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Isomorphism-free lexicographic enumeration of triangulated surfaces and 3-manifolds

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ABSTRACT

We present a fast enumeration algorithm for combinatorial 2- and 3-manifolds. In particular, we enumerate all triangulated surfaces with 11 and 12 vertices and all triangulated 3-manifolds with 11 vertices. We further determine all equivelar polyhedral maps on the non-orientable surface of genus 4 as well as all equivelar triangulations of the orientable surface of genus 3 and the non-orientable surfaces of genus 5 and 6.

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1. Introduction

Triangulations of manifolds with *few* vertices provide a valuable source of interesting and extremal testing examples for various conjectures and open problems in combinatorics, optimization, geometry, and topology:

- *How many vertices are needed to triangulate a given manifold?*
- *What do the face vectors of simplicial spheres and manifolds look like?*
- *Is a given triangulation of a sphere shellable, polytopal, does it satisfy the Hirsch conjecture?*
- *Is a given triangulated surface geometrically realizable as a polyhedron in 3-space?*
- *Do simplicial 2-spheres have polytopal realizations with small coordinates?*

According to Rado [66] and Moise [63], (closed, compact) 2- and 3-manifolds can always be triangulated as (finite) simplicial complexes. Moreover, triangulated 2- and 3-manifolds always are *combinatorial manifolds*, i.e., triangulated manifolds such that the links of all vertices are standard PL (i.e., piecewise linear) spheres. It immediately follows that triangulations of 2- and 3-manifolds *can be enumerated*: For any given positive integer n we can produce in a finite amount of time a complete list (up to combinatorial isomorphism) of all triangulated 2- respectively 3-manifolds with n vertices.

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For example, a conceptually simple, but highly inefficient enumeration approach would be to first generate all $2^{\binom{n}{3}}$ pure 2-dimensional (respectively all $2^{\binom{n}{4}}$ pure 3-dimensional) simplicial complexes. These complexes would then be tested in a second step to determine whether or not all links are triangulated circles (triangulated 2-spheres), which can be done purely combinatorially. In a third step, isomorphic copies of triangulations, i.e., copies that can be transformed into each other by relabeling the vertices, would be identified.

It is the second step which fails to work in higher dimensions: There is *no* algorithm to decide whether a given r-dimensional simplicial complex is a PL r-sphere if $r \geq 5$; cf. [83]. For $r = 4$ it is unknown whether there are algorithms to recognize PL 4-spheres. For $r = 3$ there *are* algorithms to recognize the 3-sphere (see [70,80,40,53,60,54]), however, all the known algorithms are exponential and hopeless to implement. Therefore, in principle, combinatorial 4-manifolds (with 3-dimensional links) can be enumerated, whereas the enumeration problem for combinatorial 5-manifolds is open, and there is no enumeration algorithm for combinatorial $(r + 1)$-manifolds for $r \geq 5$.

We are, of course, not only interested in the combinatorial types of triangulated manifolds: In an additional step we want to determine the topological types of the triangulations obtained by the enumeration. Algorithmically, it is easy to figure out the topological type of a triangulated surface (by computing its Euler characteristic and its orientability character). As mentioned before, there are algorithms to recognize the 3-sphere, and it is even possible to recognize Seifert manifolds [61]. For general 3-manifolds, however, there are no algorithmic tools available yet to determine their topological types (although Perelman's proof [65] of Thurston's geometrization conjecture [81] gives a complete classification of the geometric types of 3-manifolds). In particular, hyperbolic 3-manifolds are difficult to deal with. For triangulations with few vertices it turned out that heuristics (e.g., [7,55]) can be used for the recognition, thus allowing for a complete classification of the topological types of the examples obtained by the enumeration.

At present, there are three major enumeration approaches known to generate triangulated manifolds (see the overview [44]):

- *generation from irreducible triangulations* ([19,77,78], with the programs `plantri` [20] of Brinkmann and McKay and `surftri` [79] of Sulanke implementing this approach),
- *strongly connected enumeration* [52,3,8],
- and *lexicographic enumeration* [33,44,45,42,43].

A triangulation is *irreducible* if it has no *contractible edge*, i.e., if the contraction of any edge of the triangulations yields a simplicial complex, which is not homeomorphic to the original triangulation. According to Barnette and Edelson [6], every surface has only finitely many irreducible triangulations from which all other triangulations of the surface can be obtained by a suitable sequence of vertex splits. In this manner, triangulations of a particular surface with n vertices can be obtained in two steps by first generating all irreducible triangulations of the surface with up to n vertices, from which further triangulations with n vertices are obtained fast by vertex splits; see [19,77,78]. Unfortunately, every 3-manifold has infinitely many irreducible triangulations; cf. [30]. Even for surfaces, the generation of the finitely many irreducible triangulations is difficult, with complete lists available only for the 2-sphere, the 2-torus, the orientable surface of genus 2, and the non-orientable surfaces of genus up to 4; see [77,78] and the references contained therein.

Strongly connected enumeration, in particular, turned out to be successful for the enumeration of triangulated 3-manifolds with small edge degree [52], but is otherwise not very systematic.

The third approach, lexicographic enumeration, generates triangulations in *canonical form*, that is, for every fixed number n of vertices a lexicographically sorted list of triangulated manifolds is produced such that every listed triangulation with n vertices is the lexicographically smallest set of triangles (tetrahedra) combinatorially equivalent to this triangulation and is lexicographically smaller than the next manifold in the list.

In this paper, we present an improved version of the algorithm for lexicographic enumeration from [44]. The triangulations are now generated in an isomorphism-free way; see the next section for a detailed discussion. This improvement led to a substantial speed up of the enumeration. In particular, with the implementation `lextri` of the first author, we were able to enumerate all triangulated surfaces with 11 and 12 vertices (Section 3) and all triangulated 3-manifolds with 11 vertices (Section 5). Moreover, we enumerated all equivelar triangulations of the orientable surface of genus 3 and of the non-orientable surfaces of genus 4, 5, and 6 (Section 4).

2. Isomorphism-free enumeration

It is a standard problem with algorithms for the enumeration of particular combinatorial objects to avoid isomorphic copies of the objects as early as possible during their generation; see Read [67] and McKay [56] for a general discussion.

Our aim here is to give an isomorphism-free enumeration algorithm for triangulated surfaces and 3-manifolds with a fixed number n of vertices $1, 2, \ldots, n$. The algorithm is based on lexicographic enumeration as discussed in [44]. For simplicity, we describe the algorithm for surfaces, however, 3-manifolds can be generated in the same way. The basic ingredient of the algorithm is:

> Start with some triangle and add further triangles as long as no edge is contained in more than two triangles. If this condition is violated, then backtrack. A set of triangles is *closed* if each of its edges is contained in exactly two triangles. If the link of every vertex of a closed set of triangles is a circle, then this set of triangles gives a triangulated surface: OUTPUT surface.

From each equivalence class of combinatorially equivalent triangulations (with respect to relabeling the vertices) we list only the *canonical* triangulation, the labeled triangulation which has the lexicographically smallest set of triangles in this class. For every listed triangulation deg(1), the degree of vertex 1, must have minimum degree (since otherwise a lexicographically smaller set of triangles can be obtained by relabeling the vertices) and the triangulation must contain the triangles

$$123,\ 124,\ 135,\ \ldots,\ 1(\deg(1)-1)(\deg(1)+1),\ 1\deg(1)(\deg(1)+1).$$

We enumerate the canonical triangulations in *lexicographic order*, i.e., every listed triangulated surface is lexicographically smaller than the next surface in the list. With the objective to produce canonical triangulations we add the triangles during the backtracking in lexicographic order, that is, to the triangle 123 we first add 124 etc. to obtain a lexicographically ordered list of triangles.

We could wait until the list of triangles is a fully generated complex before testing whether or not there are other combinatorially equivalent triangulations with lexicographically smaller lists of triangles. However, we observe that, at each stage of adding triangles to obtain a canonical triangulation, the partial list of triangles is lexicographically at least as small as any list obtained by relabeling the vertices. We use this observation to prune the backtracking.

Whenever a new triangle is added to a partially generated complex, we test whether the new complex can be relabeled to obtain a lexicographically smaller labeling. If this is possible, then the new partial complex will not lead to a canonical triangulation and we backtrack.

If there is a closed vertex v such that deg(v) $<$ deg(1), then there is a lexicographically smaller labeling; otherwise, we search for such relabelings by

- going through all closed vertices, v, for which $\deg(v) = \deg(1)$,
- and for each edge vw we relabel v as 1 and w as 2,
- thereafter we relabel the two vertices adjacent to the edge vw to be 3 and 4 (two choices).
- Then we can extend the new labeling in a lexicographic smallest way.

Table 1 displays the backtracking in the case of $n = 6$ vertices. As a simplifying step we start not only with the triangle 123, but with the smallest possible completed vertex-star of size 3 of the vertex 1, i.e., with the triangles $123 + 124 + 134$. The next smallest triangle is 234 which closes the surface. However, the resulting surface (the boundary of the tetrahedron) has $4 < 6 = n$ vertices and is therefore discarded.

Let K be a partial complex and let k be the smallest vertex of K for which its vertex-star is not closed. Since we add the new triangles in lexicographic order, the next triangle to be added necessarily contains the vertex k. In particular, the intersection of the new triangle with the current partial complex is not empty. Therefore, *every partial complex is connected*.

In a new triangle *klm* at most the vertex m has not yet been used as a vertex in the partial complex K. (In the current vertex-star of k in K there are at least two non-closed edges, say, *kr* and *ks*. Suppose l and

Table 1
Backtracking steps in the case of $n = 6$ vertices.

Faces	Incomplete vertices					Reason for backtrack
123 + 124 + 134	2	3	4	5	6	
123 + 124 + 134 + 234				5	6	Surface complete
123 + 124 + 134	2	3	4	5	6	
123 + 124 + 134 + 235	2	3	4	5	6	
123 + 124 + 134 + 235 + 245		3	4	5	6	
123 + 124 + 134 + 235 + 245 + 345					6	Surface complete
123 + 124 + 134 + 235 + 245		3	4	5	6	
123 + 124 + 134 + 235 + 245 + 346		3	4	5	6	
123 + 124 + 134 + 235 + 245 + 346 + 356			4	5	6	
123 + 124 + 134 + 235 + 245 + 346 + 356 + 456						*Surface complete*!
123 + 124 + 134 + 235	2	3	4	5	6	
123 + 124 + 134 + 235 + 246	2	3	4	5	6	
123 + 124 + 134 + 235 + 246 + 256		3	4	5	6	
123 + 124 + 134 + 235 + 246 + 256 + 345			4	5	6	Relabeling is smaller
123 + 124 + 134 + 235 + 246 + 256		3	4	5	6	
123 + 124 + 134 + 235 + 246 + 256 + 346		3		5	6	Relabeling is smaller
123 + 124 + 134	2	3	4	5	6	
123 + 124 + 135 + 145	2	3	4	5	6	
123 + 124 + 135 + 145 + 234		3	4	5	6	Degree of 2 too small
123 + 124 + 135 + 145	2	3	4	5	6	
123 + 124 + 135 + 145 + 235	2		4	5	6	Degree of 3 too small
123 + 124 + 135 + 145	2	3	4	5	6	
123 + 124 + 135 + 145 + 236	2	3	4	5	6	
123 + 124 + 135 + 145 + 236 + 245	2	3		5	6	Degree of 4 too small
123 + 124 + 135 + 145 + 236	2	3	4	5	6	
123 + 124 + 135 + 145 + 236 + 246		3	4	5	6	
123 + 124 + 135 + 145 + 236 + 246 + 345		3	4		6	Degree of 5 too small
123 + 124 + 135 + 145 + 236 + 246		3	4	5	6	
123 + 124 + 135 + 145 + 236 + 246 + 356			4	5	6	
123 + 124 + 135 + 145 + 236 + 246 + 356 + 456						*Surface complete*!
123 + 124 + 135 + 145	2	3	4	5	6	
123 + 124 + 135 + 146 + 156	2	3	4	5	6	
123 + 124 + 135 + 146 + 156 + 234		3	4	5	6	Degree of 2 too small
123 + 124 + 135 + 146 + 156	2	3	4	5	6	
123 + 124 + 135 + 146 + 156 + 235	2		4	5	6	Degree of 3 too small
123 + 124 + 135 + 146 + 156	2	3	4	5	6	
123 + 124 + 135 + 146 + 156 + 236	2	3	4	5	6	
123 + 124 + 135 + 146 + 156 + 236 + 245	2	3	4	5	6	
123 + 124 + 135 + 146 + 156 + 236 + 245 + 256		3	4	5	6	
123 + 124 + 135 + 146 + 156 + 236 + 245 + 256 + 345		3	4		6	
123 + 124 + 135 + 146 + 156 + 236 + 245 + 256 + 345 + 346						*Surface complete*!
123 + 124 + 135 + 146 + 156 + 236	2	3	4	5	6	
123 + 124 + 135 + 146 + 156 + 236 + 246		3		5	6	Degree of 2 too small
123 + 124 + 135 + 146 + 156	2	3	4	5	6	

m have not yet been used in K and suppose krx is the triangle that closes the edge kr. Since $r, s < l, m$, it follows that krx is lexicographically smaller than klm and is thus added to K first, contradiction.) If m is a new vertex and l is an existing vertex smaller than any neighbor of k on an unclosed edge, then klm intersects K only in the vertices k and l. Thus, klm has no neighboring triangle in K. In other words, klm (temporarily) forms a new strongly connected component. For example, let the partial complex K consist of the triangles

123, 124, 135, 145, 236, 246,

to which the new component

347, 348

is added (see Fig. 1), which then is connected to the first component and closed to a triangulation of $\mathbb{R}\mathbf{P}^2$ by the triangles

357, 368, 458, 467, 567, 568.

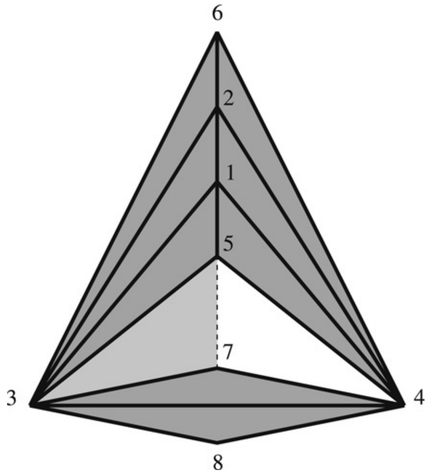

Fig. 1. Two strongly connected components that are joined by the triangle 357.

Thus, *partial complexes are not necessarily strongly connected.*

Proposition 1. *Partial complexes are strongly connected upon completion of the link of a vertex.*

Proof. Let k be the smallest vertex for which its vertex-star is not closed. We want to show that at the time the last triangle is added to close the star of k the resulting partial complex is strongly connected. For this, we proceed by induction on k.

First, we close the star of 1, which is a disc and therefore strongly connected. We next assume that the partial complex which is obtained after closing the star of the vertex k is strongly connected. Let $m > k$ be the next smallest vertex for which its vertex-star is not yet closed. At the time we will have closed the star of m, the star of m is a disc (since otherwise we would discard the respective partial complex). Since, by the induction hypothesis, the partial complex after closing the star of k was strongly connected, it follows that the partial complex after closing the star of m is also strongly connected (because the star of m contains at least one triangle that was present in the previous partial complex). $\square$

Let again k be the smallest vertex for which its vertex-star is not closed, K be the current partial complex, and klm be the next triangle that is added to K. In order for klm to start a new strongly connected component, l has to be a vertex of the boundary of K that is not (yet) adjacent to k and that is (by lexicographic minimality) smaller than all other vertices of the boundary of K to which k is already adjacent. The vertex m might be a vertex of the boundary of K or not. If m is a boundary vertex, then for klm to start a new strongly connected component the vertices k, l, and m are not pairwise adjacent in K. If m is not a boundary vertex, then we can choose (by lexicographic minimality) $m = |V(K)| + 1$, with $V(K)$ the vertex set of K. The next triangle to be added to $K + kl(|V(K)| + 1)$ is $kl(|V(K)| + 2)$. The resulting strongly connected component $kl(|V(K)| + 1) + kl(|V(K)| + 2)$ cannot grow further. In the next step, either yet another strongly connected component is started or the first strongly connected component is extended or joined to a later strongly connected component.

If during the enumeration of all 3-manifolds with n vertices the tetrahedron $klmr$ starts a new strongly connected component, then again k is the currently smallest vertex for which its vertex-star is not closed and l belongs to the boundary of K, but is not adjacent to k in K. There are three cases for the vertices m and r. Either both belong to the boundary of K, in which case k, l, m, and r are not pairwise adjacent in K, or only m belongs to the boundary of K, in which case k, l, and m are not pairwise adjacent in K and $r = |V(K)| + 1$, or both m and r do not belong to the boundary of K and $m = |V(K)| + 1$ and $r = |V(K)| + 2$. Thus, the new strongly connected component consists after its completion either of only the tetrahedron $klmr$, of the two tetrahedra $klm(|V(K)| + 1)$ and $klm(|V(K)| + 2)$, or of the join

Table 2
Total numbers of triangulated surfaces with up to 12 vertices.

n	Types
4	1
5	1
6	3
7	9
8	43
9	655
10	42 426
11	11 590 894
12	12 561 206 794

of the edge kl with a circle that consists of s edges $(|V(K)|+1)(|V(K)|+2)$, $(|V(K)|+1)(|V(K)|+3)$, $(|V(K)|+2)(|V(K)|+4), \ldots, (|V(K)|+s-1)(|V(K)|+s)$, where $|V(K)|+s \leq n$.

Proposition 1 and the above analysis of the strongly connected components explain why isomorphism-free lexicographic enumeration is fast, but not as fast as the generation of triangulations from irreducible triangulations.

In the latter approach one starts with the (finite) set of irreducible triangulations of a surface from which triangulations with more vertices are obtained by successive vertex-splitting. During this process the resulting complexes are always proper triangulations of the initial surface. In other words, we stay within the class of triangulations of the surface.

In the lexicographic approach, the partial complexes do not necessarily need to be strongly connected during the completion of the vertex-star of the pivot vertex k. The possibility of more than one strongly connected component leads to a "combinatorial explosion" of the number of choices during the completion of the vertex-star k. Fortunately, the partial complexes become strongly connected upon the completion of the vertex-star of k. Thus the combinatorial explosion happens locally, but not globally. Also, say, in the enumeration of triangulated 3-manifolds, upon the completion of the vertex-star of k we can detect whether the link of k is indeed a triangulated 2-sphere (or some other triangulated 2-manifold, in which case we discard the respective partial complex).

In the following sections we present our enumeration results and corollaries thereof. In particular, we enumerated all triangulated surfaces with 11 and 12 vertices and all triangulated 3-manifolds with 11 vertices.

The algorithm was implemented as C programs which were executed on a cluster of 2 GHz processors. The total cpu time required was 20 min to generate the surfaces with 11 vertices, 17 days for the surfaces with 12 vertices, and 170 days for the 3-manifolds with 11 vertices. See [47] for the program sources and lists of the examples.

3. Triangulated surfaces with 11 and 12 vertices

By Heawood's bound [35], at least $n \geq \left\lceil \frac{1}{2}(7+\sqrt{49-24\chi(M)}) \right\rceil$ vertices are needed to triangulate a (closed) surface of Euler characteristic $\chi(M)$. As shown by Ringel [68] and Jungerman and Ringel [39], this bound is tight, except in the cases of the orientable surface of genus 2, the Klein bottle, and the non-orientable surface of genus 3, for each of which an extra vertex has to be added.

Triangulations of surfaces with up to 8 vertices were classified by Datta [24] and Datta and Nilakantan [26]. By using (mixed) lexicographic enumeration, the second author obtained all triangulations of surfaces with 9 and 10 vertices [44].

We continued the enumeration with the isomorphism-free approach to lexicographic enumeration and were able to list all triangulated surfaces with 11 and 12 vertices. (Recently, Amendola [4] independently generated all triangulated surfaces with 11 vertices by using genus-surfaces and isomorphism-free mixed-lexicographic enumeration.)

Theorem 2. *There are precisely* 11 590 894 *(combinatorially distinct) triangulated surfaces with* 11 *vertices and there are exactly* 12 561 206 794 *triangulated surfaces with* 12 *vertices.*

Table 3
Numbers of triangulated surfaces with 11 vertices.

Genus	Orientable	Non-orientable
0	1 249	–
1	37 867	11 719
2	113 506	86 968
3	65 878	530 278
4	821	1628 504
5	–	3355 250
6	–	3623 421
7	–	1834 160
8	–	295 291
9	–	5 982

Table 4
Numbers of triangulated surfaces with 12 vertices.

Genus	Orientable	Non-orientable
0	7 595	–
1	605 496	114 478
2	7 085 444	1 448 516
3	25 608 643	16 306 649
4	14 846 522	99 694 693
5	751 593	473 864 807
6	59	1479 135 833
7	–	3117 091 975
8	–	3935 668 832
9	–	2627 619 810
10	–	711 868 010
11	–	49 305 639
12	–	182 200

The total numbers of triangulated surfaces with up to 12 vertices are given in Table 2. The numbers of triangulated surfaces with 11 and with 12 vertices are listed in detail in Tables 3 and 4, respectively.

Corollary 3. *There are* 821 *vertex-minimal triangulations of the orientable surface of genus* 4*, and there are* 295 291 *and* 5982 *vertex-minimal triangulations of the non-orientable surfaces of genus* 8 *and* 9*, respectively, with* 11 *vertices.*

With a local search, Altshuler [1] found 59 vertex-minimal *neighborly* triangulations (i.e., with complete 1-skeleton) of the orientable surface of genus 6 with 12 vertices and 40 615 neighborly triangulations with 12 vertices of the non-orientable surface of genus 12. For the orientable surface of genus 6 it was shown by Bokowski [3,8] that Altshuler's list of 59 vertex-minimal examples is complete. For the non-orientable surface of genus 12, the 40 615 examples of Altshuler make up roughly one quarter of the exact number of 182 200 vertex-minimal triangulations of this surface with 12 vertices.

Corollary 4. *There are* 751 593 *vertex-minimal triangulations of the orientable surface of genus* 5*, and there are* 711 868 010, 49 305 639*, and* 182 200 *vertex-minimal triangulations of the non-orientable surfaces of genus* 10, 11*, and* 12*, respectively, with* 12 *vertices.*

The 182 200 vertex-minimal triangulations of the non-orientable surface of genus 12 with 12 vertices were previously generated by Ellingham and Stephens [33]: They used a modified isomorphism-free lexicographic enumeration for the generation of all neighborly triangulations with 12 and 13 vertices. (There are 243 088 286 neighborly triangulations of the non-orientable surface of genus 15 with 13 vertices [33].)

Every 2-dimensional simplicial complex (with n vertices) is polyhedrally embeddable in $\mathbb{R}^5$, as it can be realized as a subcomplex of the boundary complex of the cyclic polytope $C(n, 6)$; cf. Grünbaum [34, Ex. 25, p. 67].

However, not all triangulations of orientable surfaces are geometrically realizable in $\mathbb{R}^3$, i.e., with straight edges, flat triangles, and without self-intersections: Bokowski and Guedes de Oliveira [11] showed that one of the 59 neighborly triangulations of the orientable surface of genus 6 is *not* realizable in 3-space. Recently, Schewe [71] proved non-realizability in $\mathbb{R}^3$ for all the 59 examples. Schewe further showed that for every orientable surface of genus $g \geq 5$ there are triangulations that cannot be realized in $\mathbb{R}^3$.

Realizations for all vertex-minimal triangulations of the orientable surfaces of genus 2 and 3 from [44] were obtained in [8,36,44], and realizations of these triangulations with small coordinates in [37,38]; see [36] for additional comments and further references on realizability.

The 821 vertex-minimal triangulations of the orientable surface of genus 4 from our enumeration were all found to be realizable [36] as well as at least 15 of the 751 593 vertex-minimal triangulations of the orientable surface of genus 5 with 12 vertices. These results in combination with the results of Schewe [71,72] led to:

Conjecture 5 (*Hougardy, Lutz, and Zelke, [36]*). *Every triangulation of an orientable surface of genus* $1 \leq g \leq 4$ *is geometrically realizable.*

4. Equivelar surfaces

A particularly interesting class of triangulated surfaces are *equivelar simplicial maps*, i.e., triangulations for which all vertices have the same vertex-degree q. Equivelar simplicial maps are also called *degree regular triangulations* or *equivelar triangulations*.

In general, let a *map* on a surface M be a decomposition of M into a finite cell complex and let G be the 1-skeleton of the map on M. The graph G of the map may have multiple edges, loops, vertices of degree 2, or even vertices of degree 1; for example, the embedding of a tree with n vertices and $n-1$ edges on S^2 decomposes the 2-sphere into one polygon with $2n-2$ edges, which are identified pairwise. (Sometimes the graphs of maps are required to be connected finite simple graphs, sometimes multiple edges are allowed but no loops, and vertices are often required to have at least degree 3; see [18,22,69,82].) A map is *equivelar of type* $\{p, q\}$ if M is decomposed into p-gons only with every vertex having degree q; cf. [58,59]. A map is *polyhedral* if the intersection of any two of its polygons is either empty, a common vertex, or a common edge; see the surveys [17,18]. An *equivelar polyhedral map* is a map which is both equivelar and polyhedral.

A map is *regular* if it has a flag-transitive automorphism group. Regular maps therefore provide highly symmetric examples of equivelar maps; see [21,22,85]. Vertex-transitive maps and neighborly triangulations are further classes of equivelar surfaces that have intensively been studied in the literature; cf. [1,3,33,39,42,68].

Equivelar simplicial maps (as simplicial complexes) always are polyhedral. By double counting of incidences between vertices and edges as well as between edges and triangles, we have

$$nq = 2f_1 = 3f_2 \tag{1}$$

for equivelar triangulations, with f_1 and f_2 denoting the numbers of edges and 2-faces, respectively. By Euler's equation, we further have that

$$\chi(M) = n - f_1 + f_2 = n - \frac{nq}{2} + \frac{nq}{3} = \frac{n(6-q)}{6}, \tag{2}$$

or equivalently,

$$q = 6 - \frac{6\chi(M)}{n}. \tag{3}$$

Since q is a positive integer, it follows that n has to be a divisor of $6|\chi(M)|$ if $\chi(M) \neq 0$. In particular, a surface M of Euler characteristic $\chi(M) \neq 0$ has only finitely many equivelar triangulations. Table 5 displays the possible values of (n, q) for surfaces with $\chi(M) \geq -10$.

In the case of neighborly triangulations we have $q = n - 1$ and therefore $\chi(M) = \frac{n(7-n)}{6}$. It follows that $n \equiv 0, 1, 3, 4 \mod 6$, where $n \geq 4$. In the case $n = 6k$ we have Euler characteristic

Table 5
Possible values of (n, q) for equivelar triangulations with $\chi(M) \geq -10$.

$\chi(M)$	(n, q)
2	(4,3), (6,4), (12,5)
1	(6,5)
0	$(n, 6)$, with $n \geq 7$
−1	–
−2	(12, 7)
−3	(9, 8), (18, 7)
−4	(12, 8), (24, 7)
−5	(10, 9), (15, 8), (30, 7)
−6	(12, 9), (18, 8), (36, 7)
−7	(14, 9), (21, 8), (42, 7)
−8	(12, 10), (16, 9), (24, 8), (48, 7)
−9	(18, 9), (27, 8), (54,7)
−10	(12, 11), (15, 10), (20, 9), (30, 8), (60, 7)

Table 6
Numbers of equivelar triangulations of the torus with up to 100 vertices.

k\Vertices	$10k+1$	$10k+2$	$10k+3$	$10k+4$	$10k+5$	$10k+6$	$10k+7$	$10k+8$	$10k+9$	$10(k+1)$
0	–	–	–	–	–	–	1	1	2	1
1	1	4	2	2	4	5	2	5	3	6
2	6	4	3	11	5	5	7	9	4	11
3	5	11	8	7	8	16	6	8	10	16
4	6	15	7	13	14	10	7	24	10	14
5	12	16	8	19	12	21	14	13	9	30
6	10	14	19	23	14	23	11	20	16	23
7	11	36	12	17	22	23	16	27	13	34
8	21	19	13	40	18	20	20	31	14	39
9	20	27	22	22	20	47	16	27	27	37

Table 7
Numbers of equivelar triangulations of the Klein bottle with up to 100 vertices.

k\Vertices	$10k+1$	$10k+2$	$10k+3$	$10k+4$	$10k+5$	$10k+6$	$10k+7$	$10k+8$	$10k+9$	$10(k+1)$
0	–	–	–	–	–	–	–	–	1	1
1	–	3	–	1	3	2	–	4	–	4
2	3	1	–	7	2	1	3	4	–	8
3	–	4	3	1	4	9	–	1	3	8
4	–	8	–	4	7	1	–	11	2	5
5	3	4	–	8	4	8	3	1	–	15
6	–	1	7	6	4	8	–	4	3	9
7	–	15	–	1	7	4	4	8	–	12
8	5	1	–	15	4	1	3	8	–	16
9	4	4	3	1	4	15	–	5	7	10

$\chi(M) = -6k^2 + 7k$, and if $n = 6k + 1$ then $\chi(M) = -6k^2 + 5k + 1$. If $n = 6k + 3$ we have $\chi(M) = -6k^2 + k + 2$, and if $n = 6k + 4$ then $\chi(M) = -6k^2 - k + 2$.

Equivelar triangulations with up to 11 vertices were classified by Datta and Nilakantan [27]: there are 27 such examples. Datta and Upadhyay [28] continued the classification of equivelar triangulations for the torus and the Klein bottle for up to 15 vertices. (Constructions of equivelar maps on the torus together with bounds on their number were given in [2]; for equivelar maps on the Klein bottle see [64].) All equivelar polyhedral maps on the torus are vertex-transitive [16,28]. By isomorphism-free lexicographic enumeration, we obtained all equivelar triangulations of the torus and the Klein bottle for up to 100 vertices; see Tables 6 and 7.

Theorem 6. *There are exactly* 1357 *equivelar triangulations of the torus and* 364 *equivelar triangulations of the Klein bottle with up to* 100 *vertices, respectively.*

Table 8
Numbers of simplicial equivelar maps with up to 12 vertices.

Vertices	Orient.	Genus	Types	Vertices	Orient.	Genus	Types
4	+	0	1	12	+	0	1
6	+	0	1			1	4
	−	1	1			2	6
7	+	1	1			3	34
8	+	1	1			4	112
9	+	1	2			5	103
	−	2	1			6	59
		5	2		−	2	3
10	+	1	1			4	28
	−	2	1			6	500
		7	14			8	9 273
11	+	1	1			10	48 591
						12	182 200

Recently, Brehm and Kühnel [16] gave a detailed description of all equivelar triangulations of the torus. In particular, they obtained an explicit formula for the number $T(n)$ of equivelar triangulations with n vertices (as well as for the number $Q(n)$ of equivelar polyhedral quadrangulations with n vertices).

As observed by Datta and Upadhyay [28], there is an n-vertex equivelar triangulation of the Klein bottle if and only if $n \geq 9$ is not prime.

Moreover, Datta and Upadhyay [29] determined that there are exactly six equivelar triangulations of the orientable surface of genus 2 with 12 vertices. As a consequence of Theorem 2:

Corollary 7. *There are precisely* 240 914 *equivelar triangulations with* 12 *vertices.*

Table 8 lists the numbers of simplicial equivelar maps with up to 12 vertices.

For an equivelar *polyhedral* map of type $\{p, q\}$ the same computation as in Eq. (2) gives

$$\chi(M) = n - f_1 + f_2 = n - \frac{nq}{2} + \frac{nq}{p} = nq\left(\frac{1}{p} + \frac{1}{q} - \frac{1}{2}\right). \tag{4}$$

Thus, the sign of $\chi(M)$ is determined by the sign of $\frac{1}{p} + \frac{1}{q} - \frac{1}{2}$, and vice versa.

If $\frac{1}{p} + \frac{1}{q} - \frac{1}{2} > 0$, then the only possible $\{p, q\}$-pairs are $\{3, 3\}$, $\{3, 4\}$, $\{3, 5\}$, $\{4, 3\}$, and $\{5, 3\}$ for S^2 with $\chi(S^2) = 2$, with the boundaries of the tetrahedron, the octahedron, the icosahedron, the cube, and the dodecahedron as the unique occurring examples, respectively, and $\{3, 5\}$, $\{5, 3\}$ for $\mathbb{R}\mathbf{P}^2$, with the vertex-minimal 6-vertex triangulation of $\mathbb{R}\mathbf{P}^2$ and its combinatorial dual as the only examples.

If $\frac{1}{p} + \frac{1}{q} - \frac{1}{2} = 0$, then there are infinitely many triangulations, quadrangulations, and hexangulations corresponding to the pairs $\{3, 6\}$, $\{4, 4\}$, and $\{6, 3\}$, respectively; see Brehm and Kühnel [16] for more details.

In the case $\frac{1}{p} + \frac{1}{q} - \frac{1}{2} < 0$ we write Eq. (4) as

$$q = \frac{n - \chi(M)}{n} \cdot \frac{2p}{p - 2}, \tag{5}$$

where p and q are positive integers greater than or equal to 3.

For a given surface M of Euler characteristic $\chi(M) < 0$ we next determine all triples $(p, q; n)$ which are admitted by Eq. (5). Every equivelar polyhedral map has at least one p-gon with p vertices, i.e., we always have $n \geq p$. Furthermore, a vertex has q distinct neighbors, which implies $n \geq q + 1$. The combinatorial dual of an equivelar polyhedral map of type $(p, q; n)$ is an equivelar polyhedral map of type $(q, p; \frac{nq}{p})$. Moreover, in an equivelar polyhedral map of type $(p, q; n)$ the star of any vertex contains $q(p - 3) + q + 1 = q(p - 2) + 1$ distinct vertices, from which $n \geq q(p - 2) + 1 \geq 4(p - 2) + 1 = 4p - 7 > 2p$ follows for $q \geq p \geq 4$. In the case $p = 3$ we have $q \geq 7$ for surfaces with $\chi(M) < 0$ and therefore also $n > 2p$. If $q < p$, then for the dual maps of type $(q, p; \frac{nq}{p})$ we have $\frac{nq}{p} > 2q$, and thus again $n > 2p$ for the maps of type $(p, q; n)$.

From Eq. (5) we see that n is a divisor of $(n - \chi(M))2p$ and therefore a divisor of $2|\chi(M)|p$. Let a be the gcd of n and p, and let k and l be positive integers such that $n = ka$ and $p = la$. It follows from $n \mid 2|\chi(M)|p$ that $k \mid 2|\chi|$ and from $n > 2p$ that $k > 2l$. Thus $3 \leq k \leq 2|\chi|$ and $1 \leq l \leq \lfloor \frac{k-1}{2} \rfloor$, that is, k and l can take only finitely many distinct values, where $k \mid 2|\chi|$ and $\gcd(k, l) = 1$. From

$$q = \frac{ka - \chi(M)}{ka} \cdot \frac{2la}{la - 2} = \frac{2la + \frac{2l|\chi(M)|}{k}}{la - 2} = 2 + \frac{4 + \frac{2l|\chi(M)|}{k}}{la - 2}, \tag{6}$$

we see that there are only finitely many choices for a. It follows, in particular, that for given M with $\chi(M) < 0$ there are only finitely many equivelar polyhedral maps on M.

If $\chi(M) = -1$, then there are no admissible triples $(p, q; n)$. Thus, there are no equivelar polyhedral maps on the non-orientable surface of genus 3.

For $\chi(M) = -2$ the admissible triples are $(3, 7; 12)$ and $(7, 3; 28)$. Altogether, there are 12 examples of equivelar polyhedral maps on the orientable surface of genus 2 (the six simplicial examples from above and their simple duals); see [29].

Corollary 8. *There are exactly* 56 *equivelar polyhedral maps on the non-orientable surface of genus* 4, 28 *of type* $(3, 7; 12)$ *and* 28 *of type* $(7, 3; 28)$.

None of the examples of equivelar polyhedral maps with $\chi(M) = -2$ is regular.

For $\chi(M) = -3$ the admissible triples are $(3, 8; 9)$, $(8, 3; 24)$, $(3, 7; 18)$, $(7, 3; 42)$, $(4, 5; 12)$, and $(5, 4; 15)$.

Theorem 9. *There are precisely* 1403 *equivelar triangulations of the non-orientable surface of genus* 5*, two with* 9 *vertices and* 1401 *with* 18 *vertices.*

Furthermore, there are 4 equivelar polyhedral maps on the non-orientable surface of genus 5 of type $(4, 5; 12)$ [51]. One of these examples is regular; cf. [85, p. 134].

In the case $\chi(M) = -4$ we have the possibilities $(3, 8; 12)$, $(8, 3; 32)$, $(3, 7; 24)$, $(7, 3; 56)$, $(4, 5; 16)$, and $(5, 4; 20)$.

Theorem 10. *There are precisely* 11 301 *equivelar triangulations of the orientable surface of genus* 3, 24 *with* 12 *vertices and* 11 277 *with* 24 *vertices. Moreover, there are exactly* 601 446 *equivelar triangulations of the non-orientable surface of genus* 6500 *with* 12 *vertices and* 600 946 *with* 24 *vertices.*

Exactly two of the equivelar triangulations with $\chi(M) = -4$ are regular, Dyck's regular map [31,32,9,12,13,73,76,85] of type $(3, 8; 12)$ and Klein's regular map [41,74,76,85] of type $(3, 7; 24)$.

There are 363 equivelar polyhedral maps on the non-orientable surface of genus 6 of type $(4, 5; 16)$ [51] of which one is regular; cf. [85, p. 139]. Moreover, there are 43 equivelar polyhedral maps on the orientable surface of genus 3 of type $(4, 5; 16)$, none of these are regular [51].

For neighborly triangulations of orientable surfaces the genus g grows quadratically with the number of vertices n, i.e., $g = O(n^2)$. However, the boundary of the tetrahedron and Möbius' 7-vertex torus [62] are the only examples of neighborly triangulations of orientable surfaces for which polyhedral realizations in $\mathbb{R}^3$ are known [10,23]. In contrast, as mentioned above, all 59 neighborly triangulations of the orientable surface of genus 6 with 12 vertices are not realizable [71], and it is expected that also all neighborly triangulations of orientable surfaces with more vertices never are realizable.

McMullen, Schulz, and Wills [59] constructed polyhedral realizations in $\mathbb{R}^3$ of equivelar triangulations of genus $g = O(n \log n)$, which, asymptotically, is the highest known genus $g(n)$ for geometric realizations of polyhedral maps. McMullen, Schulz, and Wills also gave infinite families of geometric realizations of equivelar polyhedral maps of the types $\{4, q\}$ and $\{p, 4\}$. For further examples of geometric realizations of equivelar polyhedral maps of these types see [12,57,58,75,86].

It is not known whether there are geometric realizations of equivelar polyhedral maps of type $\{p, q\}$ for $p, q \geq 5$; cf. [18]. Examples of equivelar polyhedral maps of type $\{5, 5\}$ and of type $\{6, 6\}$

Table 9
Combinatorial 3-manifolds with up to 11 vertices.

Vertices\Types	S^3	$S^2 \underline{\times} S^1$	$S^2 \times S^1$	$\mathbb{R}\mathbf{P}^3$	All
5	1	–	–	–	1
6	2	–	–	–	2
7	5	–	–	–	5
8	39	–	–	–	39
9	1 296	1	–	–	1 297
10	247 882	615	518	–	249 015
11	166 564 303	3 116 818	2 957 499	30	172 638 650

were first given by Brehm [14]. An infinite series of $\{k, k\}$-equivelar polyhedral maps was constructed by Datta [25].

5. Combinatorial 3-manifolds with 11 vertices

The boundary of the 4-simplex triangulates the 3-sphere with 5 vertices, and, by work of Walkup [84], the twisted sphere product $S^2 \underline{\times} S^1$, the sphere product $S^2 \times S^1$, and the real projective 3-space $\mathbb{R}\mathbf{P}^3$ can be triangulated vertex-minimally with 9, 10, and 11 vertices, respectively, while all other 3-manifolds need at least 11 vertices for a triangulation. By a result of Bagchi and Datta [5], triangulations of $\mathbb{Z}_2$-homology spheres (different from S^3) require at least 12 vertices. In particular, at least 12 vertices are needed to triangulate the lens space $L(3, 1)$. A triangulation of $L(3, 1)$ with this number of vertices was first found by Brehm [15]. For further results on minimal numbers of vertices of triangulated 3-manifolds see [46].

Triangulations of 3-manifolds with up to 10 vertices were classified previously; see [45] and the references given there. With isomorphism-free lexicographic enumeration we were able to obtain all triangulations with 11 vertices.

Theorem 11. *There are precisely* 172 638 650 *triangulated* 3*-manifolds with* 11 *vertices.*

Table 9 lists the combinatorial and topological types of the triangulations with up to 11 vertices. The numbers of triangulations with 11 vertices are displayed in detail in Table 10.

Corollary 12. *Let M be a* 3*-manifold different from S^3, $S^2 \underline{\times} S^1$, $S^2 \times S^1$, and $\mathbb{R}\mathbf{P}^3$ (which can be triangulated with* 5, 9, 10, *and* 11 *vertices, respectively), then M needs at least* 12 *vertices for a triangulation.*

Corollary 13. *There are exactly* 30 *vertex-minimal triangulations of $\mathbb{R}\mathbf{P}^3$ with* 11 *vertices.*

Corollary 14. *Walkup's triangulation of $\mathbb{R}\mathbf{P}^3$ from [84] is the unique vertex- and facet-minimal triangulation of $\mathbb{R}\mathbf{P}^3$ with $f = (11, 51, 80, 40)$.*

Corollary 15. *The minimal number of vertices for triangulations of the orientable connected sum $(S^2 \times S^1)\#(S^2 \times S^1)$ and of the non-orientable connected sum $(S^2 \underline{\times} S^1)\#(S^2 \underline{\times} S^1)$ is* 12.

Examples of triangulations of the latter two manifolds with 12 vertices are given in [48]. It is conjectured in [48] that for other 3-manifolds, different from the mentioned six examples, at least 13 vertices are necessary for a triangulation.

In [45], all triangulated 3-spheres with up to 10 vertices and all resulting simplicial 3-balls with 9 vertices were examined with respect to shellability. The respective 3-spheres all turned out to be shellable, whereas 29 vertex-minimal examples of non-shellable 3-balls were discovered with 9 vertices; see also [49].

Corollary 16. *All triangulated* 3*-spheres with* 11 *vertices are shellable.*

The smallest known example of a non-shellable 3-sphere has 13 vertices [50]. We believe that there are no non-shellable 3-spheres with 12 vertices.

Table 10
Combinatorial 3-manifolds with 11 vertices.

f-vector\Types	S^3	$S^2 \underline{\times} S^1$	$S^2 \times S^1$	$\mathbb{R}\mathbf{P}^3$	All
(11, 34, 46, 23)	131				131
(11, 35, 48, 24)	859				859
(11, 36, 50, 25)	3 435				3 435
(11, 37, 52, 26)	11 204				11 204
(11, 38, 54, 27)	31 868				31 868
(11, 39, 56, 28)	82 905				82 905
(11, 40, 58, 29)	199 303				199 303
(11, 41, 60, 30)	447 245				447 245
(11, 42, 62, 31)	939 989				939 989
(11, 43, 64, 32)	1 850 501				1 850 501
(11, 44, 66, 33)	3 413 161	448	406		3 414 015
(11, 45, 68, 34)	5 888 842	3 627	3 521		5 895 990
(11, 46, 70, 35)	9 463 527	17 065	16 559		9 497 151
(11, 47, 72, 36)	14 091 095	54 928	53 839		14 199 862
(11, 48, 74, 37)	19 288 095	137 795	134 494		19 560 384
(11, 49, 76, 38)	23 946 497	278 899	272 671		24 498 067
(11, 50, 78, 39)	26 344 282	464 328	451 126		27 259 736
(11, 51, 80, 40)	24 835 145	626 441	603 950	1	26 065 537
(11, 52, 82, 41)	19 130 339	665 845	630 869	3	20 427 056
(11, 53, 84, 42)	11 240 196	525 104	486 378	6	12 251 684
(11, 54, 86, 43)	4 457 865	272 672	244 045	8	4 974 590
(11, 55, 88, 44)	897 819	69 666	59 641	12	1 027 138
Total:	166 564 303	3116 818	2957 499	30	172 638 650

Corollary 17. *There are* 1831 363 502 *triangulated* 3*-balls with* 10 *vertices of which* 277 479 *are non-shellable.*

For all triangulated 3-spheres with up to 9 vertices and all neighborly 3-spheres with 10 vertices a classification into polytopal and non-polytopal examples was carried out mainly by Altshuler, Bokowski, and Steinberg; see [44] for a survey and references.

Problem 18. Classify all simplicial 3-spheres with 10 and 11 vertices into polytopal and non-polytopal spheres.

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