[
\tilde{L}_\epsilon = U_\epsilon^\perp = \left\{ u \in L^2(\Omega_\epsilon) : \int_{\Omega_\epsilon} u = \int_{\Omega_\epsilon} u w_\epsilon = 0 \right\},
\]

\( \tilde{H}_\epsilon = \tilde{L}_\epsilon \cap H^1(\Omega_\epsilon). \)

It is easy to see that \( -\Delta_\epsilon \) takes \( \tilde{L}_\epsilon \cap \mathcal{D}(\Delta_\epsilon) \) into \( \tilde{L}_\epsilon \) (by selfadjointness) and has a bounded inverse \( \tilde{K}_\epsilon : \tilde{L}_\epsilon \to \tilde{H}_\epsilon \) satisfying

\[
\|\tilde{K}_\epsilon\|_{\mathcal{L}(\tilde{L}_\epsilon, \tilde{H}_\epsilon)} = [\lambda^{(3)}(\Omega_\epsilon)^{-1} + \lambda^{(3)}(\Omega_\epsilon)^{-2}]^{1/2} \leq C_1 \tag{2.4}
\]

for \( \epsilon \) small, by (2.3). In the sequel, \( C_1 \) will always represent this constant.

Let \( P_\epsilon \) denote the orthogonal projection

\[
P_\epsilon : L^2(\Omega_\epsilon) \to U_\epsilon,
\]

\[
P_\epsilon f = |\Omega_\epsilon|^{-1} \int_{\Omega_\epsilon} f + w_\epsilon \int_{\Omega_\epsilon} w_\epsilon f.
\]

Then, if we write \( u = \alpha + \beta w_\epsilon + \tilde{u}, \ \alpha, \ \beta \in \mathbb{R}, \ \tilde{u} \in \tilde{H}_\epsilon, \) the equation

\(-\Delta_\epsilon u = f(\lambda, u)\)

is equivalent to

\[
\tilde{u} = \tilde{K}_\epsilon (I - P_\epsilon) f(\lambda, \alpha + \beta w_\epsilon + \tilde{u}), \ \int_{\Omega_\epsilon} f(\lambda, \alpha + \beta w_\epsilon + \tilde{u}) = 0,
\]

\[
- \beta \lambda_\epsilon + \int_{\Omega_\epsilon} w_\epsilon f(\lambda, \alpha + \beta w_\epsilon + \tilde{u}) = 0. \tag{2.5}
\]
If we now assume that, for some \( \lambda_0 > 0 \),

\[
|f_u(\lambda, u)| < k_1 < \frac{1}{2C_1} \quad \text{for} \quad |\lambda| < \lambda_0, u \in \mathbb{R},
\]

then, by the Implicit Function Theorem, for each \( \alpha, \beta \in \mathbb{R}, |\lambda| \leq \lambda_0, \epsilon \) fixed, the first equation of (2.5) has a unique solution \( \tilde{u} = \tilde{u}(\alpha, \beta, \lambda, \epsilon) \in \tilde{H}_\epsilon \). By substituting in the second and third equations of (2.5), we obtain the Bifurcation Equations:

\[
\begin{align*}
G(\alpha, \beta, \lambda, \epsilon) & \overset{\text{def}}{=} \int_{\Omega_\epsilon} f(\lambda, \alpha + \beta w_\epsilon + \tilde{u}(\alpha, \beta, \lambda, \epsilon)) = 0, \\
H(\alpha, \beta, \lambda, \epsilon) & \overset{\text{def}}{=} -\beta \lambda_\epsilon + \int_{\Omega_\epsilon} w_\epsilon f(\lambda, \alpha + \beta w_\epsilon + \tilde{u}(\alpha, \beta, \lambda, \epsilon)) = 0.
\end{align*}
\]

Summarizing: if \( \lambda \) and \( \epsilon \) are given and \((\alpha, \beta)\) solves (2.7), then \( u = \alpha + \beta w_\epsilon + \tilde{u}(\alpha, \beta, \lambda, \epsilon) \) is a solution of (2.1), and, conversely, all stationary solutions of (2.1) are obtained in this way. Thus, we have reduced our original problem to that of finding solutions \((\alpha, \beta)\) of (2.7). This is a two-dimensional bifurcation problem with two parameters \((\lambda, \epsilon)\), to which one may possibly apply scaling techniques and other methods of bifurcation theory. However, the type of dependence on \( \epsilon \) of the functions \( G, H \) defined in (2.7) is not obvious. Therefore, our first task will be to analyze this dependence (Theorem 2.2) and then proceed to the direct study of the bifurcation equations (2.7).

With the concepts and notation introduced so far, we may describe the domains \( \Omega_\epsilon \) and state the main results that will be proved in this paper.

**Definition 2.1 (Description of the domains \( \Omega_\epsilon \)).** Let \( \Omega_0^L, \Omega_0^R \) be bounded, smooth, connected domains in \( \mathbb{R}^2 \) satisfying:

(a) \( \Omega_0^L \subset \{(x, y): x < -1\}; \Omega_0^R \subset \{(x, y): x > 1\} \),

(b) \( \{(x, y): x = -1, |y| \leq 1\} \subset \partial \Omega_0^L \)
\( \{(x, y): x = 1, |y| \leq 1\} \subset \partial \Omega_0^R \),

(c) \( \{(x, y): -3 \leq x \leq -1, |y| \leq 1\} \subset \Omega_0^L \)
\( \{(x, y): 1 \leq x \leq 3, |y| \leq 1\} \subset \Omega_0^R \).

Let \( r_+, r_- \) be real-valued functions, continuous in \([-1, 1]\) and \( C^\infty \) in \((-1, 1)\), satisfying

(d) \( r_+(0) > 0, r_+(-1) = r_+(1) = 1 \),
\( r_-(-1) < 0, r_-(-1) = r_-(1) = -1 \),

(e) \( r_+(x) < 0 \) for \( x < 0 \), \( r_+(x) > 0 \) for \( x > 0 \),
\( r_-(x) > 0 \) for \( x < 0 \), \( r_-(x) < 0 \) for \( x > 0 \),
For all integers $k$,

$$\left| \frac{d^k r_+(x)}{dx^k} \right| \to \infty, \quad \left| \frac{d^k r_-(x)}{dx^k} \right| \to \infty \quad \text{as} \quad |x| \to 1.$$ 

We define:

$$\Omega_0 = \Omega_0^L \cup \Omega_0^R,$$

$$R_\epsilon = \{(x, y): \epsilon r_-(x) < y < \epsilon r_+(x), -1 \leq x \leq 1\}, \quad (2.8)$$

$$\Omega_\epsilon = \Omega_0^L \cup R_\epsilon \cup \Omega_0^R = \Omega_0 \setminus R_\epsilon. \quad (2.9)$$

**THEOREM 2.2** (Regularity of the Bifurcation Equations). Let $\Omega_\epsilon$ be the domains described in Definition 2.1 and let $f: \Lambda \times \mathbb{R} \to \mathbb{R}$ satisfy for $|\lambda| \leq \lambda_0, \lambda_0 > 0$:

(a) $f$ is $C^{k+1}$ in $\lambda, u$,

(b) $|f_u(\lambda, u)| \leq k_1 < 1/2C_1$, where $C_1$ is given in (2.4),

(c) $|f_\lambda(u)| \leq M_1(|u| + 1)$ for some constant $M_1 > 0$,

(d) all higher order derivatives up through order $k + 1$ are bounded in $|\lambda|: |\lambda| \leq \lambda_0 | \times \mathbb{R}$.

Then, the properties given by (2.2) and (2.3) are satisfied, $\hat{u}(\alpha, \beta, \lambda, \epsilon)$ is $C^k$ in $\alpha, \beta, \lambda$ for each fixed $\epsilon$, the functions $G, H$ defined in (2.7) are $C^k$ in $\alpha, \beta, \lambda$ and these functions, together with all their derivatives up through order $k$ in $\alpha, \beta, \lambda$, are continuous in $\epsilon$, provided that we define $w_\epsilon$ for $\epsilon = 0$ as:

$$w_0 \equiv C_L = \left( \frac{1}{|\Omega_0^L|} \right)^{1/2} \quad \text{on} \quad \Omega_0^L;$$

$$w_0 \equiv C_R = \left( \frac{1}{|\Omega_0^R|} \right)^{1/2} \quad \text{on} \quad \Omega_0^R.$$
An outline of the proof of this result is given in Section 7.

Let us assume that $f$ has the form $f(\lambda, u) = \lambda u - g(u)$, where $g$ is a smooth function with a zero of order $p$ at $u = 0$: $g(u) = au^p + O(|u|^{p+1})$ as $|u| \to 0$. It is easily seen that the bifurcation equations (2.7) take the form (calling $X = (X_0)$):

$$\lambda L_1(\epsilon) X + \lambda \epsilon L_2(\epsilon) X + T(X, \epsilon) + R(X, \lambda, \epsilon) = 0,$$

(2.10)

where $L_1, L_2$ are $2 \times 2$ matrices, $T(\cdot, \epsilon)$ is a homogeneous polynomial of degree $p$ and $|R(X, \lambda, \epsilon)| = O(|X|^{p+1})$.

Let us call $\lambda = \nu$ and consider it as an independent parameter. Then (2.8) is a bifurcation problem with a two-dimensional unknown $(X)$, two eigenvalue parameters $(\lambda, \nu)$ and an extra parameter $(\epsilon)$ which, as it turns out, plays no fundamental role in the analysis. In a sense, $\lambda = \nu$ carries all the "singularities" arising from the presence of $\epsilon$. Due to this fact, the basic scaling techniques discussed in Chow, Hale and Mallet-Paret [5] may be applied to (2.8) to obtain a finite number of bifurcation curves in the $(\lambda, \nu)$-plane depending on the parameter $\epsilon: \lambda = \lambda_j(\nu, \epsilon)$ ($j = 1, \ldots, r$). By resetting $\nu = \lambda$, we find a finite set of curves $\lambda = \lambda_j(\lambda, \epsilon)$ which divide a neighborhood of the origin in the $(\lambda, \epsilon)$-plane in sectors in each of which the number of equilibrium solutions of (2.1) is constant. The number and location of these curves depends on the nonlinearity $g$, as shown in the following summary:

Theorem 2.3. Let us assume that the domains $\Omega_\epsilon$ are symmetric with respect to the $y$-axis, and let $g: \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function such that $g(u) = au^p + O(|u|^{p+1})$ as $|u| \to 0$ and $g', \ldots, g^{(p+1)}$ are bounded in $\mathbb{R}$, with $|g'(u)| \leq \frac{1}{2}k_1$. Then, the function $f(\lambda, u) = \lambda u - g(u)$ satisfies the hypotheses of Theorem 2.2 for $|\lambda| \leq \frac{1}{2}k_1$ and the following properties hold:

1. If $g(-u) = -g(u)$ and $a > 0$, there are exactly three bifurcation curves:
The curves $\lambda = 0$ and $\lambda = \lambda_\epsilon$ represent the primary bifurcations at the simple eigenvalues $0, \lambda_\epsilon$, and, at the curve $\lambda = \frac{p}{p-1} \lambda_\epsilon + o(\lambda_\epsilon)$, two simultaneous secondary bifurcations occur. The bifurcation sets for $\epsilon > 0$ and $\lambda > 0$ fixed, respectively, have the following form:

\begin{equation}
\lambda = 0,
\end{equation}

\begin{equation}
\lambda = \frac{p}{p-1} \lambda_\epsilon + o(\lambda_\epsilon).
\end{equation}

(2) If $g(-u) = g(u)$, there are exactly three bifurcation curves:

\begin{equation}
\lambda = 0,
\end{equation}

\begin{equation}
\lambda = \lambda_\epsilon,
\end{equation}

\begin{equation}
\lambda = \frac{1}{p-1} \lambda_\epsilon + o(\lambda_\epsilon).
\end{equation}

$\lambda = 0, \lambda = \lambda_\epsilon$ represent primary bifurcations at simple eigenvalues, and at
BIFURCATIONS CAUSED BY PERTURBATION

\[ \lambda = -(1/p - 1) \lambda + o(\lambda) , \]  

a secondary bifurcation occurs. The bifurcation sets are (for \( a > 0 \)):

\[
\begin{align*}
\text{Figure 6} & \quad \text{Figure 7}
\end{align*}
\]

with the obvious modifications when \( a < 0 \) or \( \lambda < 0 \).

The proof of Theorem 2.3 (Section 4) shows that some of these curves and secondary bifurcations are present even if \( g \) is neither odd nor even. This is an important issue since it reflects how sensitive our diagrams are to perturbations of the nonlinearity \( g \), and it gives us some hints relative to the following question motivated by Theorem 2: How are the diagrams corresponding to \( g \) odd and \( g \) even connected? That is, if we perturb \( g \) in a continuous manner, letting it go from even to odd, how do these diagrams change?

In order to study this question, we consider the function \( g(u) = au^2 + u^4 \), which is odd for \( a = 0 \) and "even for \( a = \infty \)." We analyze this problem in exactly the same way used for Theorem 2.5, obtaining a two-dimensional problem with three parameters \( \lambda, v, a \):

\[
\lambda L_1(\varepsilon) X + v L_2(\varepsilon) X + a Q(X, \varepsilon) + C(X, \varepsilon) + R(X, \lambda, \varepsilon) = 0, \quad (2.11)
\]

where \( Q(\cdot, \varepsilon) \) and \( C(\cdot, \varepsilon) \) are quadratic and cubic polynomials, respectively, and \( R = O(|X|^4) \). Again, we apply the appropriate scaling techniques and obtain the following result:

**Theorem 2.4.** Let \( \Omega_\varepsilon \) be symmetric with respect to the y-axis and let \( f(\lambda, u) = \lambda u - au^2 - u^3 + O(|u|^4) \) satisfy the hypotheses of Theorem 2.2 for \( |\lambda| \) small. Then the bifurcation diagrams and bifurcation sets for \( \varepsilon > 0 \) fixed have the following forms:
(1) For $a^2 < 8\lambda_e$

\[ \lambda \]
\[ 9 \quad 5 \quad 3 \quad 3 \quad 1 \]
\[ \lambda_e \]

\textbf{FIGURE 8}

(2) For $8\lambda_e < a^2 < (27/2)\lambda_e$

\[ u \]
\[ \lambda \]

\textbf{FIGURE 9}

(3) For $a^2 > (27/2)\lambda_e$

\[ u \]
\[ \lambda \]

\textbf{FIGURE 10}

\textbf{FIGURE 11}

\textbf{FIGURE 12}

\textbf{FIGURE 13}
If we want to analyze the change in the bifurcation patterns for \( \lambda \), \( a \) fixed and \( \nu \) variable, that is, if we want to study how the asymmetries in the nonlinearity affect the location and structure of the bifurcations produced exclusively by varying the domain, then a different scaling is required, and the result is the following:

**Theorem 2.5.** Under the same hypotheses of Theorem 2.4, if we fix \( a \neq 0 \) and let \( \bar{\lambda} = \lambda a^{-2} \), \( \bar{\nu} = \nu a^{-2} = \lambda, a^{-2} \), the bifurcation diagram has the form shown in Fig. 14 for fixed \( a \neq 0, \bar{\lambda} = \lambda a^{-2}, \bar{\nu} = \nu a^{-2} = \lambda, a^{-2} \).

The proofs of Theorems 2.4 and 2.5, along with the precise formulas for the bifurcation curves, are given in Section 4.

*Stability of the solutions.* Due to lack of space, we cannot analyze this problem here. We will only give an example: the distribution of stable and unstable solutions in the most complicated bifurcation set of Theorem 2.4 (Fig. 13) is the following:

![Figure 14](image)
Application. The reaction–diffusion equation (2.1) arises in a natural way in many actual situations. Specifically, we have concentrated in a selection-migration model in population genetics, which is briefly discussed in Section 5.

3. THE CRITICAL CASE WITH SYMMETRIC NONLINEARITY

In this section, we prove Theorem 2.3. Let us recall that \( f(\lambda, u) = \lambda u - g(u), g(u) = au^p + O(|u|^{p+1}) \) as \( |u| \to 0 \). The bifurcation equations (2.7) take the form:

\[
G(\alpha, \beta, \lambda, \epsilon) \equiv \lambda |\Omega| a - \int_{\Omega} g(\alpha + \beta w_\epsilon + \hat{u}(\alpha, \beta, \lambda, \epsilon)) = 0,
\]

\[
H(\alpha, \beta, \lambda, \epsilon) \equiv (\lambda - \lambda_\epsilon) \beta - \int_{\partial \Omega} w_\epsilon g(\alpha + \beta w_\epsilon + \hat{u}(\alpha, \beta, \lambda, \epsilon)) = 0.
\]

**Proposition 3.1.** Let \( h(u) = au^p; r(u) = g(u) - h(u) \).

\[
L_1(\epsilon) = \begin{pmatrix} |\Omega| & 0 \\ 0 & 1 \end{pmatrix}; \quad L_2(\epsilon) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}; \quad X = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}; \quad v = \lambda_\epsilon;
\]

\[
T(\alpha, \beta, \epsilon) = -\int_{\partial \Omega} h(\alpha + \beta w_\epsilon) \begin{pmatrix} 1 \\ w_\epsilon \end{pmatrix},
\]

\[
R(\alpha, \beta, \lambda, \epsilon) = -\int_{\partial \Omega} [g(\alpha + \beta w_\epsilon + \hat{u}(\alpha, \beta, \lambda, \epsilon)) - h(\alpha + \beta w_\epsilon)] \begin{pmatrix} 1 \\ w_\epsilon \end{pmatrix}.
\]
Then, (3.1) has the form

$$\lambda L_1(\epsilon) X + vL_2(\epsilon) X + T(X, \epsilon) + R(X, \lambda, \epsilon) = 0,$$  \tag{3.2}$$

where $L_1(\epsilon), L_2(\epsilon)$ are $2 \times 2$ matrices, $T(\cdot, \epsilon)$ is a homogeneous polynomial of degree $p$, $|R(X, \lambda, \epsilon)| = O(|X|^{p+1})$ uniformly in $\lambda, \epsilon$ and all functions are continuous in $(X, \lambda, \epsilon)$.

**Proof.** It suffices to observe that $\int_{\partial a} w_{1,\epsilon}^m \to \int_{\partial a_0} w_{1,\epsilon}^m$ as $\epsilon \to \epsilon_0$ (Theorem 2.2) and that $Du(0, 0, \lambda, \epsilon) = 0$ for all $\lambda, \epsilon$ (the fact that $f(\lambda, \alpha)$ is a constant implies $u(0, 0, \lambda, \epsilon) = 0$ and $\partial_\beta u(0, 0, \lambda, \epsilon) = 0$ by uniqueness).

We now proceed to analyze (3.2) as a bifurcation problem with two eigenvalue parameters $(\lambda, v)$, following Chow, Hale and Mallet-Paret [5].

**Proposition 3.2.** (A priori bound). If $T(X, 0) = 0$ implies $X = 0$, then there exist a neighborhood $U$ of $X = 0$ and constants $\rho > 0$, $K > 0$ such that any solution $(X, \lambda, v, \epsilon)$ of (3.2) with $X \in U$, $|\lambda|, |v|, \epsilon < \rho$ must satisfy

$$|X| \leq K[|\lambda|^{1/(p-1)} + |v|^{1/(p-1)}].$$

**Proof:** If the conclusion is false, there exist sequences $X_j \to 0$, $X_j \neq 0$, $\lambda_j \to 0$, $v_j \to 0$, $\epsilon_j \to 0$ such that (3.2) holds and $|\lambda_j| + |v_j|/|X_j|^{p-1} \to 0$ as $j \to \infty$. We may assume that $X_j/|X_j| \to X^*$, with $|X^*| = 1$. Dividing (3.2) by $|X_j|^{p-1}$, using the homogeneity of $T$ and letting $j \to \infty$, we see that $T(X^*, 0) = 0$, but $|X^*| = 1$, which is a contradiction.

Our goal is to study the "small" solutions of (3.2), i.e., those solutions $X$ lying in $U$, the neighborhood of 0 given by the previous proposition, for $\lambda, v, \epsilon$ small. In particular, we are interested in finding the bifurcation values $(\lambda, v, \epsilon)$ for which there is a change in the number of solutions of (3.2).

**Proposition 3.3.** If

$$L_1(0) X + T(X, 0) = 0 \implies \det[L_1(0) + D_X T(X, 0)] \neq 0,$$

and

$$(-1)^p L_1(0) X + T(X, 0) = 0 \implies \det[(-1)^p L_1(0) + D_X T(X, 0)] \neq 0,$$

then there exist $\delta > 0$, $\mu > 0$ such that all the bifurcation values $(\lambda, v, \epsilon)$ of (3.2) with $|\lambda| < \mu, \epsilon < \mu$ satisfy $|v| \geq \delta |\lambda|$.

**Proof:** It is similar to the proof of Proposition 3.2 and will be omitted. Propositions 3.2 and 3.3 justify the following change of variables:

$$X = v^{1/(p-1)} Y, \quad \lambda = rv.$$  \tag{3.4}$$
Substituting these new variables in (3.2) and dividing by \( v^{(p-1)} \), we obtain

\[
F(Y, r, v, \epsilon) = rL_1(\epsilon) Y + L_2(\epsilon) Y + T(Y, \epsilon) + v^{-\frac{1}{p-1}} R(v^{1/(p-1)} Y, rv, \epsilon) = 0. \tag{3.5}
\]

It is clear that if \( (\lambda_j, \epsilon_j, v_j) \) are bifurcation values of (3.2) with \( v_j \to 0, \epsilon_j \to 0 \) and \( r^0 \) is any limit point of \( \{\lambda_j/v_j\} \) (which is bounded by Proposition 3.3), then there is a solution \( Y^0 \) of

\[
 r^0 L_1(0) Y^0 + L_2(0) Y^0 + T(Y^0, 0) = 0,
\]

\[
det[r^0 L_1(0) + L_2(0) + D_x T(Y^0, 0)] = 0. \tag{3.6}
\]

Thus, our next goal is to study (3.5) and (3.6) for our particular problem.

We will assume that the following symmetry hypothesis is satisfied.

**Hypothesis (S).** The domains \( \Omega_\epsilon \) are symmetric with respect to the \( y \)-axis.

Furthermore, in order to simplify the formulas, we will take \( |\Omega_0| = 1 \); we will also assume that \( a = (1/(p-1)) g^{(p)}(0) > 0 \) if \( p \) is odd.

The justification of the scaling techniques just discussed is contained in the following lemma, which shows that the hypotheses of Propositions 3.2 and 3.3 are satisfied in our problem. The proof is straightforward.

**Lemma 3.4.** The following are satisfied:

(i) \( T(X, 0) = T(\alpha, \beta, 0) = - \left[ \int_{\Omega_0} h(\alpha + \beta w_0) \right. \right. \]

\[
\left. \left. \left[ \int_{\Omega_0} w_0 \cdot h(\alpha + \beta w_0) \right] \right] \right] = - \left[ \frac{1}{2} [h(\alpha + \beta) + h(\alpha - \beta)] \right. \right. \]

\[
\left. \left. \left[ \frac{1}{2} [h(\alpha + \beta) - h(\alpha - \beta)] \right. \right. \right],
\]

(ii) \( L_1(0) = I \),

(iii) \( T(X, 0) = 0 \) implies \( X = 0 \),

(iv) \( L_1(0) X + T(X, 0) = 0 \) implies \( \det[L_1(0) + D_x T(X, 0)] \neq 0 \),

(v) \( (-1)^p L_1(0) X + T(X, 0) = 0 \) implies \( \det[(-1)^p L_1(0) + D_x T(X, 0)] \neq 0 \).

In our problem, Eqs. (3.6) have the form
BIFURCATIONS CAUSED BY PERTURBATION

\[ rY_1 - \frac{1}{2}[h(Y_1 + Y_2) + h(Y_1 - Y_2)] = 0, \]
\[ (r - 1) Y_2 - \frac{1}{2}[h(Y_1 + Y_2) - h(Y_1 - Y_2)] = 0, \]
\[ \Delta = r(r - 1) - \frac{2r - 1}{2} \left[h'(Y_1 + Y_2) + h'(Y_1 - Y_2)\right] \]
\[ + h'(Y_1 + Y_2) h'(Y_1 - Y_2) = 0, \]
where \( \Delta \) is defined as \( \text{det}[rL_1(0) + L_2(0) + D_x(Y, 0)] \).

**Proposition 3.5.** All the solutions \((Y_1, Y_2, r)\) of (3.7) satisfy \( Y_1 Y_2 = 0 \).

**Proof:** Let
\[
Z_1 = Y_1 + Y_2; \quad Y_1 = (Z_1 + Z_2)/2; \\
Z_2 = Y_1 - Y_2; \quad Y_2 = (Z_1 - Z_2)/2.
\]

If \((Y_1, Y_2, r)\) is a solution of (3.7), then, multiplying the first row of \( \Delta \) by \( Y_1 \), the second by \( Y_2 \) and using the first two equations, we obtain:
\[
4Y_1 Y_2 \Delta = h(Z_1)^2 - h(Z_2)^2 - \frac{1}{2}((Z_1 - Z_2)(h(Z_1) + h(Z_2)) \\
+ (Z_1 + Z_2)(h(Z_1) - h(Z_2)) [h'(Z_1) + h'(Z_2)] \\
+ (Z_1 + Z_2)(Z_1 - Z_2) h'(Z_1) h'(Z_2) \\
= a^2 Z_1^{2p} - a^2 Z_2^{2p} - p a^2 (Z_1^{p+1} - Z_2^{p+1})(Z_1^{p-1} + Z_2^{p-1}) \\
+ a^2 p^2 (Z_1^2 - Z_2^2) Z_1^{p-1} Z_2^{p-1}.
\]

Therefore,
\[
\frac{4Y_1 Y_2}{a^2(1 - p)} \Delta = Z_1^{2p} - Z_2^{2p} - p(Z_1^2 - Z_2^2) Z_1^{p-1} Z_2^{p-1}.
\]

If we assume \( Y_1 Y_2 \neq 0 \), then \( \Delta = 0 \) if and only if \( Z_1 = AZ_2 \), where \( A \) satisfies
\[
F(A) = A^{2p} - p A^{p+1} + p A^{p-1} - 1 = 0.
\]

By analyzing the signs of \( F' \) and \( F'' \), it is not difficult to show that \( F(A) = 0 \) if and only if \( A = \pm 1 \), which implies \( Z_1 \pm Z_2 = 0 \), i.e., \( Y_1 Y_2 = 0 \).

With the help of this proposition, it is very easy to find the solutions of (3.7). The hypothesis \( "a > 0 \text{ if } p \text{ is odd}" \) is important in order to rule out some extraneous solutions.
**Proposition 3.6.** The solutions of (3.7) are the following:

<table>
<thead>
<tr>
<th></th>
<th>$p$ even</th>
<th></th>
<th>$p$ odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>$Y_1$</td>
<td>$Y_2$</td>
<td>$r$</td>
</tr>
<tr>
<td>---------</td>
<td>----------</td>
<td>---------</td>
<td>---------</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$-\frac{1}{p-1}$</td>
<td>$-C_p$</td>
<td>0</td>
<td>$\frac{p}{p-1}$</td>
</tr>
</tbody>
</table>

where $C_p = (1/a(p-1))^{1/(p-1)}$.

These solutions represent the only possible values of $(Y_1, Y_2, r)$ from which there may be branching of solutions in our original Eq. (3.1), transformed by scaling to

$$F_1 = (r-1) Y_2 - \int_{\Omega} w_e h(Y_1 + Y_2 w_e) + v^{-p/(p-1)} R_1(v^{1/(p-1)} Y, rv, \epsilon) = 0,$$

$$F_2 = (r-1) Y_2 - \int_{\Omega} w_e h(Y_1 + Y_2 w_e) + v^{-p/(p-1)} R_2(v^{1/(p-1)} Y, rv, \epsilon) = 0.$$

Our objective is to study the behavior of (3.9) in the neighborhood of every one of the points listed in Proposition 3.6. In order to do that, we first need to derive some special properties of Eqs. (3.9) which result from the symmetry hypothesis:

**Proposition 3.7.** Let hypothesis (S) be satisfied. We define $S: L^2(\Omega_e) \to L^2(\Omega_e)$ by setting $(S g)(x, y) = g(-x, y)$. Then, for $\epsilon$ sufficiently small, the following hold:

(i) $S w_e = -w_e$, i.e., $w_e$ is odd in $x$,

(ii) $S \hat{u}(\alpha, \beta, \lambda, \epsilon) = \hat{u}(\alpha, -\beta, \lambda, \epsilon)$,

(iii) $G(\alpha, -\beta, \lambda, \epsilon) = G(\alpha, \beta, \lambda, \epsilon)$; $H(\alpha, -\beta, \lambda, \epsilon) = -H(\alpha, \beta, \lambda, \epsilon)$

\[ F_1(Y_1, -Y_2, r, v, \epsilon) = F_1(Y_1, Y_2, r, v, \epsilon); \]

\[ F_2(Y_1, -Y_2, r, v, \epsilon) = -F_2(Y_1, Y_2, r, v, \epsilon); \]

(iv) if, in addition, $f(\lambda, -u) = -f(\lambda, u)$ for all $\lambda \in A, u \in R$, then

\[ G(-\alpha, \beta, \lambda, \epsilon) = -G(\alpha, \beta, \lambda, \epsilon); H(-\alpha, \beta, \lambda, \epsilon) = H(\alpha, \beta, \lambda, \epsilon); \]

\[ F_1(-Y_1, Y_2, r, v, \epsilon) = -F_1(Y_1, Y_2, r, v, \epsilon); \]

\[ F_2(-Y_1, Y_2, r, v, \epsilon) = F_2(Y_1, Y_2, r, v, \epsilon); \]

where $G, H$ and $F_1, F_2$ are given by (3.1) and (3.9), respectively.
Proof. (i) The symmetry hypothesis and the fact that \( \lambda_1 \) is a simple eigenvalue imply that \( w_1 \) has to be either even or odd in \( x \). Since \( \int_{\Omega_{e}} w_1 \) and \( \int_{\Omega_{e}} w_1 \) are nonzero and have opposite signs, \( w_1 \) must be odd.

(ii)-(iv) If \( h_1 \in L^2(\Omega_2) \) and \( \hat{u}_e \in \hat{H}_e \) satisfy \(-\Delta \hat{u}_e = h_1 \), then \( S \hat{u}_e \in \hat{H}_e \) satisfies \(-\Delta (S \hat{u}_e) = S h_1 \), and \( \hat{u}_e/\partial n = 0 \) on \( \partial \Omega_e \) implies \( \partial (S \hat{u}_e)/\partial n = 0 \) on \( \partial \Omega_e \). Therefore, \( S(K_e h_1) = \hat{K}_e h_1 \) by uniqueness. On the other hand, \( \int_{\Omega_e} S h_1 \neq \int_{\Omega_e} h_1 \) for \( h_1 \in L^2(\Omega_2) \) and (i) imply that \( P_1 \) and \( S \) commute. Hence, for any \( g_1 \in L^2(\Omega_2) \), we have: \( S g_1 = \hat{K}_e \), which in the original variables, is just \( \lambda_1 \alpha - g(\alpha) = 0 \). These correspond to the constant solutions of \(-\Delta u = \lambda u - g(u)\).

(2) \( r^2 = 1, Y_2^0 = 0 \).

Since \( F_2(Y_1, 0, 1, 0, 0) = 0 \) and \( D_{Y_1} F_1(0, 0, 1, 0, 0) = 1 \neq 0 \), by the Implicit Function Theorem, \( F_1 = 0 \) has a unique solution \( Y_1 = Y_1(Y_2, r, v, \epsilon) \) in a neighborhood of \((0, 1, 0, 0)\).

(2a) If \( g \) is odd, then \( F_2(0, Y_2, r, v, \epsilon) = 0 \) by Proposition 3.7, which implies \( Y_1(Y_2, r, v, \epsilon) \equiv 0 \). Then

\[
F_2(0, Y_2, r, v, \epsilon) = (r - 1) Y_2 - v^{-\rho/(\rho - 1)} \int_{\Omega_e} w_1 g(v^{1/(\rho - 1)} Y_2 w_1 + \hat{u}(0, v^{1/(\rho - 1)} Y_2, rv, \epsilon)) = 0
\]

can be solved for \( r - 1 \) (once \( Y_2 \equiv 0 \) has been considered), obtaining

\[
r = r(Y_2, v, \epsilon) \sim 1 + Y_2^{p-1} \int_{\Omega_e} w_1^{p+1}.
\]

Thus, the bifurcation is supercritical.

(2b) If \( p \) is even, the situation is much more complicated. First, \( Y_1 = Y_1(Y_2, r, v, \epsilon) \) is no longer zero, but an even function of \( Y_2 \) (by Proposition 3.7) satisfying \( D_{Y_1}^k Y_1(0, r, v, \epsilon) = 0 \) for \( k = 1, 2, ..., p - 1, \)

\( D_{Y_2}^p Y_1(0, r^0, 0, 0) = a/\rho! \neq 0 \). If we define \( \tilde{F}_2(Y_2, r, v, \epsilon) = F_2(Y_1(Y_2, r, v, \epsilon), Y_2, r, v, \epsilon) \), then \( \tilde{F}_2(-Y_2, r, v, \epsilon) = \tilde{F}_2(Y_2, r, v, \epsilon) \) and \( D_{Y_2}^2 \tilde{F}_2 = 1 \). This means
that the solutions of $F_2 = 0$ are given by $Y_2 = 0$ and a unique function $r = r(Y_2, v, \varepsilon)$ which is even in $Y_2$ and has the value 1 for $Y_2 = 0$, $v = 0$, $\varepsilon = 0$. However,

$$F_2(Y_2, 1, 0, 0) = -\frac{1}{2} [h(Y_1(Y_2, 1, 0, 0) + Y_2) - h(Y_1(Y_2, 1, 0, 0) - Y_2)]$$

$$= -\frac{a}{2} [(aY_2^p + O(|Y_2|^{p+1}) + Y_2)^p - (aY_2^p + O(|Y_2|^{p+1}) - Y_2)^p]$$

$$= -a^2 p Y_2^{2p-1} + O(|Y_2|^{2p})$$

since $p$ is even. These cancellations bring some degree of degeneracy to our problem, and we will have to make use of a number of the higher-order terms of the Taylor expansion of $g$ in order to find some approximate formula for $r(Y_2, v, \varepsilon)$, and decide upon the super-, sub- or transcritical character of the bifurcation.

First of all, it is easy to see that $\tilde{u}(\alpha, \beta, \lambda, \varepsilon) = O((|\alpha| + |eta|)^p)$ uniformly in $\lambda, \varepsilon$. Then, we may write

$$\hat{u}(v^{1/(p-1)} Y_1, r v, \varepsilon) = \frac{1}{p!} v^{p/(p-1)} D_{\alpha \beta}^p \hat{u}(0, 0, r v, \varepsilon) Y_2^p + O(|v^{1/(p-1)} Y_2|^{p+1}).$$

Let us take any of the terms in the Taylor expansion of $g$, say $c u^k$ ($c = \text{constant}$). Then

$$c \int_{\mathcal{D}_e} (v^{1/(p-1)} (Y_1 + Y_2 w_{e}) + \hat{u})^k w_{e} \chi \left( \frac{\varepsilon}{A \alpha + C} \right) = c \sum_{A + B + C = k} M^{(k)}_{A B C} \int_{\mathcal{D}_e} (v^{1/(p-1)} Y_1)^A (v^{1/(p-1)} Y_2)^B (\hat{u})^C w_{e}^{B+1}.$$

where $M^{(k)}_{A B C} = k! A! B! C!$.

If $k$ is odd, the lowest-order term in $Y_2$ will be $c M^{(k)}_{0,0,0} v^{k/(p-1)} (\int_{\mathcal{D}_e} w_{e}^{k+1}) Y_2^k$. However, if $k$ is even, this term is absent, since $w_e$ is odd in $x$, and therefore $\int_{\mathcal{D}_e} w_{e}^{k+1} = 0$. Hence, in this case, the lowest-order term in $Y_2$ will correspond to $B = k - 1, A + C = 1$; i.e.,

$$c \left[ v^{k/(p-1)} M^{(k)}_{1,k-1,0} \left( \int_{\mathcal{D}_e} w_{e}^{k} \right) \right.$$

$$+ \frac{1}{p!} M^{(k)}_{0,k-1,1} v^{p/(p-1)} \int_{\mathcal{D}_e} D_{\alpha \beta}^p \hat{u}(0, 0, r v, \varepsilon) w_{e}^{k} \right] Y_2^{p+k-1} + O(|Y_2|^{p+k}).$$

We see that the smallest power of $Y_2$ in the term $c u^k$ (for $k$ even) appears when $k = p$. Since $\int_{\mathcal{D}_e} w_{e}^{2k}$ remains bounded and $D_{\alpha \beta} \hat{u}(0, 0, r v, \varepsilon) \to 0$ in $L^2$ as $\varepsilon \to 0$, the dominant term in the bracket is the first one. Hence:
(i) If \( g^{(k)}(0) \geq 0 \) for all \( k \) odd, \( p < k \leq 2p - 1 \), then \( r(Y_2, v, \epsilon) - 1 > 0 \) and the bifurcation is supercritical.

(ii) If \( g^{(k)}(0) < 0 \) for some \( k \) odd, \( p < k \leq 2p - 1 \), then

\[
r(Y_1, v, \epsilon) - 1 \sim \frac{1}{k!} g^{(k)}(0) \left( \int_{\Omega_2} w^{k+1}_k \right) Y_1^{k-1} + \ldots
\]

and the bifurcation has the form of Fig. 16.

(3) \( r^0 = p/(p - 1) \) (\( p \) odd), \( r^0 = -1/(p - 1) \) (\( p \) even).

Let us consider first \( r_0 = p/(p - 1) \), \( Y_1^0 = 0 \), \( Y_2^0 = 1 \). Since \( F_2(Y_1^0, Y_2^0, r_0, 0, 0) = 0 \) and \( D_{Y_2} F_2(Y_1^0, Y_2^0, r_0, 0, 0) = r_0 - 1 - h'(Y_2^0) \neq 0 \), \( F_2 = 0 \) has a unique solution \( Y_2 = Y_2(Y_1, r, v, \epsilon) \) such that \( Y_2(0, r_0, 0, 0) = Y_2^0 \); furthermore, it is easy to check that \( Y_2(-Y_1^0, r, v, \epsilon) = Y_2(Y_1, r, v, \epsilon) \) by Proposition 3.7, and \( D_{Y_2} F_2 = r_0 - 1 - h'(Y_2^0) = a(1 - p)(Y_2^0)^{p-1} \) implies

\[
D_{Y_1} Y_2(0, r_0, 0, 0) = 0, \quad D_{Y_1}^2 Y_2(0, r_0, 0, 0) = -p/Y_2^0,
\]

\[
D_{r, Y_2}(0, r_0, 0, 0) = \frac{1}{a(p - 1)(Y_2^0)^{p-2}}.
\]

Now we define

\[
\tilde{F}_1(Y_1, r, v, \epsilon) = F_1(Y_1, Y_2(Y_1, r, v, \epsilon), r, v, \epsilon).
\]

By implicit differentiation, we can show that

\[
D_{Y_1} \tilde{F}_1 = 0, \quad D_{Y_2} \tilde{F}_1 = 0, \quad D_{Y_1}^2 \tilde{F}_1 = 2ap(p - 1)(2p + 2)(Y_2^0)^{p-3} > 0,
\]

\[
D_{Y_2} Y_2 = 1 - p < 0 \quad \text{(all derivatives evaluated at (0, 0, 0, 0))}, \quad \text{and, by Proposition 3.7, we have} \quad \tilde{F}_1(-Y_1, r, v, \epsilon) = -\tilde{F}_1(Y_1, r, v, \epsilon), \quad \text{which means that we may write}
\]

\[
\tilde{F}_1(Y_1, r, v, \epsilon) = Y_1 \tilde{F}_1(Y_1, r, v, \epsilon).
\]

\hspace{1cm}

Figure 16
where $\mathcal{F}_1$ is smooth in $Y_1, r, v$, even in $Y_1$ and satisfies

\[ D_{r_1} \mathcal{F}_1(0, 0, 0, 0) = D_{r_{1_r}} \mathcal{F}_1 = 1 - p < 0, \]
\[ D_{Y_1} \mathcal{F}_1(0, r, v, \epsilon) = 0, \]
\[ D_{Y_1}^2 \mathcal{F}_1(0, 0, 0, 0) = \frac{1}{3} D_{Y_1}^3 \mathcal{F}_1(0, 0, 0, 0) > 0. \]

Therefore, the solutions of $F_1 = 0$ are given by $Y_1 = 0$ and a unique surface $r = r(Y_1, v, \epsilon)$ which is smooth in $Y_1, v$, continuous in $\epsilon$ and verifies

\[ r(0, 0, 0) = r^0, \quad r(-Y_1, v, \epsilon) = r(Y_1, v, \epsilon), \]

and

\[ D_{Y_1}^2 r(0, 0, 0) = -(D_{r_1} \mathcal{F}_1)^{-1} [D_{Y_1}^2 \mathcal{F}_1 + 2(D_{Y_1} r, \mathcal{F}_1)(D_{Y_1} r) + (D_{r_1} \mathcal{F}_1)(D_{r_1} r)^2] \]
\[ = \frac{1}{p - 1} \frac{1}{3} D_{Y_1}^3 \mathcal{F}_1 > 0. \]

Hence, if we define $\bar{r}(v, \epsilon) = r(0, v, \epsilon)$, we have:

(i) If $r > \bar{r}(v, \epsilon)$, $\mathcal{F}_1(Y_1, r, v, \epsilon) = 0$ has exactly three solutions near $(0, r^0, 0, 0)$, namely, $Y_1 = 0$ and $r = r(Y_1, v, \epsilon)$.

(ii) If $r < \bar{r}(v, \epsilon)$, $\mathcal{F}_1(Y_1, r, v, \epsilon) = 0$ has only one solution, $Y_1 = 0$.

This represents a supercritical bifurcation of "pitchfork" type from a nontrivial branch (it takes place at $Y_1^0 = 0$, $Y_2^0 \neq 0$); that is, a secondary bifurcation, and its existence is due to the fact that $\mathcal{F}_1$ is odd in $Y_1$, a consequence of Proposition 3.7, which, in turn, relies upon the symmetry of $\Omega_\epsilon$ and the odd character of $g$ (as a matter of fact, this secondary bifurcation may disappear if $g$ is not odd: see the next section).

If we now consider the value $Y_2^0 = -(1/a(p - 1))^{1/(p - 1)}$, we can apply again Proposition 3.7 to show that, at the same value $r = \bar{r}(v, \epsilon)$ obtained above, another secondary bifurcation appears. Briefly, the Implicit Function Theorem gives a unique solution $Y_2 = Y_2^-(Y_1, r, v, \epsilon)$ of $F_2 = 0$, with $Y_2^-(0, 0, 0, 0) = Y_2^0$, which, when substituted in $F_1$ gives a function which turns out to be the negative of $\mathcal{F}_1$ defined above.

Finally, if $p$ is even, we have to study the neighborhood of $r^0 = -1/(p - 1)$, $Y_1^0 = -(1/a(p - 1))^{1/(p - 1)}$, $Y_2^0 = 0$. By the Implicit Function Theorem, $F_1 = 0$ has a unique solution $Y_1 = Y_1(Y_2, r, v, \epsilon)$ with $Y_1(0, r^0, 0, 0) = Y_1^0$; furthermore, it satisfies $Y_1(-Y_2, r, v, \epsilon) = Y_1(Y_2, r, v, \epsilon)$. Let us define

\[ \mathcal{F}_2(Y_2, r, v, \epsilon) \triangleq F_2(Y_1(Y_2, r, v, \epsilon), Y_2, r, v, \epsilon). \]
BIFURCATIONS CAUSED BY PERTURBATION

Then \( \tilde{F}_2(-Y_2, r, v, \epsilon) = -\tilde{F}_2(Y_2, r, v, \epsilon) \) and, by implicit differentiation, it satisfies

\[
D_{y_2}^2 \tilde{F}_2(0, r^0, 0, 0) = 0, \\
D_{y_1}^2 \tilde{F}_2(0, r, v, \epsilon) = 0, \\
D_{y_1} \tilde{F}_2(0, r^0, 0, 0) = 2ap(p - 1)(2p + 2)(Y_1^0)^{p-3} < 0,
\]

and

\[
D_{y_1}^2 \tilde{F}_2(0, r^0, 0, 0) = 1 - p < 0.
\]

Therefore, we are in a situation entirely analogous to the one we have just studied, the only difference being the sign of \( D_{y_1}^2 \tilde{F}_2 \). The conclusions are thus identical (existence of a secondary bifurcation), with the remarks that the bifurcation is \textit{subcritical} in this case, and, more important, that only the hypothesis "\( p \) is even" has been used. This implies, that, unlike the case "\( g \) odd," this secondary bifurcation will be preserved under higher-order perturbations of any type.

This concludes the proof of Theorem 2.3.

4. THE CRITICAL CASE WITH ASYMMETRIC NONLINEARITY

In this section we consider the nonlinearity \( g(u) = au^2 + u^3 + \text{h.o.t} \), which was introduced in Section 2 as a "mixture" between the cases "\( g \) odd" and "\( g \) even." We will outline the proof of Theorem 2.4 and comment upon some related issues.

If we denote \( X = (\alpha, \beta), v = \lambda \), \( r(u) = g(u) - au^2 - u^3 \),

\[
Q(X, \epsilon) = -\int_{\Omega} (\alpha + \beta w_\epsilon)^2 \left( \frac{1}{w_\epsilon} \right), \\
C(X, \epsilon) = -\int_{\Omega} (\alpha + \beta w_\epsilon)^3 \left( \frac{1}{w_\epsilon} \right), \\
R(X, \lambda, \epsilon) = -\int_{\Omega} \left[ g(\alpha + \beta w_\epsilon + \tilde{u}(\alpha, \beta, \epsilon)) - a(\alpha + \beta w_\epsilon)^2 \right] \left( \frac{1}{w_\epsilon} \right), \\
L_1(\epsilon) = \left( \begin{array}{cc} |\Omega_\epsilon| & 0 \\ 0 & 1 \end{array} \right), \\
L_2(\epsilon) = \left( \begin{array}{cc} 0 & 0 \\ 0 & -1 \end{array} \right),
\]

then the Bifurcation Equations (3.1) take the form

\[
\lambda L_1(\epsilon) X + v L_2(\epsilon) X + aQ(X, \epsilon) + C(X, \epsilon) + R(X, \lambda, \epsilon) = 0, 
\]

where \( L_1(\epsilon), L_2(\epsilon) \) are 2 \times 2 matrices, \( Q(\cdot, \epsilon) \) and \( C(\cdot, \epsilon) \) are quadratic and cubic polynomials, respectively, \( |R(X, \lambda, \epsilon)| = O(|X|^4) \) uniformly in \( \lambda, \epsilon \) and all functions are continuous in \( \epsilon \). We now proceed to analyze (4.1) by the same method we have followed in Section 3: (1) obtain an a priori bound and (2) change variables by a convenient scaling.
Lemma 4.1. If \( C(X,0) = 0 \) implies \( X = 0 \), then there exist a neighborhood \( U \) of \( X = 0 \) and constants \( \rho > 0 \), \( K > 0 \) such that all solutions of (4.1) with \( X \in U \), \( |a|, |\lambda|, |v|, |\epsilon| \leq \rho \) satisfy

\[
|X| \leq K|\lambda|^{1/2} + |v|^{1/2} + |a| \quad \text{uniformly in } \epsilon.
\]

Now we change variables:

\[
X = v^{1/2}Y, \quad \lambda = rv, \quad a = bv^{1/2}.
\] (4.2)

Then (4.1) becomes

\[
F \equiv rL_1(\epsilon) Y + L_2(\epsilon) Y + bQ(Y, \epsilon) + C(Y, \epsilon) + v^{-3/2}R(v^{1/2}Y, rv, \epsilon) = 0.
\] (4.3)

As in Section 3, our first goal is to find the solutions of

\[
rL_1(0) Y + L_2(0) Y + bQ(Y, 0) + C(Y, 0) = 0,
\]

\[
det[rL_1(0) + L_2(0) + bD_1 Q(Y, 0) + D_1 C(Y, 0)] = 0
\] (4.4)

or, equivalently,

\[
\begin{align*}
& rY_1 - bY_1^2 - bY_2^2 - 3Y_1 Y_2^2 - Y_3^2 = 0, \\
& (r - 1) Y_2 - 2bY_1 Y_2 - 3Y_1^2 Y_2 - Y_3 = 0,
\end{align*}
\]

\[
\Delta \equiv \begin{vmatrix} r - 2bY_1 - 3Y_1^2 - 3Y_2^2 & -2bY_2 - 6Y_1 Y_2 \\ -2bY_2 - 6Y_1 Y_2 & r - 1 - 2bY_1 - 3Y_1^2 - 3Y_2^2 \end{vmatrix} = 0. 
\] (4.5)

Multiplying the first column of \( \Delta \) by \( Y_1 \), the second by \( Y_2 \) and utilizing the first two equations, we obtain

\[
Y_1 Y_2 \Delta = -Y_2^3 [32Y_1^3 + 22bY_1^2 + 4bY_1 + 2bY_2^2].
\]

If \( b = 0 \), then \( \Delta = 0 \) if and only if \( Y_1 Y_2 = 0 \) (this is the case studied in Section 3). If \( b \neq 0 \), then we write \( Y_1 = bY_1^* \), \( Y_2 = bY_2^* \) and conclude that \( \Delta = 0 \), \( Y_1 Y_2 \neq 0 \) if and only if

\[
16Y_1^{*3} + 11Y_1^{*2} + 2Y_1^{*} + Y_2^{*2} = 0. 
\] (4.6)

In particular, \( Y_2^{*2} = -Y_1^*(16Y_1^{*2} + 11Y_1^{*} + 2) \) and \( 16z^2 + 11z + 2 > 0 \) for all \( z \); therefore, only negative values of \( Y_1^{*} \) will produce real values of \( Y_2^{*} \).
Multiplying the first equation in (4.5) by \( Y_2 \), the second by \(-Y_1\), adding, expressing everything in terms of \( Y_1^* \), \( Y_2^* \) and assuming \( Y_1 \neq 0 \) we obtain:

\[
2Y_1^{*3} - 2Y_1^*Y_2^{*2} + Y_1^{*2} - Y_2^{*2} - \frac{1}{b^2} Y_1^* = 0. \tag{4.7}
\]

Putting (4.6) and (4.7) together, we find the condition

\[
4Y_1^{*3} + 5Y_1^{*2} + 2Y_1^* + \frac{2b^2 + 1}{8b^2} = 0. \tag{4.8}
\]

Finally, from the second equation in (4.5) we obtain

\[
\frac{r - 1}{b^2} = Y_2^{*2} + 3Y_1^{*2} + 2Y_1^* = -16Y_1^{*3} - 8Y_1^{*2} + b Y_1 \tag{4.9}
\]

(by (4.6)).

These are all the possible solutions of (4.5) which satisfy \( bY_1, Y_2 \neq 0 \). The case \( b = 0 \) was studied in Section 3, so it remains to consider the possibility \( Y_1, Y_2 = 0 \):

If \( Y_1 = 0 \), then \(-bY_2 = 0\), which implies \( Y_2 = 0 \), and then \( r = 0 \) or 1.

If \( Y_1 \neq 0 \), then \( Y_2 = 0 \) and the solutions of (4.5) must satisfy

\[
r - bY_1 - Y_1^2 = 0,
\]

\[
\Delta = 2Y_1(b + 2Y_1) \left[ \left( Y_1 + \frac{b}{4} \right)^2 + \frac{1}{2} \frac{b^2}{16} \right].
\]

Let us summarize:

**Proposition 4.2.** The solutions of (4.5) are the following:

1. For all values of \( b \neq 0 \):

<table>
<thead>
<tr>
<th>( r )</th>
<th>( Y_1 )</th>
<th>( Y_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(-b^2/4)</td>
<td>(-b/2)</td>
<td>0</td>
</tr>
<tr>
<td>( r_1(b) )</td>
<td>( b v_1(b) )</td>
<td>( \pm b u_1(b) )</td>
</tr>
</tbody>
</table>

2. Only for \( b^2 > 8 \):

\[
-\frac{1}{2} - \frac{b^2}{8} \pm \frac{b}{8} \left( b^2 - 8 \right)^{1/2} \quad -\frac{b^2}{4} \pm \frac{(b^2 - 8)^{1/2}}{4} \quad 0
\]
(3) Only for $b^2 > 27/2$:

\[
\begin{array}{ccc}
  r_2(b) & bv_2(b) & \pm bu_2(b) \\
r_3(b) & bv_3(b) & \pm bu_3(b)
\end{array}
\]

where $v_i(b)$ ($i = 1, 2, 3$) are the real roots of

\[
4v^4 + 5v^2 + 2v + \frac{2b^2 + 1}{8b^2} = 0
\]

with $v_1(b) < v_2(b) < v_3(b) < 0$; $v_2(b)$ and $v_3(b)$ exist only for $b^2 > 27/2$, $r_i(b)$ is given by either of the following formulas:

\[
r_i(b) = -8b^2[v_i(b)^3 + v_i(b)^2] + 1,
\]

\[
r_i(b) = b^2 \left[ 12v_i(b)^2 + 8v_i(b) + \frac{2b^2 + 1}{2b^2} \right] + 1
\]

and $u_i(b)$ is given by either of the following formulas:

\[
u_i(b)^2 = -16v_i(b)^3 - 11v_i(b)^2 - 2v_i(b),
\]

\[
u_i(b)^2 = -3v_i(b)^2 - 6v_i(b) + \frac{2b^2 + 1}{2b^2}.
\]

Moreover, the following properties hold:

(i) $v_i(b) < -\frac{1}{4}$ for all $b \neq 0$,

(ii) $v_1(b) < v_2(b) < v_3(b) < 0$ for $b^2 > 27/2$,

(iii) $r_1(b) > 1$ and $r_1(b) \rightarrow \frac{3}{2}$, $hv_1(b) \rightarrow 0$, $b^2u_i(b)^2 \rightarrow \frac{1}{2}$ as $b \rightarrow 0$,

(iv) if $b^2 > 27/2$, then $r_3(b) < r_2(b)$ and if $b^2$ is sufficiently close to $27/2$, then

\[
-\frac{b^2}{4} < -\frac{1}{2} - \frac{b^2}{8} - \frac{b^2}{8} \left(1 - \frac{8}{b^2}\right)^{1/2} < r_3(b) < r_2(b)
\]

\[
< -\frac{1}{2} - \frac{b^2}{8} + \frac{b^2}{8} \left(1 - \frac{8}{b^2}\right) < 0.
\]

Proof. The values appearing in the table have been obtained earlier. (i), (ii) and (iii) follow easily from the formula for $v_i(b)$.

(iv) We see that $\frac{1}{2}(v_2(b) + v_3(b)) < -\frac{1}{4}$. Then
BIFURCATIONS CAUSED BY PERTURBATION

\[
\frac{r_2(b) - r_1(b)}{4b^2} = 3v_2(b)^2 + 2v_2(b) - 3v_3(b)^2 - 2v_3(b) \\
= 6(v_2(b) - v_3(b)) \left[ \frac{1}{2} (v_2(b) + v_3(b)) + \frac{1}{3} \right] > 0.
\]

The rest of (iv) follows by continuity, since for \( b^2 = 27/2 \), we have

\[-3.375 < -3.2646 < -3 = -3 < -1.1104.\]

We now analyze each point in the table above. Recall that, for \( \varepsilon = 0 \), we have

\[
F, E (r - 1) Y, - 2b Y, Y_2 - 3Y; Y_2 - 3Y; Y_3 + Y_3; + Y_3; + Y_3; + Y_3; = 0.
\]

(1) \( r^0 = 0. \)

In the neighborhood of \( r^0 = 0, Y_1^0 = 0, Y_2^0 = 0 \), the solutions of (4.3) are given by \( Y_1 = 0 \) and \( r = bY_1 + Y_2^1 \). This represents the branch of constant solutions of \(-\Delta u = \lambda u + g(u)\).

(2) \( r^0 = 1. \)

This is the same case we saw in Section 3, where we proved that, for \( p = 2 \), the type of bifurcation is preserved under a positive cubic perturbation. Thus, we still have (locally) a supercritical primary bifurcation of pitchfork type.

(3) \( r^0 = -b^2/4. \)

By the Implicit Function Theorem, the only solution of \( F_2 = 0 \) near \( r^0 = -b^2/4, Y_1^0 = -b/2, Y_2^0 = 0 \) is \( Y_2 = 0 \). Therefore, the solutions of (4.3) satisfy

\[
F_2(Y_1, 0, r, v, \varepsilon, b) = rY_1 - bY_1^2 - Y_1^3 = 0.
\]

This represents a turning point, given by the minimum of the function \( r(Y_1) = bY_1 + Y_1^2. \)

(4) \( r^0 = r_1(b). \)

At \( Y_1^0 = bv_1(b), Y_2^0 = \pm bv_2(b) \), we have

\[
D_{Y_1}F_2 = r^0 - 1 - 2bY_1^0 - 3(Y_1^0)^2 - 3(Y_2^0)^2 \neq 0
\]

(this is because \( r_1(b) \neq -\frac{1}{3} \) for all \( b \neq 0 \)). Therefore, we obtain a unique solution \( Y_2 = Y_2(Y_1, r, v, \varepsilon, b) \) in the neighborhood of \( (Y_1^0, Y_2^0, r^0, 0, 0, b) \), satisfying \( Y_2(Y_1, r, 0, 0, b)^2 = r - 1 - 3bY_1 - 3Y_1^2 \) (this follows directly from \( F_2 = 0 \), dividing by \( Y_2 \)).
By substituting in the first equation, we define
\[
\tilde{F}_1(Y_1, r, v, \epsilon, b) = F_1(Y_1, Y_2(Y_1, r, v, \epsilon, b), r, v, \epsilon, b).
\]

It is easy to see that, when \( v = 0, \epsilon = 0 \), \( \tilde{F}_1 \) has the form
\[
\tilde{F}_1(Y_1, r, 0, 0, b) = b(1 - r) + (3 - 2r + 2b^2) Y_1 + 8bY_1^2 + 8Y_1^3.
\]

Therefore, at \((Y_1, r, v, \epsilon, b) = (Y_1^0, r^0, 0, 0, b)\) we have
\[
\tilde{F}_1 = 0, \quad D_r \tilde{F}_1 = -b - 2Y_1^0, \quad D_{Y_1} \tilde{F}_1 = 0, \quad D_{Y_1}^2 \tilde{F}_1 = 16b[1 + 3\nu_i(b)];
\]
and since \(-b - 2Y_1^0 = -b[1 + 2\nu_i(b)] \neq 0\) for \( b \neq 0 \), we obtain \( r = r^*(Y_1^0, v, \epsilon, b) \) as the unique solution of \( \tilde{F}_1 = 0 \). This function satisfies \( D_{Y_1} r^*(Y_1^0, 0, 0, b) = 0 \) and
\[
D_{Y_1}^2 r^*(Y_1^0, 0, 0, b) = \frac{16b[1 + 3\nu_i(b)]}{b[1 + 2\nu_i(b)]} \begin{cases} > 0 & \text{for } i = 1, 3 \\ < 0 & \text{for } i = 2. \end{cases}
\]

Hence, by another application of the Implicit Function Theorem we find \( Y_1 = Y_1^*(v, \epsilon, b) \) such that \( D_{Y_1} r^*(Y_1^*(v, \epsilon, b), v, \epsilon, b) = 0 \) and then,
\[
\tilde{r}(v, \epsilon, b) \overset{\text{def}}{=} r^*(Y_1^*(v, \epsilon, b), v, \epsilon, b)
\]
represents a turning point, supercritical for \( i = 1, 3 \) and subcritical for \( i = 2 \).

\[
(5) \quad r^0 = -\frac{1}{2} - \frac{b^2}{8} \pm \frac{b(b^2 - 8)^{1/2}}{8}.
\]

At \( Y_1^0 = -b/4 \pm \frac{1}{4}(b^2 - 8)^{1/2}, \ Y_2^0 = 0, \) we have:

\[
\begin{align*}
F_1 &= 0, \quad D_{Y_1} F_1 = r^0 - 2bY_1^0 - 3(Y_1^n)^2 = 1, \\
D_{Y_2} F_1 &= 0, \quad D_{Y_2}^2 F_1 = -2b - 6Y_1^n, \quad D_r F_1 = Y_1^n.
\end{align*}
\]

By the Implicit Function Theorem, we have a unique solution \( Y_1 = Y_1(Y_2, r, v, \epsilon, b) \) of \( F_1 = 0 \), which, by Proposition 3.7, is even in \( Y_2 \). This function also satisfies (at \((Y_1^0, 0, r^0, 0, 0, b))\):
\[
D_r Y_1 = -Y_1^0, \quad D_{Y_2} Y_1 = 0, \quad D_{Y_2}^2 Y_1 = 2b + 6Y_1^0.
\]

The function
\[
\tilde{F}_2(Y_2, r, v, \epsilon, b) = F_2(Y_1(Y_2, r, v, \epsilon, b), Y_2, r, v, \epsilon, b)
\]
is odd in $Y_2$, so we may write $\tilde{F}_2 = Y_2\bar{f}_2(Y_2, r, v, \epsilon, b)$. At $v = 0$, $\epsilon = 0$, $\bar{f}_2$ has the following form:

$$\bar{f}_2(Y_2, r, 0, 0, b) = r - 1 - 2bY_1 - 3Y_1^2 - Y_2^2.$$ 

Hence, at $Y_2^0 = 0$, $r = r^0$, $v = 0$, $\epsilon = 0$, we have

$$D_r\bar{f}_2 = 1 - 2bD_rY_1 - 6Y_1D_rY_1 = 1 - (2b + 6Y_1^0)(-Y_1^0)$$
$$= 1 + bY_1^0 + 2(Y_1^0)^2 + bY_1^0 + 4(Y_1^0)^2 = bY_1^0 + 4(Y_1^0)^2 \neq 0$$

(where we have used the formula for $Y_1^0$, i.e., $1 + bY_1^0 + 2(Y_1^0)^2 = 0$).

$$D_{r^2}\bar{f}_2 = -2bD_{r^2}Y_1 - 6Y_1D_{r^2}Y_1 - 2 = (-2b + 6Y_1^0)^2 - 2.$$ 

Since $D_r\bar{f}_2 \neq 0$, we obtain a unique solution $r = r^*(Y_2, v, \epsilon, b)$ with $r^*(0, 0, 0, b) = r^0$, which is also even in $Y_2$, so $D_{r^2}r^* = 0$. Furthermore,

$$D_{r^2}^2r^* = -(D_r\bar{f}_2)^{-1}D_{r^2}\bar{f}_2 = \frac{2 + (2b + 6Y_1^0)^2}{Y_1^0(b + 4Y_1^0)}.$$ 

Therefore:

1. If $Y_1^0 = \frac{1}{4}(-b + (b^2 - 8)^{1/2})$, then $Y_1^0 < 0$ and $b + 4Y_1^0 > 0$. Thus, $D_{r^2}^2r^* < 0$ and the bifurcation at $r^*(0, v, \epsilon, b)$ is subcritical. This is precisely the secondary bifurcation that we found when $g$ is even, which, as mentioned there, is preserved under higher-order perturbations.

2. If $Y_1^0 = \frac{1}{4}(-b - (b^2 - 8)^{1/2})$, then $Y_1^0 < 0$ and $b + 4Y_1^0 < 0$. Thus, $D_{r^2}^2r^* > 0$ and the bifurcation at $r^*(0, v, \epsilon, b)$ is supercritical.

Putting all these results together, we obtain the pictures of Theorem 2.4.

If we want to analyze the changes in the bifurcation pattern when the function $f(\lambda, u)$ is fixed and $v$ (i.e., the domains) vary, the appropriate scaling is given by

$$\lambda = a^2\bar{\lambda}, \quad v = a^2\bar{v}, \quad X = a\bar{X} \quad (X = (a, \beta)).$$

Then, (4.1) becomes

$$\lambda L_1(\epsilon) \bar{X} + \bar{v}L_2(\epsilon) \bar{X} + Q(\bar{X}, \epsilon) + C(\bar{X}, \epsilon) + a^{-3}R(a\bar{X}, a^2\bar{\lambda}, \epsilon) = 0. \quad (4.10)$$

If we define $\sigma = \bar{\lambda} - \frac{1}{4}\bar{v}, Z_1 = \bar{a} + \bar{\beta}, Z_2 = \bar{a} - \bar{\beta}$, (4.10) at $a = 0$, $\epsilon = 0$ takes the form
\[ \Phi_1 \equiv \sigma Z_1 + \frac{\bar{v}}{2} Z_2 - Z_1^2 - Z_1^3 = 0, \]
\[ \Phi_2 \equiv \frac{\bar{v}}{2} Z_1 + \sigma Z_2 - Z_2^2 - Z_2^3 = 0, \]
\[ \Delta \equiv (\sigma - 2Z_1 - 3Z_1^2)(\sigma - 2Z_2 - 3Z_2^2) - \frac{\bar{v}^2}{4} = 0, \]
where
\[ \Delta \overset{\text{def}}{=} \det \frac{\partial (\Phi_1, \Phi_2)}{\partial (Z_1, Z_2)}. \]

The solutions of (4.11) for \( \bar{v} = 0 \) are

\[
\begin{align*}
\lambda : & \quad 0 \quad 0 \quad 0 \quad -\frac{1}{4} \quad -\frac{1}{4} \quad -\frac{1}{4} \\
Z_1 : & \quad 0 \quad 0 \quad -1 \quad 0 \quad -\frac{1}{4} \quad -\frac{1}{4} \\
Z_2 : & \quad 0 \quad -1 \quad 0 \quad -\frac{1}{2} \quad 0 \quad -\frac{1}{2}
\end{align*}
\]

(1) \( \bar{\lambda} = 0, Z_1 = 0, Z_2 = 0 \) corresponds precisely to the problem we have dealt with in Section 3, when \( g \) is even. As we saw there, when the first nonvanishing term of \( g \) is quadratic, the bifurcation curves are preserved under positive cubic and higher-order perturbations. Therefore, we have exactly three bifurcation curves:

\[ \bar{\lambda} = \bar{v}, \quad \bar{\lambda} = 0, \quad \bar{\lambda} = -\bar{v} + o(\bar{v}). \]

(2) \( \bar{\lambda} = 0, Z_1 = 0, Z_2 = -1 \). At this point, \( \partial (\Phi_1, \Phi_2, \Delta)/\partial (\sigma, Z_1, Z_2, \bar{v}) \) has rank three, and the Implicit Function Theorem gives us a unique solution \( (Z_1, Z_2, \bar{v}) = (Z_1, Z_2, \bar{v})(\sigma, \epsilon, a) \), and one can see (by implicit differentiation) that \( \bar{v}(\sigma, \epsilon, a) \) has the form

\[ \bar{v} = \frac{1}{2} \sigma^2 + \text{h.o.t.} \]

which, in the original variables, reads

\[ \bar{\lambda} = \pm (2\bar{v})^{1/2} + o((\bar{v})^{1/2}). \]

The point \( \bar{\lambda} = 0, Z_1 = -1, Z_2 = 0 \) gives rise to the same bifurcation curves, because of symmetry.

(3) \( \bar{\lambda} = -\frac{1}{4}, Z_1 = 0, Z_2 = -\frac{1}{4} \). Now we obtain \( (\sigma, Z_1, Z_2) = (\sigma, Z_1, Z_2)(\bar{v}, \epsilon, a) \), and \( \sigma \) has the expression \( \sigma = -\frac{1}{4} + o(\bar{v}) \), which means \( \bar{\lambda} = -\frac{1}{4} + \bar{v}/2 + o(\bar{v}) \). The same curve arises from \( \bar{\lambda} = -\frac{1}{4}, Z_1 = -\frac{1}{4}, Z_2 = 0 \).
(4) \( \lambda = -\frac{1}{4}, Z_1 = -\frac{1}{4}, Z_2 = -\frac{1}{4} \). In this case, \( \partial(\Phi_1, \Phi_2, A)/\partial(\sigma, Z_1, Z_2, \Phi) \) has rank two, and a direct analysis is required. We obtain two bifurcation curves, \( \lambda = -\frac{1}{4} \) (or \( \lambda = -a^2/4 \), the turning point of the branch of nonzero constant solutions of \( -Au = \lambda u - au^2 - u^3 \) and \( \lambda = -\frac{1}{4} + 2\Phi^2 + o(\Phi^2) \).

This completes the proof of Theorem 2.5.

5. Application to a Selection–Migration Model

Let the two-dimensional domain \( \Omega \) represent an island inhabited by a large, isolated population. We consider a particular genetic characteristic (a "locus") governed by the combinations of two alleles, \( A \) and \( a \). The population is thus divided into three possible genotypes: \( aa, aA \) and \( AA \); let their corresponding death rates be denoted by \( r_1, r_2, r_3 \), respectively.

We assume: (1) the population is uniformly distributed in \( \Omega \) and its size is constant in time; (2) its individuals migrate isotropically within the island at a constant rate \( r \); (3) \( r_1, r_2, r_3 \) are constant in time and space. Let \( p \) be the frequency of gene \( A \) in the population. Then, it can be shown (see, for instance, Fife [7], Ewens [6], Aronson and Weinberger [2]) that \( p \) satisfies

\[
\frac{\partial p}{\partial t} = rAp + p(1-p)[(r_1 - r_2)(1-p) - (r_3 - r_2)p] \quad \text{in } \Omega,
\]

\[
\frac{\partial p}{\partial n} = 0 \quad \text{on } \partial \Omega.
\]

Let \( \mu = (r_2 - r_1)/(2r_2 - r_1 - r_3) \). Then, \( 0 \leq \mu \leq 1 \) if \( r_3 < r_1 < r_2 \) or \( r_1 < r_3 < r_2 \), which indicates heterozygote inferiority: natural selection acts against the mixed race \( aA \) and favors the pure races \( aa \) and \( AA \). If this is the case, we define:

\[
b = \frac{1}{2} - \mu, \quad p - \frac{1}{2} + b = \delta u, \quad \delta^2 = r, \quad \lambda = (\frac{1}{4} - b^2)(1/r) > 0.
\]

Then, if \( p(t, x) = p(x) \) is a stationary solution of (5.1), \( u \) satisfies

\[
-Au = \lambda u + 2b(\frac{1}{4} - b^2)^{-1/2}\lambda^{1/2}u^2 - u^3 \quad \text{in } \Omega,
\]

\[
\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega.
\]

If \( b \) (which represents a genetic property) is fixed and the migration rate \( r \) is large, then \( \lambda \) is small and our results can be applied; we may interpret them as follows:
Let us consider two islands \( \Omega_0^R \) and \( \Omega_0^L \) where the process of natural selection has acted so that in \( \Omega_0^R \) all individuals are of type \( AA \), whereas in \( \Omega_0^L \) all individuals are of type \( aa \) (this is a realistic situation since the hypothesis of heterozygote inferiority implies that the population will tend to stabilize around one of the pure races). If we build a narrow bridge joining both islands, the genotype distribution will be practically the same: "almost all" individuals in \( \Omega_0^R \) are \( AA \), and "almost all" individuals in \( \Omega_0^L \) are \( aa \). As the bridge becomes wider, this distribution loses its stability and, even though some nonhomogeneous distributions persist, they are unstable, and, eventually, we will observe a unique homogeneous distribution of a certain pure race.

6. Remarks

(1) If the hypothesis of symmetry of the domains \( \Omega_\epsilon \) is dropped, the transition from five to nine solutions in all bifurcation diagrams will be made, in general, through an intermediate region of seven solutions. Whether or not the secondary bifurcations persist is an open question, although it is very likely that they do not.

(2) The boundary condition \( \frac{\partial u}{\partial n} = 0 \) may be generalized to \( \frac{\partial u}{\partial n} + yu = 0 \). It can be shown that the Bifurcation Equations are still continuous with respect to this new parameter. However, in general, we will not be dealing with a problem of bifurcation from a double eigenvalue (since, in general, the first eigenvalue of \( A \) on \( \Omega \) with the boundary condition \( \frac{\partial u}{\partial n} + yu = 0 \) depends on the geometry of the domain \( \Omega \)).

(3) In Hale and Vegas [9] a set of conditions was given on the domains \( \Omega_\epsilon \) that ensured the continuity of the Bifurcation Equations and their first derivatives in \( \epsilon \). However, it seems that these conditions are not sufficient to prove Theorem 2.2, and it is not clear what type of restrictions would have to be imposed so that properties of the type \( \int_{\Omega_\epsilon} \frac{W_\epsilon}{e} \to 0 \) as \( \epsilon \to 0 \) hold.

7. Regularity of the Bifurcation Equations

Theorem 2.2 is the most technical result in this paper, and its complete proof is very lengthy. Hale and Vegas [9] proved that the bifurcation functions \( G, H \) and their first derivatives are continuous in \( \epsilon \). However, some serious difficulties appear when the higher-order derivatives are considered; these problems can be easily overcome when \( \epsilon \) remains bounded away from zero (after all, the perturbation of the boundary is \( C^1 \) in that case), but they
require a much more careful study of the behavior of the quantities involved as \( \epsilon \) approaches zero. We now present a brief account of the main estimates and the proof of continuity at \( \epsilon = 0 \).

The first proposition establishes an estimate for the norms of the Sobolev imbeddings \( H^1(\Omega_\epsilon) \hookrightarrow L^p(\Omega_\epsilon) \):

**Proposition 7.1.** For every \( p > 2 \) there exists a constant \( M_p \) such that, for all \( \epsilon > 0, \epsilon \leq 1, u \in H^1(\Omega_\epsilon) \), the estimate

\[
\| u \|_{L^p(\Omega_\epsilon)} \leq M_p \epsilon^{1/p-1/2} \| u \|_{H^1(\Omega_\epsilon)}
\]

holds.

**Proof:** Define \( \bar{\Omega}_\epsilon = \Omega_\epsilon \cap \{(x,y): |x| \leq 3, |y| \leq \epsilon\} \). Fix \( \epsilon = 1 \). Then, \( \bar{\Omega}_1 \) is a domain satisfying the cone property. Therefore, the Sobolev Imbedding Theorem can be applied to it: \( H^1(\bar{\Omega}_1) \hookrightarrow L^p(\bar{\Omega}_1) \) (see Adams [11]); let \( M'_p \) denote the norm of this imbedding.

Let now \( u \in H^1(\Omega_\epsilon) \); define \( v(x', y') = u(x', \epsilon y') \). Then \( v \) is defined in \( \epsilon^{-1}\Omega_\epsilon \) and therefore in \( \bar{\Omega}_1 \), so \( \| v \|_{L^p(\bar{\Omega}_1)} \leq M'_p \| v \|_{H^1(\bar{\Omega}_1)} \). Now,

\[
\| u \|_{L^p(\Omega_\epsilon)} = \epsilon \| v \|_{L^p(\bar{\Omega}_1)} \leq \epsilon (M'_p)^p \| v \|_{H^1(\bar{\Omega}_1)} \leq \epsilon^{1-p/2}(M'_p)^p \| u \|_{H^1(\Omega_\epsilon)}
\]

and the result follows immediately.

The next proposition is essential in the understanding of this perturbation problem since it provides estimates for solutions of elliptic problems in \( \Omega_\epsilon \) in terms (partly) of their values in the fixed domain \( \Omega_0 \).

**Proposition 7.2.** Let \( Q_\epsilon \) be the subset of \( \Omega_0 \) defined as follows:

\[
Q^L_\epsilon = \{(x,y): -3 \leq x \leq -1, \epsilon r_-(2-x) < y < \epsilon r_+(2-x)\} \subset \Omega^L_0,
\]

\[
Q^R_\epsilon = \{(x,y): 1 \leq x \leq 3, \epsilon r_-(2-x) < y < \epsilon r_+(2-x)\} \subset \Omega^R_0,
\]

\[
Q_\epsilon = Q^L_\epsilon \cup Q^R_\epsilon.
\]

Then there exists a constant \( M > 0 \) such that, for every \( \epsilon > 0 \) and every \( u \in C^2(\bar{\Omega}_\epsilon) \) such that \( \partial u / \partial n = 0 \) on \( \partial \Omega_\epsilon \), we have

\[
\| u \|_{H^1(\Omega_\epsilon)} \leq M[\| \Delta u \|_{L^2(\Omega_\epsilon)} + \| u \|_{H^1(\Omega_\epsilon)}].
\]

**Proof:** See Hale and Vegas [9].

Next, we present the most important properties concerning eigenvalues and eigenfunctions of \( \Delta_\epsilon \).
Proposition 7.3.

(i) \( \lambda_\epsilon = \text{def } \lambda^{(2)}(\Omega_\epsilon) = O(\epsilon) \) as \( \epsilon \to 0 \).

(ii) \( \lambda^{(3)}(\Omega_\epsilon) \) is bounded away from zero as \( \epsilon \to 0 \).

(iii) If \( w_\epsilon \) is the eigenfunction corresponding to \( \lambda_\epsilon \) satisfying \( \int_{\Omega_\epsilon} w_\epsilon^2 = 1 \) and \( \int_{\Omega_\epsilon} w_\epsilon > 0 \), and \( w_0 \) is defined as

\[
 w_0 = C_L = - \left( \frac{1}{|\Omega_0|} \frac{|\Omega_0^R|}{|\Omega_0^L|} \right)^{1/2} \quad \text{on } \Omega_0^L;
\]

\[
 w_0 = C_R = \left( \frac{1}{|\Omega_0|} \frac{|\Omega_0^L|}{|\Omega_0^R|} \right)^{1/2} \quad \text{on } \Omega_0^R,
\]

then

\[
 \int_{\Omega_\epsilon} w_\epsilon^2 + \int_{\Omega_0} (w_\epsilon - w_0)^2 = O(\epsilon) \quad \text{as } \epsilon \to 0.
\]

Proof. (i) and (ii) have been proved in Hale and Vegas [9], where, moreover, it is shown that, if \( \epsilon \) is sufficiently small, \( \int_{\Omega_\epsilon} w_\epsilon \) is different from zero; therefore, the eigenfunction \( w_\epsilon \) in (iii) is well defined.

(iii) Let \( m'_\epsilon = |\Omega_0^R|^{-1} \int_{\Omega_\epsilon} w_\epsilon \) \((I = L, R)\). By the Poincaré Inequality,

\[
 \int_{\Omega_\epsilon} (w_\epsilon - m'_\epsilon)^2 \leq \lambda_{\epsilon}^{(2)}(\Omega_\epsilon)^{-1} \int_{\Omega_\epsilon} |\nabla w_\epsilon|^2 \leq (\text{const.}) \lambda_\epsilon = O(\epsilon).
\]

By Proposition 7.2,

\[
 \| w_\epsilon \|_{H^1(R, I)}^2 \leq M[\lambda_\epsilon^2 \| w_\epsilon \|_{H^1(R, I)}^2 + \| w_\epsilon \|_{H^2(I)}^2]
\]

\[
 \leq M \left[ \lambda_\epsilon^2 + \sum_{I = L, R} (\| x_\epsilon - m'_\epsilon \|_{H^1(I)}^2 + \| m'_\epsilon \|_{H^1(I)}^2) \right]
\]

\[
 \leq M[\lambda_\epsilon^2 + 2\lambda_\epsilon + O(\epsilon)] = O(\epsilon).
\]

Therefore,

\[
 \int_{R_\epsilon} w_\epsilon^2 = O(\epsilon) \quad \text{and} \quad \left| \int_{R_\epsilon} w_\epsilon \right| \leq |R_\epsilon|^{1/2} \left( \int_{R_\epsilon} w_\epsilon^2 \right)^{1/2} = O(\epsilon).
\]

On the other hand, by using the definition of \( w_0 \) one can see that

\[
 \int_{\Omega_\epsilon} w_\epsilon w_0 = \left( \frac{|\Omega_0^R|}{|\Omega_0^R|} \frac{|\Omega_0^L|}{|\Omega_0^L|} \right)^{1/2} \int_{\Omega_\epsilon} w_\epsilon - \left( \frac{|\Omega_0^R|}{|\Omega_0^R|} \frac{|\Omega_0^L|}{|\Omega_0^L|} \right)^{1/2} \int_{\Omega_\epsilon} w_\epsilon
\]

\[
 = \left( \frac{|\Omega_0^R|}{|\Omega_0^R|} \frac{|\Omega_0^L|}{|\Omega_0^L|} \right)^{1/2} \int_{\Omega_\epsilon} w_\epsilon + O(\epsilon).
\]
Therefore,

\[ \int_{\Omega_0^e} \left( w_e - |\Omega_0|^{-1} \int_{\Omega_0^e} w_e \right)^2 \]

\[ = \int_{\Omega_0^e} \left[ w_e - \frac{1}{|\Omega_0|} \left( \frac{|\Omega_0^e|}{|\Omega_0|} \right)^{1/2} \int_{\Omega_0} w_e w_0 + O(\epsilon) \right]^2 \]

\[ = \int_{\Omega_0^e} \left[ w_e - w_0 \int_{\Omega_0} w_e w_0 + O(\epsilon) \right]^2. \]

We obtain a similar estimate for \( \Omega_0^e \), and then we conclude

\[ \left\| w_e - \left( \int_{\Omega_0} w_e w_0 \right) w_0 \right\|_{L^2(\Omega_0)}^2 = O(\epsilon). \]

This implies \(-\left( \int_{\Omega_0} w_e w_0 \right)^2 = O(\epsilon) - \| w_e \|^2_{L^2(\Omega_0)} \). Then, since

\[ 0 \leq \int_{\Omega_0} w_e w_0 \leq \| w_e \|_{L^2(\Omega_0)} \| w_0 \|_{L^2(\Omega_0)} \leq 1, \]

we have (all norms are \( L^2(\Omega_0) \) norms):

\[ \| w_e - w_0 \|^2 = \| w_e \|^2 + \| w_0 \|^2 - 2 \int_{\Omega_0} w_e w_0 \]

\[ \leq \| w_e \|^2 + 1 - 2 \left( \int_{\Omega_0} w_e w_0 \right)^2 \leq \| w_e \|^2 + 1 + O(\epsilon) - 2 \| w_e \|^2 \]

\[ = -\| w_e \|^2 + 1 + O(\epsilon) = \int_{\Gamma_e} w_e^2 + O(\epsilon) = O(\epsilon). \]

This concludes the proof of (iii).

Since \( \lambda^{(3)}(\Omega_\epsilon) \) is bounded away from zero as \( \epsilon \to 0 \), the operator \( \tilde{K}_e \)

defined in Section 2 is well-defined and \( \| \tilde{K}_e \| \leq C_1 \) for a fixed constant \( C_1 \).

We now examine its dependence on \( \epsilon \):

PROPOSITION 7.4. Let \( g_\epsilon \in L^2(\Omega_\epsilon) \), \( q_\epsilon \in L^\infty(\Omega_\epsilon) \) be given for \( \epsilon > 0 \) with

\[ \| q_\epsilon \|_{L^\infty(\Omega_\epsilon)} \leq k_1 < C_1^{-1}, \]

let \( P_\epsilon \) denote the projection operator defined in

Section 2, and assume that \( g_0 \) is locally constant. Then, \( (I - P_0) g_0 = 0 \) and the following estimates hold:

(i) \[ \| P_\epsilon g_\epsilon - P_0 g_0 \|_{L^2(\Omega_\epsilon)} \leq M \epsilon^{1/2} \| g_\epsilon \|_{L^2(\Omega_\epsilon)} + \| g_\epsilon - g_0 \|_{L^2(\Omega_\epsilon)}, \]

(ii) \[ \| \tilde{K}_e (I - P_\epsilon) g_\epsilon \|_{H^1(\Omega_\epsilon)} \leq M \epsilon^{1/2} \| g_\epsilon \|_{L^2(\Omega_\epsilon)} + 2C_1 \| g_\epsilon - g_0 \|_{L^2(\Omega_\epsilon)} + C_1 \| g_\epsilon \|_{L^2(\Gamma_e)}. \]
(iii) If $\hat{\vartheta}_e$ is the unique solution of $\hat{\vartheta}_e = \hat{K}_e(I - P_e)[g_e + q_e \hat{\vartheta}_e]$, then
\[
\| \hat{\vartheta}_e \|_{H^1(\Omega_e)} \leq (1 - k_1 C_1)^{-1} \{ C_1 \| g_e \|_{L^2(\Omega_e)} + 2C_1 \| g_e - g_0 \|_{L^2(\Omega_e)} + Me^{1/2} \| g_e \|_{L^2(\Omega_e)} \}
\]
for a certain absolute constant $M > 0$.

Proof. (i) $P_e g_e - P_0 g_0 = (P_e - P_0) g_e + P_0 (g_e - g_0)$. Now,
\[
(P_e - P_0) g_e = \Omega_e^{-1} \int_{\Omega_e} g_e - \Omega_0^{-1} \int_{\Omega_0} g_e + w_e \int_{\Omega_e} w_e g_e - w_0 \int_{\Omega_0} w_0 g_e;
\]
\[
\left| \Omega_e^{-1} \int_{\Omega_e} g_e - \Omega_0^{-1} \int_{\Omega_0} g_e \right| \leq \frac{1}{|\Omega_e| |\Omega_0|} \left| \int_{\Omega_0} g_e (|\Omega_0| - |\Omega_e|) + \int_{\Omega_e} g_e |\Omega_0| \right|
\leq M' \| g_e \|_{L^2(\Omega_0)} |R_e|^{1/2} + \| g_e \|_{L^2(\Omega_e)} |R_e|^{1/2} \leq M' \| g_e \|_{L^2(\Omega_e)} ;
\]
\[
\left\| w_e \int_{\Omega_e} w_e g_e - w_0 \int_{\Omega_0} w_0 g_e \right\|_{L^2(\Omega_e)} = \left\| w_e \left( \int_{\Omega_0} (w_e - w_0) g_e + \int_{\Omega_e} w_e g_e \right) + \int_{\Omega_0} w_0 g_e \right\|_{L^2(\Omega_e)}
\leq \| g_e \|_{L^2(\Omega_0)} \| w_e - w_0 \|_{L^2(\Omega_e)} \leq \| g_e \|_{L^2(\Omega_0)} \| w_e - w_0 \|_{L^2(\Omega_e)}
\]
and $\| P_0 (g_e - g_0) \|_{L^2(\Omega_0)} \leq \| g_e - g_0 \|_{L^2(\Omega_0)}$. The result then follows from Proposition 7.3(iii).

(ii) Call $\bar{u}_e = \hat{K}_e(I - P_e) g_e$. Then
\[
\| \bar{u}_e \|_{H^1(\Omega_e)} \leq C_1 \| (I - P_e) g_e \|_{L^2(\Omega_e)}
\]
\[
\cdot \| (I - P_e) g_e \|_{L^2(\Omega_e)}
\leq \| g_e \|_{L^2(\Omega_e)} + |\Omega_e|^{-1} \left| \int_{\Omega_e} g_e \right| |R_e|^{1/2} + \| w_e \|_{L^2(\Omega_e)} \left| \int_{\Omega_e} w_e \right| g_e \right|
\leq \| g_e \|_{L^2(\Omega_e)} + M_2 \| g_e \|_{L^2(\Omega_e)} \epsilon^{1/2}.
\]
\[
\| (I - P_e) g_e \|_{L^2(\Omega_0)} = \| (I - P_e) g_e - (I - P_0) g_0 \|_{L^2(\Omega_0)}
\leq \| g_e - g_0 - (P_e g_e - P_0 g_0) \|_{L^2(\Omega_0)} \leq 2 \| g_e - g_0 \|_{L^2(\Omega_0)} + M_3 \epsilon^{1/2} \| g_e \|_{L^2(\Omega_0)},
\]
by part (i) above. This proves (ii).
(iii) $\tilde{\vartheta}_\epsilon$ satisfies $\tilde{\vartheta}_\epsilon - \tilde{K}_\epsilon(I - P_\epsilon) q_{\vartheta}_\epsilon = \tilde{K}_\epsilon(I - P_\epsilon) g_{\vartheta}_\epsilon$. Applying (ii) the result follows immediately.

Our final task is to analyze how these properties of $\tilde{K}_\epsilon$ are reflected in the behavior of $\tilde{u}(\alpha, \beta, \lambda, \epsilon)$ as a function of $\epsilon$; let us recall that $\tilde{u}(\alpha, \beta, \lambda, \epsilon)$ is defined as the unique solution of

$$\tilde{u}(\alpha, \beta, \lambda, \epsilon) = \tilde{K}_\epsilon(I - P_\epsilon) f(\lambda, \alpha + \beta w_\epsilon + \tilde{u}(\alpha, \beta, \lambda, \epsilon)).$$

**Proposition 7.5.** If $f$ satisfies the hypotheses of Theorem 2.2, then the map $\mathbb{R} \times \mathbb{R} \times \Lambda \to H^1(\Omega_\epsilon)$ given by $(\alpha, \beta, \lambda) \mapsto \tilde{u}(\alpha, \beta, \lambda, \epsilon)$ is $C^k$, and all its derivatives up through order $k$ are continuous in $(\alpha, \beta, \lambda, \epsilon)$.

Moreover, as $\epsilon \to 0$, for every $p > 2$ and every derivative of order $|r| \leq k$, we have

1. $\|D^r \tilde{u}(\alpha, \beta, \lambda, \epsilon)\|_{H^1(\Omega_\epsilon)} = O(\epsilon^{1/p})$ as $\epsilon \to 0$,

2. $\|D^r \tilde{u}(\alpha, \beta, \lambda, \epsilon)\|_{L^p(\Omega_\epsilon)} \to 0$ as $\epsilon \to 0$,

uniformly for $(\alpha, \beta, \lambda)$ in bounded sets. If $|r| = 0$ or 1, (i) also holds for $p = 2$.

**Proof.** The boundedness hypotheses on the derivatives of $f$ imply that $\tilde{u}$ is $C^k$ in $(\alpha, \beta, \lambda)$ for each fixed $\epsilon$. Let us proceed to prove the estimate:

1. $r = 0$: Let us call $\tilde{u}_\epsilon = \tilde{u}(\alpha, \beta, \lambda, \epsilon)$, and let $\tilde{w}_0 \in H^1(\mathbb{R}^2)$ be any extension of $w_0$. Then, by the Mean Value Theorem,

$$\tilde{u}_\epsilon = \tilde{K}_\epsilon(I - P_\epsilon)[f(\lambda, \alpha + \beta \tilde{w}_0) + f_u(\lambda, \alpha + \beta \tilde{w}_0 + \tau) (\beta(w_\epsilon - \tilde{w}_0) + \tilde{u}_\epsilon)],$$

where $\tau_\epsilon$ is some function. If we define $q_\epsilon = f_u(\lambda, \alpha + \beta \tilde{w}_0 + \tau)$, $g_\epsilon = f(\lambda, \alpha + \beta \tilde{w}_0) + f_u(\lambda, \alpha + \beta \tilde{w}_0 + \tau) (\beta(w_\epsilon - \tilde{w}_0)$ and $g_0 = f(\lambda, \alpha + \beta w_\epsilon)$, we may apply Proposition 7.4(iii), Proposition 7.3(iii) and the fact that $\| f(\lambda, \alpha + \beta \tilde{w}_0) \|_{L^2(\mathbb{R}^2)} = O(\epsilon^{1/2})$ (since $\tilde{w}_0$ is a fixed function) to obtain

$$\| \tilde{u}_\epsilon \|_{H^1(\Omega_\epsilon)} = O(\epsilon^{1/2}).$$

2. $|r| = 1$: Let $\tilde{u}_\epsilon - \tilde{u}(\alpha, \beta, \lambda, \epsilon), \tilde{\vartheta}_\epsilon = D \tilde{u}_\epsilon$. Then, $\| \tilde{\vartheta}_\epsilon \|_{H^1(\Omega_\epsilon)} = O(\epsilon^{1/2})$ follows easily from Proposition 7.4(iii), using the facts that $f_u$ is bounded, $f_\lambda$ grows linearly and $\| \tilde{u}_\epsilon \|_{H^1(\Omega_\epsilon)} = O(\epsilon^{1/2})$ (just proved).

3. The general case: Let us assume that the estimate $\| D^r \tilde{u}(\alpha, \beta, \lambda, \epsilon) \|_{H^1(\Omega_\epsilon)} = O(\epsilon^{1/p})$ holds for every $p > 2$ and $0 \leq j \leq k - 1$. By Proposition 7.1, $\| D^r \tilde{u} \|_{L^2(\Omega_\epsilon)} \leq M_p \epsilon^{1/p - 1/2} \| D^j \tilde{u} \|_{H^1(\Omega_\epsilon)}$, and $\| w_\epsilon - w_0 \|_{L^p(\Omega_\epsilon)} = O(\epsilon^{1/p})$ (this follows from the facts $\| w_\epsilon \|_{L^p(\Omega_\epsilon)} = O(\epsilon)$, $\tilde{w}_0 = O(\epsilon)$ and the estimates in $L^2(\tilde{R}_\epsilon)$ obtained in the proof of Proposition 7.1). Hence $\| w_\epsilon \|_{L^p(\Omega_\epsilon)}$ is bounded as $\epsilon \to 0$ for any fixed $p > 2$ and, for $j = 1, \| D \tilde{u} \|_{L^p(\Omega_\epsilon)} = O(\epsilon^{1/p})$. 
Let us fix \( \bar{p} > 2 \). The general form of a derivative of order \( k \) (which we represent by \( D^k u \)) is
\[
D^k \hat{u} = \tilde{K}_\epsilon (I - P_\epsilon)[\sum g_\epsilon + f_\epsilon(\lambda, \alpha + \beta w_\epsilon + \hat{u}) D^k \hat{u}],
\]
where the \( g_\epsilon \)'s are terms of the type
\[
f_\lambda \ldots u_\ldots u(\lambda, \alpha + \beta w_\epsilon + \hat{u})(1 + D_\alpha \hat{u})^{m_1}(w_\epsilon + D_\beta \hat{u})^{m_2}(D_{\alpha_1} \hat{u})^{s_1}(D_{\beta_2} \hat{u})^{s_2}(D_{\alpha_3} \hat{u})^{s_3},
\]
where \( m_1 + m_2 + r_1 s_1 + r_2 s_2 + r_3 s_3 = k \).

(a) Terms with \( r_1 s_1 + r_2 s_2 + r_3 s_3 \geq 1 \), say \( r_2 s_2 \geq 1 \). We may write
\[
g_\epsilon = (D^2_{\beta_2} \hat{u}) \tilde{g}_\epsilon \quad \text{and} \quad \| g_\epsilon \|_{L^2(\Omega)} \leq \| D^2_{\beta_2} \hat{u} \|_{L^p(\Omega)} \| \tilde{g}_\epsilon \|_{L^p(\Omega)}
\]
where \( 1/p + 1/p_2 = \frac{1}{2} \).

Let \( \bar{p} > 2 \) be fixed and choose \( p_1, p_2^* \) such that \( 2 < p_1 < \bar{p} \) and \( 2 < p_2 < (\bar{p} - 1 + 1/p_2 + \frac{1}{2})^{-1} \), then, \( \| \tilde{g}_\epsilon \|_{L^p(\Omega)} \) is bounded (by the induction hypothesis) for any \( p_1 > 2 \) (observe that all the derivatives of \( \hat{u} \) that appear in \( g_\epsilon \) are of order \( \leq k - 1 \)); now,
\[
\| g_\epsilon \|_{L^2(\Omega)} \leq \| \tilde{g}_\epsilon \|_{L^p(\Omega)} \| D^2_{\beta_2} \hat{u} \|_{L^p(\Omega)}
\]
and \( \| D^2_{\beta_2} \hat{u} \|_{H^1(\Omega)} \leq M_{p_2}^* \leq \frac{1}{p} \) by the induction hypothesis. Therefore, with the choice of \( p_2, p_2^* \) above, there exists a constant \( M_{\bar{p}} \), depending only on \( \bar{p} \), such that \( \| g_\epsilon \|_{L^2(\Omega)} \leq M_{\bar{p}} \), where \( a = 1/p_2 - \frac{1}{2} + 1/p^* > 1/\bar{p} \).

(b) Terms with \( r_1 s_1 + r_2 s_2 + r_3 s_3 = 0 \). In this case we may apply Proposition 7.4(iii) with
\[
g_\epsilon = f_\lambda \ldots u_\ldots u(\lambda, \alpha + \beta w_\epsilon + \hat{u})(1 + D_\alpha \hat{u})^{m_1}(w_\epsilon + D_\beta \hat{u})^{m_2},
\]
\[
g_0 = f_\lambda \ldots u_\ldots u(\lambda, \alpha + \beta w_0) w_0^{m_2}.
\]

We have
\[
\| g_\epsilon \|_{L^2(\Omega)} \leq M_1 \| 1 + D_\alpha \hat{u} \|_{L^{p_1 m_1}(\Omega)} \| w_\epsilon + D_\beta \hat{u} \|_{L^{p_2 m_2}(\Omega)}
\]
\[
= M_1 [O(\epsilon^{1/p_1 m_1})]^{m_1} [O(\epsilon^{1/p_2 m_2})]^{m_2} = O(\epsilon^{1/2})
\]
since \( 1/p_1 + 1/p_2 = \frac{1}{2} \).
\[
\| g_\epsilon - g_0 \|_{L^2(\Omega)}
\]
\[
\leq M_2 \| f_\lambda \ldots u_\ldots u(\lambda, \alpha + \beta w_\epsilon + \hat{u}) - f_\lambda \ldots u_\ldots u(\lambda, \alpha + \beta w_0) w_0^{m_2} \|_{L^2(\Omega)}
\]
\[
+ M_2 \| f_\lambda \ldots u_\ldots u(\lambda, \alpha + \beta w_\epsilon + \hat{u})[(1 + D_\alpha \hat{u})^{m_1}(w_\epsilon + D_\beta \hat{u})^{m_2} - w_0^{m_2}] \|_{L^2(\Omega)}
\]
\[
\leq M_3 \| w_0^{m_2} \|_{L^{\infty}(\Omega)} \| \beta(w_\epsilon - w_0) + \hat{u} \|_{L^2(\Omega)}
\]
\[
+ M_4 \| (1 + D_\alpha \hat{u})^{m_1}[(w_\epsilon + D_\beta \hat{u})^{m_2} - w_0^{m_2}] + [(1 + D_\beta \hat{u})^{m_2} - 1] w_0^{m_2} \|_{L^2(\Omega)}.
\]
We know that \( \|\beta(w_e - w_0) + \hat{u}\|_{L^p(\Omega_0)} = O(\epsilon^{1/p}) \) \( (p > 2) \). Now,

\[
\|(1 + D_\alpha \hat{u})^{m_1}(w_e + D_\beta \hat{u})^{m_2} - w_0^{m_2}\|_{L^2(\Omega_0)} \\
= \left\| (1 + D_\alpha \hat{u})^{m_1}(w_e + D_\beta \hat{u} - w_0) \left[ \sum_{j=1}^{m_2-1} (w_e + D_\beta \hat{u})^j w_0^{m_2-j} \right] \right\|_{L^2(\Omega_0)}.
\]

We observe that \( (1 + D_\alpha \hat{u})^{m_1} \) and \( \sum_{j=1}^{m_2-1} (w_e + D_\beta \hat{u})^j w_0^{m_2-j} \) are bounded as \( \epsilon \to 0 \) in \( L^p(\Omega_0) \), for any \( p > 2 \), and the whole expression is majorized by

\[
\|(1 + D_\alpha \hat{u})^{m_1} \left[ \sum_{j=1}^{m_2-1} (w_e + D_\beta \hat{u})^j w_0^{m_2-j} \right] \|_{L^p(\Omega_0)} \|\beta(w_e - w_0) + \hat{u}\|_{L^p(\Omega_0)} \\
= O(\epsilon^{1/p}).
\]

Now take \( p_2 = \bar{p} \). The other term is handled similarly.

Adding all the terms and applying Proposition 7.4 we finally obtain

\[
\|D^k \hat{u}\|_{H^k(\Omega, \lambda)} = O(\epsilon^{1/p}) \quad \text{and the proof of (i) is finished.}
\]

In order to prove (ii), fix \( 2 < p^* < 2\bar{p}/(\bar{p} - 2) \) and apply Proposition 7.5 and (i) above for \( p = p^* \).

**Proof of Theorem 2.2.** With the help of the previous propositions, all we have to do in order to complete the proof of Theorem 2.2 is to observe that the kth-order derivatives of

\[
(\zeta_{\eta}) (\alpha, \beta, \lambda, \epsilon) = P_\epsilon f(\lambda, \alpha + \beta w_e + d(\alpha, \beta, \lambda, \epsilon))
\]

have the general form

\[
D^k(\zeta_{\eta}) = \sum P_\epsilon f_\alpha f_{\alpha + \beta w_e + \hat{u}}(1 + D_\alpha \hat{u})^{m_1}(w_e + D_\beta \hat{u})^{m_2} \cdot (D_{\alpha_1}^{{r_1}} \hat{u})^{s_1}(D_{\beta_2}^{{r_2}} \hat{u})^{s_2}(D_{\lambda_3}^{{r_3}} \hat{u})^{s_3}
\]

with \( m_1 + m_2 + r_1 s_1 + r_2 s_2 + r_3 s_3 = k \). By Proposition 7.5(ii),

\[
\lim_{\epsilon \to 0} D^k \left( \frac{G}{H} \right)(\alpha, \beta, \lambda, \epsilon) = \sum_{s_1 + s_2 + s_3 = 0} P_\epsilon f_\alpha f_{\alpha + \beta w_0} w_0^{m_2},
\]

which is precisely \( D^k(\zeta_{\eta})(\alpha, \beta, \lambda, 0) \). Since this limit is uniform for \( \alpha, \beta, \lambda \) in bounded sets, we have \( (\zeta_{\eta})(\alpha', \beta', \lambda', \epsilon) \to (\zeta_{\eta})(\alpha, \beta, \lambda, 0) \) as \( \alpha \to \alpha', \beta \to \beta', \lambda \to \lambda', \epsilon \to 0^- \). This concludes the proof.

**References**

2. D. G. Aronson and H. F. Weinberger, "Nonlinear diffusion in population genetics,


